# Explicit Noether-Lefschetz for arbitrary threefolds [1]

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*Abstract*

We study the Noether-Lefschetz locus of a very ample line bundle $L$ on an arbitrary smooth threefold $Y$. Building on results of Green, Voisin and Otwinowska, we give explicit bounds, depending only on the Castelnuovo-Mumford regularity properties of $L$, on the codimension of the components of the Noether-Lefschetz locus of $|L|$ .

## 1. *Introduction.*

It is well-known in algebraic geometry that the geometry of a given variety is influenced by the geometry of its subvarieties. It less common, but not unusual, that a given ambient variety forces to some extent the geometry of its subvarieties.

A particularly nice case of the latter is given by line bundles, whose properties do very much influence the geometry.

If $Y$ is a smooth variety and $i : X \hookrightarrow Y$ is a smooth divisor, there is then a natural restriction map

$$i^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$$

given by pull-back of line bundles.

Now suppose that $X$ is very ample. By the Lefschetz theorem $i^*$ is injective if $\dim Y \geq 3$. On the other hand, it was already known to the Italian school (Severi [**18**], Gherardelli [**6**]), that $i^*$ is surjective when $\dim Y \geq 4$. Simple examples show that in the case where $\dim Y = 3$ we cannot hope for surjectivity unless a stronger restriction is considered.

For the case $Y = \mathbb{P}^3$, this is also a classical problem, first posed by Noether and solved in the case of generic $X$ by Lefschetz who showed that

**Theorem (Noether-Lefschetz)** *For $X$ a generic surface of degree $d \geq 4$ in $\mathbb{P}^3$ we have $\mathrm{Pic}(X) \cong \mathbb{Z}$.*

Here and below by generic we mean outside a countable union of proper subvarieties.

Suppose now that a smooth threefold $Y$ and a line bundle $L$ on $Y$ are given. We will say that a Noether-Lefschetz theorem holds for the pair $(Y, L)$, if

$$i^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$$

is a surjection for a generic smooth surface $X \subset Y$ such that $\mathcal{O}_Y(X) = L$.

The following result of Moishezon ([**14**], see also the argument given in Voisin [**21**, Thm. 15.33]) establishes the exact conditions under which a Noether-Lefschetz theorem holds for $(Y, L)$.

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**Theorem (Moishezon)** *If $(Y, L)$ are such that $L$ is very ample and*

$$h_{ev}^{0,2}(X, \mathbb{C}) \neq 0$$

*for a generic smooth $X$ such that $\mathcal{O}_Y(X) = L$, then a Noether-Lefschetz theorem holds for the pair $(Y, L)$.*

Here, $h_{ev}^{0,2}$ denotes the evanescent $(2, 0)$-cohomology of $X$: see below for a precise definition.

More precisely, we denote by $U(L)$ the open subset of $\mathbb{P}H^0(L)$ parameterizing smooth surfaces in the same equivalence class as $L$. We further denote by $\mathrm{NL(L)}$ (*the Noether-Lefschetz locus of* $L$) the subspace parameterizing surfaces $X$ equipped with line bundles which are not produced by pull-back from $Y$. The above theorem then admits the following alternative formulation.

**Theorem (Moishezon)** *If $(Y, L)$ are such that $L$ is very ample and*

$$h_{ev}^{0,2}(X, \mathbb{C}) \neq 0$$

*for a generic smooth $X$ such that $\mathcal{O}_Y(X) = L$, then the Noether-Lefschetz locus* $\mathrm{NL(L)}$ *is a countable union of proper algebraic subvarieties of $U(L)$.*

These proper subvarieties will henceforth be referred to as *components of the Noether-Lefschetz locus*.

A Noether-Lefschetz theorem for a pair $(Y, L)$ essentially says that for a generic surface $X$ such that $\mathcal{O}_Y(X) = L$, the set of line bundles on $X$ is well-understood and as simple as possible. A natural follow-up question is: how rare are surfaces with badly behaved Picard groups? Or alternatively: how large can the components of the Noether-Lefschetz locus be in comparison with $U(L)$? This leads us to attempt to prove what we will call *explicit Noether-Lefschetz theorems*. An explicit Noether-Lefschetz theorem (the terminology is due to Green) says that the codimension of $\mathrm{NL(L)} \subset U(L)$ is bounded below by some number $n_L$ depending non-trivially on the positivity of $L$. The first known example of these was the following theorem, established independently by Voisin and Green, [**8**], [**20**], which gives an explicit Noether-Lefschetz theorem for $\mathbb{P}^3$.

**Theorem (Green, Voisin)** *Let $Y = \mathbb{P}^3$ and $L = \mathcal{O}_{\mathbb{P}^3}(d)$. Let $\Sigma_L \subset U(L)$ be any component of the Noether-Lefschetz locus. Then $\operatorname{codim} \Sigma_L \geq d - 3$, with equality being achieved only for the component of surfaces containing a line.*

In this theorem we see also another of the reigning principles of the study of components of the Noether-Lefschetz locus, namely that components of small codimension should parameterize surfaces containing low-degree curves.

Recently, the subject has been much advanced by the following result of Otwinowska, ([**17**], see also [**15**] and [**16**]) which implies an explicit Noether-Lefschetz theorem for analogues of Noether-Lefschetz loci for highly divisible line bundles on varieties of arbitrary odd dimension. (For ease of presentation, we give a weakened version of the result proved).

**Theorem (Otwinowska)** *Let $Y$ be a projective variety of dimension $2n + 1$, let $\mathcal{O}_Y(1)$ be a very* *ample line bundle on $Y$ and let $\Sigma_L \subset U(\mathcal{O}_Y(d))$ be any component of the Noether-Lefschetz locus. Let $X$ be a hypersurface contained in $\Sigma_L$. For $d$ large enough, if*

$$\operatorname{codim} \Sigma_L \leq \frac{d^n}{n!}$$

*then $X$ contains a $n$-dimensional linear space.*

In fact, Otwinowska also gives a numerical criterion on $d$ and the codimension of $\Sigma_L$ under which $X$ necessarily contains a degree-$b$ n-dimensional subvariety.
We recall also the results of Joshi [**13**] and Ein-Lazarsfeld [**5**, Prop. 3.4].

The aim in this paper will be to shed light on the fact that it is the *Castelnuovo-Mumford regularity properties* of a line bundle that insure that an explicit Noether-Lefschetz theorem holds, independently on the divisibility properties.
To state our first result we suppose that $Y$ is a smooth threefold and $H$ is a very ample line bundle on $Y$. We define numbers $\alpha_Y$ and $\beta_Y$ as follows.

DEFINITION 1. *The integer $\alpha_Y$ is defined to be the minimal positive integer such that $K_Y + \alpha_Y H$ is very ample. The integer $\beta_Y$ is defined to be the minimal integer such that $(\beta_Y - \alpha_Y)H - K_Y$ is nef.*

We recall that, by the results of adjunction theory [**19**], if $(Y, H) \neq (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$, we have that $\alpha_Y \leq 4$ with equality if and only if either $Y$ is a $\mathbb{P}^2$-bundle over a smooth curve and the restriction of $H$ to the fibers is $\mathcal{O}_{\mathbb{P}^2}(1)$ (*we will refer later to this case as a linear $\mathbb{P}^2$-bundle*) or $(Y, H) = (Q, \mathcal{O}_Q(1))$ where $Q \subset \mathbb{P}^4$ is a smooth quadric hypersurface. On the other hand $\beta_Y \geq 1$ with equality if $Y$ is subcanonical and nonpositive (that is if $K_Y = eH$ for some integer $e \leq 0$).
We have

THEOREM 1. *Let $Y$ be a smooth threefold, $Y \neq \mathbb{P}^3$ and let $H$ be a very ample divisor on $Y$. Let $L$ be a (-d)-regular line bundle with respect to $H$. We suppose that either $H^1(\Omega^2_Y \otimes L) = 0$ or $d \geq 3\beta_Y - 3\alpha_Y + 13$. Let $\Sigma_L$ be any component of the Noether-Lefschetz locus* $\mathrm{NL(L)}$. *The following bounds hold:*

(i) *If $(Y, H)$ is not a linear $\mathbb{P}^2$-bundle then*

$$\operatorname{codim} \Sigma_L \geq \begin{cases} d - 5 + \alpha_Y - 2\beta_Y & \text{if } \beta_Y \geq 2 \text{ and } d \geq \frac{\beta_Y^2(\beta_Y+5)}{2} \\ d - 6 + \alpha_Y & \text{if } \beta_Y = 1 \end{cases}.$$

(ii) *If $(Y, H)$ is a linear $\mathbb{P}^2$-bundle then*

$$\operatorname{codim} \Sigma_L \geq \begin{cases} d - 2 - 2\beta_Y & \text{if } \beta_Y \geq 2 \text{ and } d \geq \frac{\beta_Y^2(\beta_Y+5)}{2} \\ d - 3 & \text{if } \beta_Y = 1 \end{cases}.$$

We can do a little bit better in the case of the Noether-Lefschetz locus of adjoint line bundles.

We now define numbers $a_Y$ and $b_Y$ as follows.

DEFINITION 2. *The integer $a_Y$ is defined to be the minimal integer such that $K_Y + a_Y H$ is very ample. The integer $b_Y$ is defined to be the minimal integer such that $(b_Y - a_Y)H - K_Y$ is nef.*

As above, if $(Y,H) \neq (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$, we have that $a_Y \leq 4$ with equality if and only if either $(Y,H)$ is a linear $\mathbb{P}^2$-bundle or $(Y,H) = (Q, \mathcal{O}_Q(1))$ and again $b_Y \geq 1$ with equality if $Y$ is subcanonical.

THEOREM 2. *Let $Y$ be a smooth threefold, $Y \neq \mathbb{P}^3$ and let $H$ be a very ample divisor on $Y$. Let*

$$L = K_Y + dH + A,$$

*where $A$ is numerically effective. We suppose that either $H^1(\Omega_Y^2 \otimes L) = 0$ or $d \geq 2b_Y - 2a_Y + 13$. Let $\Sigma_L$ be any component of the Noether-Lefschetz locus* $\mathrm{NL(L)}$. *The following bounds hold:*

(i) *If $(Y,H)$ is not a linear $\mathbb{P}^2$-bundle then*

$$\operatorname{codim} \Sigma_L \geq \begin{cases} d - 5 - b_Y & \text{if } b_Y \geq 2 \text{ and } d \geq \frac{b_Y(b_Y^2 + 7b_Y - 6)}{2} \\ d - 5 & \text{if } b_Y = 1 \end{cases}.$$

(ii) *If $(Y,H)$ is a linear $\mathbb{P}^2$-bundle then*

$$\operatorname{codim} \Sigma_L \geq \begin{cases} d - 6 - b_Y & \text{if } b_Y \geq 2 \text{ and } d \geq \frac{b_Y(b_Y - 1)(b_Y + 8)}{2} \\ d - 6 & \text{if } b_Y = 1 \end{cases}.$$

We also note the following application that generalises [**2**] (see also [**3**]).

COROLLARY 1. *Let $Y$ be a smooth threefold such that $Y \neq \mathbb{P}^3$ and $\operatorname{Pic}(Y) \cong \mathbb{Z}H$ where $H$ is a very ample line bundle and let $K_Y = eH$. We suppose that either $H^1(\Omega_Y^2(d)) = 0$ or $d \geq 3e + 13$. Let $P_1, \ldots, P_k$ be $k$ general points in $Y$ and $\pi : \widetilde{Y} \to Y$ be the blow-up of $Y$ at these points with exceptional divisors $E_1, \ldots, E_k$.*
*If $d \geq 7 + e$ then*

$$d\pi^*(H) - E_1 - \ldots - E_k \quad \text{is ample on } \widetilde{Y} \;\Leftrightarrow\; d^3H^3 > k.$$

We outline our approach to the study of the Noether-Lefschetz locus.
In section 2, we will give the standard expression of this problem in terms of variation of Hodge structure of $X$. We will then recall the classical results of Griffiths, Carlson et. al. which allow us to express variation of Hodge structure of $X$ in terms of multiplication of sections of line bundles on $X$.

We define $\sigma$ to be the section of $L$ defining $X$. The tangent space of a component of the Noether-Lefschetz locus is naturally a subspace of $H^0(L)/\langle\sigma\rangle$, and we will denote its preimage in $H^0(L)$ by $T$. If we suppose that $H^1(\Omega_Y^2 \otimes L) = 0$, then $T$ has the following property: The natural multiplication map

$$T \otimes H^0(K_Y \otimes L) \to H^0(K_Y \otimes L^2) \tag{1·1}$$

is not surjective.
A full proof of this fact is given in section 3.
In section 3, we also explain Green's methods for proving the explicit Noether-Lefschetz theorem for $\mathbb{P}^3$ using Koszul cohomology to prove that equation (1·1) cannot be satisfied if $T$ is too large. Green's method does not immediately apply to our case, since it requires $T$ to be base-point free— which is only guaranteed if the tangent bundle of $Y$ is globally generated, hence only for a few threefolds. However, we show in section 4 that there exists $W \subset H^0(K_Y \otimes L(3))$ such

that $W$ is base-point free and

$$\{T \otimes H^0(K_Y \otimes L)\} \oplus \{W \otimes H^0(L(-3))\} \to H^0(K_Y \otimes L^2)$$

is not surjective. Results of Ein and Lazarsfeld [**5**] then imply a lower bound on the codimension of

$$\{T \otimes H^0(K_Y(3))\} \oplus W \subset H^0(K_Y \otimes L(3))$$

and more particularly on the codimension of

$$T \otimes H^0(K_Y(3)) \subset H^0(K_Y \otimes L(3)).$$

In introducing $W$, we get around the base-point free problems, but introduce others. In particular, we now need a method for extracting a lower bound on $\mathrm{codim}\, T$ from a lower bound for $\mathrm{codim}\, (T \otimes H^0(K_Y(3)))$. When $Y = \mathbb{P}^3$, this is a simple application of a classical inequality in commutative algebra due to Macaulay and Gotzmann. In section 5 we extend the Macaulay-Gotzmann inequality to sections of any Castelnuovo-Mumford regular sheaf. In section 6, we pull all of the above together to prove the theorem.

## 2. *Preliminaries.*

In this section we recall the classical results of Griffiths, Carlson et. al. on which our work will be based. We will show how a component $\Sigma_L$ of the Noether-Lefschetz locus $\mathrm{NL(L)}$ can be locally expressed as the zeros of a certain section of a vector bundle over $U(L)$. We will then use this expression— together with the work of Griffiths from the 60s, relating variation of Hodge structure with deformations of $X$ to multiplication of sections of line bundles on $X$— to relate the codimension of $\Sigma_L$ to cohomological questions on $X$.

### 2·1. $\mathrm{NL}$ *expressed as the zero locus of a vector bundle section.*

We note first that by the Lefschetz theorem the map $\mathrm{Pic}_0(Y) \to \mathrm{Pic}_0(X)$ is necessarily an isomorphism. It follows that the map $i^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ fails to be surjective if and only if the $(1,1)$ integral evanescent cohomology is non-trivial: $H^{1,1}_{\mathrm{ev}}(X,\mathbb{Z}) \neq 0$. (We recall that the subspace $H^{1,1}_{\mathrm{ev}}(X,\mathbb{C}) \subset H^{1,1}(X,\mathbb{C})$ is defined by $\gamma \in H^{1,1}_{\mathrm{ev}}(X,\mathbb{C}) \Leftrightarrow \langle i^*\beta, \gamma\rangle = 0$ for all $\beta \in H^2(Y,\mathbb{C})$.)

In particular, we can therefore define $\mathrm{NL(L)}$ as follows

$$X \in \mathrm{NL(L)} \Leftrightarrow H^{1,1}_{\mathrm{ev}}(X,\mathbb{Z}) \neq 0.$$

This is the definition of $\mathrm{NL(L)}$ which we will use henceforth, since it is much more manageable. In particular, it is this description which will allow us to write any component of $\mathrm{NL(L)}$ as the zero locus of a special section of a vector bundle.

Henceforth, we will assume that $X$ is contained in $\mathrm{NL(L)}$ and $\gamma$ will be a non-trivial element of $H^{1,1}_{\mathrm{ev}}(X,\mathbb{Z})$. The point in $U(L)$ corresponding to $X$ will be denoted by $0$. We will now define what we mean by the *Noether-Lefschetz locus associated to* $\gamma$, which we denote by $NL(\gamma)$. Since we will be interested in the local geometry of $\mathrm{NL(L)}$, we fix for simplicity a contractible neighbourhood of $0$, $O$. Henceforth, all our calculations will be made over $O$. We form a vector bundle $\mathcal{H}^2_{\mathrm{ev}}$ over $O$, defined by

$$\mathcal{H}^2_{\mathrm{ev}}(u) = H^2_{\mathrm{ev}}(X_u,\mathbb{C}).$$

The vector bundle contains holomorphic sub-bundles $\mathcal{F}^i(\mathcal{H}^2_{\mathrm{ev}})$ given by

$$\mathcal{F}^i(\mathcal{H}^2_{\mathrm{ev}})(u) = F^i(H^2_{\mathrm{ev}}(X_u, \mathbb{C})).$$

We define bundles $\mathcal{H}^{i,2-i}_{\mathrm{ev}}$ by $\mathcal{H}^{i,2-i}_{\mathrm{ev}} = \mathcal{F}^i(\mathcal{H}^2_{\mathrm{ev}})/\mathcal{F}^{i+1}(\mathcal{H}^2_{\mathrm{ev}})$.

(The fibre of $\mathcal{H}^{i,2-i}_{\mathrm{ev}}$ at the point $u$ is isomorphic to $H^{i,2-i}_{\mathrm{ev}}(X_u)$: however, $\mathcal{H}^{i,2-i}_{\mathrm{ev}}$ does not embed naturally into $\mathcal{H}^2_{\mathrm{ev}}$ as a holomorphic sub-bundle.) The bundle $\mathcal{H}^2_{\mathrm{ev}}$ is equipped with a natural flat connexion, the Gauss-Manin connexion, which we denote by $\nabla$. We now define $\overline{\gamma}$ to be the section of $\mathcal{H}^2_{\mathrm{ev}}$ produced by flat transport of $\gamma$.

We define $\overline{\gamma}^{0,2}$, a section of $\mathcal{H}^{0,2}_{\mathrm{ev}}$, to be the image of $\overline{\gamma}$ under the projection

$$\pi : \mathcal{H}^2_{\mathrm{ev}} \to \mathcal{H}^{0,2}_{\mathrm{ev}}.$$

We are now in a position to define $\mathrm{NL}(\gamma)$.

DEFINITION 3. *The Noether-Lefschetz locus associated to* $\gamma$, $\mathrm{NL}(\gamma)$, *is given by*

$$\mathrm{NL}(\gamma) = \mathrm{zero}(\overline{\gamma}^{0,2}).$$

Informally, $\mathrm{NL}(\gamma)$ parameterizes the small deformations of $X$ on which $\gamma$ remains of Hodge type $(1,1)$. Any component of $\mathrm{NL(L)}$ is locally equal to $\mathrm{NL}(\gamma)$ for some $\gamma$.

The tangent space $T\mathrm{NL}(\gamma)$ at $X$ is a subspace of $H^0(L)/\langle\sigma\rangle$, where $\sigma$ is the section of $L$ defining $X$. We will denote its preimage in $H^0(L)$ by $T$.

2·2. *IVHS and residue maps.*

We will now explain the classical work of Griffiths which makes the section $\overline{\gamma}^{0,2}$ particularly manageable.

Let $\mathcal{H}^2_{\mathrm{ev}}$ be as above. For the purposes of this section we will consider the holomorphic sub-vector bundle $\mathcal{F}^i_{\mathrm{ev}}$ to be a holomorphic map $\mathcal{F}^i_{\mathrm{ev}} : O \to \mathrm{Grass}(f_i, \mathcal{H}^2_{\mathrm{ev}})$ where $f_i$ is the dimension of $F^i H^2_{\mathrm{ev}}(X, \mathbb{C})$. The Gauss-Manin connexion gives us a canonical isomorphism $\mathcal{H}^2_{\mathrm{ev}} \cong H^2_{\mathrm{ev}}(X, \mathbb{C}) \times O$, from which we deduce a canonical isomorphism

$$\mathrm{Grass}(f_i, \mathcal{H}^2_{\mathrm{ev}}) \cong O \times \mathrm{Grass}(f_i, H^2_{\mathrm{ev}}(X, \mathbb{C})).$$

In particular, $\mathcal{F}^i_{\mathrm{ev}}$ is now expressed as a map from $O$ to the constant space $\mathrm{Grass}(f_i, H^2_{\mathrm{ev}}(X, \mathbb{C}))$, and as such can be derived. We obtain a derivation map, which we denote by IVHS (for Infinitesimal Variation of Hodge Structure)

$$\mathrm{IVHS}^i : TO \to \mathrm{Hom}(F^i(H^2_{\mathrm{ev}}), H^2_{\mathrm{ev}}/F^i(H^2_{\mathrm{ev}})).$$

Griffiths proved the following result in [**10**].

**Theorem (Griffiths' Transversality)** *The image of* $\mathrm{IVHS}^i$ *is contained in* $\mathrm{Hom}(H^{i,2-i}_{\mathrm{ev}}, H^{i-1,3-i}_{\mathrm{ev}})$.

The importance of this work for our purposes is the following lemma.

LEMMA 1. *For any* $v \in TO$*, we have that* $d_v(\overline{\gamma}^{0,2}) = -\mathrm{IVHS}^1(v)(\gamma)$.

**Proof.** The isomorphism $f : T_W \mathrm{Grass(n,V)} \cong \mathrm{Hom}(W, V/W)$ is given by

$$f(v) : w \to \frac{\partial}{\partial v}(\tilde{w})_{|V/W}$$

where $w \in W$ and $\tilde{w}$ is any local section of the tautological bundle over the Grassmannian such that $\tilde{w}_W = w$.
In the case in hand, we choose a lifting of $\overline{\gamma}^{0,2}$ to a section of $\mathcal{H}^2_{\mathrm{ev}}$, which we denote by $\overline{\gamma}^{0,2}_{\mathrm{lift}}$. By definition of $\overline{\gamma}^{0,2}$, we then have that $\overline{\gamma} - \overline{\gamma}^{0,2}_{\mathrm{lift}} \in \mathcal{F}^1(\mathcal{H}^2_{\mathrm{ev}})$ and it follows that $\mathrm{IVHS}^1(v)(\gamma) = \frac{\partial}{\partial v}(\overline{\gamma} - \overline{\gamma}^{0,2}_{\mathrm{lift}})_{|_{H^{0,2}_{\mathrm{ev}}}}$ and now, since by definition $\overline{\gamma}$ is constant, $\mathrm{IVHS}^1(v)(\gamma) = -d_v(\overline{\gamma}^{0,2}_{\mathrm{lift}})_{|_{H^{0,2}_{\mathrm{ev}}}} = -d_v(\overline{\gamma}^{0,2})$. □

We will also need the work of Carlson and Griffiths relating the residue maps to Hodge structure of varieties ([**1**]). Suppose given, for $i = 1, 2$, a section

$$s \in H^0(K_Y \otimes L^i).$$

This can be thought of as a holomorphic 3-form on $Y$ with a pole of order $i$ along $X$, and as such defines a cohomology class in $H^3(Y \setminus X, \mathbb{C})$. The group $H^3(Y \setminus X, \mathbb{C})$ maps to $H^2_{\mathrm{ev}}(X, \mathbb{C})$ via residue, and hence there is an induced residue map

$$\mathrm{res_i} : H^0(K_Y \otimes L^i) \to H^2_{\mathrm{ev}}(X, \mathbb{C}).$$

The relevance of this map to variation of Hodge structure comes from the following theorem, which is proved by Griffiths in [**11**].

**Theorem** *The image of* $\mathrm{res_i}$ *is contained in* $F^{3-i}(H^2_{\mathrm{ev}})$.

Henceforth, we will denote by $\pi_i$ the induced projection map

$$\pi_i : H^0(K_Y \otimes L^i) \to H^{3-i,i-1}_{\mathrm{ev}}(X, \mathbb{C}).$$

In this representation, the map $\mathrm{IVHS}^{3-i}$ has a particularly nice form ([**1**], page 70).

**Theorem (multiplication)** *Consider $v \in TO$. Let $\tilde{v}$ be a lifting of $v$ to $H^0(L)$. Then for any $P \in H^0(K_X \otimes L^i)$, we have that*

$$\mathrm{IVHS}^{3-i}(v)(\pi_i(P)) = \pi_{i+1}(\tilde{v} \otimes P)$$

*up to multiplication by some nonzero constant.*

The only fly in the ointment is that in general we cannot be sure that the map $\pi_i$ is surjective onto $H^{3-i,i-1}_{\mathrm{ev}}(X, \mathbb{C})$. It is precisely for this reason that we will be obliged to suppose that $H^1(\Omega^2_Y \otimes L) = 0$.
The following lemma will be crucial.

LEMMA 2. *Consider $\gamma \in H^{1,1}_{\mathrm{ev}}(X)$ and $\omega \in H^{2,0}_{\mathrm{ev}}(X)$. For any vector $v \in TO$ we have*

$$\langle \mathrm{IVHS}^1(v)(\gamma), \omega \rangle + \langle \gamma, \mathrm{IVHS}^2(v)(\omega) \rangle = 0.$$

**Proof.** We note that $d_v(\langle \overline{\gamma}, \overline{\omega} \rangle) = 0$. We note that we can write

$$\overline{\gamma} = \overline{\gamma}^1 + \overline{\gamma}^2$$

where $\overline{\gamma}^1 \in \mathcal{F}^1_{\rm ev}$ and $\overline{\gamma}^2(0) = 0$. Similarly, we can write $\overline{\omega} = \overline{\omega}^1 + \overline{\omega}^2$ where $\overline{\omega}^1 \in \mathcal{F}^2_{\rm ev}$ and $\overline{\omega}^2(0) = 0$. We note that for Hodge theoretic reasons $< \overline{\omega}^1, \overline{\gamma}^1 >= 0$ and hence

$$d_v(\langle \overline{\gamma}, \overline{\omega} \rangle) = \langle d_v(\overline{\gamma}^2), \omega \rangle + \langle \gamma, d_v(\overline{\omega}^2) \rangle.$$

Here, of course, it makes sense to talk about $d_v(\overline{\omega}^2)$ and $d_v(\overline{\gamma}^2)$ only because $\overline{\omega}^2(0) = 0$ and $\overline{\gamma}^2(0) = 0$. Since $\langle \mathcal{F}^1, \mathcal{F}^2 \rangle = 0$, we have that

$$\langle d_v(\overline{\gamma}^2), \omega \rangle = \langle d_v(\overline{\gamma}^2)^{0,2}, \omega \rangle = \langle -\mathrm{IVHS}^1(v)(\gamma), \omega \rangle$$

and similarly

$$\langle \gamma, d_v(\overline{\omega}^2) \rangle = \langle \gamma, (d_v \overline{\omega}^2)^{1,1} \rangle = \langle \gamma, -\mathrm{IVHS}^2(v)(\omega) \rangle.$$

So it follows immediately from $d_v(\langle \overline{\gamma}, \overline{\omega} \rangle) = 0$ that

$$\langle \mathrm{IVHS}^1(v)(\gamma), \omega \rangle + \langle \gamma, \mathrm{IVHS}^2(v)(\omega) \rangle = 0. \qquad \square$$

## 3. *Strategy and overview.*

The basic idea of this proof is that used by Green in [**8**]. We summarise his proof and explain why it cannot be immediately applied to the situation in hand.

First some notation. Given any pair of coherent sheaves on $X$, $L$ and $M$ we denote by $\mu_{L,M}$ the multiplication map

$$\mu_{L,M} : H^0(L) \otimes H^0(M) \to H^0(L \otimes M).$$

Where there is no risk of confusion, we will write $\mu$ for $\mu_{L,M}$.

The starting point of Green's work is the following lemma.

LEMMA 3. *Suppose that $T \subset H^0(\mathcal{O}_{\mathbb{P}^3}(d))$ is the preimage of $TNL(\gamma)$. Then the inclusion*

$$\mu(T \otimes H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-4))) \subset H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2d-4))$$

*is a strict inclusion.*

**Proof.** In the case of $Y = \mathbb{P}^3$, we have that $\pi_i : H^0(K_Y \otimes L^i) \to H^{3-i,i-1}_{\rm ev}(X)$ is a surjection. (See, for example, [**21**, proof of Thm. 18.5, page 420]). By Lemma 2, if $v \in TNL(\gamma)$ and $P \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-4))$ then

$$\langle \gamma, \mathrm{IVHS}^2(v)(\pi_1(P)) \rangle = -\langle \mathrm{IVHS}^1(v)(\gamma), \pi_1(P) \rangle = 0$$

from which we conclude that $\mathrm{IVHS}^2(v)(\pi_1(P)) \in \gamma^\perp$, where $\gamma^\perp$ is the orthogonal to $\gamma$, and in particular is a proper subspace. By the multiplication theorem it follows that $\pi_2(\mu(\tilde{v} \otimes P)) \in \gamma^\perp$ or alternatively

$$\mu(\tilde{v} \otimes P) \in \pi_2^{-1}(\gamma^\perp).$$

Since $\pi_2$ is surjective, $\pi_2^{-1}(\gamma^\perp)$ is a proper subspace. $\square$

Green then proves the following theorem via the vanishing of certain Koszul cohomology groups.

**Theorem (Green)** *Let $T \subset H^0(\mathcal{O}_{\mathbb{P}^r}(d))$ be a base-point free linear system of codimension*

*$c$. Then the Koszul complex*

$$\bigwedge^{p+1} T \otimes H^0(\mathcal{O}_{\mathbb{P}^r}(k-d)) \to \bigwedge^{p} T \otimes H^0(\mathcal{O}_{\mathbb{P}^r}(k)) \to \bigwedge^{p-1} T \otimes H^0(\mathcal{O}_{\mathbb{P}^r}(k+d))$$

*is exact in the middle provided that $k \geq p + d + c$.*

In the case in hand, on setting $r = 3, p = 0$ and $k = 2d - 4$ we see that the multiplication map

$$T \otimes H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-4)) \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2d-4))$$

is surjective if $2d - 4 \geq d + c$. But we have already observed that this multiplication map is necessarily non-surjective, from which it follows that $c \geq d - 3$.

In Lemma 4 below we will see that, provided $H^1(\Omega^2_Y \otimes L) = 0$, it is still true that the multiplication map $T \otimes H^0(K_Y \otimes L) \to H^0(K_Y \otimes L^2)$ is non-surjective. One might therefore reasonably entertain the hope of adapting Green's methods to arbitrary varieties. The difficulty is that in order to apply Green's result, $T$ must be base-point free. This was immediate when $Y = \mathbb{P}^3$, since, if $X$ was given by $F \in H^0(\mathcal{O}_{\mathbb{P}^3}(d))$, $T$ then automatically contained $H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \times \langle \frac{\partial F}{\partial X_i} \rangle$. However if $T_Y$ is not globally generated, there is no reason why this should hold in general. The rest of this paper will be concerned with finding ways around this difficulty.

LEMMA 4. *Let $L$ be very ample and such that $H^1(\Omega^2_Y \otimes L) = 0$.*
*Let $T \subset H^0(L)$ be the preimage in $H^0(L)$ of the tangent space to $\mathrm{NL}(\gamma)$. Then*

$$\mu(T \otimes H^0(K_Y \otimes L)) \subset H^0(K_Y \otimes L^2)$$

*is a strict inclusion.*

**Proof.** We note that by the argument given in the proof of Lemma 3,

$$\pi_2(\mu(T \otimes H^0(K_Y \otimes L))) \neq H^{1,1}_{\mathrm{ev}}(X, \mathbb{C}).$$

Now it just remains to observe that, by [**21**, proof of Thm. 18.5, page 420],

$$\pi_2 : H^0(K_Y \otimes L^2) \to H^{1,1}_{\mathrm{ev}}(X, \mathbb{C})$$

is a surjection, since $H^1(\Omega^2_Y(X)) = 0$. □

So, we would now like to apply Green's argument; unfortunately, $T$ may have base points. Our strategy for getting around this problem will be as follows.

(i) First of all, we will construct $W \subset H^0(K_Y \otimes L(3))$ with the following good properties.
 (a) $W$ is base-point free,
 (b) $\pi_2(\mu(W \otimes H^0(L(-3)))) = 0$.
(ii) The result proved by Ein and Lazarsfeld in [**5**] then gives us a lower bound on the codimension of $\mu(T \otimes H^0(K_Y(3)))$ in $H^0(K_Y \otimes L(3))$.
(iii) We will then extract from the lower bound on $\operatorname{codim} \mu(T \otimes H^0(K_Y(3)))$ a lower bound on the codimension of $T$ in $H^0(L)$.

## 4. *Constructing $W$.*

We henceforth let $Y$ be a smooth threefold, $Y \neq \mathbb{P}^3$ and $H$ be a very ample divisor on $Y$.

PROPOSITION 1. *There is a subspace $W \subset H^0(K_Y \otimes L(3))$ such that*
(i) *The map $\pi_2 \circ \mu : W \otimes H^0(L(-3)) \to H^{1,1}_{\mathrm{ev}}(X, \mathbb{C})$ is identically zero.*

(ii) $W$ *is base-point free.*

**Proof.** We denote the image of $\mu : W \otimes H^0(L(-3)) \to H^0(K_Y \otimes L^2)$ by $\langle W \rangle$. Consider the map

$$d : H^0(\Omega^2_Y \otimes L) \to H^0(K_Y \otimes L^2)$$

which sends a two-form on $Y$ with a simple pole along $X$ to its derivation. We note that for any $\omega \in H^0(\Omega^2_Y \otimes L)$ we have that $d\omega \in \mathrm{Ker}(\mathrm{res}_2)$, because $d\omega$, being exact, defines a null cohomology class on $Y \setminus X$.
The space $W$ will be chosen in such a way that

$$\langle W \rangle_{|X} \subset \mathrm{Im}(d)_{|X}.$$

The map $d$ is difficult to deal with because it is not a map of $\mathcal{O}_Y$-modules: the value of $d\omega$ at a point $x$ is not determined by the value of $\omega$ at $x$. In particular, it is not possible to form a tensor product map

$$d \otimes (L^{-1}(3)) : H^0(\Omega^2_Y(3)) \to H^0(K_Y \otimes L(3)).$$

Our first step will be to show that, even if $d$ does not come from an underlying map of $\mathcal{O}_Y$-modules, the restriction

$$d_X : H^0(\Omega^2_Y \otimes L) \to H^0(K_X \otimes L_{|X})$$

does.

LEMMA 5. *Let the map* $r : \Omega^2_Y \otimes L \to K_X \otimes L$ *be given by tensoring with* $L$ *the pull-back* $i^* : \Omega^2_Y \to \Omega^2_X (\cong K_X)$. *Then we have that* $d_X = -H^0(r)$.

**Proof.** We calculate in analytic complex coordinates near a point $p \in X$. Let $f$ be a function defining $X$ in a neighbourhood of $p$ and let $x, y$ be coordinates chosen in such a way that $(f, x, y)$ form a system of coordinates for $Y$ close to $p$. If $\nu \in H^0(\Omega^2_Y \otimes L)$, then in a neighbourhood of $p$ we can write

$$\nu = \frac{f_1 dx \wedge dy + f_2 dx \wedge df + f_3 dy \wedge df}{f}$$

where $f_1, f_2, f_3$ are holomorphic functions on a neighbourhood of $p$.
Differentiating and restricting to $X$, we get that

$$d\nu_{|X} = \frac{-f_1 dx \wedge dy \wedge df}{f^2}.$$

As an element of $H^0((K_Y \otimes L) \otimes L)$, this is represented by

$$\frac{-f_1 dx \wedge dy \wedge df}{f} \otimes 1/f.$$

Under the canonical isomorphism $(K_Y \otimes L)_{|X} \to K_X$, we have that

$$\frac{-f_1 dx \wedge dy \wedge df}{f} \to -f_1 dx \wedge dy.$$

Hence, under the canonical isomorphism $(K_Y \otimes L^2)_{|X} \to K_X \otimes L_{|X}$, we have that

$$(d\nu)_{|X} \to \frac{-f_1 dx \wedge dy}{f} = -r(\nu).$$

This concludes the proof of Lemma 5. $\square$

We now proceed with the proof of Proposition 1.
The map $d_X$, which is a map of $\mathcal{O}_Y$-modules, has the advantage that we can form tensor products. We consider the map induced by tensor product with $L^{-1}(3)$

$$d_X^{L^{-1}(3)} : H^0(\Omega^2_Y(3)) \to H^0(K_X(3)).$$

We define $W$ by

$$W = \{w \in H^0(K_Y \otimes L(3)) : w_{|X} \in \mathrm{Im}(d_X^{L^{-1}(3)})\}.$$

We will prove first that

LEMMA 6. *For any $w \in W$ and $P \in H^0(L(-3))$, we have that*

$$\pi_2(\mu(P \otimes w)) = 0.$$

**Proof.** Since $w \in W$ there exists $s \in H^0(\Omega^2_Y(3))$ such that $w_{|X} = d_X^{L^{-1}(3)} s$ and hence

$$(Pw)_{|X} = d_X(Ps) = d(Ps)_{|X}.$$

From this it follows that there exists $s' \in H^0(K_Y \otimes L)$ such that

$$Pw = d(Ps) + \sigma s'.$$

We observed above that $\pi_2(d(Ps)) = 0$. We note that $\mathrm{res}_2(\sigma s') = \mathrm{res}_1(s')$ and hence $\mathrm{res}_2(\sigma s') \in F^2 H^2_{\mathrm{ev}}(X, \mathbb{C})$, from which it follows that $\pi_2(\sigma s') = 0$. Whence $\pi_2(Pw) = 0$. This concludes the proof of Lemma 6. $\square$

To conclude the proof of Proposition 1 it remains only to show that $W$ is base-point free. Since $Y \neq \mathbb{P}^3$ we have ([**4**]) that $K_Y(3)$ is globally generated. Also

$$\mu(\mathbb{C}\sigma \otimes H^0(K_Y(3))) \subset W$$

therefore the only possible base points of $W$ are the points of $X$. Consider an arbitrary point $p \in X$. Now if $\mathbb{P}^N = \mathbb{P}H^0(Y, H)$ we have that $\Omega^2_Y(3)$ is globally generated since $\Omega^2_{\mathbb{P}^N}(3)$ is such and there is a surjection $\Omega^2_{\mathbb{P}^N}(3) \twoheadrightarrow \Omega^2_Y(3)$. Whence there exists a section $s \in H^0(\Omega^2_Y(3))$ such that $d_X^{L^{-1}(3)}(s)(p) \neq 0$. From the short exact sequence

$$0 \to K_Y(3) \to K_Y \otimes L(3) \to K_X(3) \to 0$$

and Kodaira vanishing we see that there exists $w \in H^0(K_Y \otimes L(3))$ such that $w_{|X} = d_X^{L^{-1}(3)}(s)$. It follows that $w \in W$, and

$$w(p) = d_X^{L^{-1}(3)}(s)(p) \neq 0.$$

Hence $p$ is not a base-point of $W$. This completes the proof of Proposition 1. $\square$

To get lower bounds on the codimension we will apply the following result of Ein and Lazarsfeld, [**5**, Prop. 3.1].

**Theorem (Ein, Lazarsfeld)** *Let $H$ be a very ample line bundle and $B, C$ be nef line bundles on a smooth complex projective $n$-fold $Z$. We set*

$$F_f = K_Z + fH + B \text{ and } G_e = K_Z + eH + C.$$

*Let $V \subset H^0(Z, F_f)$ be a base-point free subspace of codimension $c$ and consider the Koszul-type complex*

$$\bigwedge^{p+1} V \otimes H^0(G_e) \to \bigwedge^{p} V \otimes H^0(F_f + G_e) \to \bigwedge^{p-1} V \otimes H^0(2F_f + G_e).$$

*If $(Z, H, B) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n})$, $f \geq n+1$ and $e \geq n+p+c$, then this complex is exact in the middle.*

In order to apply this to our situation, we set $p = 0$, and, in case $L = K_Y + dH + A$ we choose $f = d, e = d-3$, $B = A + K_Y + 3H$ (note that $B$ is nef since $K_Y + 3H$ is globally generated) and $C = A$. In the case $L$ (-d)-regular we have $L = M(d)$ for a Castelnuovo-Mumford regular line bundle $M$ and we choose $f = d+3, e = d-3+\alpha_Y - \beta_Y$, $B = M$ and $C = M + (\beta_Y - \alpha_Y)H - K_Y$, so that $B$ is nef since $M$ is globally generated and also $C$ is nef by definition of $\alpha_Y$ and $\beta_Y$ (see Definition 1). We then have that

$$F_f = K_Y \otimes L(3) \text{ and } G_e = L(-3)$$

and the theorem in this particular case says that:

PROPOSITION 2. *Suppose that $d \geq 4$ and $Y \neq \mathbb{P}^3$. Let $V$ be a base-point free linear system in $H^0(K_Y \otimes L(3))$ with the property that*

$$\mu(V \otimes H^0(L(-3))) \subset H^0(K_Y \otimes L^2)$$

*is a strict inclusion. Then the codimension $c$ of $V$ satisfies the inequality*

$$c \geq \begin{cases} d - 5 + \alpha_Y - \beta_Y & \text{if L is } (-\mathrm{d}) - \text{regular} \\ d - 5 & \text{if } L = K_Y + dH + A \end{cases}.$$

In general, pulling together the results of sections 3 and 4, we have the following bound.

PROPOSITION 3. *Suppose that $Y \neq \mathbb{P}^3$ and $H^1(\Omega^2_Y \otimes L) = 0$. Then the codimension of the image of*

$$\mu : T \otimes H^0(K_Y(3)) \to H^0(K_Y \otimes L(3))$$

*is at least $d - 5 + \alpha_Y - \beta_Y$ if $L$ is (-d)-regular or at least $d - 5$ if $L = K_Y + dH + A$.*

**Proof.** For simplicity, we set

$$\tilde{T} := W + \mu(T \otimes H^0(K_Y(3))) \subset H^0(K_Y \otimes L(3)).$$

Notice that the multiplication map

$$\tilde{\mu} : \tilde{T} \otimes H^0(L(-3)) \to H^0(K_Y \otimes L^2)$$

cannot be surjective, otherwise, as in the proof of Lemma 4, we get that

$$\pi_2 \circ \tilde{\mu}(\tilde{T} \otimes H^0(L(-3))) = H^{1,1}_{\mathrm{ev}}(X, \mathbb{C})$$

and, given the first property of $W$, the latter equality implies the contradiction

$$\pi_2 \circ \mu(T \otimes H^0(K_Y \otimes L))) = H^{1,1}_{\mathrm{ev}}(X, \mathbb{C}).$$

Now, by Proposition 2, we get that

$$\operatorname{codim} \mu(T \otimes H^0(K_Y(3))) \geq \begin{cases} d - 5 + \alpha_Y - \beta_Y & \text{if L is } (-\mathrm{d}) - \text{regular} \\ d - 5 & \text{if } L = K_Y + dH + A \end{cases}.$$

□

Therefore it will be enough to devise a mechanism for extracting codimension bounds for $T$ from codimension bounds for $\mu(T \otimes H^0(K_Y(3)))$. This is the subject of the next section.
We end the section by studying the vanishing of $H^1(\Omega^2_Y \otimes L)$.

REMARK 1. *If $d \geq 3\beta_Y - 3\alpha_Y + 13$ and $L$ is (-d)-regular or if $d \geq 2b_Y - 2a_Y + 13$ and $L = K_Y + dH + A$, then $H^1(\Omega^2_Y \otimes L) = 0$.*
*Proof.* We just apply Griffiths' vanishing theorem [**12**] to the globally generated vector bundle $E = \Omega^2_Y(3)$. We write

$$\Omega^2_Y \otimes L = E(\det E + K_Y + B)$$

whence we just need to prove that $B = L - 12H - 3K_Y$ is ample. By definition of $a_Y, b_Y, \alpha_Y$ and $\beta_Y$ we can write $-K_Y = (a-b)H + A'$, where $A'$ is nef and $a = \alpha_Y, b = \beta_Y$ if $L$ is (-d)-regular, while $a = a_Y, b = b_Y$ if $L = K_Y + dH + A$. Hence $B = (d - 12 - ub + ua)H + A''$, where $A''$ is nef and $u = 2$ if $L = K_Y + dH + A$, $u = 3$ if $L$ is (-d)-regular. Therefore $B$ is ample. □

REMARK 2. Notice that if $Y$ is a quadric hypersurface in $\mathbb{P}^4$, since $K_Y = -3H$, if $L = (d-3)H$, we have that $H^1(\Omega^2_Y \otimes L) = 0$ for $d \geq 7$, whence

$$\operatorname{codim} T \geq d - 5.$$

## 5. *Macaulay-Gotzmann for CM regular sheaves.*

We start by reviewing the situation for $\mathbb{P}^n$, which we will then generalise to arbitrary varieties.

**Definition of $c^{<d>}$ and $c_{<d>}$.** Given integers $c \geq 1, d \geq 1$, there exists a unique sequence of integers $k_d, k_{d-1}, \ldots, k_f$ with $d \geq f \geq 1$ ($f$ is uniquely determined by $c$ and $d$) such that

(i) $k_d > k_{d-1} > \ldots > k_f \geq f$,

(ii) $c = \sum_{i=d}^{f} \binom{k_i}{i}$.

Here and below we use the convention $\binom{m}{p} = 0$ if $m < p$. We define

$$c^{<d>} := \sum_{i=d}^{f} \binom{k_i + 1}{i + 1}, \; c_{<d>} := \sum_{i=d}^{f} \binom{k_i - 1}{i}.$$

When $c = 0$ we set $c^{<d>} = c_{<d>} = 0$.
We have the following result of Macaulay and Gotzmann, which can be found in [**7**], pages 64-65.

**Theorem (Macaulay, Gotzmann)** *Let $V \subset H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ be a subspace of codimension $c$. Then the subspace*

$$\mu(V \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))) \subset H^0(\mathcal{O}_{\mathbb{P}^n}(d+1))$$

*is of codimension at most $c^{<d>}$.*

Gotzmann proved the Macaulay-Gotzmann inequality using combinatorial algebraic techniques. Green gave a geometric proof in [**9**]. We will now generalise the argument given by Green in order to prove that the Macaulay-Gotzmann inequality is valid for arbitrary Castelnuovo-Mumford regular sheaves.

THEOREM 3. *Let $M$ be a Castelnuovo-Mumford regular coherent sheaf on a projective space $\mathbb{P}^N$. For $d \geq 1$ let $V \subset H^0(M(d))$ be a subspace of codimension $c$, and define $V^{d+1} \subset H^0(M(d+1))$ by $V^{d+1} = \mu(V \otimes H^0(\mathcal{O}_{\mathbb{P}^N}(1)))$. Then*

$$\text{codim } V^{d+1} \leq c^{<d>}.$$

The Theorem will follow from the following proposition.

PROPOSITION 4. *Suppose that $V$, $M$ and $d$ are as above. Let $H$ be a generic hyperplane of $\mathbb{P}^N$ and denote by $M_H$ the restriction of $M$ to $H$. We further denote the restriction of $V$ to $H^0(M_H(d))$ by $V_H$. Then*

$$\text{codim } V_H \leq c_{<d>}.$$

**Proof.** We shall proceed by a double induction on the dimension of the support of $M$ and the number $d$. We assume now that $d \geq 2, \dim\text{Supp}(M) \geq 1$. The proof of the Proposition for $d = 1$ or for sheaves with zero-dimensional supports is to be found in subsections 5·0·1 and 5·0·2.

Let $H$ and $H'$ be two generic hyperplanes. We define the spaces $V^H$ (respectively $V^{H'}$) in the following way. Let $L_H$ (resp. $L_{H'}$) be a linear polynomial defining $H$ (resp. $H'$). We define $V^H \subset H^0(M(d-1))$ by

$$v \in V^H \Leftrightarrow L_H \times v \in V.$$

(Similarly, $V^{H'}$ is defined by $v \in V^{H'} \Leftrightarrow L_{H'} \times v \in V$.) We now consider the following exact sequence

$$0 \to H^0(M(d-1)) \overset{\times L_H}{\to} H^0(M(d)) \overset{\text{res}}{\to} H^0(M_H(d)) \to 0.$$

Here, of course, we have right exactness of the sequence only because $M$ is a Castelnuovo-Mumford regular sheaf. There is an induced exact sequence

$$0 \to V^H \to V \to V_H \to 0$$

whence we see that

$$\text{codim } V = \text{codim } V^H + \text{codim } V_H.$$

We now consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (V^{H'})^H & \longrightarrow & V^{H'} & \longrightarrow & (V^{H'})_H & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \times L_{H'}} & & \downarrow{\scriptstyle \times L_{H\cap H'}} & & \\
0 & \longrightarrow & V^H & \longrightarrow & V & \longrightarrow & V_H & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \text{res}} & & \\
0 & \longrightarrow & (V_{H'})^{H\cap H'} & \longrightarrow & V_{H'} & \longrightarrow & (V_{H'})_{H\cap H'} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

In the above diagram, all the rows are exact (since $M_H$ is Castelnuovo-Mumford regular on $H$), as is the middle column. It is not immediate that the right-hand column is exact, but we will be able to show that it is close enough to exact for our purposes.

More precisely,

$$(V_{H'})_{H\cap H'} = V_{|_{H\cap H'}} = (V_H)_{H\cap H'}$$

and hence the restriction map $V_H \rightarrow (V_{H'})_{H\cap H'}$ is a surjection. We have automatically that $(V^{H'})_H \subset (V_H)^{H\cap H'}$ and hence the composition of the maps $\times L_{H\cap H'}$ and $\mathrm{res}$ is zero. It follows that

$$\mathrm{codim}\ V_H \leq \mathrm{codim}\ (V_{H'})_{H\cap H'} + \mathrm{codim}\ (V^{H'})_H.$$

We denote by $c'$ the codimension of $V_H$ for generic $H$. Hence, since $H'$ has been chosen generic, $\mathrm{codim}\ V_{H'} = c'$. We have that $\mathrm{codim}\ V^{H'} = c - c'$. We note that

(i) $V^{H'} \subset H^0(M(d-1))$ and hence by the induction hypothesis

$$\mathrm{codim}\ (V^{H'})_H \leq (c-c')_{<d-1>}.$$

(ii) The dimension of the support of $M_{H'}$ is strictly less than the dimension of the support of $M$ and hence by the induction hypothesis

$$\mathrm{codim}\ (V_{H'})_{H\cap H'} \leq c'_{<d>}.$$

It follows that

$$c' \leq c'_{<d>} + (c-c')_{<d-1>}.$$

Green shows in [**9**], pages 77-78, that this inequality implies that $c' \leq c_{<d>}$.

It remains only to prove the Proposition for zero-dimensional sheaves or for $d = 1$.

5·0·1. *The case d=1.*

For any $c \neq 0$ we have that $c_{<1>} = c - 1$. We suppose first that $V \neq H^0(M(1))$. If for generic $H$ we have $\mathrm{codim}\ V_H > c_{<1>}$, then, for generic $H$, $V^H = H^0(M)$. In other words, for generic $H$

$$L_H \times H^0(M) \subset V.$$

It follows that

$$\mu(H^0(M) \otimes H^0(\mathcal{O}_{\mathbb{P}^N}(1))) \subset V.$$

Since $M$ is Castelnuovo-Mumford regular, it follows that $V = H^0(M(1))$ which contradicts our supposition that $V \neq H^0(M(1))$.

But if $c = 0$ then $c_{<1>} = 0$ and Proposition 4 is immediate. This completes the proof of the Proposition in the case where $d = 1$.

5·0·2. *The case where the dimension of the support of $M$ is zero.*

In this case, for generic $H$, $H^0(M_H(d)) = 0$, and hence $\mathrm{codim}\ V_H = 0$. This completes the proof of the Proposition in the case where the dimension of the support of $M$ is zero.

This completes the proof of Proposition 4. □

We now show how Proposition 4 implies Theorem 3. We proceed by induction on the dimension of the support of $M$. We consider the following exact sequence, where $H$ is once again a generic hyperplane in $\mathbb{P}^N$,

$$0 \to (V^{d+1})^H \to V^{d+1} \to (V^{d+1})_H \to 0$$

from which it follows that

$$\text{codim } V^{d+1} = \text{codim } (V^{d+1})^H + \text{codim } (V^{d+1})_H.$$

We note that $V \subset (V^{d+1})^H$ and $(V_H)^{d+1} \subset (V^{d+1})_H$ from which it follows that

$$\text{codim } V^{d+1} \leq c + (c_{<d>})^{<d>} \leq c^{<d>}.$$

This completes the proof of Theorem 3. □

## 6. *Proof of the main theorems.*

We will now show how all this ties together to give a proof of the main theorems. We henceforth set

$$a = \begin{cases} \alpha_Y & \text{if L is } (-\text{d}) - \text{regular} \\ a_Y & \text{if } L = K_Y + dH + A \end{cases}, \quad b = \begin{cases} \beta_Y & \text{if L is } (-\text{d}) - \text{regular} \\ b_Y & \text{if } L = K_Y + dH + A \end{cases}$$

where $\alpha_Y, \beta_Y, a_Y$ and $b_Y$ are as in Definitions 1 and 2.
It is now that we will use the supposition that $(Y, H)$ is not a linear $\mathbb{P}^2$-bundle, hence $K_Y(3)$ is very ample, or, alternatively, that $a \leq 3$ (the case of the quadric is done by Remark 2). The case $a = 4$ will be dealt with at the end of the article.
We start with the following lemma.

LEMMA 7. *Suppose $d \geq 5$ and let $T \subset H^0(L)$ be of codimension $c \leq d - 4$. Define*

$$T' := \mu(T \otimes H^0(\mathcal{O}_Y(3 - a))) \subset H^0(L(3 - a)).$$

*Then*

$$\text{codim } T' \leq c$$

**Proof.** When $L$ is (-d)-regular we can write $L = M(d)$, where $M$ is a Castelnuovo-Mumford regular sheaf. Also when $L = K_Y + dH + A$, since $M := K_Y + 4H + A$ is Castelnuovo-Mumford regular, we can write $L = M(d - 4)$, where $M$ is a Castelnuovo-Mumford regular sheaf. Applying Theorem 3, $(3 - a)$-times, we obtain the result. □

We denote now by $n$ the integer $\lfloor \frac{d+3-a}{b} \rfloor - 4$. We will also denote the very ample line bundle $K_Y(a)$ by $P$, and the bundle $L(3 - a)$ by $L'$. We have the following lemma.

LEMMA 8. *The line bundle $L'$ can be written in the form*

$$L' = M_P + nP$$

*where $M_P$ is a sheaf which is Castelnuovo-Mumford regular with respect to the projective embedding defined by P.*

**Proof.** We know by definition of $a$ and $b$ that there is a nef line bundle $N$ such that $bH = K_Y + aH + N$, from which it follows that

$$(d + 3 - a)H = (n + 4)P + (n + 4)N + rH$$

for some $r \geq 0$, hence

$$(d+3-a)H = (n+4)P + A'$$

where $A'$ is a nef line bundle. Now

$$M_P := L' - nP = \begin{cases} 4P + A_1 & \text{if L is } (-\text{d}) - \text{regular} \\ K_Y + 4P + A_2 & \text{if } L = K_Y + dH + A \end{cases}$$

for some nef line bundles $A_1, A_2$. This clearly implies, by Kodaira vanishing, that $M_P$ is Castelnuovo-Mumford regular with respect to $P$ in the case $L = K_Y + dH + A$. But also in the other case, for each $1 \leq i \leq 3$, we can write

$$M_P - iP = K_Y + aH + (3-i)P + A_1$$

whence again we have Castelnuovo-Mumford regularity by Kodaira vanishing since now $a = \alpha_Y > 0$ by definition. □

We are now in a position to prove the following proposition.

PROPOSITION 5. *Suppose $d \geq 5$ and let $T \subset H^0(L)$ be of codimension $c \leq d-4$. Define*

$$\overline{T} := \mu(T \otimes H^0(K_Y(3)) \subset H^0(K_Y \otimes L(3)).$$

*Then*

$$\operatorname{codim} \overline{T} \leq c^{<n>}.$$

**Proof.** With $T'$ as in Lemma 7, we note that

$$\mu(T' \otimes H^0(K_Y(a)) \subset \overline{T}.$$

We know by Lemma 7 that $\operatorname{codim} T' \leq c$. We know further by Lemma 8 that $L' = M_P + nP$ and hence Theorem 3 applied to the map

$$\mu : T' \otimes H^0(P) \to H^0(K_Y \otimes L(3))$$

gives us that $\operatorname{codim} \mu(T' \otimes H^0(K_Y(a)) \leq c^{<n>}$. From this it follows that

$$\operatorname{codim} \overline{T} \leq c^{<n>}.$$ □

By Proposition 3 we know that

$$\operatorname{codim} \overline{T} \geq \begin{cases} d - 5 + \alpha_Y - \beta_Y & \text{if L is } (-\text{d}) - \text{regular} \\ d - 5 & \text{if } L = K_Y + dH + A \end{cases}$$

and hence either $c \geq d-3$ or

$$c^{<n>} > \begin{cases} d - 6 + \alpha_Y - \beta_Y & \text{if L is } (-\text{d}) - \text{regular} \\ d - 6 & \text{if } L = K_Y + dH + A \end{cases}.$$

The following elementary lemma will allow us to control the growth of $c^{<n>}$.

LEMMA 9. *If there exists an integer $e \geq 0$ such that*

$$c < \sum_{i=0}^{e} (n+1-i)$$

*then $c^{<n>} \leq c + e$.*

**Proof.** The Lemma being obvious for $c = 0$ we suppose $c \geq 1$ and $c = \sum_{i=n}^{f} \binom{k_i}{i}$. Observe that

$$\sum_{i=0}^{e}(n+1-i) \leq \frac{(n+1)(n+2)}{2}.$$

Now suppose $k_i = i$ for $f \leq i \leq f_1$ for some $f-1 \leq f_1 \leq n$, $k_i = i+1$ for $f_1+1 \leq i \leq f_2$ for some $f_2$ such that $f_1 \leq f_2 \leq n$ and $k_i \geq i+2$ for $f_2+1 \leq i \leq n$ (the case $f-1 = f_1$ simply means that no $k_i$ is equal to $i$, and similarly for $f_2$). Then, if $f_2 < n$, we have

$$c \geq \binom{k_n}{n} \geq \binom{n+2}{2} = \frac{(n+1)(n+2)}{2}$$

contradicting the hypothesis. Therefore $f_2 = n$ and $c^{<n>} = c + n - f_1$ and it remains to show that $n - f_1 \leq e$. Since we can write $c = \sum_{i=0}^{n-f_1}(n+1-i) - f$ if $n - f_1 \geq e+1$ we deduce the contradiction $c \geq \sum_{i=0}^{e}(n+1-i)$. □

In particular, it follows that

LEMMA 10. *Suppose* $L = K_Y + dH + A$, $b_Y \geq 2$ *and* $d - 6 - b_Y < \sum_{i=0}^{b_Y}(n+1-i)$. *Then*

$$\operatorname{codim} T > d - 6 - b_Y.$$

*If* $b_Y = 1$, *then*

$$\operatorname{codim} T > d - 6.$$

**Proof.** By Lemma 9, if $b_Y \geq 2$, we have $d - 6 - b_Y < \sum_{i=0}^{b_Y}(n+1-i)$ and $c = \operatorname{codim} T \leq d - 6 - b_Y$ whence, by Proposition 5,

$$\operatorname{codim} \overline{T} \leq c^{<n>} \leq d - 6.$$

But this is impossible by Proposition 3. If $b_Y = 1$ and $c \leq d - 6$ we have $c \leq n$ hence

$$\operatorname{codim} \overline{T} \leq c^{<n>} = c \leq d - 6,$$

again impossible by Proposition 3. □

Similarly we have

LEMMA 11. *Suppose* $L$ *is (-d)-regular,* $\beta_Y \geq 2$ *and*

$$d - 6 + \alpha_Y - 2\beta_Y < \sum_{i=0}^{\beta_Y}(n+1-i).$$

*Then*

$$\operatorname{codim} T > d - 6 + \alpha_Y - 2\beta_Y.$$

*If* $\beta_Y = 1$, *then*

$$\operatorname{codim} T > d - 7 + \alpha_Y.$$

We now require only the following lemma.

LEMMA 12. *If* $b_Y \geq 2$ *and* $d \geq \frac{b_Y(b_Y^2 + 7b_Y - 6)}{2}$ *then*

$$d - 6 - b_Y < \sum_{i=0}^{b_Y}(n+1-i). \tag{6·1}$$

*If* $\beta_Y \geq 2$ *and* $d \geq \frac{\beta_Y^2(\beta_Y+5)}{2}$ *then*

$$d - 6 + \alpha_Y - 2\beta_Y < \sum_{i=0}^{\beta_Y}(n+1-i).$$

**Proof.** We note first that $n \geq \lfloor \frac{d}{b_Y} \rfloor - 4$ and it follows that $b_Y(n+1) > d - 4b_Y$. Hence we have that

$$\sum_{i=0}^{b_Y}(n+1-i) > d - 4b_Y + (n+1) - \frac{b_Y(b_Y+1)}{2}.$$

In particular, if

$$d - 6 - b_Y \leq d - 4b_Y + (n+1) - \frac{b_Y(b_Y+1)}{2}$$

then (6·1) is immediately satisfied. This inequality is equivalent to

$$-7 + 3b_Y \leq n - \frac{b_Y(b_Y+1)}{2}$$

and since $n \geq \lfloor \frac{d}{b_Y} \rfloor - 4$, (6·1) will be satisfied provided that

$$-7 + 3b_Y \leq \lfloor \frac{d}{b_Y} \rfloor - 4 - \frac{b_Y(b_Y+1)}{2}$$

which is equivalent to $-3 + 3b_Y + \frac{b_Y(b_Y+1)}{2} \leq \lfloor \frac{d}{b_Y} \rfloor$, which is equivalent to

$$\frac{b_Y(b_Y^2 + 7b_Y - 6)}{2} \leq d.$$

The second assertion of the Lemma is proved similarly. □

**Completion of the proof of Theorems 1 and 2.**
The results proved so far (together with Remark 2) give a proof of the Theorems under the hypothesis that $(Y, H)$ is not a linear $\mathbb{P}^2$-bundle. In the latter case since $K_Y(4)$ is very ample, repeating verbatim the whole proof replacing everywhere $K_Y(3)$ with $K_Y(4)$ and using $a_Y = \alpha_Y = 4$ we get the desired bound. □

**Proof of Corollary 1.**
This is a straightforward generalisation of [**2**] given the following two facts :

(i) a lower bound on the codimension on the components of the Noether-Lefschetz locus $\mathrm{NL}(\mathcal{O}_Y(d))$ that insures that they have codimension at least two (our hypothesis $d \geq 7 + e$);

(ii) the fact that, on a general surface $X$ not in $\mathrm{NL}(\mathcal{O}_Y(d))$ we have that if a complete intersection of $X$ with another surface in $|\mathcal{O}_Y(d)|$ is reducible then its irreducible components are also complete intersection of $X$ with another surface in $|\mathcal{O}_Y(s)|$ for some $s$ (this is needed in the proof of [**2**, Prop. 2.1] and is insured, in our case, by the hypothesis $\mathrm{Pic}(Y) \cong \mathbb{Z}$). □

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