ON THE IRREDUCIBILITY OF SECANT CONES, AND AN APPLICATION TO LINEAR NORMALITY

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Abstract. Given a smooth subvariety of dimension $> \frac{2}{3}(r-1)$ in $\mathbb{P}^r$, we show that the double locus (upstairs) of its generic projection to $\mathbb{P}^{r-1}$ is irreducible. This implies a version of Zak's Linear Normality theorem.

A classical, and recently revisited (see [GP, L, Pi] and references therein), method for studying the geometry of a subvariety

$$Y \subset \mathbb{P}^r$$

is to project Y generically to a lower-dimensional projective space, for example so that Y maps birationally to a (singular) hypersurface

$$\bar{Y} \subset \mathbb{P}^{m+1}.$$

To make use of this method, it is usually important to have precise control over the singularities of $\bar{Y}$ and in particular over the entire singular (i.e. double) locus D_Y of $\bar{Y}$ and its inverse image C_Y in Y. As the dimension of these is easily determined, a natural question is: are C_Y and D_Y irreducible? This question plays an important role, for instance, in Pinkham's work on regularity bounds for surfaces [Pi]. The purpose of this note is to show that this irreducibility holds provided the codimension of Y is sufficiently small compared to its dimension (see Theorems 1,2 and Corollary 3 below). As an application we give a proof of Zak's linear normality theorem (in a slightly restricted range, see Corollary 5 below). Indeed the results seem closely related as our argument ultimately depends on having a bound on the dimension of singular loci of hyperplane sections, manifested in the form of the integer $\sigma(Y)$ (see Thm. 1 below), and it is Zak's theorem on tangencies- also a principal ingredient in other proofs of linear normality- that gives us good control over $\sigma(Y)$.

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We begin with some definitions. Let Y denote an irreducible $m-$ dimensional subvariety of $\mathbb{P}^r$. As usual, we mean by a *secant line* of Y a limit of lines in $\mathbb{P}^r$ spanned by pairs of distinct points of Y. The union of all secant lines is denoted by $\mathrm{Sec}(Y)$. Y is said to be *J-projectable* if $\mathrm{Sec}(Y) \subsetneq \mathbb{P}^r$. Y is said to be *projectable* if it is J-projectable and moreover the union of all embedded Zariski tangent spaces is contained in a proper subvariety of $\mathbb{P}^r$. Of course, if Y is smooth and J-projectable then it is projectable. For any linear subspace $Q \subset \mathbb{P}^r$, we let

$$\pi_Q : \mathbb{P}^r - Q \to \mathbb{P}^{r - \dim Q - 1}$$

denote the associated projection.

For a nondegenerate projective variety Y, let $\sigma(Y)$ denote the maximum dimension of a (locally closed) subvariety $Z \subset Y_{\text{smooth}}$ such that

(i) Z contains a generic point of some divisor on Y;

(ii) the tangent planes T_yY for all $y \in Z$ are contained in a fixed hyperplane H (i.e. Z is contained in the singular locus of $Y \cap H$).

Note that if Y is nonsingular in codimension 1 then assumption (i) above for a subvariety $Z \subset Y$ already implies that $Z \cap Y_{\text{smooth}}$ is dense in Z . Zak's Tangency theorem (cf. [F, Z1]) implies that if Y is smooth then

$$\sigma(Y) \le r - m - 1. \tag{1}$$

Theorem 1. *Let*

$$Y \subset \mathbb{P}^r$$

be m-dimensional, normal, irreducible and non-J-projectable, and let $Q \subset \mathbb{P}^r$ be a generic (resp. arbitrary) linear subspace disjoint from Y. Assume that $\dim Q < r - m - 1$ and that

$$2m > r + \sigma(Y) - 1. \tag{2}$$

Then the double locus of $\pi_{Q|Y}$ (equal to the locus of points $y \in Y$ such that $\pi_Q^{-1}(\pi_Q(y)) \cap Y \neq \{y\}$ as schemes) is irreducible (resp. connected).

Now the case where $\dim Q > 0$ (no other hypotheses needed but non-J-projectability) is an easy and well known consequence, due to Franchetta, of Bertini's Theorem (see [Mo], p.115 or [Pi] or below for a proof due to D. Mumford), so the only new conclusion is when Q is a point and as usual, the case Q arbitrary follows easily by connectedness principles from the case Q generic. For this case, it is convenient to shift our viewpoint slightly, as follows.

Let $Y^{[2]}$ denote the normalization of the blow-up of $Y \times Y$ along the diagonal Δ_Y, with exceptional divisor E_Y. Let I_Y denote the tautological $\mathbb{P}^1$-bundle (or incidence variety) over $Y^{[2]}$ (equal to the pullback of the analogous object over $(\mathbb{P}^r)^{[2]}$). Being a $\mathbb{P}^1-$ bundle over a normal variety, I_Y is also normal and we have a diagram

$$\begin{array}{ccc} I_Y & \xrightarrow{f} & \mathbb{P}^r \\ \pi \downarrow & & \\ Y^{[2]}, & & \end{array} \tag{3}$$

where non-J-projectability means f is surjective. Now it follows easily from the classical trisecant Lemma (most secants are not multisecant) that for a generic linear subspace Q, $f^{-1}(Q)$ is birational to the double locus of $\pi_{Q|Y}$. If $\dim(Q) > 0$, then $f^{-1}(Q)$ is automatically irreducible by Bertini's theorem, which already proves Theorem 1 for this case. This result is due originally to Franchetta [Fr2]; the foregoing argument is due to Mumford and is given in [Mo].

Let us say that Y has the *irreducible secant cone* (ISC) property if a generic fibre of f is irreducible. Then the remaining case $\dim(Q) = 0$ of Theorem 1 follows from (indeed, is equivalent to) the following

Theorem 2. *Hypotheses as in Theorem 1, Y has the ISC property.*

In view of Zak's Tangency theorem, Theorem 2 implies the following result (which will shortly be improved below):

Corollary 3(temporary version). *If $Y \subset \mathbb{P}^r$ is smooth, non-projectable with*

$$\dim(Y) > \frac{2}{3}(r-1),$$

then Y has the ISC property.

Next, we note an interesting connection between the ISC property and linear normality which will imply a version of Zak's theorem on linear normality and hence allow us to remove the nonprojectibility hypothesis from the above corollary.

Proposition 4. *Let $X \subset \mathbb{P}^N$ be an irreducible variety and $M \subseteq \mathbb{P}^N$ a generic linear subspace and set $Y = X \cap M$. Assume that Y is nonsingular and, as subset of M, is nonprojectable and has the ISC property. Then X is J-linearly normal, i.e. not the bijective projection of a nondegenerate subvariety of $\mathbb{P}^{N+1}$. In particular, if X itself is nonsingular nonprojectable and has the ISC property, then X is linearly normal.*

Proof. Assume to begin with that $M \subsetneq \mathbb{P}^N$. Suppose for contradiction that X is the bijective projection of a nondegenerate variety

$$\tilde{X} \subset \mathbb{P}^{N+1}$$

from a point $Q \in \mathbb{P}^{N+1} - \tilde{X}$. Our assumptions about the inclusion $Y \subset M$ mean that the secant variety $\mathrm{Sec}(Y)$ coincides with M 'with multiplicity 1', in the sense that, for a generic linear space

$$L = \mathbb{P}^c \subset \mathbb{P}^N,$$

where $c = N - \dim(M)$, the scheme-theoretic inverse image $f^{-1}(L)$ is a reduced irreducible subvariety of codimension $N - c$ lying over a single point of L (viz. $L \cap M$); i.e. $f^{-1}(L)$ coincides with the fibre of f considered as a map

$$I_Y \to \mathrm{Sec}(Y) = M$$

over (the point) $L \cap \mathrm{Sec}(Y)$ which is a general point of $\mathrm{Sec}(Y)$.

On the other hand, note that M is the projection of a unique codimension-c linear subspace of $\mathbb{P}^{N+1}$ containing Q, say A, and $Y \simeq \tilde{X} \cap A$. Consequently, Y can be viewed as a specialization of a smooth subvariety

$$Y' \subset X,$$

which is the (isomorphic) projection of a generic codimension-c linear space section

$$\tilde{X} \cap A' \subset \mathbb{P}^{N+1}, Q \notin A'.$$

Note that Y' spans a $\mathbb{P}^{N-c+1}$ which we denote by M'. By semi-continuity, similar assertions as for f_Y must hold also for the analogously-defined map

$$f_{Y'} : I_{Y'} \to \mathbb{P}^N;$$

thus $f_{Y'}^{-1}(L)$ is reduced, irreducible and of codimension $N-c = \mathrm{codim}(L, \mathbb{P}^N)$. This implies firstly that $\mathrm{Sec}(Y')$ is $(N-c)$-dimensional; then since $f_{Y'}^{-1}(L)$ has a component over each point of $L \cap \mathrm{Sec}(Y')$, the only way $f_{Y'}^{-1}(L)$ can be irreducible is if $\mathrm{Sec}(Y')$ is a linear $\mathbb{P}^{N-c}$, which contradicts the fact that Y' spans M' of dimension $N-c+1$. This completes the proof in case $M \subsetneq \mathbb{P}^N$.

Finally in the case where $M = \mathbb{P}^N$, we replace X by its cone $CX \subset \mathbb{P}^{N+1}$, viewing M as a hyperplane in $\mathbb{P}^{N+1}$. By the case already proven, we conclude that CX is J-linearly normal, hence so is X. $\square$

As an application of Proposition 4 together with Corollary 3 (temp), we obtain a proof of a version of Zak's linear normality theorem (cf. [Z2], Thm. II.2.14):

Corollary 5. *Let $X \subset \mathbb{P}^N$ be irreducible, nondegenerate, and set $b = 0$ if X is smooth and otherwise $b = \dim \mathrm{Sing}(X)$. Assume that*

$$\dim(X) > \frac{1}{3}(2N + b - 1).$$

Then X is J-linearly normal, i.e. not the image of the bijective projection of a nondegenerate subvariety of $\mathbb{P}^{N+1}$.

Proof. We use induction on $\dim(X)$. We apply Proposition 4 in the case where $M \subset \mathbb{P}^N$ be a generic $\mathbb{P}^{N-b-1}$ and $Y = X \cap M$. Thus Y is smooth and spans M. If Y is nonprojectable within M we are done, so suppose Y is projectable within M and let Y', M' be as in the proof of Proposition 4. Then the nondegenerate nonsingular subvariety

$$Y' \subset M' = \mathbb{P}^{N-b}$$

of dimension $\dim(X) - b - 1$ is projectable to $\mathbb{P}^{N-b-2}$, which contradicts our induction hypothesis. $\square$

Remarks. (1) For X smooth, Zak's linear normality theorem covers the larger range $\dim(X) > \frac{1}{3}(2N-2)$.

(2) The basic idea of the foregoing argument goes back to [R1], and a similar idea was recently used by Brandigi [B] to prove linear normality in the range $\dim(X) \geq \frac{3}{4}N$.

(3) Corollary 5 is sharp: to see this let Z be a smooth Severi variety (see [Z2]) in $\mathbb{P}^r$, that is Z is projectable and $\dim(Z) = \frac{2}{3}(r-2)$. Set $N = r + b$ and let $Z' \subset \mathbb{P}^{N+1}$ be the cone over Z with vertex a $\mathbb{P}^b$, $X \subset \mathbb{P}^N$ its generic (isomorphic) projection. Then $\dim(X) = \frac{1}{3}(2N + b - 1)$ and dim $\mathrm{Sing}(X) = b$.

Given Corollary 5, we can sharpen slightly the statement of Corollary 3(temp):

Corollary 3. *Let $Y \subset \mathbb{P}^r$ be smooth nondegenerate with*

$$\dim(Y) > \frac{2}{3}(r-1).$$

Then Y is non-projectable and has the ISC property.

Proof. By Corollary 3(temp), it suffices to prove that Y is non-projectable. If not, apply Corollary 5 to the generic projection of Y to $\mathbb{P}^{r-1}$ to deduce a contradiction. $\square$

Remark 6. Again Corollary 3 is sharp: for this let $Y \subset \mathbb{P}^r$ be the generic projection of a Severi variety (cf. Remark (3) following Corollary 5). Then Y is smooth, non-projectable and does not have the ISC property (e.g. because the cone on Y is not linearly normal). Also, the Corollary fails without the smoothness hypothesis on Y, a counterexample being provided by any surface in $\mathbb{P}^4$ with a double or multiple line (e.g. a surface arising as the projection of a surface on $\mathbb{P}^5$ containing a plane curve from a point on the plane).

It is amusing, perhaps, to translate the irreducibility conclusion of Corollary 3 into cohomology (taking for granted the nonprojectability conclusion). Let F denote a general fibre of f. Then F is smooth, nonempty and $(2\dim(Y)+1-r)$-dimensional. Clearly irreducibility (i.e. connectedness) of F is equivalent, provided $q(Y)=h^1(\mathcal{O}_Y)=0$, to the vanishing

$$H^1(I_Y, \mathcal{I}_F)=0, \tag{4}$$

where $\mathcal{I}_F$ denotes the ideal sheaf of F; in the dimension range in question, $q(Y)=0$ automatically by the Barth-Larsen Theorem. Pulling back the Koszul resolution of the ideal sheaf of a point in $\mathbb{P}^r$ and using standard vanishing results (e.g. [SS], Thm. 7.1) which imply that

$$H^i(I_Y, f^*(\mathcal{O}(-j)))=0, \forall i<r, j>0,$$

we see easily that (4) is equivalent to the vanishing

$$H^r(I_Y, f^*(\mathcal{O}(-r)))=0. \tag{5}$$

Let $\mathcal{E}$ denote the tautological subbundle on $Y^{[2]}$, so that $I_Y=\mathbb{P}(\mathcal{E})$. Then by standard computations the vanishing (5) reduces to the vanishing on $Y^{[2]}$:

$$H^{r-1}(Y^{[2]}, \mathrm{Sym}^{r-2}(\mathcal{E}^v)\otimes\det(\mathcal{E}^v))=0. \tag{6}$$

Corollary 7. *With hypotheses as in Corollary 3, the vanishing (6) holds.*

Trying to find a direct proof of Corollary 7 might seem like a promising route to a cohomological proof of Corollary 3, but we were unable to find such a direct proof. This still looks like an intriguing, though difficult problem.

The following Corollary was graciously pointed out by the referee.

Corollary 8. *Let*

$$Y\subset\mathbb{P}^r$$

be an irreducible, smooth, non-projectable m-dimensional subvariety with normal bundle N such that $2m+1>r$ and $N(-1)$ is ample (e.g. $N(-2)$ is globally generated). Then Y has the ISC property.

Proof. In view of Theorem 1, this follows immediately from the well-known fact that in this case we have $\sigma(Y)=0$ (cf. [FL], p. 60). $\square$

As a (surprisingly nonobvious, perhaps) special case, we conclude the following

Corollary 9. *Let $Y \subset \mathbb{P}^r$ be an $m-$dimensional smooth complete intersection with $2m+1 > r$. Then Y has the ISC property.*

Proof. With no loss of generality we may assume Y is nondegenerate, i.e. a complete intersection of hypersurfaces of degree > 1. Then it is well known that Y has a maximal dimensional secant variety: one way to see this is to first observe that it suffices to prove it for Y a *generic* complete intersection and then to specialize Y to the intersection of hypersurfaces each of which is a generic union of hyperplanes, for which the assertion is obvious. Thus Y is nonprojectable, so the result follows from Corollary 8. □

We now give the proof of Theorem 2, letting notations and assumptions be as there. The basic idea is the following. Consider a Stein factorization of f:

$$I_Y \to Z \xrightarrow{g} \mathbb{P}^r,$$

where Z is normal and g is generically finite and surjective. Now it is a general fact that if $h : W \to T$ is a morphism of irreducible varieties and W is normal, then so is a general fibre of h: this can be seen, e.g. using Serre's criterion, or alternatively, use [G], Thm 12.2.4, which says, in the scheme-theoretic context, that the set $N(h)$ of points $t \in T$ such that $h^{-1}(t)$ is normal is open; when W is normal, $N(h)$ contains the generic point (in the scheme-theoretic sense) of T, hence also a nonempty open set of closed points. In our case, since I_Y is normal, it follows that so is a generic fibre of f, therefore the irreducible and connected components of this fibre coincide (cf. [E], Thm. 18.12). Consequently the degree of g coincides with the number of irreducible components (which equals the number of connected components) of the general fibre of f, so the Theorem's assertion is that g is birational.

Then there is a Zariski open $U \subset \mathbb{P}^r$ such that $\mathbb{P}^r - U$ has codimension > 1 and $g^{-1}(U) \to U$ is finite, and we may assume $g^{-1}(U)$ is smooth as well. Since U, like $\mathbb{P}^r$, is simply connected, it follows that if $\deg(g) > 1$ then g, hence f is *ramified in codimension 1*, i.e. there is a prime divisor $F \subset I_Y$ such that $f(F) \subset \mathbb{P}^r$ is a divisor and f is ramified on F. We proceed to show that the latter conclusion leads to a contradiction.

Now it is an easy consequence of the Fulton-Hansen Connectedness Theorem (cf. [FL], Corollary 5.5) that in our case we have

$$f(\pi^{-1}(E_Y)) = \mathbb{P}^r,$$

hence $F \neq \pi^{-1}(E_Y)$, and therefore a general point of F is of the form (x, y, z) where $x, y \in Y$ are distinct and

$$z \in < x, y >$$

($< x, y >$ denotes the line spanned by x, y). Now a standard computation known as Terracini's Lemma [FR] says that

$$\text{im } df_{(x,y,z)} =< T_xY, T_yY >,$$

and in particular this image is independent of $z \in < x, y >$. It follows that F is the pullback of a divisor $D \subset Y^{[2]}$, where a general point $(x, y) \in D$ has the property that $x \neq y$ and

$$\rho := \dim < T_x Y, T_y Y > < r.$$

We may assume that the projection map $p_1 : D \to Y$ is surjective, and let $D_x \subset Y$ denote the image of its general fibre under p_2, which is a divisor on Y. Setting $W = T_x Y$, note that a general $y \in D_x$ is smooth on Y and we have

$$\rho - 1 \leq \dim < T_y D_x, W > \leq \rho. \tag{7}$$

Now consider the following diagram, with vertical arrows only rational maps induced by projection from W:

$$\begin{array}{ccccc} S_{x,v} & \subset & D_x & \subset & \mathbb{P}^r \\ \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ v & \in & V_x & \subset & \mathbb{P}_x^{c-1}. \end{array} \quad \pi_W \tag{8}$$

Here $c = r - m, V_x$ is the (closure of the) image of D_x, $v \in V_x$ is a general point and $S_{x,v} = \pi_W^{-1}(v)$, which we may assume contains a general point of D_x. By (7), the dimension of V_x is either $\rho - m - 1$ or $\rho - m - 2$, and in these respective cases we have $\dim S_{x,v} = 2m - \rho$ (resp. $2m - \rho + 1$)(note that the second case is evidently impossible if $r = m + 2$). Though not essential for our purposes, it is interesting to note that when x is viewed as variable, a general hyperplane in $\mathbb{P}_x^{c-1}$ corresponds in $\mathbb{P}^r$ to a general tangent hyperplane to Y, i.e. a general element of the dual variety Y^*.

Now suppose that V_x is of dimension $\rho - m - 1$, so that $S_{x,v}$ is of dimension $2m - \rho$. Note that by (7) this implies that for general $y \in D_x$,

$$< T_y Y, W > = < T_y D_x, W >,$$

which projects modulo W to $T_v V_x, v = \pi_W(y)$. Now for a linear subspace $U \subset \mathbb{P}_x^{c-1}$, we denote by $\pi_W^*(U)$ the unique linear subspace of $\mathbb{P}^r$ which contains W and projects to U (this is uniquely determined by U). Then we conclude that for general $y \in S_{x,v}$, we have

$$T_y Y \subset \pi_W^*(T_v V_x).$$

Thus the linear space $\pi_W^*(T_v V_x)$ of dimension $\rho \leq r - 1$ is tangent to Y along a locus of dimension at least $2m - \rho \geq 2m - r + 1$, contradicting (2).

Suppose now that V_x is of dimension $\rho - m - 2$, so $S_{x,v}$ is of dimension $2m - \rho + 1$. Then for $y \in S_{x,v}$, the projection of $T_y Y$ to $\mathbb{P}_x^{c-1}$ is a $\mathbb{P}^{\rho - m - 1}$

containing $T_vV_x = \mathbb{P}^{\rho-m-2}$, and the set of all these linear subspaces is a $\mathbb{P}^{r-\rho}$, so imposing such a subspace to stay fixed is $r-\rho$ conditions. Thus, pulling back to $\mathbb{P}^r$, we can find a subvariety T of codimension at most $r-\rho$ in $S_{x,v}$, containing a general point of $S_{x,v}$ (hence of D_x), such that T_yY is contained in a fixed $\mathbb{P}^\rho$ for all $y \in T$. Since

$$\dim T \geq 2m - \rho + 1 - (r - \rho) = 2m - r + 1,$$

this again contradicts (2). $\square$

Remark 10. Note that the foregoing argument is unaffected by removing from Y a subset of codimension 2. Therefore, rather than assume $Y \subset \mathbb{P}^r$, it suffices to assume we have a morphism $Y \to \mathbb{P}^r$ that is an embedding through codimension 1.

Examples 11. By Corollary 3, any smooth 3-fold in $\mathbb{P}^5$ has the ISC property. A specific example is provided by the Segre variety

$$Y = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$$

(cf. [SR]). Any $\mathbb{P}^1 \times \mathbb{P}^1$ is embedded as a quadric in a solid, and there is precisely one such solid through a general point $Q \in \mathbb{P}^5$. Projection from Q maps Y to a cubic hypersurface whose double locus is a double plane that is the image of the aforementioned quadric .

On the other hand, if Y is a smooth surface in $\mathbb{P}^4$, Theorem 2 says that Y has the ISC property *unless* $\sigma(Y) = 1$, i.e. unless Y admits a hyperplane section with a multiple component. For example, the projected Veronese surface $Y \subset \mathbb{P}^4$ admits a 1-parameter family of multiple hyperplane sections: namely the Veronese in $\mathbb{P}^5$ has a 2-parameter family of double-conic hyperplane sections, corresponding to double lines in $\mathbb{P}^2$, and ∞^1 of these descend to $\mathbb{P}^4$. Indeed Y does not have the ISC property; in fact, the double curve of its generic projection to $\mathbb{P}^3$ consists of 3 conics on Y mapping 2:1 to 3 concurrent lines on $\bar{Y} \subset \mathbb{P}^3$ (the latter is known as the Steiner surface). On the other hand a cone in $\mathbb{P}^4$ over a curve in $\mathbb{P}^3$ has $\sigma = 1$ and does not have the ISC. See [GH], pp. 628-635 for this and other interesting examples.

In fact, Franchetta [Fr1], [En] (see [MP] for a modern proof) proved that for any smooth (or even immersed) surface in $\mathbb{P}^4$ or higher, other than the Veronese, the double curve of its generic projection to $\mathbb{P}^3$ is irreducible. In this generality, this assertion does not follow from our results because there are many immersed surfaces in $\mathbb{P}^4$ (e.g. scrolls) with $\sigma = 1$ (although all known (to us) examples with $\sigma = 1$ other than the Veronese and some scrolls actually come from $\mathbb{P}^5$, so the ISC for them follows from the easy Mumford-Bertini argument mentioned above).

In the case of codimension 2, our result can be improved somewhat:

Theorem 12. *Let Y be a smooth, irreducible, $m-$dimensional variety and*

$$f : Y \to \mathbb{P}^{m+2}$$

a morphism that is an embedding off a codimension-2 subset. Then $f(Y)$ has the ISC property unless Y admits a nontrivial algebraic family of divisors D_α such that for general $y \in D_\alpha$, $T_{f(y)}f(Y)$ is contained in a fixed hyperplane H_α.

Proof. Going back to the proof of Theorem 2, note that in our case V_x is evidently a point (i.e. of dimension $\rho - m - 1 = 0$), so

$$S_{x,v} = D_x.$$

In the above proof it was already shown that for general $y \in S_{x,v} = D_x$, $T_{f(y)}f(Y)$ is contained in a fixed hyperplane, so it suffices to prove that the divisor D_x cannot be fixed independent of x, unless $f(Y)$ is degenerate. Suppose $D_x = D$ for all x. Recall that for $y \in D$ general we have (dropping the f for convenience of notation)

$$\dim(< T_xY, T_yY >) \ < \ r = m + 2,$$

i.e.

$$\dim(T_xY \cap T_yY) \ \geq \ m - 1.$$

In particular, we have

$$\dim(T_{y_1}Y \cap T_{y_2}Y) \geq m - 1, \ y_1, y_2 \in D.$$

By our assumptions on the singularity of f, the Gauss map associated to Y (and f) is defined in a neighborhood of a complete curve C in general position on D. By results on the structure of the Gauss mapping of Y (cf. [R2]), this mapping cannot be constant on C without being constant on Y, which would make $f(Y)$ a linear space, which is not the case. Hence we may assume $T_{y_1}Y \neq T_{y_2}Y$ for general $y_1, y_2 \in D$. Therefore either

$$T_xY \supset T_{y_1}Y \cap T_{y_2}Y$$

or

$$T_xY \subset < T_{y_1}Y, T_{y_2}Y > \subsetneq \mathbb{P}^r.$$

In the first case we get that all tangent spaces to Y contain a fixed $m-1$-plane so Y itself is an $m-$plane. In the second case we get that Y is in a hyperplane. This completes the proof. $\square$

Remark. The Veronese and some scrolls are the only immersed surfaces we know in $\mathbb{P}^4$ which admit a 1-parameter family of nonreduced hyperplane sections. If these could be shown to be the only surfaces with this property then Theorem 12 might imply Franchetta's assertion that any immersed surface besides the Veronese has the ISC property.

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