ON THE EXISTENCE OF ULRICH VECTOR BUNDLES ON SOME SURFACES OF MAXIMAL ALBANESE DIMENSION

ANGELO FELICE LOPEZ*

Abstract. We establish the existence of simple Ulrich vector bundles on surfaces $S \subset \mathbb{P}^N$ of maximal Albanese dimension with $\chi(\mathcal{O}_S) = 0$ and $H^1(\mathcal{O}_S(1)) = 0$.

1. Introduction

An Ulrich vector bundle on a smooth projective variety $X \subseteq \mathbb{P}^N$ is a bundle $\mathcal{E}$ on X such that $H^i(\mathcal{E}(-p)) = 0$ for all $i \geq 0$ and $1 \leq p \leq \dim X$. Even though Ulrich bundles (and the natural consequences on X that they bring - such as determinantal representation or properties of the Chow form), have received a lot of attention in recent years (see for example [ES, B1, CKM]), the basic existence question on any X is still open even in dimension two (while on a curve X, of genus g, there is always an Ulrich line bundle: il $g = 0$ take $\mathcal{O}_X$, if $g > 0$ take $\mathcal{O}_X(1) \otimes L$ where L is a general line bundle of degree $g - 1$).

We mention here that several classes of surfaces do carry an Ulrich vector bundle, for example K3 surfaces [AFO, F], abelian surfaces [B2], bielliptic surfaces [B1] and surfaces with $p_g = 0, q = 1$ [C4, C4e], Enriques surfaces [B3, C1, C1e, BN], del Pezzo surfaces [ES, CH, B1, C3, PT], several regular surfaces [C1, C1e, C2], hypersurfaces and complete intersections [HUB] and several ruled surfaces [ACM, B1]. In particular we observe that, aside from the mentioned surfaces, not so many results are known for irregular surfaces.

In this paper we show the following

Theorem 1. *Let S be a surface with maximal Albanese dimension and $\chi(\mathcal{O}_S) = 0$. Let H be a very ample divisor on S such that $H^1(H) = 0$.*

Then there exists a simple rank two Ulrich vector bundle $\mathcal{E}$ for the pair (S, H).

Note that in general on a surface as in the theorem there are no Ulrich line bundles (see Remark 3.1), whence rank two is the least one can expect.

Examples of surfaces with maximal Albanese dimension and $\chi(\mathcal{O}_S) = 0$ are any surface birational to an abelian surface, the products $E \times C$, with E elliptic and any smooth C and the quotients $(E \times C)/G$, where G is a finite group acting without fixed points.

We work over the complex numbers.

2. The vector bundle

As in [B2], we construct a vector bundle that will be later shown to be Ulrich.

Lemma 2.1. *Let S be a surface with maximal Albanese dimension and $\chi(\mathcal{O}_S) = 0$. Let H be a very ample divisor on S such that $H^1(H) = 0$. Let $\eta \in \operatorname{Pic}^0(S)$ be general.*

Then there exists a rank two simple vector bundle $\mathcal{G}_\eta$ on S with

$$c_1(\mathcal{G}_\eta) = H + K_S + 2\eta, \quad c_2(\mathcal{G}_\eta) = \frac{1}{2} H \cdot (H + K_S) \tag{2.1}$$

and

$$H^0(\mathcal{G}_\eta) = H^0(\mathcal{G}_\eta(-2\eta)) = 0. \tag{2.2}$$

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Proof. First note that

$$H^2(H) = 0. \tag{2.3}$$

In fact if $H^2(H) \neq 0$, then, by Serre duality $H^0(K_S - H) \neq 0$, whence K_S is big, that is S is of general type. But then $\chi(\mathcal{O}_S) > 0$ by [B4, Thm. X.4], a contradiction.

Next observe that

$$h^i(\pm\eta) = 0 \text{ for } i \geq 0. \tag{2.4}$$

In fact $\eta \neq \mathcal{O}_S$, whence $h^0(\pm\eta) = 0$. We have $h^1(\pm\eta) = 0$ by [GL, Thm. 1] and, since $\chi(\mathcal{O}_S) = 0$, we get (2.4) by Riemann-Roch.

Set $N + 1 = h^0(H)$. We claim that

$$H^i(H \pm \eta) = 0 \text{ for } i = 1, 2 \tag{2.5}$$

and

$$N + 1 = \frac{1}{2} H \cdot (H - K_S) = h^0(H \pm \eta). \tag{2.6}$$

To see them note that it follows by semicontinuity (see for example [C4, Proof of Cor. 3.3]) that the subsets

$$Z_i := \{L \in \mathrm{Pic}^0(S) : h^i(H \pm L) > 0\}$$

are closed for $i = 1, 2$. On the other hand, by hypothesis and (2.3), $\mathcal{O}_S \notin Z_i$ for $i = 1, 2$, whence we get that $H^i(H \pm \eta) = 0$ for $i = 1, 2$ and $\eta \in \mathrm{Pic}^0(S)$ general. This gives (2.5) and then (2.6) follows by Riemann-Roch.

Let $C \in |H|$ be a smooth irreducible curve defined by $s \in H^0(H)$ and let $P_i, 1 \leq i \leq N + 1$ be general points on C. Since C is non-degenerate, it follows that

(2.7) $Z = P_1 + \ldots + P_{N+1}$ satisfies the Cayley-Bacharach property with respect to $|H|$

and

$$H^0(\mathcal{I}_{Z/S}(H)) = \mathbb{C}s. \tag{2.8}$$

By (2.7) and [HL, Thm. 5.1.1] we get a rank two vector bundle $\mathcal{F}$ with $c_1(\mathcal{F}) = H - K_S, c_2(\mathcal{F}) = N + 1$ and sitting in an exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{F} \to \mathcal{I}_{Z/S}(H - K_S) \to 0. \tag{2.9}$$

For later use we record that

$$H^0(\mathcal{F}(K_S - H)) = 0. \tag{2.10}$$

In fact $H^0(K_S - H) = H^2(H)^* = 0$ by (2.3). From (2.9) we get the exact sequence

$$0 \to K_S - H \to \mathcal{F}(K_S - H) \to \mathcal{I}_{Z/S} \to 0$$

whence $H^0(\mathcal{F}(K_S - H)) = 0$. This proves (2.10).

Let $\mathcal{G}_\eta = \mathcal{F}(K_S + \eta)$. It follows by (2.6) that

$$c_1(\mathcal{G}_\eta) = H + K_S + 2\eta, \quad c_2(\mathcal{G}_\eta) = \frac{1}{2} H \cdot (H + K_S) \tag{2.11}$$

so that (2.1) is proved, and that there are two exact sequences

$$0 \to K_S + \eta \to \mathcal{G}_\eta \to \mathcal{I}_{Z/S}(H + \eta) \to 0 \tag{2.12}$$

and

$$0 \to K_S - \eta \to \mathcal{G}_\eta(-2\eta) \to \mathcal{I}_{Z/S}(H - \eta) \to 0. \tag{2.13}$$

Next we prove that

$$H^0(\mathcal{I}_{Z/S}(H \pm \eta)) = 0 \tag{2.14}$$

where, as above, $\eta \in \mathrm{Pic}^0(S)$ is general. From the exact sequence

$$0 \to \pm\eta \to H \pm \eta \to (H \pm \eta)_{|C} \to 0$$

using (2.4) and (2.5), we get that $h^0((H \pm \eta)_{|C}) = h^0(H \pm \eta) = N + 1$, and this gives that

$$h^0(\mathcal{I}_{Z/C}(H \pm \eta) = h^0((H \pm \eta)_{|C}) - N - 1 = 0.$$

Now the exact sequence

$$0 \to \pm\eta \to \mathcal{I}_{Z/S}(H \pm \eta) \to \mathcal{I}_{Z/C}(H \pm \eta) \to 0$$

and (2.4) show that $H^0(\mathcal{I}_{Z/S}(H \pm \eta)) = 0$. Thus (2.14) is proved.

We deduce by (2.12), (2.13), (2.4) and (2.14) that (2.2) holds.

Finally let us prove that $\mathcal{G}_\eta$ is simple.

First we can assume that $K_S \neq 0$. In fact if $K_S = 0$ then S is an abelian surface and $\mathcal{G}_\eta$ is the same vector bundle constructed in [B2, Thm. 1]. This is simple by [B2, Rmk. 2].

Let us now show that

$$H^0(\mathcal{I}_{Z/S}(H - K_S)) = 0. \tag{2.15}$$

Since $q(S) \geq 2$ and $\chi(\mathcal{O}_S) = 0$ we have that $p_g(S) \geq 1$. Let $\tau \in H^0(K_S), \tau \neq 0$. Let $\sigma \in H^0(\mathcal{I}_{Z/S}(H - K_S))$. If $\sigma \neq 0$ then $0 \neq \sigma\tau \in H^0(\mathcal{I}_{Z/S}(H))$, hence, by (2.8), $\sigma\tau = \lambda s$, for some $\lambda \in \mathbb{C}^*$. Let D be the divisor associated to τ. Note that D is effective non-zero because $K_S \neq 0$. But $D \subseteq C$ and therefore $D = C$, that is $K_S \sim H$, giving the contradiction $0 = h^1(H) = h^1(K_S) = q(S)$. Therefore $\sigma = 0$ and (2.15) is proved.

Tensoring (2.9) by $\mathcal{F}^* \cong \mathcal{F}(K_S - H)$ we get the exact sequence

$$0 \to \mathcal{F}(K_S - H) \to \mathcal{F} \otimes \mathcal{F}^* \to \mathcal{I}_{Z/S} \otimes \mathcal{F} \to 0.$$

Now $H^0(\mathcal{F}(K_S - H)) = 0$ by (2.10) and therefore, using (2.9) and (2.15) we have

$$h^0(\mathcal{F} \otimes \mathcal{F}^*) \leq h^0(\mathcal{I}_{Z/S} \otimes \mathcal{F}) \leq h^0(\mathcal{F}) = 1$$

that is $\mathcal{F}$ is simple, and then so is $\mathcal{G}_\eta$. □

3. Proof of theorem 1

Proof. Let $\mathcal{G} := \mathcal{G}_\eta$ be the vector bundle constructed in Lemma 2.1. We will prove that

$$\mathcal{E} := \mathcal{G}(H)$$

is a simple Ulrich vector bundle.

To this end observe that $\mathcal{G}$ is simple and then so is $\mathcal{E}$. Moreover we have, by (2.2), that

$$H^0(\mathcal{G}) = H^0(\mathcal{G}(-2\eta)) = 0$$

and by (2.1) that

$$c_1(\mathcal{G}) = H + K_S + 2\eta \text{ and } c_2(\mathcal{G}) = \frac{1}{2}H \cdot (H + K_S)$$

so that

$$c_1(\mathcal{E}) \cdot H = (c_1(\mathcal{G}) + 2H) \cdot H = 3H^2 + H \cdot K_S$$

and

$$c_2(\mathcal{E}) = c_2(\mathcal{G}) + c_1(\mathcal{G}) \cdot H + H^2 = \frac{5}{2}H^2 + \frac{3}{2}H \cdot K_S = \frac{1}{2}(c_1(\mathcal{E})^2 - c_1(\mathcal{E}) \cdot K_S) - 2(H^2 - \chi(\mathcal{O}_S))$$

and therefore $\mathcal{E}$ satisfies the conditions (2.2) in [C1, Prop. 2.1]. Also

$$H^0(\mathcal{E}(-H)) = H^0(\mathcal{G}) = 0$$

and

$$H^0(\mathcal{E}^*(2H + K_S)) = H^0(\mathcal{G}^*(H + K_S)) = H^0(\mathcal{G}(-c_1(\mathcal{G}))(H + K_S)) = H^0(\mathcal{G}(-2\eta)) = 0$$

and we are done by [C1, Prop. 2.1]. □

Remark 3.1. Let S be a K3 or abelian surface having rank one. We claim that given any very ample line bundle H on S, there are no Ulrich line bundles for (S, H). In particular a very general K3 or abelian surface has no Ulrich line bundles.

To see this let A be an ample generator of $N^1(S)$. Note that for any $s \geq 1$ and any line bundle $M \equiv sA$ we have that $h^0(M) > 0$. In fact M is ample, whence $h^i(M) = 0$ for $i = 1, 2$ and then Riemann-Roch gives that $h^0(M) = \chi(\mathcal{O}_S) + \frac{1}{2}M^2 > 0$. Now let L be an Ulrich line bundle for (S, H). We have $L \equiv lA$ and $H \equiv hA$ for some $l, h \in \mathbb{Z}$ with $h \geq 1$. Since $H^0(L - H) = 0$ we get that $l - h \leq 0$ while, since $H^2(L - 2H) = 0$, we get that $H^0(2H - L) = 0$, whence $2h - l \leq 0$. But then $2h \leq l \leq h$, a contradiction.

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Dipartimento di Matematica e Fisica, Università di Roma Tre, Largo San Leonardo Murialdo 1, 00146, Roma, Italy. e-mail `lopez@mat.uniroma3.it`