FOCAL LOCI OF FAMILIES AND THE GENUS OF CURVES ON SURFACES

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ABSTRACT: In this article we apply the classical method of focal loci of families to give a lower bound for the genus of curves lying on general surfaces. First we translate and reprove Xu's result that any curve C on a general surface in $\mathbb{P}^3$ of degree $d \geq 5$ has geometric genus $g > 1 + \deg C(d-5)/2$. Then we prove a similar lower bound for the curves lying on a general surface in a given component of the Noether-Lefschetz locus in $\mathbb{P}^3$ and on a general projectively Cohen-Macaulay surface in $\mathbb{P}^4$.

1. INTRODUCTION

The theory of singular curves lying in a projective variety X has been extensively studied from the beginnings of Algebraic Geometry; however, even when X is a smooth surface, many basic questions still remain open. Recently, the interest about these arguments grew, essentially for two reasons: on the one hand, the theory of strings of nuclear physicists deals with the enumerative geometry of rational curves contained in some projective threefolds; on the other, the study of singular curves is naturally related with the hyperbolic geometry of complex projective varieties. Let us recall briefly this last setting. A compact complex manifold M is said to be hyperbolic (in the sense of Kobayashi [K] or Brody [B]) if there are no nonconstant entire holomorphic maps $f : \mathbb{C} \to M$. An intriguing question that lies in the intersection between differential and algebraic geometry is to characterize which projective algebraic varieties X over the complex field are hyperbolic. One necessary condition, in the above case, is that there are no nonconstant holomorphic maps $f : A \to X$ from an abelian variety A to X and it has been conjectured by Kobayashi [K] and Lang [La] that this condition is in fact sufficient.

An approach to the themes around this conjecture was given by Demailly in his Santa Cruz notes [D], in which he proposed an intermediate step:

Definition (1.1). A smooth projective variety X is said to be *algebraically hyperbolic*

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if there exists a real number $\epsilon > 0$ such that every algebraic curve $C \subset X$ of geometric genus g and degree d satisfies $2g - 2 \geq \epsilon d$.

Demailly proved that hyperbolic implies algebraically hyperbolic and that the latter does not allow the existence of nonconstant holomorphic maps from an abelian variety to X. Then, in view of Kobayashi-Lang's conjecture, it becomes relevant to check which projective varieties are algebraically hyperbolic. Already in the case of surfaces a complete answer appears far from reach. In $\mathbb{P}^3$, the works of Brody, Green [BG], Nadel [N] and El Goul [EG] showed that for all integers $d \geq 14$ there exist hyperbolic surfaces of degree d. Very recently Demailly and El Goul [DEG] proved that a *general* surface (in the countable Zariski topology) in $\mathbb{P}^3$ of degree at least 42 is hyperbolic. On the other hand, Clemens [Cl] proved that a *general* surface of degree at least 6 in $\mathbb{P}^3$ is algebraically hyperbolic. Clemens' argument was extended by Ein ([E1], [E2]) to the case of complete intersections in higher dimensional varieties and recently improved and simplified by Voisin in [V] (see also [CLR] for a very recent improvement). In 1994, Xu [X1] provided a more precise lower bound for the geometric genus of curves in any linear system over a general surface in $\mathbb{P}^3$, which is in fact sharp in some cases. His proof is based on a clever investigation of the equations defining singular curves on surfaces moving in $\mathbb{P}^3$; however it is involved in explicit hard computations with local coordinates. It was when trying to understand Xu's method from a global point of view, that we realized its connection with the focal locus of a family of curves. In general, the theory of "focal loci" for a family of varieties was classically developed by C.Segre [S] and recently rephrased in a modern language by Ciliberto and Sernesi [CS] (see also [CC]). These loci play an important role in differential projective geometry, and hence they have a quite natural involvement in the study of algebraic hyperbolicity of projective varieties.

Our first task, in the present article, has been to translate Xu's local analysis with a global property of the focal locus of a family of curves (Proposition (2.4)). This property turns out to be simple and powerful enough to get interesting applications.

In the case of general surfaces in $\mathbb{P}^3$, we give a short proof of one of the main theorems of Xu [X1, Theorem 2.1], only by means of focal loci.

Theorem (1.2). *On a general surface S of degree $d \geq 5$ in $\mathbb{P}^3$, there are no reduced irreducible curves C of geometric genus $g \leq 1 + degC(d-5)/2$. In particular, for $d \geq 6$, S is algebraically hyperbolic.*

An interesting consequence is that we reobtain a proof of Harris' conjecture: a general quintic surface in $\mathbb{P}^3$ does not contain rational or elliptic curves (see also Remark (3.2)).

It should be noted that from the articles of Ein [E1], [E2] and Voisin [V], it follows that on a general surface of degree $d \geq 5$ there are no reduced irreducible curves with $g \leq \deg C(d-5)/2$.

The next possibility for surfaces in $I\!P^3$, which was also a starting point of our work, is to analyze the question of algebraic hyperbolicity for surfaces that are general in a given component of the Noether-Lefschetz locus, that is the locus of smooth surfaces of degree $d \geq 4$ in $I\!P^3$ whose Picard group is *not* generated by the hyperplane bundle. The problem was partly motivated by the simple observation that a surface of general type which is not algebraically hyperbolic and does not contain rational or elliptic curves, would be a counterexample to Kobayashi-Lang's conjecture. The simplest examples of surfaces in a component of the Noether-Lefschetz locus are those containing a fixed curve $D \subset I\!P^3$. The method of focal loci proved useful also in this case:

Theorem (1.3). *Let D be an integral curve in $I\!P^3$ and let s, d be two integers such that $d \geq s+4$ and*

(i) there exists a surface $Y \subset I\!P^3$ of degree s containing D,

(ii) the general element of the linear system $|\mathcal{O}_Y(dH-D)|$ is smooth and irreducible. Let S be a general surface of degree d in $I\!P^3$, containing D. Then S contains no reduced irreducible curves $C \neq D$ of geometric genus $g < 1 + \deg C(d-s-5)/2$. In particular, for $d \geq s+6$ and $g(D) \geq 2$, S is algebraically hyperbolic.

This result should be compared with what can be obtained adapting the methods of [V], namely that on S there are no reduced irreducible curves with $g < 1 + \deg C(d-\alpha-4)/2$, where α is a degree in which the homogeneous ideal of D is generated [Voisin, *priv. comm.*].

Finally we give another application to the case of projectively Cohen-Macaulay surfaces in $I\!P^4$, where the methods of Ein and Voisin do not seem to apply easily, because they are not complete intersections. To establish the notation, let $S \subset I\!P^4$ be a projectively Cohen-Macaulay surface and consider the minimal free resolution of its ideal sheaf $\mathcal{I}_S$

$$0 \to \bigoplus_{i=1}^{m} \mathcal{O}_{I\!P^4}(-d_{2i}) \xrightarrow{\phi} \bigoplus_{j=1}^{m+1} \mathcal{O}_{I\!P^4}(-d_{1j}) \xrightarrow{\psi} \mathcal{I}_S \to 0$$

where we assume $d_{2i} \geq d_{2,i+1}, d_{1j} \geq d_{1,j+1}$. Now set $u_{ij} = d_{2i} - d_{1j}$; note that the order chosen implies $u_{i+1,j} \leq u_{ij} \leq u_{i,j+1}$. We have

Theorem (1.4). *On a general projectively Cohen-Macaulay surface S in $I\!P^4$ such that $u_{m,m+1} \geq 6$ there are no reduced irreducible curves C of geometric genus $g < 1 + \deg C(u_{m,m+1}-7)/2$. In particular, for $u_{m,m+1} \geq 8$, S is algebraically hyperbolic.*

2. SOME BASIC FACTS ABOUT FOCAL SETS

In this section we recall the construction of the focal set of a projective family of curves. We refer to [CC] for more details.

Let $\mathcal{X} \to B$ be a family of hypersurfaces $X_t \subset \mathbb{P}^n, t \in B$ and let $\mathcal{C} \to B$ be a family of curves of geometric genus g such that $C_t \subset X_t$, for every $t \in B$. Let $\pi : \mathcal{C} \to \mathbb{P}^n$ be the corresponding map and denote by $z(\mathcal{C})$ the dimension of the image of π. Without loss of generality we can assume, by shrinking B, the existence of a global desingularization

$$\begin{array}{c} \sigma : \widetilde{\mathcal{C}} \to \mathcal{C} \\ \searrow \swarrow \\ B \end{array}$$

of all the fibers of $\mathcal{C}$. Let $0 \in B$ be a general point and denote by $X_0, C_0, \widetilde{C}_0$ and $\sigma_0 : \widetilde{C}_0 \to C_0$ the corresponding fibers. We have the basic

Proposition (2.1). *Let $s : \widetilde{\mathcal{C}} \to \mathcal{C} \hookrightarrow B \times \mathbb{P}^n$ be the composition, $\mathcal{N}$ the cokernel of the induced map $T_{\widetilde{\mathcal{C}}} \to s^*(T_{B\times\mathbb{P}^n})$ (the normal sheaf to s) and let N_0 be the restriction of $\mathcal{N}$ to the fiber $\widetilde{\mathcal{C}}_0$. Then*

(a) N_0 is the normal bundle to the composition $s_0 = \pi \circ \sigma_0$, i.e. the cokernel of the map $T_{\widetilde{C}_0} \to s_0^*(T_{\mathbb{P}^n})$;

(b) the family $\mathcal{C} \to B$ induces a characteristic map

$$\chi_0 : T_B \otimes \mathcal{O}_{\widetilde{C}_0} \to N_0$$

where T_B is the tangent space to B at 0; its rank at a general point of $\widetilde{C}_0$ is $z(\mathcal{C}) - 1$.

Proof: The first fact is Proposition 1.4 of [CC]; the second fact is shown in [CC, p.98]. ■

Definition (2.2). The *global focal set* F_0 of the family $\mathcal{C} \to B$ is the locus defined on $\widetilde{C}_0$ by $\bigwedge^{n-1} \chi_0 = 0$.

The global focal set F_0 has the following simple but useful properties.

Proposition (2.3). *Let $P \in \widetilde{C}_0$ be a point such that $Q = \sigma_0(P)$ is a smooth point of C_0 and X_0. We have*

(i) $F_0 = \widetilde{C}_0$ if and only if $z(\mathcal{C}) < n$;

(ii) if $z(\mathcal{C}) = n$ and Q is a fixed point of $\mathcal{X}$, then $P \in F_0$.

Proof: (i) is a consequence of Proposition (2.1)(b). To see (ii) observe that s factorizes via the inclusion $\mathcal{X} \hookrightarrow B \times \mathbb{P}^n$; hence we have a map

$$N_0 \to \sigma_0^*(N_{X_0/\mathbb{P}^n} \otimes \mathcal{O}_{C_0})$$

whose restriction to Q is surjective, since Q is a smooth point of X_0. Now, if Q is a fixed point of $\mathcal{X}$, then $\chi_0(T_B \otimes \mathcal{O}_{\widetilde{C}_0,P})$ is in the rank $n-2$ kernel of the previous map. ∎

The following result is the interpretation of Xu's method ([X1], [X2], [X3]) in terms of focal sets.

Proposition (2.4). *Let F_s be the subset of the global focal set F_0, supported at the points $P \in \widetilde{C}_0$ which map to the regular locus of C_0 and assume $z(\mathcal{C}) = n$. Then*

$$deg F_s \leq 2g - 2 + (n+1) deg C_0.$$

Proof: The first Chern class of N_0 is $2g-2+(n+1)\mathrm{deg}C_0$, by construction. In general, N_0 may have some torsion subsheaf $\mathcal{T}$, but in any case $\mathcal{T}$ is supported at the *cuspidal points* of $\widetilde{C}_0$, which map to the singular locus of C_0 (see [CC, 1.6]). Let $N_0' = N_0/\mathcal{T}$ and consider the composition $\chi_0' : T_B \otimes \mathcal{O}_{\widetilde{C}_0} \to N_0 \to N_0'$. By our assumptions on $z(\mathcal{C})$ this map is generically surjective and F_s is contained in the locus where χ_0' drops rank. It follows that $\mathrm{deg}F_s \leq c_1(N_0') \leq c_1(N_0)$. ∎

3. THE GENUS OF EFFECTIVE DIVISORS VIA THE FOCAL SET

The strategy that we will employ to show that on a general member of a given family of surfaces in $\mathbb{P}^n$, $\mathcal{S} \to B$, there are no curves of "low" geometric genus is the following. Supposing there is a family $\mathcal{C} \to B$ of curves on each surface, first find a collection of subfamilies $B(U) \subset B$ such that their tangent spaces span T_B and that there is a family of hypersurfaces $\mathcal{X}(U)$ containing the curves and with many fixed points. Then find one such U so that the corresponding family of curves $\mathcal{C}(U)$ satisfies $z(\mathcal{C}(U)) = n$ and apply Propositions (2.3) and (2.4) to get a lower bound for the geometric genus.

We first record an elementary lemma on vector spaces that we will use in all the proofs.

Lemma (3.1). *Let $f : V \to W$ be a linear map of finite dimensional vector spaces and suppose $\mathrm{dim}f(V) > z$. Let $\{V_i\}$ be a family of subspaces of V, such that V is generated by $\bigcup V_i$ and assume that for any pair of subspaces V_i, V_j in the family, we can find a chain of subspaces of the family $V_i = V_{i_1}, V_{i_2}, \ldots, V_{i_k} = V_j$ such that for all h, $\mathrm{dim}f(V_{i_h} \cap V_{i_{h+1}}) \geq z$. Then there is a V_i in the family with $\mathrm{dim}f(V_i) > z$.*

Proof: Assuming that $\mathrm{dim}f(V_i) \leq z$ for all i we will prove that, for any V_i, $f(V_i)$ generates $f(V)$, a contradiction. Indeed, for any j, take a chain $V_i = V_{i_1}, V_{i_2}, \ldots, V_{i_k} = V_j$ which links V_i and V_j. We have $\mathrm{dim}f(V_i) \leq z \leq \mathrm{dim}f(V_i \cap V_{i_2}) \leq \mathrm{dim}f(V_{i_2}) \leq z$ and also $z \leq \mathrm{dim}f(V_i \cap V_{i_2}) \leq \mathrm{dim}f(V_i) \leq z$; hence $f(V_i) = f(V_i \cap V_{i_2}) = f(V_{i_2})$. Repeating the argument, we finally get $f(V_i) = f(V_j)$. Since $\bigcup f(V_i)$ generate $f(V)$, we are done. ∎

Our first application will be to give a new proof of Xu's theorem ([X1, Theorem 2.1]).
Proof of Theorem (1.2): Take a family $\mathcal{S} \to B$ of surfaces of degree d in $\mathbb{P}^3$, with B dense in $\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^3}(d))$. Let $\mathcal{C} \to B$ be a family of reduced irreducible curves, such that for all $t \in B$, the fiber $C_t \subset S_t$ has geometric genus g and let $\widetilde{\mathcal{C}} \to B$ be a global desingularization of the fibers. Fix $0 \in B$ general and suppose, without loss of generality, that the authomorphisms of $\mathbb{P}^3$ act on B and moreover that S_0 does not contain any line. The action of PGL(3) on B shows that the image of $\mathcal{C}$ in $\mathbb{P}^3$ cannot be contained in any fixed surface; hence $z(\mathcal{C}) = 3$ and therefore the characteristic map of $\widetilde{\mathcal{C}}$ has rank 2 at a general point of $\widetilde{\mathcal{C}}$ by Proposition (2.1). For all surfaces U of degree $d-1$, transversal to C_0, and for $P \in C_0 - U$ general, let $B(U)$ be the subvariety of B parametrizing surfaces which contain $U \cap S_0$ and let $B(U,P)$ be the subvariety of $B(U)$ parametrizing surfaces passing through P. Denote by $T(U)$ and $T(U,P)$ their tangent spaces at 0 and by $\widetilde{\mathcal{C}}(U), \widetilde{\mathcal{C}}(U,P)$ the corresponding families of curves. Note that $\dim B(U) \geq 3$, $\dim B(U,P) \geq 2$. We claim that the characteristic map of $\widetilde{\mathcal{C}}(U,P)$ has rank 2 for U, P general. To prove this, let U, U' be two monomials of degree $d-1$ which differ only in degree one, that is, $M =$ l.c.m.(U, U') has degree d; then $B(U) \cap B(U')$ contains the pencil $S_0 + \lambda M$, which defines a non-trivial deformation of C_0, since C_0 is not in the base locus. It follows that any pair $T(U), T(U')$, with U, U' monomials, can be linked by a chain of subspaces of this type, in such a way that the intersection of two consecutive elements of the chain has non-trivial image under the characteristic map. Since T_B is generated by the tangent vectors to the varieties $B(U)$, with U monomial, and the characteristic map of $\widetilde{\mathcal{C}}$ has rank 2 at a general point, we get by Lemma (3.1) that the characteristic map of $\widetilde{\mathcal{C}}(U)$ has also rank 2 for some U. Similarly, fixing a general U, for any smooth points $P, P' \in C_0 - U$, the intersection $B(U,P) \cap B(U,P')$ contains a pencil $S_0 + \lambda M$ with $M = U\cdot$(some plane through PP'), which induces a non trivial deformation of C_0 (as C_0 is not a line). Hence applying Lemma (3.1) again one gets the claim.
Now look at the focal locus of $\widetilde{\mathcal{C}}(U,P)$: it contains the inverse image of $C_0 \cap U$ and P, and therefore, by Propositions (2.3) and (2.4), we have

$$2g - 2 \geq (d-1)\deg C_0 + 1 - 4\deg C_0$$

that is the required inequality on g, as $\deg C_0$ is a multiple of d by the Noether-Lefschetz theorem. ∎

(3.2) *Remark.* Note that the above proof makes use of the Noether-Lefschetz theorem only at the very last line, and in fact only for d even, $\deg C$ odd. In particular our proof of Harris' conjecture is independent of the Noether-Lefschetz theorem, while Xu's proof makes essential use of the fact that on a general surface every curve is a complete intersection. However Xu gets $g \geq 3$ for a curve on a general quintic.

We now consider a fixed integral curve $D \subset \mathbb{P}^3$ and the family of surfaces of degree $d \geq s+4$ containing it, where s and d are as in the hypotheses (i) and (ii) of Theorem (1.3).

Proof of Theorem (1.3): Act with PGL(3) on D and let $\mathcal{D}$ be the corresponding family of curves. Let $\mathcal{S} \to B$ be the family of surfaces of degree d in $\mathbb{P}^3$, containing some curve in $\mathcal{D}$ and let, as above, $\mathcal{C} \to B$ be a family of reduced irreducible curves, such that for all $t \in B$, the fiber $C_t \subset S_t$ has geometric genus g; let $\widetilde{\mathcal{C}} \to B$ be a global desingularization of the fibers. Fix $0 \in B$ general, call S_0 and C_0 the corresponding fibers and suppose again that PGL(3) acts on B. Let Y_0 be the surface as in (i) containing D_0. Suppose first that $C_0 \not\subset Y_0$. The action of PGL(3) gives, as above, $z(\mathcal{C}) = 3$; hence the characteristic map of $\widetilde{\mathcal{C}}$ has rank 2 by Proposition (2.1). For all surfaces U of degree $d-s-1$ transversal to D_0 and C_0, let $B(U)$ be the subvariety of B parametrizing surfaces which contain $U \cap S_0$; let $T(U)$ be its tangent space at 0 and $\widetilde{\mathcal{C}}(U)$ the corresponding family of curves. Note that $\dim B(U) \geq 2$, for D_0 is contained in many surfaces of degree $s+1$. As in the proof of Theorem (1.2) let U, U' be two monomials of degree $d-s-1$ which differ only in degree one, that is, $M = \text{l.c.m.}(U, U')$ has degree $d-s$. Then $B(U) \cap B(U')$ contains the pencil $S_0 + \lambda Y_0 M$, which defines a non-trivial deformation of C_0, since C_0 is not in the base locus. Therefore, by Lemma (3.1), we get that the characteristic map of $\widetilde{\mathcal{C}}(U)$ has rank 2 for U general. Now look at the focal locus of $\widetilde{\mathcal{C}}(U)$: it contains the inverse image of $C_0 \cap U$; hence by Propositions (2.3) and (2.4), we get

$$2g - 2 \geq (d-s-1)\deg C_0 - 4\deg C_0.$$

On the other hand if $C_0 \subset Y_0$, then, by (ii), C_0 is the smooth complete intersection of S_0 and Y_0; hence $2g-2 = (d+s-4)\deg C_0$. We have then proved that for a general curve $D_0 \in \mathcal{D}$, a general surface of degree d containing D_0 satisfies the assertion of the theorem. Since all the curves in $\mathcal{D}$ are projectively isomorphic to D, the theorem follows. ∎

(3.3) *Remark.* The bound of Theorem (1.3) can be improved in some cases by fixing some base points in the family $B(U)$, as we did in the proof of Theorem (1.2).

(3.4) *Remark.* When D is rational or elliptic, we still have the algebraic hyperbolicity of the open surface $S - D$, for $d \geq s+6$. More than that, we know that every nonconstant map $\mathbb{C} \to S$ has image contained in D.

(3.5) *Remark.* The proofs of Theorems (1.2) and (1.3) can be easily extended both to the case of general complete intersection surfaces $S \subset \mathbb{P}^r$ and to the case of general complete intersection surfaces $S \subset \mathbb{P}^r$ containing a fixed curve D.

Finally we deal with the case of projectively Cohen-Macaulay surfaces $S \subset \mathbb{P}^4$. Let us recall some notation from the introduction. We denote by $M_S = [A_{ij}]$ and $[F_1, \ldots, F_{m+1}]$ the matrices of the maps ϕ and ψ, respectively, appearing in the resolution of the ideal sheaf of S. As is well known, either A_{ij} is a polynomial of degree u_{ij} if $u_{ij} \geq 0$ or $A_{ij} = 0$ if $u_{ij} < 0$ and, by the Hilbert-Burch theorem, we can assume that F_j is the determinant of the minor obtained by removing the j-th column from M_S.

Proof of Theorem (1.4): Take a family $\mathcal{S} \to B$ of projectively Cohen-Macaulay surfaces $S_t \subset \mathbb{P}^4, t \in B$ with B dense and let $\mathcal{X} \to B$ be the corresponding family of hypersurfaces $X_t \subset \mathbb{P}^4$ defined by the minors $F_{1,t}$ of the matrix M_{S_t}. Note that by [Ch] we can assume that $\dim Sing(X_t) = 0$, as $u_{ii} > 0$ by minimality. Let $\mathcal{C} \to B$ a family of reduced irreducible curves, such that for all $t \in B$, the fiber C_t has geometric genus g and is contained in S_t and let $\widetilde{\mathcal{C}} \to B$ be a global desingularization of the fibers. Fix $0 \in B$ general and suppose that PGL(4) acts on B. The action of PGL(4) on B shows that the image of $\mathcal{C}$ in $\mathbb{P}^4$ cannot be contained in any fixed proper subvariety; hence $z(\mathcal{C}) = 4$ and the characteristic map of $\widetilde{\mathcal{C}}$ has rank 3 by Proposition (2.1). For all hypersurfaces U of degree $u = u_{m,m+1} - 2$, transversal to C_0 and such that $U \cap C_0 \cap Sing(X_0) = \emptyset$, let $B_1(U)$ be the subvariety of $t \in B$ parametrizing projectively Cohen-Macaulay surfaces such that the hypersurface X_t contains $U \cap C_0$. Now let $B(U)$ be an irreducible component of $B_1(U)$ containing 0 and the points in $B_1(U)$ parametrizing projectively Cohen-Macaulay surfaces whose matrix is of type

$$\begin{bmatrix} A_{11} & \ldots & A_{1m} & UG_1 \\ \vdots & & & \vdots \\ A_{m1} & \ldots & A_{mm} & UG_m \end{bmatrix}.$$

Let $T(U)$ be the tangent space to $B(U)$ at 0 and $\widetilde{\mathcal{C}}(U), \mathcal{X}(U)$ the corresponding families of curves and threefolds. We will prove that the characteristic map of $\widetilde{\mathcal{C}}(U)$ has rank 3 for U general. To prove this using Lemma (3.1), we will show first that if U, U' are monomials which differ only in degree one, i.e. $V = \text{l.c.m.}(U, U')$ has degree $u + 1$, then the characteristic map on $B(U) \cap B(U')$ has rank at least 2. In fact by Proposition (2.1) it is enough to show that the corresponding curves $C_t, t \in B(U) \cap B(U')$, fill up a variety $\overline{\Sigma} \subset \mathbb{P}^4$ of dimension at least 3. Suppose to the contrary $\dim\overline{\Sigma} \leq 2$. As we can assume that V is transversal to C_0, it is necessarily $\dim\overline{\Sigma} \cap V \leq 1$, since $C_0 \subset \overline{\Sigma}$. What we need follows then by the

Claim (3.6). For all monomials U, U', V as above and for every variety $\Sigma \subset \mathbb{P}^4$ such that $\dim\Sigma \leq 2$ and $\dim\Sigma \cap V \leq 1$, there exists $t \in B(U) \cap B(U')$ such that $\dim S_t \cap \Sigma = 0$.

Proof of Claim (3.6): We choose $t \in B(U) \cap B(U')$ so that the matrix of S_t is a general one of type

$$\begin{bmatrix} A_{11} & \dots & A_{1m} & VH_1 \\ \vdots & & & \vdots \\ A_{m1} & \dots & A_{mm} & VH_m \end{bmatrix}$$

that is, the polynomials A_{ij} and H_i (of degree $u_{i,m+1} - u - 1$) are general. Let T be the projectively Cohen-Macaulay surface defined by the vanishing of the $(m-1) \times (m-1)$ minors of the matrix $[A_{ij}, 1 \le i \le m, 2 \le j \le m]$ and F_1, F_{m+1} the two distinguished generators of the ideal of S_t. Clearly $\dim T \cap \Sigma = 0, \dim F_{m+1} \cap \Sigma = 1$ and F_{m+1} does not contain any component of $V \cap \Sigma$, as the entries A_{ij} are general. Now suppose that $S_t \cap \Sigma$ has a component D of dimension at least one and let $x \in D$ be a general point. Then $x \notin V$. As $D \not\subset T$, there is an $(m-1) \times (m-1)$ minor of the matrix of T not vanishing on x; hence $x \notin F_1$, by a general choice of the H_i. This contradiction proves Claim (3.6). To finish the proof of the theorem just proceed in analogy with the previous proofs. Since T_B is generated by the tangent vectors to the varieties $B(U)$, with U monomial and the characteristic map of $\widetilde{\mathcal{C}}$ has rank 3 at a general point, we get by Lemma (3.1) that the characteristic map of $\widetilde{\mathcal{C}}(U)$ has also rank 3. Now look at the focal locus of $\widetilde{\mathcal{C}}(U)$: it contains the inverse image of $C_0 \cap U$, as they are fixed points of the family of hypersurfaces $\mathcal{X}(U)$; hence by Propositions (2.3) and (2.4),

$$2g - 2 \ge u\deg C_0 - 5\deg C_0. \quad \blacksquare$$

(3.7) *Remark.* It is clear that if we drop the hypothesis $u_{m,m+1} \ge 8$ in Theorem (1.4) there can be rational or elliptic curves on a general projectively Cohen-Macaulay surface S in $\mathbb{P}^4$. For example the Castelnuovo surface contains elliptic curves.

(3.8) *Remark.* The Picard group of the surfaces considered in our theorems is, in many cases, particularly simple. In the case of Theorem (1.2) it is just generated by the hyperplane bundle, by the Noether-Lefschetz theorem. In the hypothesis of Theorem (1.3) it follows by [Lo, Corollary II.3.8] that S has Picard group generated by the hyperplane bundle H and by the line bundle associated to D. For the general projectively Cohen-Macaulay surface $S \subset \mathbb{P}^4$ the situation is quite different. If $u_{ij} > 0$ for every i, j, the Picard group is generated by the hyperplane bundle H and by the canonical bundle of S [Lo, Theorem III.4.2], while in general it can have large rank. Despite of the simplicity of the Picard groups in many cases, there does not seem to be a way to make use of it, as the theorems are concerned with the possible singularities of the curves. As a matter of fact our proofs are independent of the knowledge on the Picard group.

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