

SUBVARIETIES OF GENERIC HYPERSURFACES IN ANY VARIETY

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0. Introduction

The geometry of a desingularization Y^m of an arbitrary subvariety of a generic hypersurface X^n in an ambient variety W (e.g. $W = \mathbb{P}^{n+1}$) has received much attention over the past decade or so. Clemens [5] has proved that for $m = 1, n = 2, W = \mathbb{P}^3$ and X of degree d , Y has genus $g \geq 1 + d(d-5)/2$; Xu [13], [14] improved this to $g \geq d(d-3)/2 - 2$ for $d \geq 5$ and showed that if equality holds for $d \geq 6$ then Y is planar; he also gave some lower bounds on the geometric genus $p_g(Y)$ in case $m = n - 1$. Voisin [10], [11] proved that, for X of degree d in $\mathbb{P}^{n+1}, n \geq 3, m \leq n - 2$ then $p_g(Y) > 0$ if $d \geq 2n+1-m$ and K_Y separates generic points if $d \geq 2n+2-m$ (see also [4], [1]). For X^n a generic complete intersection of type (d_1, \dots, d_k) in any smooth polarized $(n+k)$ -fold M , Ein [6], [7] proved that $p_g(Y) > 0$ if $d_1 + \dots + d_k \geq 2n+k-m+1$ and Y is of general type if $d_1 + \dots + d_k \geq 2n+k-m+2$.

In [2] the first two authors applied the classical method of focal loci of families as in C. Segre [8] and Ciliberto-Sernesi [3] (really normal bundle considerations) to give a new proof of one of the main results of Xu in [13], [14] and to extend the lower bounds for the genus in the cases of general surfaces in a component of the Noether-Lefschetz locus in \mathbb{P}^3 and of general projectively Cohen-Macaulay surfaces in \mathbb{P}^4 .

In this paper we introduce some notions of 'filling' subvarieties and use them to prove two new results which *inter alia* give another perspective on the above results, especially the genus bounds. The general philosophy is as usual that as Y moves with X , sections of K_Y (or some twist) can be produced through differentiation; but here this is implemented by exploiting and extending an elementary but perhaps surprising technique in the spirit of classical projective geometry which goes back to [2] and which gives a useful lower bound, depending on the dimension of the projective span of Y , on the number of independent sections of K_Y produced by the differentiation process.

As to our results, in Theorem 1 below we extend and refine in the above sense most of the above-quoted results (not including Voisin's), by giving a lower bound on the number of sections of a certain twist of K_Y involving K_W (for a recent extension, including the proof of a conjecture of Clemens, see [15]). Our second result (Theorem 2), based on the notion of ' r -filling' subvariety, indicates an apparently new and unexpected direction as it deals with some higher-order tensors on Y , manifested in the form of an effective divisor on a Cartesian product Y^r having certain vanishing order on a diagonal locus as well as on a 'double point' locus associated to the map $Y \rightarrow X$. As one application, we conclude a lower bound on the number of quadrics (and higher-degree hypersurfaces) containing certain 'adjoint-type' projective images of Y (and even on the dimension of the kernel of certain 'symmetric Gaussian' maps) (Corollary 2.1 below). Our feeling, however, is that the latter is only the tip of an iceberg, and we hope to explore further in this direction in the future.

1. Filling subvarieties

Fix an $(n + 1)$ -fold W (with isolated singularities) and a free linear system $\mathcal{L} \subseteq H^0(L)$ for some line-bundle L , with associated map $\varphi_{\mathcal{L}}$ to a projective space; set \mathcal{L}_d for the image of $\mathcal{L}^{\otimes d}$ under the multiplication map $H^0(L)^{\otimes d} \rightarrow H^0(L^d)$, and let $X \in \mathcal{L}_d$ be a generic member. We consider generically finite maps

$$f : Y \rightarrow X \subset W$$

from a smooth m -fold. Such a map is said to be *filling* if (f, Y, X) deforms in a family $\{(f_t, Y_t, X_t) : t \in T\}$ such that $X_t \in \mathcal{L}_d$ for all t (X_t not fixed), the natural map $T \rightarrow \mathcal{L}_d$ is generically finite, dominant and $\bigcup_{t \in T} f_t(Y_t)$ is dense in W . f is said to be $(1, b)$ -filling if, in addition, (f, Y, X) moves in a family $\{(f_s, Y_s, X) : s \in S\}$ with X fixed such that $\bigcup_{s \in S} f_s(Y_s)$ is at least $(m + b)$ -dimensional. For instance, if W is homogeneous then any f is filling.

Our main result in the filling case is the following which, though a special case of Theorem 2 below, we have chosen to state and prove separately in order to make the argument easier to follow.

Theorem 1. *Let $X \in \mathcal{L}_d$ be generic, $f : Y \rightarrow X$ a $(1, b)$ -filling, generically finite map from a smooth irreducible m -fold, and suppose $d \geq n - m - b$, $b \leq n - m$, $1 \leq m \leq n - 1$, and that $\varphi_{\mathcal{L}}(f(Y))$ spans a \mathbb{P}^{n+1} . Then*

$$(1) \quad h^0(K_Y - (d - n + m + b)f^*L - f^*K_W) \geq 1 + p(n - m - b).$$

If equality holds in (1) and $n - m - b \geq 1$, then f is not $(1, b + 1)$ -filling; in particular, if in addition $b = 0$ then f has no deformations which move $f(Y)$ with X fixed.

Before proving the Theorem, let us give a sampling of examples and applications, beginning with the most popular case $W = \mathbb{P}^{n+1}$. Set $d = 2n + 2 - m + j$ and let $b = 0$ unless otherwise mentioned. Then we get:

$$(2) \quad h^0(K_Y(-j)) \geq 1 + p(n - m),$$

with equality only if f has no deformations which move $f(Y)$ with X fixed.

If $j \geq 0$ this bound implies a similar one for K_Y itself, while Voisin shows in this case that φ_{K_Y} is generically 1-1 provided $m \leq n - 2$.

For $m = n - 1$ and $j > 0$, Xu gets a bound on p_g while the above shows $h^0(K_Y(-j)) \geq 1 + p$, which implies some bounds on p_g somewhat like Xu's; for a very lazy such bound, note that our bound implies $p_g(Y) \geq 1 + h^0(\mathcal{O}_Y(j))$ and the latter may be estimated using Koszul, assuming (as we may) that $f(Y)$ is a (d, e) complete intersection, yielding

$$(3) \quad p_g(Y) \geq 1 + \binom{n+1+j}{j} - \binom{n+1+j-d}{j} - \binom{n+1+j-e}{j} + \binom{n+1+j-d-e}{j}.$$

Perhaps better bounds can be obtained by estimating more carefully the relevant multiplication map.

For $m = 1$, i.e. Y a curve, and $b \geq 0$, the basic estimate is

$$(4) \quad h^0(K_Y(-j)) \geq 1 + p(n - 1 - b),$$

with equality only if $f(Y)$ is not $(1, b + 1)$ -filling. For instance, a genus-3 nonplanar curve on a generic quintic surface is rigid (we do not know if such curves exist); on a generic sextic 3-fold X any nonrigid curve $f(Y)$ must have genus at least $1 + p = \dim \text{span} f(Y)$, and curves for which equality holds don't fill up X (note that 4-tangent plane sections have genus 6 and do fill up X); the genus of a curve on a generic septic 3-fold is at least $1 + 2p$ (note that a 6-tangent-plane section has genus 9).

If W is an abelian variety and L is twice a principal polarization (well-known to be free), and again taking $b = 0$, note that $p > 0$ if $m > 1$, so if in addition $d \geq n - m$, we conclude that $h^0(K_Y) > 1$; in particular

- X does not contain any (translated) abelian subvariety of dimension $m > 1$.

If $d > n - m, m \geq 1$, we conclude as a special case that

- X contains only subvarieties of general type; in particular it does not contain any translated abelian subvariety of dimension $m \geq 1$.

In Ein's general situation, if (M, L) is a smooth polarized $(n + k)$ -fold and L is assumed base-point free, we may take as W any smooth complete intersection of type (d_2, \dots, d_k) . By adjunction, $K_W = (K_M + (d_2 + \dots + d_k)L)|_W$. Letting X be a generic member of $H^0(L^d)|_W$, Theorem 1 yields

- $K_Y - (f^*K_M + (d_1 + \dots + d_k - n + m + b)f^*L)$ is effective.

Now by Kodaira vanishing and Riemann-Roch, $K_M + (n + k + 1)L$ is effective (else the degree- $(n + k)$ polynomial $\chi(K_M + tL)$ would vanish at the $n + k + 1$ points $1, \dots, n + k + 1$); moreover it is well-known [9] that $K_M + (n + k)L$ is effective unless $M = \mathbb{P}^{n+k}$. Note that as f is filling, its image may be assumed not contained in any fixed divisor, hence f^* preserves effectiveness. Thus Ein's result that $p_g(Y) > 0$ if $d_1 + \dots + d_k \geq 2n + k - m + 1$ follows, assuming f is filling; with Theorem 1 we conclude further, in this case, that

- If $(M, L) \not\cong (\mathbb{P}^{n+k}, \mathcal{O}(1))$ then $p_g(Y) > 0$ provided $d_1 + \dots + d_k \geq 2n + k - m$.

If f is not filling, then by considering a desingularization of the subvariety of M filled up by $f(Y)$ we actually get a better bound. Note that Sommese [9] has classified the polarized pairs (M, L) such that $K_M + (n + k - 2)L$ is not \mathbb{Q} -effective, so excluding those and taking for simplicity $k = 1$, for any $f : Y \rightarrow X \subset M$ filling, we have that the Kodaira dimension of Y is nonnegative (resp. Y is of general type) provided $d \geq 2n - 1 - m$ (resp. $2n - m$). Moreover note that in fact Ein proves $h^0(K_Y - (d + m - 2n - 2 + b)f^*L) \neq 0$ when f is $(1, b)$ -filling, hence in Theorem 1 we improve Ein's result in several directions: we give a lower bound on the dimension that depends upon how much $\varphi_{\mathcal{L}}(f(Y))$ spans, and we give information on the extremal cases; the twist is strictly better unless $M = \mathbb{P}^{n+k}$; and now W can be any variety with isolated singularities and not only a general complete intersection in M .

Proof of Theorem 1. Let $f : Y \rightarrow X \subset W$ be a generic member of a $(1, b)$ -filling family as above and let $N_f, N_{f/X}$ denote the normal sheaves. Note that in the definition of filling, there is no loss of generality in assuming that the map $T \rightarrow \mathcal{L}_d$ is quasi-finite

and unramified at the point corresponding to X ; likewise in the definition of $(1, b)$ -filling there is no loss of generality in assuming that the natural map $\coprod Y_s \rightarrow X$ has (differential) rank at least $m + b$ at a generic point of Y . Denote by N^0 the subsheaf of $N_{f/X}$ generated by b generic global sections. Thus N^0 has rank b and $c_1(N^0)$ is effective. Now set

$$N'_{f/X} = N_{f/X}/N^0, N'_f = N_f/N^0,$$

$$(5) \quad D = \left(\bigwedge^{n-m+1-b} N'_f \right)^{**}.$$

Thus

$$(6) \quad c_1(N_f) = K_Y - f^*K_W = D + D', D' \text{ effective.}$$

(In fact $D' = 0$ iff f is unramified in codimension 1.) Now consider the exact sequences

$$0 \rightarrow N_{f/X} \rightarrow N_f \rightarrow f^*(L^d) \rightarrow 0,$$

$$0 \rightarrow N'_{f/X} \rightarrow N'_f \rightarrow f^*(L^d) \rightarrow 0.$$

Our genericity assumption on X implies that any infinitesimal deformation of X in \mathcal{L}_d carries an infinitesimal deformation of f , hence the natural map $\mathcal{L}_d \rightarrow H^0(f^*L^d)$ admits a lifting

$$(7) \quad \phi : \mathcal{L}_d \rightarrow H^0(N_f),$$

whence sheaf maps

$$(8) \quad \psi : \mathcal{L}_d \otimes \mathcal{O}_Y \rightarrow N_f,$$

$$(9) \quad \psi' : \mathcal{L}_d \otimes \mathcal{O}_Y \rightarrow N'_f$$

and pointwise maps at a general point $y \in Y$

$$(10) \quad \psi_y : \mathcal{L}_d \rightarrow N_{f,y} = \mathbb{C}^{n+1-m},$$

$$(11) \quad \psi'_y : \mathcal{L}_d \rightarrow N'_{f,y} = \mathbb{C}^{n+1-m-b}.$$

Our filling hypothesis implies that $f(Y)$ moves, filling up W , hence that the map ψ_y is surjective, hence so is ψ'_y .

The following technical Lemma will easily imply Theorem 1. We will prove later a more general version, useful also for the applications in §2.

Lemma 1.1. *Assumptions as in Theorem 1. Let k be an integer such that $0 \leq k \leq d$ and let*

$$\Psi : \mathcal{L}_d \rightarrow \mathbb{C}^{k+1}$$

be a surjection of vector spaces. Then for a general choice of elements $h_{k+1}, \dots, h_d \in \mathcal{L}$ and for general subsets $Y_1, \dots, Y_k \subset Y$ each of cardinality $p = \dim \text{span} \varphi_{\mathcal{L}}(f(Y)) - 1$, the restriction of Ψ to the subspace $\mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_k) h_{k+1} \cdots h_d$ surjects.

Let's first see that Lemma 1.1 implies Theorem 1. We apply it to the map $\Psi = \psi'_y$ above with $k = n - m - b$. It yields the existence of a generic subset $S \subset Y$ of cardinality $p(n - m - b)$ such that $\psi'_y|_{\mathcal{L}_k(-S)h_{k+1} \cdots h_d}$ is surjective. We conclude that the restriction of ψ'_y on a subbundle of the form

$$\mathcal{L}_{(n-m-b)}(-S) \left(\prod_{a=1}^{d-(n-m-b)} h_a \right) \otimes \mathcal{O}_Y$$

is generically surjective. Consequently, for a generic $(n - m + 1 - b)$ -dimensional subspace $V \subset \mathcal{L}_{(n-m-b)}$ we get an injective generically surjective map $\rho : V(\prod h_a) \otimes \mathcal{O}_Y \rightarrow N'_f$ which evidently drops rank on $(\prod f^*(h_a))_0$, as well as on S .

Since $\bigwedge^{n-m+1-b} \rho$ yields a section of $\bigwedge^{n-m+1-b} N'_f$ vanishing on a generically placed divisor of class $(d - (n - m - b))f^*L$ as well as on $p(n - m - b)$ many generic points, it follows that

$$h^0(c_1(N'_f) - (d - (n - m - b))f^*L) \geq 1 + p(n - m - b)$$

and Theorem 1 follows in light of Eq. (5)(6). \square

Now Lemma 1.1 is the special case $r = 1$ of the following result that we are going to use in its full generality in the next section.

Lemma 1.2. *Assumptions as in Theorem 1. Fix integers r, k with $0 \leq r - 1 \leq k \leq d$. Let $\mathcal{L}_d \rightarrow \mathbb{C}^N$ be a linear map of vector spaces such that for y_1, \dots, y_{r-1} general points of Y , the restriction to $\mathcal{L}_d(-y_1 - \cdots - y_{r-1})$ induces a surjection*

$$\Psi : \mathcal{L}_d(-y_1 - \cdots - y_{r-1}) \rightarrow \mathbb{C}^{k+1}.$$

Then for a general choice of elements $h_{k+1}, \dots, h_d \in \mathcal{L}$ and for general subsets $Y_1, \dots, Y_k \subset Y$ each of cardinality p , with $Y_i \ni y_i$, $i = 1, \dots, r - 1$, the restriction of Ψ to the subspace $\mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_k) h_{k+1} \cdots h_d$ surjects.

Proof. Fix an integer j such that $0 \leq j \leq k$. We say that $U \subset \mathcal{L}_d(-y_1 - \cdots - y_{r-1})$ is a subspace of type j if

$$U = U(Y_1, \dots, Y_j, h_{j+1}, \dots, h_d) := \mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_j) h_{j+1} \cdots h_d$$

for some subsets $Y_1, \dots, Y_j \subset Y$ of cardinality p and elements h_{j+1}, \dots, h_d in \mathcal{L} , such that there exists an integer q with $0 \leq q \leq \min\{j, r - 1\}$ satisfying $y_1 \in Y_1, \dots, y_q \in Y_q, h_{j+1} \in \mathcal{L}(-y_{q+1}), \dots, h_{r+j-1-q} \in \mathcal{L}(-y_{r-1})$. Given U , the maximal such integer q will be called the *index* of U . Observe that a general subspace of type $j < d$ and index q is contained in a general subspace of type $j + 1$, index $\geq q$. Here of course the word 'general' means: in a general choice of the linear forms h_i and the subsets Y_i , subject only to the restrictions determined by the points y_1, \dots, y_{r-1} and the index q . Let us now record the following

Claim 1. *If for a general subspace of type j and index $q < \min\{j, r-1\}$ one has $\dim \Psi(U) = x$, then for a general subspace U' of type j and index $q+1$ one has $\dim \Psi(U') \geq x$.*

Proof of Claim 1. Observe indeed that the points y_1, \dots, y_{r-1} are chosen generically. On the other hand, if $U = \mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_j) h_{j+1} \cdots h_d$ is of type j and index q for y_1, \dots, y_{r-1} , then since Y_{q+1} is general, without any loss of generality we may replace y_{q+1} with some point of Y_{q+1} and similarly replace the h 's; hence U becomes of index $q+1$. This proves Claim 1.

The proof of Lemma 1.2 is based on an inductive construction of general subspaces U_0, U_1, \dots, U_k of $\mathcal{L}_d(-y_1 - \cdots - y_{r-1})$, such that for all j , U_j is of type j and the image under Ψ has dimension $\geq j+1$.

Since $\mathcal{L}_d(-y_1 - \cdots - y_{r-1})$ is generated by monomials $h_1 \cdots h_d$ and Ψ surjects, then for a general choice of $U = \mathbb{C}h_1 \cdots h_d$ of type 0, we get $\dim \Psi(U) = 1$. Assume now that, for some $j < k$, we have constructed a general subspace U of type j and maximal index $q = \min\{j, r-1\}$ whose image under Ψ is of dimension $\geq j+1$. If this dimension is actually bigger than $j+1$, then a general subspace of type $j+1$ and maximal index has image of dimension $\geq j+2$ and the induction goes on. Hence we can assume that for a general U of type j we have $\dim \Psi(U) = j+1$. Since $j < k$, and Ψ is surjective we know that for a general monomial $M \notin U$ we have $\Psi(M) \notin \Psi(U)$.

Claim 2. *For any $j = 0, \dots, k$, let $U = \mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_j) h_{j+1} \cdots h_d$ be a general subspace of type j , index $q = \min\{j, r-1\}$ and let $M' = h'_1 \cdots h'_d$ be a general monomial in $\mathcal{L}_d(-y_1 - \cdots - y_{r-1})$. Then there exists a chain of subspaces $U^0 = U, U^1, \dots, U^a$ of type j , such that $M' \in U^a$ and each consecutive pair U^i, U^{i+1} is contained in the same subspace of type $j+1$ and index $\geq q$. Furthermore as U, M' move generically, we may assume that any subspace of the chain is general.*

Proof of Claim 2. Without any loss of generality, we may assume that $h'_i \in \mathcal{L}(-y_i)$, for all $i = 1, \dots, r-1$. Let us now construct a chain as above, which links U and the subspace $\mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_j) h'_{j+1} h_{j+2} \cdots h_d$. First suppose $j \geq r-1$. Pick a general subset $\{A_1, \dots, A_p\} \subset Y$, over which $f^* h_{j+1}$ vanishes, and a general subset $\{B_1, \dots, B_p\} \subset Y$ contained in the locus where $f^* h'_{j+1} = 0$. Take, for $c = 1, \dots, p$, a general element $z_c \in \mathcal{L}(-B_1 - \cdots - B_c - A_c - \cdots - A_p)$, which exists by our definition of p . Then put $z_0 = h_1, z_{p+1} = h'_1$ and define for all $c = 0, \dots, p+1$, $U^c = \mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_j) z_c h_{j+2} \cdots h_d$. It is clear that U^c, U^{c+1} both are contained in $\mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_j) \mathcal{L}(-B_1 - \cdots - B_c - A_{c+1} - \cdots - A_p) h_{j+2} \cdots h_d$, which is in fact general of type $j+1$, for h_{j+1}, h'_{j+1} and the points A_i 's, B_i 's are general. When $j < r-1$, i.e. when both h_{j+1}, h'_{j+1} lie in $\mathcal{L}(-y_{j+1})$, we argue as above, except that we take $B_p = A_p = y_{j+1}$. We repeat then the previous procedure, considering h_{j+2}, h'_{j+2} instead of h_{j+1}, h'_{j+1} and so on. We get finally a chain with the required properties, that links U with the subspace $\mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_j) h'_{j+1} \cdots h'_d$. Therefore we may replace, from now on, U with the subspace $\mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_j) h'_{j+1} \cdots h'_d$. Choose now a general subset $Y' \subset Y$ of cardinality p , containing y_{j+1} if $j+1 < r$ and such that $h'_{j+1} \in \mathcal{L}(-Y')$. Choose also a general $h \in \mathcal{L}(-Y_j)$. Define $U^* = \mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_{j-1}) \mathcal{L}(-Y') h h'_{j+2} \cdots h'_d$. By our generality assumptions on Y' and h , also U^* is a general subspace of type j and maximal index and both U and U^* are contained in $\mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_j) \mathcal{L}(-Y') h'_{j+2} \cdots h'_d$, which is of type $j+1$. Now use the procedure at the beginning of the proof, acting on h and h'_j , to find a chain of the required type which links U^* with the subspace $\mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_{j-1}) \mathcal{L}(-Y') h'_j h'_{j+2} \cdots h'_d$. At the end of this procedure, we see that

we may link U with the new subspace $\mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_{j-1})\mathcal{L}(-Y')h'_j h'_{j+2} \cdots h'_d$; notice that renumbering Y' as Y_j and swapping h'_j with h'_{j+1} , we linked U with the subspace $\mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_j)h'_{j+1} \cdots h'_d$, but with the good news that now $h'_j \in \mathcal{L}(-Y_j)$. Repeat this last construction, using $h \in \mathcal{L}(-Y_{j-1})$ and so on. At the very end we see that we constructed a chain with the required properties, linking our original U with a subspace of type $\mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_j)h'_{j+1} \cdots h'_d$ for which $h'_1 \in \mathcal{L}(-Y_1), \dots, h'_j \in \mathcal{L}(-Y_j)$. This is our U^a and Claim 2 is proved.

Let us go back to the proof of Lemma 1.2. By Claim 2, we get a chain as above of subspaces of type j joining U with M ; all the elements of the chain are general, so replacing U with some element of the chain, we may assume that U, M both belong to the same subspace U' of type $j+1$. Since U is general, U' is also general (of its index) and on the other hand $\dim \Psi(U') > \dim \Psi(U)$. Therefore we have the inductive step. \square

Remark. The case $k=1$ is in [2]; the general case is not much more difficult. Another proof of the above lemma can be found in [15].

2. r -filling subvarieties

We continue with the notations of the previous section. The map f is said to be r -filling if (f, Y, X) deforms in a family $\{(f_t, Y_t, X_t) : t \in T\}$ such that $X_t \in \mathcal{L}_d$ for all t (X_t not fixed), the natural map $T \rightarrow \mathcal{L}_d$ is generically finite, dominant and $\bigcup_{t \in T} f_t(Y_t)^r$ is dense in the Cartesian product W^r . f is said to be (r, b) -filling if it is r -filling and $(1, b)$ -filling. For instance,

- if $W = \mathbb{P}^{n+1}$ and $f(Y)$ spans (at least) a \mathbb{P}^{r-1} , then clearly f is r -filling (since linearly independent r -tuples are projectively equivalent);
- if W is a Grassmannian of lines in \mathbb{P}^a then, because all pairs of disjoint lines are projectively equivalent, it follows that any f is 2-filling unless $f(Y)$ is contained in a (Plücker-) linear subvariety $U \subset W$, and it is easy to see that any such U must be of the form either { lines through a point } or { lines in a \mathbb{P}^2 }.

The following result extends Theorem 1 to the r -filling case:

Theorem 2. *Let $X \in \mathcal{L}_d$ be generic, $f : Y \rightarrow X$ an (r, b) -filling, generically finite map from a smooth irreducible m -fold, and suppose $d \geq r(n-m-b)$, $r \leq n-m+1-b$, $1 \leq m \leq n-1$, and that $\varphi_{\mathcal{L}}(f(Y))$ spans a \mathbb{P}^{p+1} . Let $\Delta_Y^r \subset Y^r$ be the big diagonal (= locus of nondistinct r -tuples) and $D_f^r \subset Y^r$ the double locus of f (= closure of locus of distinct r -tuples (y_1, \dots, y_r) such that $f(y_1), \dots, f(y_r)$ are not distinct). Then the linear system of S_r -invariant sections in $H^0(K_Y - (d-r(n-m-b))f^*L - f^*K_W)^{\boxtimes r}$ having multiplicity at least $n-m+1$ on Δ_Y^r and multiplicity at least n on D_f^r is of dimension at least $2-r+pr(n-m-b)$.*

To get an idea of the significance of this result for $r > 1$ consider now the case $r=2, n-m-b \geq 1$. Set $M = K_Y - f^*K_W - (d-2(n-m-b))f^*L$, and assume f is 2-filling (which automatically holds if $W = \mathbb{P}^{n+1}$ in which case $M = K_Y(3n-2m-d-2b+2)$). The Theorem 2 yields a space of symmetric elements $\tau \in H^0(M)^{\otimes 2}$ vanishing to order $n-m+1 \geq 2$ on the diagonal. These in turn yield, in particular, elements of

$$I_2(M) = \ker(\text{sym}^2(H^0(M)) \rightarrow H^0(M^{\otimes 2}))$$

i.e. quadrics containing $\varphi_M(Y)$ (in fact, if $n - m > 1$ they belong to the kernel of a certain 'higher-order symmetric Gaussian', cf. [12]); thus

Corollary 2.1. *In the above situation, we have*

$$\dim I_2(M) \geq 2p(n - m - b).$$

Clearly similar results can be stated for $r > 2$; we shall leave this to the reader.

Proof of Theorem 2. We continue with the notations used in the proof of Theorem 1. Now set $\bar{y} = \{y_1, \dots, y_{r-1}\}$ for generic $y_1, \dots, y_{r-1} \in Y$. As before we have sheaf maps $\psi : \mathcal{L}_d \otimes \mathcal{O}_Y \rightarrow N_f$, $\psi' : \mathcal{L}_d \otimes \mathcal{O}_Y \rightarrow N'_f$ and pointwise maps $\psi_y : \mathcal{L}_d \rightarrow N_{f,y} = \mathbb{C}^{n+1-m}$, $\psi'_y : \mathcal{L}_d \rightarrow N'_{f,y} = \mathbb{C}^{n+1-m-b}$, $y \in Y$ general (cf. (7)-(11)). Our r -filling hypothesis implies that $f(Y)$ moves, filling up W while fixing $f(y_1), \dots, f(y_{r-1})$. We claim that this (and the hypothesis on d), implies that the restricted map

$$\Psi_y : \mathcal{L}_d(-\bar{y}) \rightarrow N_{f,y} \oplus N_{f/X,\bar{y}}$$

is surjective, where $\mathcal{L}_d(-\bar{y})$ denotes the set of elements of \mathcal{L}_d vanishing on $f(y_1), \dots, f(y_{r-1})$ and $N_{f/X,\bar{y}} = \bigoplus_{i=1}^{r-1} N_{f/X,y_i}$. Indeed this follows immediately by the 'snake lemma' from the fact that we have a surjection $\mathcal{L}_d \rightarrow N_{f,y} \oplus N_{f,\bar{y}}$ (r -fillingness) which induces an isomorphism $L^{\otimes d} \otimes \mathcal{O}_{\bar{y}} \rightarrow N_{f,\bar{y}}/N_{f/X,\bar{y}}$. Or more concretely, given any vector of type $(v, 0) \in N_{f,y} \oplus N_{f/X,\bar{y}}$ we can find an element of $\mathcal{L}_d(-\bar{y})$ inducing the deformation v on y , hence the vectors $(v, 0)$ lie in the image of Ψ_y . Similarly given any vector of type $(0, w_i) \in N_{f,y} \oplus N_{f/X,\bar{y}}$ with $w_i \neq 0$ only on its i -th component, again by r -fillingness, we can find a vector $w \in N_{f,y}$ such that (w, w_i) lies in the image of Ψ_y . Therefore Ψ_y is surjective and likewise the map $\Psi'_y : \mathcal{L}_d(-\bar{y}) \rightarrow N'_{f,y} \oplus N'_{f/X,\bar{y}}$ is surjective as well. Now consider the Cartesian product

$$f^r : Y^r \rightarrow X^r \subset W^r.$$

We have an exact S_r -invariant sequence

$$0 \rightarrow N_{f^r/X^r} \rightarrow N_{f^r} \rightarrow f^*(L^d)^{\boxplus r} \rightarrow 0$$

and $N_{f^r} = N_f^{\boxplus r}$ etc. Likewise

$$0 \rightarrow N'_{f^r/X^r} \rightarrow N'_{f^r} \rightarrow f^*(L^d)^{\boxplus r} \rightarrow 0,$$

$N'_{f^r} = (N'_f)^{\boxplus r}$. Now there is a natural S_r -invariant (i.e. symmetric) map

$$\mathcal{L}_d \rightarrow H^0((f^*L^d)^{\boxplus r}) = \oplus H^0(f^*(L^d))$$

and by the r -filling assumption this lifts to a symmetric map $\mathcal{L}_d \rightarrow H^0(N_{f^r})$, whence a map $\phi^r : \mathcal{L}_d \rightarrow H^0(N'_{f^r})$ whose corresponding sheaf map $\psi^r : \mathcal{L}_d \otimes \mathcal{O}_{Y^r} \rightarrow N'_{f^r}$ is generically surjective and symmetric.

We now apply Lemma 1.2 to the map $\Psi = \Psi'_y$ above with $k = r(n - m - b)$. It yields the existence of a generic (independently of \bar{y}) subset $S \subset Y$ of cardinality

$pr(n-m-b) - r + 1$ such that $\Psi'_y|_{\mathcal{L}_k(-S)h_{k+1}\cdots h_d}$ is surjective. We conclude that the restriction of ψ^r on a subbundle of the form

$$\mathcal{L}_{r(n-m-b)}(-S)\left(\prod_{a=1}^{d-r(n-m-b)} h_a\right) \otimes \mathcal{O}_{Y^r}$$

is generically surjective. Consequently, for a generic $r(n-m+1-b)$ -dimensional subspace $V \subset \mathcal{L}_{r(n-m-b)}$ we get an injective generically surjective map

$$\rho^r : V\left(\prod h_a\right) \otimes \mathcal{O}_{Y^r} \rightarrow N'_{f_r}$$

and since the h_i are chosen generally, independently of the y_j , ρ^r is symmetric as well. On the other hand ρ^r evidently drops rank on $(\prod f^*(h_a))_0$, as well as on slices $Y^{r-1} \times S$ (and their S_r -orbits).

Now let

$$b_r : Y_r \rightarrow Y^r$$

be the blowing-up of the components of the big diagonal in some order, with exceptional divisor $E_{Y,r}$, and likewise for X, W (same order). We have a natural map $f_r : Y_r \rightarrow X_r \subset W_r$, whence an exact sequence

$$0 \rightarrow N_{f_r/X_r} \rightarrow N_{f_r} \rightarrow f^*(L^d)^{\boxplus r} \rightarrow 0.$$

Let $N'_{f_r} = N_{f_r}/b_r^*(N^{0\boxplus r})$ and note that $c_1(N_{f_r}) = c_1(N'_{f_r}) + A$, A effective. Since f_r moves with f , there is as above a generically surjective map $\psi_r : \mathcal{L}_d \otimes \mathcal{O}_{Y_r} \rightarrow N'_{f_r}$ whence a generic isomorphism $\rho_r : V \otimes \mathcal{O}_{Y_r} \rightarrow N'_{f_r}$ compatible with ρ^r . Now we compute

$$\begin{aligned} c_1(N_{f_r}) &= b_r^*(K_Y - f^*K_W)^{\boxtimes r} + (m-1)E_{Y,r} - nf_r^*E_{W,r} \\ &= b_r^*(K_Y - f^*K_W)^{\boxtimes r} - (n-m+1)E_{Y,r} - nR, \end{aligned}$$

where R is effective and contains the blowup of some scheme supported on the locus of distinct r -tuples mapped by f to nondistinct ones. Since $\bigwedge^{r(n-m+1-b)} \rho^r$ yields a symmetric section of $\bigwedge^{r(n-m+1-b)} N_{f_r}^{**}$ vanishing on a generically placed symmetric divisor of class $(d-r(n-m-b))L^{\boxtimes r}$ as well as on $pr(n-m-b) - r + 1$ many slices $Y^{r-1} \times \text{pt.}, \text{pt.} \in S$, the Theorem 2 follows by projecting back to Y^r . \square

Remark. Rather than blow up the big diagonal we could equally well blow up other 'diagonal strata', thus leading to analogues of the Theorem 2 which we will let the reader state; though we do not pursue them here, they might prove useful.

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