NON-EXISTENCE OF LOW RANK ULRICH BUNDLES ON VERONESE VARIETIES

ANGELO FELICE LOPEZ AND DEBADITYA RAYCHAUDHURY

Abstract. We show that Veronese varieties of dimension $n \geq 4$ do not carry any Ulrich bundles of rank $r \leq 3$. In order to prove this, we prove that a Veronese embedding of a complete intersection of dimension $m \geq 4$, which if $m = 4$ is either $\mathbb{P}^4$ or has degree $d \geq 2$ and is very general and not of type $(2), (2, 2)$, does not carry any Ulrich bundles of rank $r \leq 3$.

1. Introduction

In the theory of vector bundles on a smooth irreducible variety $X \subset \mathbb{P}^N$, an open problem [ES] that has attracted attention lately is whether X carries an Ulrich bundle $\mathcal{E}$, that is such that $H^i(\mathcal{E}(-p)) = 0$ for $i \geq 0, 1 \leq p \leq \dim X$. Once existence is proved, the next question is what is the Ulrich complexity of $(X, \mathcal{O}_X(1))$, namely the lowest possible rank of an Ulrich bundle.

Perhaps one of the simplest but interesting cases to be considered is when $X = \mathbb{P}^n$ and the embedding in $\mathbb{P}^N$ is given by $\mathcal{O}_{\mathbb{P}^n}(a)$, for some integer $a \geq 1$. Existence of Ulrich bundles is known for high rank: $n!$ by [B, Thm. 3.1] and $a^{\binom{n}{2}}$ by [ES, Cor. 5.7]. As for lower rank, to set up the picture, let $\mathcal{E}$ be a rank r Ulrich bundle for $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a))$. If $a = 1$ it is well-known [ES, Prop. 2.1], [B, Thm. 2.3] that $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus r}$. Also, if $n = 1$ one easily sees that $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a-1)^{\oplus r}$. Things are different for $n \geq 2, a \geq 2$. First of all, one has that $r \geq 2$ and there are some strong numerical constraints, since (see for example [ES, Thm. 5.1]) we have that

$$\chi(\mathcal{E}(\ell)) = \frac{r}{n!}(\ell + a) \cdots (\ell + na) \in \mathbb{Z} \text{ for every } \ell \in \mathbb{Z}. \tag{1.1}$$

It follows by [ES, Cor. 5.3] that if p is any prime such that $p|a$ and $p^t|n!$, then $p^t|r$.

The necessary condition (1.1) is easily translated into $2|r(a-1)$ when $n = 2$ and $6|r(a^2-1)$ when $n = 3$. If $n = 2$, it follows by [CMR1, Thm. 1] (see also [CG, Thms. 6.1 and 6.2]) that it is in fact sufficient. If $n = 3$, it was conjectured in [CMR2, Conj. 1.1] that it is again sufficient and this has been recently proved in [FP, Thm. 1].

On the other hand, when $n \geq 4$, there seems to be an important difference. For example, consider the case $n = 4$. We get by [ES, Cor. 5.3] that $\gcd(a, 6) = 1$ for $r = 2$ and $\gcd(a, 2) = 1$ for $r = 3$. Similarly, if $n = 5$, we find that $\gcd(a, 30) = 1$ for $r = 2$ and $\gcd(a, 10) = 1$ for $r = 3$. But assuming that the latter non-divisibility conditions on a are satisfied, we have that (1.1) holds unconditionally.

Despite the fact that this seems to suggest that, when $n \geq 4$, Ulrich bundles of rank 2 or 3 might exist on Veronese varieties, we show that this is not the case. In fact we have:

Theorem 1. *Let $n \geq 4$ and $a \geq 2$. Then $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a))$ does not carry Ulrich vector bundles of rank $r \leq 3$.*

The strategy to prove the above theorem is to consider the Veronese embedding $v_a(\mathbb{P}^n) \subset \mathbb{P}^N$, take hyperplane sections and use the fact that the restriction of an Ulrich bundle to the hyperplane section remains Ulrich. One then gets an Ulrich bundle on a Veronese embedding of a complete intersection of type $(a, \dots, a)$ in $\mathbb{P}^n$. In order to handle these, we use Ulrich subvarieties (see Section 3) to show the following generalization of [LR2, Thm. 2]:

Theorem 2. *Let $s \geq 1, a \geq 2, m \geq 4$ and let $X \subset \mathbb{P}^{m+s}$ be a smooth m-dimensional complete intersection of hypersurfaces of degrees $(d_1, \dots, d_s)$ with $d_i \geq 1, 1 \leq i \leq s$ and degree d. Assume that one of the following holds:*

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(a) $m \geq 5$, *or*
(b) $m = 4$ *and* $d = 1$, *or*
(c) $m = 4, d \geq 2$, X *is very general and* $(d_1, \dots, d_s) \notin \{(2, \underbrace{1, \dots, 1}_{s\text{-}1}), (2, 2, \underbrace{1, \dots, 1}_{s\text{-}2})\}$ *(up to permutation).*

Then there are no rank $r \leq 3$ *Ulrich vector bundles with respect to* $(X, \mathcal{O}_X(a))$.

2. Preliminaries

2.1. Notation and conventions.

Throughout the paper we work over the complex numbers.

We will use the convention $\binom{\ell}{m} = \frac{\ell(\ell-1)\dots(\ell-m+1)}{m!}$ for $m \geq 1, \ell \in \mathbb{Z}$. Note that $\binom{-\ell}{m} = (-1)^m \binom{\ell+m-1}{m}$ and $\chi(\mathcal{O}_{\mathbb{P}^m}(\ell)) = \binom{\ell+m}{m}$.

2.2. Generalities on (Ulrich) vector bundles.

We will need the following statement on vanishing of cohomology.

Lemma 2.1. *Let* $a \geq 1$ *be an integer, let* $X \subset \mathbb{P}^N$ *be a smooth irreducible variety of dimension* $n \geq 1$ *and let* $\mathcal{F}, \mathcal{G}$ *be two vector bundles on* X. *We have:*
(i) *If* $H^0(\mathcal{G}(2)) = H^1(\mathcal{G}(1)) = 0$, *then* $H^1(\mathcal{G}) = 0$.
(ii) *If* $H^0(\mathcal{F}(-a)) = H^1(\mathcal{F}(-2a)) = 0$, *then* $H^1(\mathcal{F}(-j)) = 0$ *for all* $j \geq 2a$.

Proof. Let $Y \in |\mathcal{O}_X(1)|$. To see (i), observe that the exact sequence

$$0 \to \mathcal{G}(1) \to \mathcal{G}(2) \to \mathcal{G}(2)_{|Y} \to 0$$

implies that $H^0(\mathcal{G}(2)_{|Y}) = 0$. In particular we have that $\dim Y \geq 1$ and, since $H^0(\mathcal{G}(1)_{|Y}) \subseteq H^0(\mathcal{G}(2)_{|Y}) = 0$, we deduce that $H^0(\mathcal{G}(1)_{|Y}) = 0$. Then, the exact sequence

$$0 \to \mathcal{G} \to \mathcal{G}(1) \to \mathcal{G}(1)_{|Y} \to 0$$

implies that $H^1(\mathcal{G}) = 0$. This proves (i). We now show (ii) by induction on j. If $j = 2a$, then $H^1(\mathcal{F}(-j)) = 0$ by hypothesis. If $j \geq 2a+1$, set $\mathcal{G} = \mathcal{F}(-j)$. Then $H^1(\mathcal{G}(1)) = H^1(\mathcal{F}(-j+1)) = 0$ by induction. Also, since $-j+2 \leq 1-2a \leq -a$ we have that

$$H^0(\mathcal{G}(2)) = H^0(\mathcal{F}(-j+2)) \subseteq H^0(\mathcal{F}(-a)) = 0.$$

Therefore (i) implies that $H^1(\mathcal{F}(-j)) = 0$ and this proves (ii). □

We will often use the following well-known properties of Ulrich bundles.

Lemma 2.2. *Let* $X \subset \mathbb{P}^N, L = \mathcal{O}_X(1)$ *and let* $\mathcal{E}$ *be a rank* r *Ulrich bundle. We have:*
(i) $\mathcal{E}_{|Y}$ *is Ulrich on a smooth hyperplane section* Y *of* X.
(ii) *If* $n \geq 2$, *then* $c_2(\mathcal{E})L^{n-2} = \frac{1}{2}[c_1(\mathcal{E})^2 - c_1(\mathcal{E})K_X]L^{n-2} + \frac{r}{12}[K_X^2 + c_2(X) - \frac{3n^2+5n+2}{2}L^2]L^{n-2}$.

Proof. See for example [LR1, Lemma 3.2]. □

3. Ulrich subvarieties

Ulrich subvarieties were defined in [LR2]. We now recall some of the properties that they enjoy.

First, we give a simplified version of [LR2, Lemma 3.2], adapted to our purposes.

Lemma 3.1. *Let* $n \geq 2$, *let* $X \subset \mathbb{P}^N$ *be a smooth irreducible* n*-dimensional variety and let* $\mathcal{E}$ *be a rank* $r \geq 2$ *Ulrich bundle with* $\det \mathcal{E} = \mathcal{O}_X(D)$. *Then there is a subvariety* $Z \subset X$ *such that, if* $Z \neq \emptyset$, *we have:*
(i) $[Z] = c_2(\mathcal{E})$.
(ii) *If* $r = 2$, *then* $\omega_Z \cong \mathcal{O}_Z(K_X + D)$ *and* $c_2(Z) = c_2(X)_{|Z} - c_2(\mathcal{E})_{|Z} + K_Z^2 - K_Z K_{X|Z}$.
(iii) *If* $r = 3$, *then*
$$c_2(Z) = c_2(X)_{|Z} - c_2(\mathcal{E})_{|Z} - c_1(\mathcal{E})^2_{|Z} + K_Z K_{X|Z} - K^2_{X|Z} + 2K_Z c_1(\mathcal{E})_{|Z} - 2K_{X|Z} c_1(\mathcal{E})_{|Z}.$$

Proof. See [LR2, Lemma 3.2]. □

Next, we recall the statement of [LR2, Thm. 1]. Let $Z \subset X$ be a Cohen-Macaulay, pure codimension 2 subvariety and let D be a divisor on X. The short exact sequence

$$0 \to \mathcal{J}_{Z/X}(K_X + D) \to \mathcal{O}_X(K_X + D) \to \mathcal{O}_Z(K_X + D) \to 0$$

determines a coboundary map

$$\gamma_{Z,D} : H^{n-2}(\mathcal{O}_Z(K_X + D)) \to H^{n-1}(\mathcal{J}_{Z/X}(K_X + D))$$

whose dual, by Serre duality, is

$$\gamma^*_{Z,D} : \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{J}_{Z/X}(D), \mathcal{O}_X) \to H^0(\omega_Z(-K_X - D)).$$

Then we have

Theorem 3.2. *Let $X \subset \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 2$, degree $d \geq 2$ and let D be a divisor on X. Then $(X, \mathcal{O}_X(1))$ carries a rank $r \geq 2$ Ulrich vector bundle $\mathcal{E}$ with $\det \mathcal{E} = \mathcal{O}_X(D)$ if and only if there is a subvariety $Z \subset X$ such that:*

(a) *Z is either empty or of pure codimension 2,*
(b) *if $Z \neq \emptyset$ and either $r = 2$ or $n \leq 5$, then Z is smooth (possibly disconnected),*
(c) *if $Z \neq \emptyset$ and $n \geq 6$, then Z is either smooth or is normal, Cohen-Macaulay, reduced and with $\dim \operatorname{Sing}(Z) = n - 6$,*

and there is a $(r-1)$-dimensional subspace $W \subseteq \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{J}_{Z/X}(D), \mathcal{O}_X)$ such that the following hold:

(i) *If $Z \neq \emptyset$, then $\gamma^*_{Z,D}(W)$ generates $\omega_Z(-K_X - D)$ (that is the multiplication map $\gamma^*_{Z,D}(W) \otimes \mathcal{O}_Z \to \omega_Z(-K_X - D)$ is surjective).*
(ii) *$H^0(K_X + nH - D) = 0$.*
(iii) *$H^0(\mathcal{J}_{Z/X}(D - H)) = 0$.*
(iv) *If $n \geq 3$, then $H^i(\mathcal{J}_{Z/X}(D - pH)) = 0$ for $1 \leq i \leq n-2, 1 \leq p \leq n$.*
(v) *$(-1)^{n-1}\chi(\mathcal{J}_{Z/X}(D - pH)) = (r-1)\chi(K_X + pH)$, for $1 \leq p \leq n$.*
(vi) *$\delta_{Z,W,-nH} : H^{n-1}(\mathcal{J}_{Z/X}(D - nH)) \to W^* \otimes H^n(-nH)$ is either injective or surjective.*

Moreover the following exact sequences hold

$$0 \to W^* \otimes \mathcal{O}_X \to \mathcal{E} \to \mathcal{J}_{Z/X}(D) \to 0 \tag{3.1}$$

and, if $Z \neq \emptyset$,

$$0 \to \mathcal{O}_X(-D) \to \mathcal{E}^* \to W \otimes \mathcal{O}_X \to \omega_Z(-K_X - D) \to 0.$$

Then we have

Definition 3.3. Let $n \geq 2, d \geq 2, r \geq 2$ and let D be a divisor on X. An *Ulrich subvariety of X* is a subvariety $Z \subset X$ carrying a $(r-1)$-dimensional subspace $W \subseteq \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{J}_{Z/X}(D), \mathcal{O}_X)$ such that properties (a)-(c) and (i)-(vi) of Theorem 3.2 hold.

Hence, if $n \geq 2, d \geq 2, r \geq 2$, to each rank r Ulrich vector bundle $\mathcal{E}$ with $\det \mathcal{E} = \mathcal{O}_X(D)$ one can associate as in Theorem 3.2 an Ulrich subvariety Z. In particular, by its construction, Z satisfies the properties of Lemma 3.1.

4. Veronese embeddings of complete intersections

In this section we study Ulrich bundles on Veronese embeddings of complete intersections.

Given integers $d_i \geq 1, 1 \leq i \leq s$, we set

$$d = \prod_{i=1}^{s} d_i, S = \sum_{i=1}^{s} d_i \text{ and } S' = \begin{cases} 0 & \text{if } s = 1 \\ \sum\limits_{1 \leq i < j \leq s} d_i d_j & \text{if } s \geq 2 \end{cases}.$$

Then we have

Lemma 4.1. *Let $s \geq 1, r \geq 2, a \geq 2, m \geq 3$ and let $X \subset \mathbb{P}^{m+s}$ be a smooth m-dimensional complete intersection of hypersurfaces of degrees $(d_1, \dots, d_s)$ with $d_i \geq 1, 1 \leq i \leq s$ and degree d. Let $H \in |\mathcal{O}_X(1)|$. Let $\mathcal{E}$ be a rank r Ulrich vector bundle for $(X, \mathcal{O}_X(a))$ and let $Z \subset X$ be the associated Ulrich subvariety, as in Theorem 3.2 applied to the Veronese embedding $v_a(X) \subset \mathbb{P}^N$. Then Z is irreducible, of dimension $m-2$, smooth when $r = 2$ or when $m \leq 5$ and:*

(i) $K_X = (S - s - m - 1)H$.

(ii) $c_2(X) = \left[\binom{m+s+1}{2} + S(S-s-m-1) - S'\right] H^2$.

(iii) $c_1(\mathcal{E}) = uH$ *where* $u = \frac{r}{2}[(m+1)(a-1) + S - s]$.

(iv) $$\deg_H(Z) = \frac{rd}{24}[-4+6a-2a^2-7m+12am-5a^2m-3m^2+6am^2-3a^2m^2+3r-6ar+3a^2r+ +6mr-12amr+6a^2mr+3m^2r-6am^2r+3a^2m^2r-7s+6as-6ms+6ams+ +6rs-6ars+6mrs-6amrs-3s^2+3rs^2+6S-6aS+6mS-6amS-6rS+ +6arS-6mrS+6amrS+6sS-6rsS-2S^2+3rS^2-2S')].$$

(v) $$\chi(\mathcal{O}_Z(\ell)) = \binom{\ell+m+s}{m+s} + (-1)^{m+1}\frac{rd}{m!}(u-\ell-a)\cdots(u-\ell-ma) + (-1)^{m+s}(r-1)\binom{u-\ell-1}{m+s} + \sum_{k=1}^{s}(-1)^{k+m+s}\sum_{1\le i_1<\ldots<i_k\le s}\left[\binom{d_{i_1}+\ldots+d_{i_k}-\ell-1}{m+s} + (r-1)\binom{d_{i_1}+\ldots+d_{i_k}+u-\ell-1}{m+s}\right].$$

Moreover suppose that one of the following holds:

(1) $m \ge 5$, *or*

(2) $m = 4, d = 1$, *or*

(3) $m = 4$, $X = X' \cap F$, *where* $X' \subset \mathbb{P}^{5+s}$ *is a smooth complete intersection,* $F \subset \mathbb{P}^{5+s}$ *is a hypersurface of degree* a *and* $\mathcal{E} = \mathcal{E}'_{|X}$, *where* $\mathcal{E}'$ *is a vector bundle on* X', *or*

(4) $m = 4, d \ge 2$, X *is very general and* $(d_1,\ldots,d_s) \notin \{(2,\underbrace{1,\ldots,1}_{s\text{-}1}), s\ge 1; (2,2,\underbrace{1,\ldots,1}_{s\text{-}2}), s \ge 2\}$ *(up to permutation).*

Then

(vi) $c_2(\mathcal{E}) = eH^2$ *with*
$$e = \frac{r}{24}[-4+6a-2a^2-7m+12am-5a^2m-3m^2+6am^2-3a^2m^2+3r-6ar+3a^2r+ +6mr-12amr+6a^2mr+3m^2r-6am^2r+3a^2m^2r-7s+6as-6ms+6ams+ +6rs-6ars+6mrs-6amrs-3s^2+3rs^2+6S-6aS+6mS-6amS-6rS+ +6arS-6mrS+6amrS+6sS-6rsS-2S^2+3rS^2-2S')].$$

Proof. (i) and (ii) follow from the tangent and Euler sequence of $X \subset \mathbb{P}^{m+s}$. By Lefschetz's theorem (see for example [H, Thm. 2.1]), we have that $\mathrm{Pic}(X) \cong \mathbb{Z}H$. Then (iii) follows by [L, Lemma 3.2]. Note that $\deg v_a(X) = (aH)^m = a^m d \ge 2$, thus [LR2, Rmk. 4.3] applies. Since $H^1(\mathcal{O}_X(-u)) = 0$ we have that $Z \ne \emptyset$ by [LR2, Rmk. 4.3(i)], hence Z is of dimension $m-2$, smooth when $r = 2$ or when $m \le 5$. Also, $u \ge 2a$ and $H^i(\mathcal{E}(-pa)) = 0$ for $i \ge 0, 1 \le p \le m$, hence $H^1(\mathcal{E}(-u)) = 0$ by Lemma 2.1(ii). On the other hand, $H^2(\mathcal{O}_X(-u)) = 0$, hence Z is irreducible by [LR2, Rmk. 4.3(vi)]. Next, Lemma 3.1(i) gives that $[Z] = c_2(\mathcal{E})$, hence (iv) follows by (i)-(iii) and Lemma 2.2(ii) with $L = aH$. As for (v), observe first that $\chi(\mathcal{E}(\ell))$ is a polynomial in ℓ of degree m with leading coefficient $\frac{rd}{m!}$. On the other hand, as $\mathcal{E}$ is Ulrich for $(X, \mathcal{O}_X(a))$, we have that $\chi(\mathcal{E}(-pa)) = 0$ for $1 \le p \le m$. Therefore
$$\chi(\mathcal{E}(\ell)) = \frac{rd}{m!}(\ell+a)\cdots(\ell+ma).$$
Now, the exact sequence (3.1) twisted by $\mathcal{O}_X(\ell-u)$ gives
$$\chi(\mathcal{O}_Z(\ell)) = \chi(\mathcal{O}_X(\ell)) - \chi(\mathcal{J}_{Z/X}(\ell) = \chi(\mathcal{O}_X(\ell)) - \chi(\mathcal{E}(\ell-u)) + (r-1)\chi(\mathcal{O}_X(\ell-u))$$
and computing $\chi(\mathcal{O}_X(\ell))$ and $\chi(\mathcal{O}_X(\ell-u))$ with the Koszul resolution of $\mathcal{J}_{X/\mathbb{P}^{m+s}}$, we get (v). Finally, to see (vi), we claim that under any of the hypotheses (1), (2), (3) or (4), the following holds:
$$\exists e \in \mathbb{Z} \text{ such that } c_2(\mathcal{E}) = eH^2. \tag{4.1}$$
In fact, if $m \ge 5$, we have by Lefschetz's theorem (see for example [H, Thm. 2.1]) that $H^4(X,\mathbb{Z}) \cong \mathbb{Z}H^2$. If $m = 4$ and $d = 1$, we have that $X \cong \mathbb{P}^4$. Hence (4.1) holds under either one of the hypotheses (1) or (2). Under hypothesis (3), we have as above that $H^4(X',\mathbb{Z}) \cong \mathbb{Z}(H')^2, H' \in |\mathcal{O}_{X'}(1)|$, hence $c_2(\mathcal{E}') = e(H')^2$ on X', for some $e \in \mathbb{Z}$. Therefore $c_2(\mathcal{E}) = c_2(\mathcal{E}'_{|X}) = eH^2$, so that (4.1) holds under hypothesis (3). Also, under hypothesis (4), we know again by Noether-Lefschetz's theorem (see for example [S, Thm. 1.1]) that every algebraic cohomology class of codimension 2 in X is in $\mathbb{Z}H^2$. Since

$[Z] = c_2(\mathcal{E})$ by Lemma 3.1(i), we have that (4.1) holds under hypothesis (4). Now, since $[Z] = c_2(\mathcal{E})$ using (iv) and (4.1), we have that $e = \frac{\deg_H(Z)}{d}$ is as stated in (vi). □

With the above lemma at hand, we now show Theorem 2.

Proof of Theorem 2. First, we dispose of the case $r = 1$. Since $\mathrm{Pic}(X) \cong \mathbb{Z}H$ by Lefschetz's theorem, let $b \in \mathbb{Z}$ and let $\mathcal{O}_X(b)$ be an Ulrich bundle with respect to $(X, \mathcal{O}_X(a))$. Then $H^0(\mathcal{O}_X(b-a)) = H^m(\mathcal{O}_X(b-ma)) = 0$, that is, by Lemma 4.1(i), $H^0(\mathcal{O}_X(b-a)) = H^0(\mathcal{O}_X(S-s-m-1-b+ma)) = 0$. But this gives the contradiction $m(a-1)+S-s \le b \le a-1$.

Hence, from now on, we can assume that $r \in \{2,3\}$.

Note that by adding some 1's to $(d_1,\dots,d_s)$, we can always assume that $s \ge 4$.

Suppose that we have an Ulrich bundle of rank r for $(X, \mathcal{O}_X(a))$ and consider the Veronese embedding $v_a(X) \subset \mathbb{P}^N$. If $m \ge 5$, taking hyperplane sections and using Lemma 2.2(i), it will be enough to show that, on the 4-dimensional section of $v_a(X)_4$ of $v_a(X)$, there are no Ulrich bundles of rank r.

Now $v_a(X)_4$ is isomorphic to a smooth complete intersection $\widetilde{X} \subset \mathbb{P}^{4+\tilde{s}}$ of type $(d_1,\dots,d_{\tilde{s}}) = (d_1,\dots,d_s,a,\dots,a)$, with $\tilde{s} = s+m-4$ and we have an Ulrich bundle of rank r for $(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(a))$.

With an abuse of notation, let us call again X the above 4-dimensional section $\widetilde{X}$ and $(d_1,\dots,d_s)$ its degrees. Hence we have that $X \subset \mathbb{P}^{4+s}$ is a smooth complete intersection of hypersurfaces of degrees $(d_1,\dots,d_s)$ with $s \ge 4, d_i \ge 1, 1 \le i \le s$.

Let $\mathcal{E}$ be an Ulrich bundle of rank r for $(X, \mathcal{O}_X(a))$ and let $Z \subset X$ be the associated smooth irreducible surface by Lemma 4.1.

Observe that, under hypothesis (a) (respectively (b), resp. (c)) of the theorem, we have that condition (3) (respectively (2), resp. (4)) of Lemma 4.1 hold. In any case, we deduce that Lemma 4.1(vi) holds.

Let $H \in |\mathcal{O}_X(1)|$ and set $H_Z = H_{|Z}$.

Assume that $r = 2$.

By Lemma 4.1(iii), (iv), (v) and (vi) we see that

$$u = 5(a-1)+S-s \tag{4.2}$$

$$e = \frac{1}{12}(70-150a+80a^2+29s-30as+3s^2-30S+30aS-6sS+4S^2-2S') \tag{4.3}$$

$$\deg_H(Z) = H_Z^2 = \frac{d}{12}(70-150a+80a^2+29s-30as+3s^2-30S+30aS-6sS+4S^2-2S') \tag{4.4}$$

and

$$\begin{aligned}\chi(\mathcal{O}_Z) =& 1 - \frac{d}{12}(u-a)\cdots(u-4a) + (-1)^s\binom{u-1}{s+4} + \\ &+ \sum_{k=1}^{s}(-1)^{k+s}\sum_{1\le i_1<\ldots<i_k\le s}\left[\binom{d_{i_1}+\ldots+d_{i_k}-1}{s+4} + \binom{d_{i_1}+\ldots+d_{i_k}+u-1}{s+4}\right]\end{aligned}.$$

In the notation (A.1) of the functions in the appendix, this is just

$$\chi(\mathcal{O}_Z) = f_{a,4,s,2,0}(d_1,\dots,d_s). \tag{4.5}$$

Next, we have by Lemma 3.1(ii), (4.2) and Lemma 4.1(i) that

$$K_Z = [2S-2s+5(a-2)]H_Z \tag{4.6}$$

so that

$$K_Z^2 = (100-100a+25a^2+40s-20as+4s^2-40S+20aS-8sS+4S^2)\deg_H(Z). \tag{4.7}$$

Using Lemma 3.1(ii), Lemma 4.1(i)-(ii), (4.2), (4.3), (4.6) and (4.7) we get

$$c_2(Z) = \frac{1}{12}(650-750a+220a^2+265s-150as+27s^2-270S+150aS-54sS+32S^2-10S')\deg_H(Z). \tag{4.8}$$

Then Noether's formula Z, $\chi(\mathcal{O}_Z) = \frac{1}{12}[K_Z^2 + c_2(Z)]$ gives, using (4.4), (4.7) and (4.8), that

$$\begin{aligned}\chi(\mathcal{O}_Z) = \frac{5d}{1728}[&25900 - 82800a + 95380a^2 - 46800a^3 + 8320a^4 + 21160s - 50220as + 38336a^2s \\ & - 9360a^3s + 6481s^2 - 10152as^2 + 3852a^2s^2 + 882s^3 - 684as^3 + 45s^4 - 21600S+ \\ & + 50760aS - 38520a^2S + 9360a^3S - 13140sS + 20412asS - 7704a^2sS - 2664s^2S+ \\ & + 2052as^2S - 180s^3S + 7100S^2 - 10800aS^2 + 4036a^2S^2 + 2860sS^2 - 2160asS^2+ \\ & + 288s^2S^2 - 1080S^3 + 792aS^3 - 216sS^3 + 64S^4 - 880S' + 1080aS' - 368a^2S' \\ & - 356sS' + 216asS' - 36s^2S' + 360SS' - 216aSS' + 72sSS' - 40S^2S' + 4(S')^2].\end{aligned}$$

In the notation (A.8) of the appendix, this is just

$$\chi(\mathcal{O}_Z) = g_{a,4,s}(d_1, \dots, d_s). \tag{4.9}$$

Therefore (4.5) and (4.9) imply that

$$g_{a,4,s}(d_1, \dots, d_s) - f_{a,4,s,2,0}(d_1, \dots, d_s) = 0$$

that is, by Lemma A.6(1) of the appendix,

$$m_{1^s}(s)(d_1, \dots, d_s) v_{s,a,8}(d_1, \dots, d_s) = 0$$

or, equivalently,

$$d v_{s,a,8}(d_1, \dots, d_s) = 0$$

contradicting Lemma A.7.

Next, assume that $r = 3$.

By Riemann-Roch we see that

$$K_Z H_Z = -2\chi(\mathcal{O}_Z(1)) + 2\chi(\mathcal{O}_Z) + \deg_H(Z). \tag{4.10}$$

Now Lemma 4.1 gives, in the notation (A.1) of the functions in the appendix, that

$$\chi(\mathcal{O}_Z(\ell)) = f_{a,4,s,3,\ell}(d_1, \dots, d_s) \tag{4.11}$$

and that

$$\deg_H(Z) = \frac{d}{8}(145 - 300a + 155a^2 + 59s - 60as + 6s^2 + (-60 + 60a - 12s)S + 7S^2 - 2S')$$

that is, in the notation (A.8), that

$$\deg_H(Z) = \delta_s(d_1, \dots, d_s) \tag{4.12}$$

and therefore, in the notation (A.8), (4.10) becomes

$$K_Z H_Z = -2f_{a,4,s,3,1}(d_1, \dots, d_s) + 2f_{a,4,s,3,0}(d_1, \dots, d_s) + \delta_s(d_1, \dots, d_s) = h_s(d_1, \dots, d_s). \tag{4.13}$$

On the other hand, we have by [LR2, Rmk. 4.3(ix)] and Lemma 4.1 that $[K_Z - \frac{5}{2}(S-s+3a-5)H_Z]^2 = 0$, so that, using (4.12), (4.13) and the notation (A.8), we get

$$\begin{aligned} K_Z^2 &= 5(S - s + 3a - 5)K_Z H_Z - \frac{25}{4}(S - s + 3a - 5)^2 \deg_H(Z) = \\ &= 5(S - s + 3a - 5)h_s(d_1, \dots, d_s) - \frac{25}{4}(S - s + 3a - 5)^2 \delta_s(d_1, \dots, d_s) = k_s(d_1, \dots, d_s). \end{aligned} \tag{4.14}$$

Next, we get by Lemma 3.1(iii) and Lemma 4.1(i)-(iii) and (vi), using also the notation (A.8), that

(4.15)

$$\begin{aligned} c_2(Z) = &\frac{1}{8}[-1315 + 1800a - 605a^2 - 523s + 360as - 52s^2 + 520S - 360aS + 104sS - 49S^2 - 6S']H_Z^2 \\ & + (4S - 4s - 20 + 15a)K_Z H_Z = \\ & = \frac{1}{8}[-1315 + 1800a - 605a^2 - 523s + 360as - 52s^2 + 520S - 360aS + 104sS \\ & \quad - 49S^2 - 6S']\delta_s(d_1, \dots, d_s) + (4S - 4s - 20 + 15a)h_s(d_1, \dots, d_s) = \\ & = c_s(d_1, \dots, d_s). \end{aligned}$$

Hence (4.14), (4.15) and Noether's formula, using also the notation (A.8), give

$$\chi(\mathcal{O}_Z)=\frac{1}{12}(K_Z^2+c_2(Z))=\frac{k_s(d_1,\dots,d_s)+c_s(d_1,\dots,d_s)}{12}=\chi'_s(d_1,\dots,d_s). \tag{4.16}$$

Thus we get, by (4.11) and (4.16) we have that

$$\chi'_s(d_1,\dots,d_s)-f_{a,4,s,3,0}(d_1,\dots,d_s)=0$$

that is, using Lemma A.6(2),

$$m_{1^s}(s)(d_1,\dots,d_s)v_{s,a,9}(d_1,\dots,d_s)=0$$

or, equivalently,

$$dv_{s,a,9}(d_1,\dots,d_s)=0$$

contradicting Lemma A.7.

This concludes the proof in the case $r=3$ and therefore also ends the proof of the theorem. □

5. Proof of Theorem 1

In this section we prove our main theorem.

Proof of Theorem 1. Let $\overline{\mathcal{E}}$ be an Ulrich bundle of rank $r\le 3$ for $(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(a))$. By Theorem 2(b) we see that the case $n=4$ cannot occur. Hence we assume from now on that $n\ge 5$.

We can consider $\overline{\mathcal{E}}$ as an Ulrich bundle for $(v_a(\mathbb{P}^n),\mathcal{O}_{v_a(\mathbb{P}^n)}(1))$, where $v_a(\mathbb{P}^n)\subset\mathbb{P}^N$ is the a-Veronese embedding. Choosing $n-4$ general hyperplanes H_i in $\mathbb{P}^N$, we get, by Lemma 2.2(i), a rank r Ulrich bundle $\mathcal{E}'=\overline{\mathcal{E}}_{|X'}$ on $X'=v_a(\mathbb{P}^n)\cap H_1\cap\ldots\cap H_{n-4}$ with respect to $\mathcal{O}_{X'}(1)=\mathcal{O}_{\mathbb{P}^N}(1)_{|X'}$. On the other hand, X' is isomorphic to a general 4-dimensional smooth complete intersection $X\subset\mathbb{P}^n$ of type $(a,\dots,a)$ and, via this isomorphism, $\mathcal{E}'$ corresponds to a rank r Ulrich bundle $\mathcal{E}$ on X with respect to $\mathcal{O}_X(a)$. We have then obtained a nonempty open subset U in the parameter space M of complete intersections $X\subset\mathbb{P}^n$ of type $(a,\dots,a)$. Since U cannot be contained in a countable union of proper closed subvarieties of M, we deduce that we can find a very general 4-dimensional smooth complete intersection $X\subset\mathbb{P}^n$ of type $(a,\dots,a)$ carrying a rank r Ulrich bundle $\mathcal{E}$ with respect to $\mathcal{O}_X(a)$.

Therefore, setting $s=n-4$, we have that $X\subset\mathbb{P}^{4+s}$ is a very general 4-dimensional smooth complete intersection of type $(d_1,\dots,d_s)=(a,\dots,a)$ carrying a rank r Ulrich bundle $\mathcal{E}$ with respect to $\mathcal{O}_X(a)$.

Then we get a contradiction by Theorem 2 unless $a=2$ and $s=1,2$, that is $n=5,6$. But in the latter two cases we have a contradiction by [ES, Cor. 5.3].

This concludes the proof. □

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Appendix A. Symmetric functions associated to Veronese embeddings of complete intersections

Given a smooth complete intersection $X \subset \mathbb{P}^{m+s}$ of hypersurfaces of degrees $(d_1, \dots, d_s)$, a rank $r \ge 2$ Ulrich vector bundle $\mathcal{E}$ on X with respect to $\mathcal{O}_X(a)$ and an Ulrich subvariety Z associated to $\mathcal{E}$, we have some natural symmetric functions of $(d_1, \dots, d_s)$ as in Lemma 4.1 and as in the proof of Theorem 2. In this section we will lay out the necessary calculations related to them. Several calculations have been performed by Mathematica. The corresponding codes can be found in [LR].

Definition A.1. Given integers $m \ge 1, s \ge 1, r \ge 2, \ell$, consider the polynomials in $\mathbb{Q}[x_1, \dots, x_s]$ given by

$$a_s(\ell, x_1, \dots, x_s) = \binom{\ell+m+s}{m+s} + \sum_{k=1}^{s} (-1)^{k+m+s} \sum_{1 \le i_1 < \dots < i_k \le s} \binom{x_{i_1} + \dots + x_{i_k} - \ell - 1}{m+s}$$

and

$$b_s(x_1, \dots, x_s) = (-1)^{m+1} \frac{r}{m!} (\prod_{i=1}^{s} x_i) \prod_{j=1}^{m} \left[\frac{r}{2}[(m+1)(a-1) + \sum_{i=1}^{s} x_i - s] - \ell - ja \right]$$

Next, we set

$$f_{a,m,s,r,\ell} = a_s(\ell, x_1, \dots, x_s) + (r-1) a_s(\ell - \frac{r}{2}[(m+1)(a-1) + \sum_{i=1}^{s} x_i - s], x_1, \dots, x_s) + b_s(x_1, \dots, x_s).$$

Explicitly we have

$$\begin{aligned} f_{a,m,s,r,\ell} = & \binom{\ell+m+s}{m+s} + (-1)^{m+1} \frac{r}{m!} (\prod_{i=1}^{s} x_i) \prod_{j=1}^{m} \left[\frac{r}{2}[(m+1)(a-1) + \sum_{i=1}^{s} x_i - s] - \ell - ja \right] + \\ & + (-1)^{m+s}(r-1) \binom{\frac{r}{2}[(m+1)(a-1) + \sum_{i=1}^{s} x_i - s] - \ell - 1}{m+s} + \\ & + \sum_{k=1}^{s} (-1)^{k+m+s} \sum_{1 \le i_1 < \dots < i_k \le s} \binom{x_{i_1} + \dots + x_{i_k} - \ell - 1}{m+s} + \\ & + (r-1) \sum_{k=1}^{s} (-1)^{k+m+s} \sum_{1 \le i_1 < \dots < i_k \le s} \binom{x_{i_1} + \dots + x_{i_k} + \frac{r}{2}[(m+1)(a-1) + \sum_{i=1}^{s} x_i - s] - \ell - 1}{m+s}. \end{aligned}$$

Lemma A.2.

(1) *$f_{a,m,s,r,\ell}$ is symmetric in $x_1, \dots, x_s$.*

(2) *For any $1 \le k \le s$, the following identity holds in $\mathbb{Q}[x_1, \dots, x_k]$:*

$$f_{a,m,k,r,\ell}(x_1, \dots, x_k) = f_{a,m,s,r,\ell}(x_1, \dots, x_k, 1, \dots, 1).$$

(3) *$x_i \mid f_{a,m,s,r,\ell}$ for all $1 \le i \le s$.*

Proof. Same as [LR2, Lemmas A.2, A.3]. □

We will now express the symmetric polynomials $f_{a,m,s,r,\ell}$ in terms of monomial symmetric polynomials. For this we will use some properties of them, for which we refer for example to [Eg, §1].

Definition A.3. Let $s \ge 1$ be an integer and let $x_1, \dots, x_s$ be indeterminates. Given a partition $\lambda = \{\lambda_1, \dots, \lambda_k\}$ with $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_k \ge 1$, if $k \le s$ we let $m_\lambda(s)$ be the monomial symmetric polynomial in $x_1, \dots, x_s$ corresponding to λ, while if $k > s$ we set $m_\lambda(s) = 0$.

We will also write $m_\lambda(s) = m_{\lambda_1 \dots \lambda_k}$. We denote by $\{1^k\}$ the partition $\{1, \dots, 1\}$ of k and we set $m_{1^0}(s) = 1$. For example

$$m_h(s) = \sum_{i=1}^{s} x_i^h \text{ for } h \ge 1 \text{ and } m_{1^s}(s) = \prod_{i=1}^{s} x_i.$$

We will consider below the following $\mathbb{Q}$-basis of the vector space of symmetric polynomials with rational coefficients and of degree at most 4 in s variables:

$$e = \{m_4(s), m_{31}(s), m_{22}(s), m_{211}(s), m_{1111}(s), m_3(s), m_{21}(s), m_{111}(s), m_2(s), m_{11}(s), m_1(s), 1\}. \tag{A.1}$$

We can now express any of the functions $f \in \{f_{a,4,s,2,0}, f_{a,4,s,3,0}, f_{a,4,s,3,1}\}$ in terms of the above basis as

$$\begin{aligned} f = \frac{m_{1^s}(s)}{M}[&a_1m_4(s) + a_2m_{31}(s) + a_3m_{22}(s) + a_4m_{211}(s) + a_5m_{1111}(s) + a_6m_3(s) + a_7m_{21}(s)+ \\ &+ a_8m_{111}(s) + a_9m_2(s) + a_{10}m_{11}(s) + a_{11}m_1(s) + a_{12}] \end{aligned} \tag{A.2}$$

with the coefficients given by the following

Lemma A.4. *Let $f \in \{f_{a,4,s,2,0}, f_{a,4,s,3,0}, f_{a,4,s,3,1}\}$. For all $s \geq 4$ the coefficients in (A.2) are:*

(i) *For $f_{a,4,s,2,0}$ we have $M = 360$ and*
  (1) $a_1 = 66, a_2 = 225, a_3 = 320, a_4 = 600, a_5 = 1125$
  (2) $a_6 = 75(-15 + 11a - 3s), a_7 = 150(-20 + 15a - 4s)$
  (3) $a_8 = 225(-25 + 19a - 5s)$
  (4) $a_9 = 10(740 - 1125a + 420a^2 + 298s - 225as + 30s^2)$
  (5) $a_{10} = \frac{75}{2}(370 - 570a + 214a^2 + 149s - 114as + 15s^2)$
  (6) $a_{11} = \frac{75}{2}(-600 + 1410a - 1070a^2 + 260a^3 - 365s + 567as - 214a^2s - 74s^2 + 57as^2 - 5s^3)$
  (7) $a_{12} = \frac{1}{8}(215760 - 690000a + 795000a^2 - 390000a^3 + 69240a^4 + 176302s - 418500as + 319500a^2s - 78000a^3s + 54005s^2 - 84600as^2 + 32100a^2s^2 + 7350s^3 - 5700as^3 + 375s^4)$.

(ii) *For $f_{a,4,s,3,0}$ we have $M = 1920$ and*
  (1) $a_1 = 1683, a_2 = 6060, a_3 = 8770, a_4 = 16860, a_5 = 32400$
  (2) $a_6 = -30300 + 24600a - 6060s, a_7 = -84300 + 69000a - 16860s$
  (3) $a_8 = -162000 + 133200a - 32400s$
  (4) $a_9 = 209050 - 345000a + 140850a^2 + 83960s - 69000as + 8430s^2$
  (5) $a_{10} = 401700 - 666000a + 272700a^2 + 161340s - 133200as + 16200s^2$
  (6) $a_{11} = -658500 + 1653000a - 1363500a^2 + 369000a^3 - 398400s + 663600as - 272700a^2s - 80340s^2 + 66600as^2 - 5400s^3$
  (7) $a_{12} = 802635 - 2715000a + 3386250a^2 - 1845000a^3 + 371115a^4 + 650302s - 1641000as + 1359000a^2s - 369000a^3s + 197555s^2 - 330600as^2 + 136350a^2s^2 + 26670s^3 - 22200as^3 + 1350s^4$.

(iii) *For $f_{a,4,s,3,1}$ we have $M = 1920$ and*
  (1) $a_1 = 1683, a_2 = 6060, a_3 = 8770, a_4 = 16860, a_5 = 32400$
  (2) $a_6 = -32580 + 24600a - 6060s, a_7 = -90420 + 69000a - 16860s$
  (3) $a_8 = -173520 + 133200a - 32400s$
  (4) $a_9 = 240490 - 371400a + 140850a^2 + 90080s - 69000as + 8430s^2$
  (5) $a_{10} = 460740 - 716400a + 272700a^2 + 172860s - 133200as + 16200s^2$
  (6) $a_{11} = -807900 + 1912200a - 1473300a^2 + 369000a^3 - 457080s + 714000as - 272700a^2s - 86100s^2 + 66600as^2 - 5400s^3$
  (7) $a_{12} = 1051035 - 3375000a + 3953850a^2 - 2001000a^3 + 371115a^4 + 797782s - 1899000as + 1468800a^2s - 369000a^3s + 226715s^2 - 355800as^2 + 136350a^2s^2 + 28590s^3 - 22200as^3 + 1350s^4$.

Proof. We sketch the proof, since it is similar to the one in [LR2, Lemma A.7].

By Lemma A.2(1) and (3) we see that there exists a symmetric polynomial $p_s \in \mathbb{Q}[x_1, \dots, x_s]$ of degree at most 4 such that

$$f = \frac{m_{1^s}(s)}{M}p_s.$$

Thus, we can express p_s through the basis (A.1) as follows:

$$\begin{aligned} p_s =& a_1m_4(s)+a_2m_{31}(s)+a_3m_{22}(s)+a_4m_{211}(s)+a_5m_{1111}(s)+a_6m_3(s)+ \\ &+a_7m_{21}(s)+a_8m_{111}(s)+a_9m_2(s)+a_{10}m_{11}(s)+a_{11}m_1(s)+a_{12} \end{aligned} \tag{A.3}$$

with $a_1,\dots,a_{12}\in\mathbb{Q}$. By Lemma A.2(2), we have that

$$p_4(x_1,\dots,x_4)=p_s(x_1,\dots,x_4,1\dots,1). \tag{A.4}$$

On the other hand, direct calculations show that:

(A.5)

$$\begin{aligned} f_{a,4,4,2,0} = \frac{m_{1^4}(4)}{360}[&66m_4(4)+225m_{31}(4)+320m_{22}(4)+600m_{211}(4)+1125m_{1111}(4) \\ &+(825a-2025)m_3(4)+(2250a-5400)m_{21}(4)+(4275a-10125)m_{111}(4) \\ &+(4200a^2-20250a+24120)m_2(4)+(8025a^2-38475a+45225)m_{11}(4) \\ &+(9750a^3-72225a^2+172125a-133650)m_1(4)+8655a^4-87750a^3+323325a^2 \\ &-510300a+293931]. \end{aligned}$$

(A.6)

$$\begin{aligned} f_{a,4,4,3,0} =& \frac{m_{1^4}(4)}{1920}[1683m_4(4)+6060m_{31}(4)+8770m_{22}(4)+16860m_{211}(4)+32400m_{1111}(4) \\ &+(-54540+24600a)m_3(4)+(-151740+69000a)m_{21}(4)+(-291600+133200a)m_{111}(4) \\ &+(679770-621000a+140850a^2)m_2(4)+(1306260-1198800a+272700a^2)m_{11}(4) \\ &+(-3883140+5373000a-2454300a^2+369000a^3)m_1(4) \\ &+8617203-15989400a+11003850a^2-3321000a^3+371115a^4]. \end{aligned}$$

(A.7)

$$\begin{aligned} f_{a,4,4,3,1} =& \frac{m_{1^4}(4)}{1920}[1683m_4(4)+6060m_{31}(4)+8770m_{22}(4)+16860m_{211}(4)+32400m_{1111}(4) \\ &+(-56820+24600a)m_3(4)+(-157860+69000a)m_{21}(4)+(-303120+133200a)m_{111}(4) \\ &+(735690-647400a+140850a^2)m_2(4)+(1411380-1249200a+272700a^2)m_{11}(4) \\ &+(-4359420+5833800a-2564100a^2+369000a^3)m_1(4) \\ &+10044963-18084600a+12010650a^2-3477000a^3+371115a^4]. \end{aligned}$$

Hence, replacing $x_5=\dots=x_s=1$ in (A.3) and using [LR2, Lemma A.6] for $G=p_s$, we get an expression for $p_s(x_1,\dots,x_4,1\dots,1)$ in terms of the basis (A.1) whose coefficients must coincide, by (A.4), with the ones in (A.5)-(A.7). Solving the corresponding linear systems in the a_j's, we get (1)-(7) in (i)-(iii). $\square$

Consider now the following polynomials in $\mathbb{Q}[x_1,\dots,x_s]$:

$$g_{a,4,s} = \frac{m_{1^s}(s)}{1728}[25900 - 82800a + 95380a^2 - 46800a^3 + 8320a^4 + 21160s - 50220as + 38336a^2s - 9360a^3s + 6481s^2 - 10152as^2 + 3852a^2s^2 + 882s^3 - 684as^3 + 45s^4 + (-21600 + 50760a - 38520a^2 + 9360a^3 - 13140s + 20412as - 7704a^2s - 2664s^2 + 2052as^2 - 180s^3)m_1(s) + (7100 - 10800a + 4036a^2 + 2860s - 2160as + 288s^2)m_1(s)^2 + (-1080 + 792a - 216s)m_1(s)^3 + 64m_1(s)^4 + (-880 + 1080a - 368a^2 - 356s + 216as - 36s^2)m_{11}(s) + (360 - 216a + 72s)m_1(s)m_{11}(s) - 40m_1(s)^2m_{11}(s) + 4m_{11}(s)^2]$$

$$\delta_s = \frac{m_{1^s}(s)}{8}[145 - 300a + 155a^2 + 59s - 60as + 6s^2 + (-60 + 60a - 12s)m_1(s) + 7m_1(s)^2 - 2m_{11}(s)] \tag{A.8}$$

$$h_s = -2f_{a,4,s,3,1} + 2f_{a,4,s,3,0} + \delta_s$$

$$k_s = 5(m_1(s) - s + 3a - 5) - 5)h_s - \frac{25}{4}(m_1(s) - s + 3a - 5)^2\delta_s$$

$$c_s = \frac{1}{8}[-1315 + 1800a - 605a^2 - 523s + 360as - 52s^2 + (520 - 360a + 104s)m_1(s) - 49m_1(s)^2 - 6m_{11}(s)]\delta_s + [4m_1(s) - 4s - 20 + 15a]h_s$$

$$\chi'_s = \frac{k_s + c_s}{12}.$$

Then we have

Lemma A.5. *For all $s \geq 1$ the following identities hold:*

(1) $g_{a,4,s} = \frac{5m_{1^s}(s)}{1728}[64m_4(s) + 216m_{31}(s) + 308m_{22}(s) + 576m_{211}(s) + 1080m_{1111}(s) + (-1080 + 792a - 216s)m_3(s) + (-2880 + 2160a - 576s)m_{21}(s) + (-5400 + 4104a - 1080s)m_{111}(s) + (7100 - 10800a + 4036a^2 + 2860s - 2160as + 288s^2)m_2(s) + (13320 - 20520a + 7704a^2 + 5364s - 4104as + 540s^2)m_{11}(s) + (-21600 + 50760a - 38520a^2 + 9360a^3 - 13140s + 20412as - 7704a^2s - 2664s^2 + 2052as^2 - 180s^3)m_1(s) + 25900 - 82800a + 95380a^2 - 46800a^3 + 8320a^4 + 21160s - 50220as + 38336a^2s - 9360a^3s + 6481s^2 - 10152as^2 + 3852a^2s^2 + 882s^3 - 684as^3 + 45s^4]$

(2) $\delta_s = \frac{m_{1^s}(s)}{8}[145 - 300a + 155a^2 + 59s - 60as + 6s^2 + (-60 + 60a - 12s)m_1(s) + 7m_2(s) + 12m_{11}(s)]$

(3) $h_s = \frac{m_{1^s}(s)}{8}[19m_3(s) + 51m_{21}(s) + 96m_{111}(s) + (-255 + 220a - 51s)m_2(s) + (-480 + 420a - 96s)m_{11}(s) + (1185 - 2100a + 915a^2 + 477s - 420as + 48s^2)m_1(s) - 1925 + 5200a - 4575a^2 + 1300a^3 - 1170s + 2090as - 915a^2s - 237s^2 + 210as^2 - 16s^3]$

(4) $k_s =\frac{5m_{1^s}(s)}{32}[41m_4(s) + 150m_{31}(s) + 218m_{22}(s) + 422m_{211}(s) + 816m_{1111}(s)$
$+ (-750 + 598a - 150s)m_3(s) + (-2110 + 1702a - 422s)m_{21}(s)$
$+ (-4080 + 3312a - 816s)m_{111}(s)$
$+ (5240 - 8510a + 3410a^2 + 2103s - 1702as + 211s^2)m_2(s)$
$+ (10130 - 16560a + 6670a^2 + 4066s - 3312as + 408s^2)m_{11}(s)$
$+ (-16650 + 41170a - 33350a^2 + 8830a^3 - 10060s + 16514as - 6670a^2s - 2026s^2 + 1656as^2$
$- 136s^3)m_1(s) + 20375 - 67850a + 83000a^2 - 44150a^3 + 8625a^4 + 16475s - 40940as$
$+ 33275a^2s - 8830a^3s + 4995s^2 - 8234as^2 + 3335a^2s^2 + 673s^3 - 552as^3 + 34s^4]$

(5) $c_s =\frac{m_{1^s}(s)}{64}[265m_4(s) + 924m_{31}(s) + 1330m_{22}(s) + 2524m_{211}(s) + 4800m_{1111}(s)$
$+ (-4620 + 3860a - 924s)m_3(s) + (-12620 + 10580a - 2524s)m_{21}(s)$
$+ (-24000 + 20160a - 4800s)m_{111}(s)$
$+ (31210 - 52900a + 22250a^2 + 12552s - 10580as + 1262s^2)m_2(s)$
$+ (59380 - 100800a + 42380a^2 + 23876s - 20160as + 2400s^2)m_{11}(s)$
$+ (-96900 + 249500a - 211900a^2 + 59300a^3 - 58760s + 100300as - 42380a^2s - 11876s^2$
$+ 10080as^2 - 800s^3)m_1(s) + 117325 - 407500a + 524450a^2 - 296500a^3 + 62225a^4 + 95380s$
$- 247000as + 210840a^2s - 59300a^3s + 29073s^2 - 49900as^2 + 21190a^2s^2 + 3938s^3 - 3360as^3$
$+ 200s^4]$

(6) $\chi'_s =\frac{m_{1^s}(s)}{768}[675m_4(s) + 2424m_{31}(s) + 3510m_{22}(s) + 6744m_{211}(s) + 12960m_{1111}(s)$
$+ (-12120 + 9840a - 2424s)m_3(s) + (-33720 + 27600a - 6744s)m_{21}(s)$
$+ (-64800 + 53280a - 12960s)m_{111}(s)$
$+ (83610 - 138000a + 56350a^2 + 33582s - 27600as + 3372s^2)m_2(s)$
$+ (160680 - 266400a + 109080a^2 + 64536s - 53280as + 6480s^2)m_{11}(s)$
$+ (-263400 + 661200a - 545400a^2 + 147600a^3 - 159360s + 265440as - 109080a^2s$
$- 32136s^2 + 26640as^2 - 2160s^3)m_1(s) + 321075 - 1086000a + 1354450a^2 - 738000a^3$
$+ 148475a^4 + 260130s - 656400as + 543590a^2s - 147600a^3s + 79023s^2 - 132240as^2$
$+ 54540a^2s^2 + 10668s^3 - 8880as^3 + 540s^4].$

Proof. Immediate from [LR2, Lemma A.5]. $\square$

We wish to compare all of the above functions.
In order to do this, for any $a, b \in \mathbb{Z}$, define the polynomial $v_{s,a,b} \in \mathbb{Q}[x_1, \dots, x_s]$ by

$$v_{s,a,b} = bm_4(s)+10m_{22}(s)+(50a^2-10s-50)m_2(s)-250a^2-50a^2s+5s^2+150+(55-b)s-5b+(100+5b)a^4.$$

We have

Lemma A.6. *For all $s \geq 4$ the following identities hold:*
(i) $g_{a,4,s} - f_{a,4,s,2,0} = \frac{m_{1^s}(s)}{4320} v_{s,a,8}$.
(ii) $\chi'_s - f_{a,4,s,3,0} = \frac{m_{1^s}(s)}{3840} v_{s,a,9}$.

Proof. Immediate from Lemmas A.4 and A.5. $\square$

We will also need the following crude estimate:

Lemma A.7. *Let $b \in \{8, 9\}$, let $a \geq 2$ and let $s \geq 2$ be integers. Then for all integers $d_i \geq 1, 1 \leq i \leq s$ we have that $v_{s,a,b}(d_1, \dots, d_s) > 0$.*

Proof. Note that

$$v_{s,1,b} = bm_4(s) + 10m_{22}(s) - 10sm_2(s) + s(5s - b + 5).$$

In the notation in [LR2, Lemma A.10] we have that $v_{s,1,b} = q_{s,b}$ and it follows by [LR2, Lemma A.10] that $v_{s,1,b}(d_1, \dots, d_s) > 0$ if $\prod_{i=1}^s d_i \geq 2$. On the other hand, if $\prod_{i=1}^s d_i = 1$, then one easily checks that $v_{s,1,b}(1, \dots, 1) = 0$. Thus

$$v_{s,1,b}(d_1, \dots, d_s) \geq 0. \tag{A.9}$$

Now observe that

(A.10)

$$v_{s,a,b}(d_1, \dots, d_s) = v_{s,a-1,b}(d_1, \dots, d_s) + 5(2a-1)[10(m_2(s)(d_1, \dots, d_s) - s) + b(2a^2 - 2a + 1) + 10(4a^2 - 4a - 3)].$$

Finally, we proceed by induction on a. If $a = 2$, (A.10) and (A.9) give

$$v_{s,2,b}(d_1, \dots, d_s) = v_{s,1,b}(d_1, \dots, d_s) + 75[2(m_2(s)(d_1, \dots, d_s) - s) + b + 10] > 0.$$

If $a \geq 3$ we have by induction that $v_{s,a-1,b}(d_1, \dots, d_s) > 0$ and therefore also $v_{s,a,b}(d_1, \dots, d_s) > 0$ by (A.10). □

References

[Eg] E. S. Egge. *An introduction to symmetric functions and their combinatorics.* Stud. Math. Libr., **91**. American Mathematical Society, Providence, RI, 2019, xiii+342 pp. 8

[LR] Mathematica codes available at http://ricerca.mat.uniroma3.it/users/lopez/Mathematica-for-Veronese.pdf https://rcdeba.github.io/research.html. 8

Angelo Felice Lopez, Dipartimento di Matematica e Fisica, Università di Roma Tre, Largo San Leonardo Murialdo 1, 00146, Roma, Italy. e-mail angelo.lopez@uniroma3.it

Debaditya Raychaudhury, Department of Mathematics, University of Arizona, 617 N Santa Rita Ave., Tucson, AZ 85721, USA. email: draychaudhury@math.arizona.edu