ON THE POSITIVITY OF THE FIRST CHERN CLASS OF AN ULRICH VECTOR BUNDLE

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† Dedicated to all medical workers and people that sacrificed their lives during the Covid-19 pandemic

ABSTRACT. We study the positivity of the first Chern class of a rank r Ulrich vector bundle $\mathcal{E}$ on a smooth n-dimensional variety $X \subseteq \mathbb{P}^N$. We prove that $c_1(\mathcal{E})$ is very positive on every subvariety not contained in the union of lines in X. In particular if X is not covered by lines we have that $\mathcal{E}$ is big and $c_1(\mathcal{E})^n \geq r^n$. Moreover we classify rank r Ulrich vector bundles $\mathcal{E}$ with $c_1(\mathcal{E})^2 = 0$ on surfaces and with $c_1(\mathcal{E})^2 = 0$ or $c_1(\mathcal{E})^3 = 0$ on threefolds (with some exceptions).

1. INTRODUCTION

Let $X \subseteq \mathbb{P}^N$ be a smooth projective variety. A classical method to study the geometry of X is to produce interesting vector bundles with some kind of positivity on X. These in turn give rise to meaningful subvarieties shedding light on the geometrical properties of X.

Among such bundles, in recent years a distinguished role has been achieved by Ulrich vector bundles, that is vector bundles $\mathcal{E}$ such that $H^i(\mathcal{E}(-p)) = 0$ for all $i \geq 0$ and $1 \leq p \leq \dim X$.

The study of Ulrich vector bundles touches and is intimately related with several areas of commutative algebra and algebraic geometry, such as determinantal representation, Chow forms, Boij-Söderberg theory and so on (see for example [ES, Be, Fl] and references therein), even though their basic existence question is still open (see for example [HUB, Be, F, C1, C2, CoHu, Lo] in the case of surfaces).

The starting observation of this paper is that an Ulrich vector bundle is globally generated, whence it must positive in some sense. The question is: how much is it positive? We will investigate in this paper the positivity of its first Chern class.

Let $\mathcal{E}$ be a rank r Ulrich vector bundle on X. It is a simple observation (see Lemma 2.1) that $c_1(\mathcal{E}) = 0$ if and only if $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$.

Aside from this case, we have that $c_1(\mathcal{E})$ is globally generated and non-zero, so the question becomes: How positive is $c_1(\mathcal{E})$?

The first answer, surprisingly related to the projective geometry of X, is given in Theorem 1 below. It shows that whenever $X \subset \mathbb{P}^N$ is not covered by lines then $c_1(\mathcal{E})$ is quite positive and that $\mathcal{E}$ is big. More precisely we have

Theorem 1.
Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 1$ and degree d. Let $t \geq 1$ and let $\mathcal{E}$ be a rank r Ulrich vector bundle for $(X, \mathcal{O}_X(t))$. Then $c_1(\mathcal{E})$ has the following positivity property:

For every $x \in X$ and for every subvariety $Z \subseteq X$ of dimension $k \geq 1$ passing through x, we have that

$$(1.1) \qquad c_1(\mathcal{E})^k \cdot Z \geq r^k \mathrm{mult}_x(Z)$$

holds if either $t \geq 2$ or $t = 1$ and x does not belong to a line contained in X.

In particular $c_1(\mathcal{E})$ is ample if either $t \geq 2$ or $t = 1$ and there is no line contained in X.

Moreover assume that either $t \geq 2$ or $t = 1$ and X is not covered by lines. Then $\mathcal{E}$ is big,

$$(1.2) \qquad c_1(\mathcal{E})^n \geq r^n$$

and, if $r \geq 2$,

$$(1.3) \qquad c_1(\mathcal{E})^n \geq r(d-1).$$

Other positivity properties of Chern and dual Segre classes of $\mathcal{E}$ actually hold, see Remark 7.3.

It is easy to see that the results of Theorem 1 do not hold on varieties covered by lines. In particular (1.2) is sharp, in the sense that for every $n \geq 2$ there are many n-dimensional varieties such that (1.2) does not hold: for example products $\mathbb{P}^1 \times B$, scrolls or quadrics and complete intersections in any of them (see Lemmas 3.2 and 4.1 and Remark 4.3).

On the other hand, varieties covered by lines are classified in dimension $n \leq 3$ and one could try to similarly classify their Ulrich vector bundles with $c_1(\mathcal{E})$ not positive. We will concentrate on the case when $c_1(\mathcal{E})^n = 0$.

We first recall the following

Definition 1.1. Let $X \subset \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 1$. We say that $(X, \mathcal{O}_X(1))$ is a *linear $\mathbb{P}^{n-1}$-bundle over a smooth curve* B if there are a rank n vector bundle $\mathcal{F}$ and a line bundle L on B such that $X \cong \mathbb{P}(\mathcal{F})$ and $\mathcal{O}_X(1) \cong \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \otimes \pi^*L$, where $\pi : \mathbb{P}(\mathcal{F}) \to B$ is the projection.

With this in mind, we now classify all rank r Ulrich vector bundles on surfaces with $c_1(\mathcal{E})^2 = 0$.

Theorem 2.

Let $S \subseteq \mathbb{P}^N$ be a smooth irreducible surface. Let $\mathcal{E}$ be a rank r Ulrich vector bundle on S.

Then $c_1(\mathcal{E})^2 = 0$ if and only if $(S, \mathcal{O}_S(1), \mathcal{E})$ is one of the following:

(i) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}^{\oplus r})$;

(ii) $(\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \otimes \pi^*L, \pi^*(\mathcal{G}(L + \det \mathcal{F})))$, a linear $\mathbb{P}^1$-bundle over a smooth curve B, with $\mathcal{G}$ a rank r vector bundle on B such that $H^q(\mathcal{G}(-L)) = 0$ for $q \geq 0$.

Note that the vector bundle $\mathcal{G}$ above is not necessarily Ulrich. See also Remark 4.2 for other observations.

In the case of threefolds, let us recall the following

Definition 1.2. We say that $(X, \mathcal{O}_X(1))$ is a *linear quadric fibration over a smooth curve* B, if there is a morphism $\pi : X \to B$ such that each fiber Q is isomorphic to a quadric and $\mathcal{O}_X(1)_{|Q} \cong \mathcal{O}_Q(1)$.

Again we have a classification, though with some exceptions, of rank r Ulrich vector bundles on threefolds first when $c_1(\mathcal{E})^2 = 0$ and then when only $c_1(\mathcal{E})^3 = 0$.

Theorem 3.

Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible threefold. Let $\mathcal{E}$ be a rank r Ulrich vector bundle on X.

Assume that $(X, \mathcal{O}_X(1))$ is neither a linear quadric fibration nor a linear $\mathbb{P}^1$-bundle over a smooth surface with $\Delta(\mathcal{F}) = 0$.

Then $c_1(\mathcal{E})^2 = 0$ if and only if $(X, \mathcal{O}_X(1), \mathcal{E})$ is one of the following:

(i) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1), \mathcal{O}_{\mathbb{P}^3}^{\oplus r})$;

(ii) $(\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \otimes \pi^*L, \pi^*(\mathcal{G}(2L + \det \mathcal{F})))$, a linear $\mathbb{P}^2$-bundle over a smooth curve B, with $\mathcal{G}$ a rank r vector bundle on B such that $H^q(\mathcal{G}(-L)) = 0$ for $q \geq 0$;

(iii) $(\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \otimes \pi^*L, \pi^*(\mathcal{G}(L + \det \mathcal{F})))$, a linear $\mathbb{P}^1$-bundle over a smooth surface B, with $\mathcal{G}$ a rank r vector bundle on B such that $H^q(\mathcal{G}(-L)) = H^q(\mathcal{G}(-2L) \otimes \mathcal{F}^*) = 0$ for $q \geq 0$ and $c_1(\mathcal{G}(L + \det \mathcal{F}))^2 = 0$.

We do not know if the case (iii) actually occurs.

Theorem 4.

Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible threefold. Let $\mathcal{E}$ be a rank r Ulrich vector bundle on X.

Assume that $(X, \mathcal{O}_X(1))$ is neither a linear quadric fibration, nor a linear $\mathbb{P}^1$-bundle over a smooth surface, nor a linear $\mathbb{P}^2$-bundle over a smooth curve with $\mathcal{F}$ normalized and $\deg \mathcal{F} \geq 0$, unless $(X, \mathcal{O}_X(1))$ is also a Del Pezzo threefold.

Then $c_1(\mathcal{E})^3 = 0, c_1(\mathcal{E})^2 \neq 0$ if and only if $(X, \mathcal{O}_X(1), \mathcal{E})$ is one of the following:

(i) $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1), (\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(2))^{\oplus s} \oplus (\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))^{\oplus(r-s)})$ for $0 \leq s \leq r$;

(ii) $(\mathbb{P}(T_{\mathbb{P}^2}), \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^2})}(1), \pi^*(\mathcal{O}_{\mathbb{P}^2}(2))^{\oplus r})$, where π is one of the two $\mathbb{P}^1$-bundle structures on $\mathbb{P}(T_{\mathbb{P}^2})$.

In the case of a linear quadric fibration different from $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ we do not know if there are Ulrich vector bundles $\mathcal{E}$ with $c_1(\mathcal{E})^3 = 0$, but we suspect there aren't (see Proposition 6.1 and the discussion just before).

Throughout the whole paper we work over the complex numbers.

2. ULRICH VECTOR BUNDLES WITH $c_1 = 0$

It is easy, and probably well-known, to classify Ulrich vector bundles with $c_1 = 0$. We give a proof for a lack of reference and for self-containedness.

Lemma 2.1. *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 1$. Let $\mathcal{E}$ be a rank r Ulrich vector bundle on X. Then $c_1(\mathcal{E}) = 0$ if and only if $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$.*

Proof. If $(X, \mathcal{O}_X(1)) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ it is well-known [ES, Prop. 2.1], [Be, Thm. 2.3], that $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}^{\oplus r}$ and of course $c_1(\mathcal{E}) = 0$.

Viceversa suppose that $\mathcal{E}$ is a rank r Ulrich vector bundle with $c_1(\mathcal{E}) = 0$ on X. It follows by [Be, (3.4)] that for any smooth $Y \in |\mathcal{O}_X(1)|$ we have that $\mathcal{E}_{|Y}$ is an Ulrich vector bundle for $Y \subseteq \mathbb{P}^{N-1}$ and of course $c_1(\mathcal{E}_{|Y}) = 0$. Therefore if $C = Y_1 \cap \ldots \cap Y_{n-1}$ is a smooth curve section of X, with $Y_i \in |\mathcal{O}_X(1)|$ for $1 \leq i \leq n-1$, then $\mathcal{E}_{|C}$ is an Ulrich vector bundle for $(C, \mathcal{O}_C(1))$ and $\deg(\mathcal{E}_{|C}) = 0$. By [CaHa, Prop. 2.3] we get that

$$0 = \deg(\mathcal{E}_{|C}) = r(\deg C + g(C) - 1).$$

Thus $\deg C = 1 - g(C)$, hence $\deg C = 1$ and then $\deg X = 1$. Therefore $(X, \mathcal{O}_X(1)) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. It follows by [ES, Prop. 2.1] (or [Be, Thm. 2.3]) that $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}^{\oplus r}$. $\square$

3. VARIETIES OF PICARD RANK 1

It has been observed by several authors that on varieties of Picard rank 1 more information can be given on Ulrich vector bundles. We concentrate here on the first Chern class.

Notation 3.1. We will denote by Q_n an n-dimensional smooth quadric in $\mathbb{P}^{n+1}$.

We have

Lemma 3.2. *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 2$ such that $\operatorname{Pic} X \cong \mathbb{Z}$. Let $\mathcal{E}$ be a rank r Ulrich vector bundle on X. Let H be the hyperplane divisor. Then*

$$c_1(\mathcal{E}) = \frac{r}{2}(K_X + (n+1)H).$$

In particular if $(X, \mathcal{O}_X(1)) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ we have

$$c_1(\mathcal{E})^n = \begin{cases} \frac{r^n}{2^{n-1}} < r^n & \text{if } (X, \mathcal{O}_X(1)) = (Q_n, \mathcal{O}_{Q_n}(1)), n \geq 3 \\ \frac{r^n}{2^n}(K_X + (n+1)H)^n > r^n & \text{if } (X, \mathcal{O}_X(1)) \neq (Q_n, \mathcal{O}_{Q_n}(1)) \end{cases}.$$

Proof. Let $H_1, \ldots, H_{n-2}$ be general in $|H|$ and let $S = X \cap H_1 \cap \ldots \cap H_{n-2} \subseteq \mathbb{P}^{N-n+2}$. It follows by [Be, (3.4)] that $\mathcal{E}_{|S}$ is a rank r Ulrich vector bundle on S and [C1, Prop. 2.2(3)] gives

$$(3.1) \qquad c_1(\mathcal{E}_{|S}) \cdot H_{|S} = \frac{r}{2}(3H_{|S}^2 + K_S \cdot H_{|S}).$$

Let A be an ample generator of $\operatorname{Pic} X$, so that we have $H = hA, K_X = kA$ and $c_1(\mathcal{E}) = aA$, for some $h, k, a \in \mathbb{Z}$. Replacing in (3.1) we get

$$ahA_{|S}^2 = \frac{rh}{2}(k + (n+1)h)A_{|S}^2$$

from which we deduce that $a = \frac{r}{2}(k + (n+1)h)$ and therefore

$$c_1(\mathcal{E}) = \frac{r}{2}(K_X + (n+1)H).$$

Moreover if $(X, \mathcal{O}_X(1)) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)), (Q_n, \mathcal{O}_{Q_n}(1))$, it follows by [BS2, Prop. 7.2.2, 7.2.3 and 7.2.4]) that $D := K_X + (n-1)H$ is nef, hence

$$c_1(\mathcal{E})^n = \frac{r^n}{2^n}(D + 2H)^n \geq r^n H^n > r^n.$$

If $(X, \mathcal{O}_X(1)) = (Q_n, \mathcal{O}_{Q_n}(1))$, then $n \geq 3$, since $\operatorname{Pic} X \cong \mathbb{Z}$. In the latter case we have that $K_X = -nH$ and therefore $c_1(\mathcal{E})^n = \frac{r^n}{2^{n-1}} < r^n$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

On the other hand, when the Picard rank is not 1, one can achieve that $c_1(\mathcal{E})^n = 0$ even though $(X, \mathcal{O}_X(1)) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Several examples are given in the next section.

4. ULRICH VECTOR BUNDLES ON SCROLLS

A big source of examples are given by scrolls. These are treated, to mention a few, in [Be, Prop. 4.1(ii)] and in [Ho, FLCPL, AHMPL].

We study here the case when $c_1(\mathcal{E})^n = 0$.

Lemma 4.1. *Let B be a smooth irreducible variety of dimension $b \geq 1$ and let L be a divisor on B. Let n be an integer such that $n \geq b+1$ and let $\mathcal{F}$ be a rank $n-b+1$ vector bundle on B. Let $X = \mathbb{P}(\mathcal{F})$ with tautological line bundle ξ and projection $\pi : X \to B$. Let $H = \xi + \pi^* L$ and assume that H is very ample. Let $\mathcal{G}$ be a rank r vector bundle on B and let*

$$\mathcal{E} = \pi^*(\mathcal{G}((n-b)L + \det \mathcal{F})).$$

Then $\mathcal{E}$ is an Ulrich vector bundle for (X, H) if and only if

$$(4.1) \qquad H^q(\mathcal{G}(-(k+1)L) \otimes S^k \mathcal{F}^*) = 0 \text{ for } q \geq 0 \text{ and } 0 \leq k \leq b-1.$$

In particular $\mathcal{E}$ is an Ulrich vector bundle for (X, H) if

  (i) *$b = 1$ and $H^q(\mathcal{G}(-L)) = 0$ for $q \geq 0$ (in particular if L is very ample and $\mathcal{G}$ is an Ulrich vector bundle for (B, L)), or*
  (ii) *$b \geq 2$, $\mathcal{F} = \mathcal{O}_B^{\oplus(n-b+1)}$ and $H^q(\mathcal{G}(-kL)) = 0$ for $q \geq 0$ and $1 \leq k \leq b$ (in particular if L is very ample and $\mathcal{G}$ is an Ulrich vector bundle for (B, L)).*

Moreover $c_1(\mathcal{E})^s = 0$ for every integer $s \geq b+1$.

Proof. Let p be an integer such that $1 \leq p \leq n$. Let $j \geq 0$. By [Ha, Ex. III.8.4] we see that

$$(4.2) \qquad R^j \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-p) = \begin{cases} 0 & \text{if } j \neq n-b \text{ or } j = n-b \text{ and } 1 \leq p \leq n-b \\ S^{p-n+b-1} \mathcal{F}^*(-\det \mathcal{F}) & \text{if } j = n-b \text{ and } n-b+1 \leq p \leq n \end{cases}.$$

Now

$$(4.3) \qquad R^j \pi_* \mathcal{E}(-pH) \cong R^j \pi_* \mathcal{E}(-p\xi)(-pL) \cong \mathcal{G}((n-b-p)L + \det \mathcal{F}) \otimes R^j \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-p)$$

whence (4.2) and the Leray spectral sequence give that

$$(4.4) \qquad H^i(\mathcal{E}(-pH)) = 0 \text{ for } i \leq n-b-1 \text{ and for } i = n-b \text{ and } p \leq n-b.$$

On the other hand, for $n-b \leq i \leq n$ and $n-b+1 \leq p \leq n$, again using (4.2), (4.3) and the Leray spectral sequence, we have

$$H^i(\mathcal{E}(-pH)) \cong H^{i-n+b}(\mathcal{G}((n-b-p)L) \otimes S^{p-n+b-1} \mathcal{F}^*)$$

that is the equivalence with (4.1) holds.

Now in cases (i) or (ii) one eassly sees that (4.1) is equivalent to assert that $H^q(\mathcal{G}(-kL)) = 0$ for $q \geq 0$ and $1 \leq k \leq b$. Finally since $c_1(\mathcal{E}) = \pi^*(c_1(\mathcal{G}) + r(n-b)L + r \det \mathcal{F})$ it is clear that $c_1(\mathcal{E})^s = 0$ for every integer $s \geq b+1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remarks 4.2. (about Lemma 4.1)

  (1) The condition that $H = \xi + \pi^* L$ is very ample can be easily achieved, for example if $\mathcal{F}$ is very ample and L is globally generated.
  (2) In order to have that $\mathcal{E}$ is an Ulrich vector bundle, the conditions in (i) or (ii) do not require L to be any positive. For example choose $\mathcal{F}$ very ample, $L = 0$, $b = 1$ and $\mathcal{G}$ is a general line bundle on B of degree $g(B) - 1$ or $b \geq 2$, $B = \mathbb{P}^b$ and $\mathcal{G} = \mathcal{O}_{\mathbb{P}^b}(-1)$.
  (3) In case (ii) we have that $X = \mathbb{P}^{n-b} \times B$ and $H = \pi_1^* \mathcal{O}_{\mathbb{P}^{n-b}}(1) + \pi_2^* L$. This shows that in products $\mathbb{P}^{n-b} \times B$ there are always Ulrich vector bundles as long as there are on B.

Remark 4.3. Let Y be any smooth variety of dimension $n + k$ carrying an Ulrich vector bundle $\mathcal{E}$ for (Y, H) such that $c_1(\mathcal{E})^s = 0$ for some integer s. Let X be a smooth complete intersection of $H_1, \ldots, H_k \in |H|$. Then $\mathcal{E}_{|X}$ is an Ulrich vector bundle for $(X, H_{|X})$ by [Be, (3.4)] and again $c_1(\mathcal{E}_{|X})^s = 0$.

Using this one can produce several examples of varieties with Ulrich vector bundles $\mathcal{E}$ with $c_1(\mathcal{E})^s = 0$. For example one can take complete intersections in products. One special case will be studied in section 6.

5. A BASIC LEMMA

We record a useful result that will be applied several times.

Lemma 5.1. *Let $\pi : X \to Y$ be a morphism between two smooth projective varieties such that $\pi_* \mathcal{O}_X = \mathcal{O}_Y$. Let $\mathcal{E}$ be a rank r globally generated vector bundle on X. If there is a line bundle M on Y such that $\det \mathcal{E} = \pi^* M$, then there is a rank r vector bundle $\mathcal{H}$ on Y such that $\mathcal{E} = \pi^* \mathcal{H}$.*

Proof. Consider the map
$$\lambda_{\mathcal{E}} : \Lambda^r H^0(\mathcal{E}) \to H^0(\det \mathcal{E}).$$
The linear system $|\mathrm{Im}\lambda_{\mathcal{E}}|$ is base-point free: In fact, given any point $x \in X$, there are $s_1, \ldots s_r \in H^0(\mathcal{E})$ such that $s_1(x), \ldots s_r(x)$ are linearly independent in $\mathcal{E}_x$. Hence the section $\lambda_{\mathcal{E}}(s_1 \wedge \ldots \wedge s_r) \in \mathrm{Im}\lambda_{\mathcal{E}}$ does not vanish at x. Now $\mathcal{E}$ is globally generated, whence it defines a morphism
$$\Phi_{\mathcal{E}} : X \to G(H^0(\mathcal{E}), r)$$
such that
$$\mathcal{E} = \Phi_{\mathcal{E}}^* \mathcal{U},$$
where $\mathcal{U}$ is the tautological bundle on $G(H^0(\mathcal{E}), r)$. Moreover we have a commutative diagram (see for example [M, §3])

$$
\begin{array}{ccc}
X & \xrightarrow{\Phi_{\mathcal{E}}} & G(H^0(\mathcal{E}), r) \\
\downarrow{\varphi_{|\mathrm{Im}\lambda_{\mathcal{E}}|}} & & \uparrow{P} \\
\mathbb{P}\mathrm{Im}\lambda_{\mathcal{E}}^* & \hookrightarrow & \mathbb{P}\Lambda^r H^0(\mathcal{E})^*
\end{array}
$$

where P is the Plücker embedding. Then $Z := \Phi_{\mathcal{E}}(X) = \varphi_{|\mathrm{Im}\lambda_{\mathcal{E}}|}(X)$ and we deduce that there is a morphism $\varphi_{|\mathrm{Im}\lambda_{\mathcal{E}}|} : X \to Z$ and a rank r vector bundle $\mathcal{G} := \mathcal{U}_{|Z}$ on Z such that $\mathcal{E} \cong \varphi_{|\mathrm{Im}\lambda_{\mathcal{E}}|}^* \mathcal{G}$.

If $W = \varphi_{\det \mathcal{E}}(X)$, using the fact that $\mathrm{Im}\lambda_{\mathcal{E}} \subseteq H^0(\det \mathcal{E})$, we also have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi_{\det \mathcal{E}}} & W \\
& {\varphi_{|\mathrm{Im}\lambda_{\mathcal{E}}|}} \searrow & \downarrow{f} \\
& & Z
\end{array}
$$

and we find that $\mathcal{E} \cong \varphi_{\det \mathcal{E}}^*(f^* \mathcal{G})$.

Since $\det \mathcal{E} = \pi^* M$ is globally generated and $\pi_* \mathcal{O}_X = \mathcal{O}_Y$, we have that M is globally generated and there is a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\pi} & Y \\
{\varphi_{\det \mathcal{E}}} \searrow & & \downarrow{\varphi_M} \\
& & W
\end{array}
\quad .
$$

Therefore $\mathcal{E} \cong \pi^* \mathcal{H}$ where $\mathcal{H}$ is the rank r vector bundle $\varphi_M^*(f^* \mathcal{G})$. $\qquad \square$

The above lemma allows to classify Ulrich vector bundles $\mathcal{E}$ on a linear $\mathbb{P}^{n-1}$-bundle over a curve when $c_1(\mathcal{E})^s = 0$ for $2 \le s \le n - 1$.

Lemma 5.2. *Let B be a smooth irreducible curve and let L be a divisor on B. Let n be an integer such that $n \ge 3$ and let $\mathcal{F}$ be a rank n vector bundle on B. Let $X = \mathbb{P}(\mathcal{F})$ with tautological line bundle ξ and projection $\pi : X \to B$. Let $H = \xi + \pi^* L$ and assume that H is very ample. Let $\mathcal{E}$ be an Ulrich*

vector bundle for (X, H) *with* $c_1(\mathcal{E})^s = 0$ *for some* s *with* $2 \leq s \leq n - 1$. *Then there exists a rank* r
vector bundle $\mathcal{G}$ *on* B *such that* $H^q(\mathcal{G}(-L)) = 0$ *for* $q \geq 0$ *and*

$$\mathcal{E} = \pi^*(\mathcal{G}((n-1)L + \det \mathcal{F})).$$

Proof. We can write $\det \mathcal{E} = a\xi + \pi^*M$ for some $a \in \mathbb{Z}$ and some line bundle M on B, so that there is
$b \in \mathbb{Z}$ such that $c_1(\mathcal{E}) = a\xi + bf$, where f is a fiber. Then

$$0 = c_1(\mathcal{E})^s = a^s\xi^s + sa^{s-1}b\xi^{s-1} \cdot f.$$

Since ξ^s and $\xi^{s-1} \cdot f$ are numerically independent, we deduce that $a = 0$ and then $\det \mathcal{E} = \pi^*M$.
Now Lemma 5.1 implies that there is a rank r vector bundle $\mathcal{H}$ on B such that $\mathcal{E} \cong \pi^*\mathcal{H}$. Setting
$\mathcal{G} = \mathcal{H}((1-n)L - \det \mathcal{F})$, we find that $\mathcal{E} \cong \pi^*(\mathcal{G}((n-1)L + \det \mathcal{F}))$ and Lemma 4.1(i) implies that
$H^q(\mathcal{G}(-L)) = 0$ for $q \geq 0$. $\square$

6. ULRICH VECTOR BUNDLES ON DEL PEZZO THREEFOLDS COVERED BY LINES

The goal of this section is to study Ulrich vector bundles $\mathcal{E}$ with $c_1(\mathcal{E})^3 = 0$ on $C \times \mathbb{P}^1 \times \mathbb{P}^1$, where C
is a curve and on $\mathbb{P}(T_{\mathbb{P}^b})$. In particular when $C = \mathbb{P}^1$ and $b = 2$ we get the two examples [LP, Thm. 1.4],
[Fu, Introduction] of Del Pezzo threefolds covered by lines.

We start with the simplest linear quadric fibration. The lemma will show that we cannot hope to
construct Ulrich vector bundles $\mathcal{E}$ with $c_1(\mathcal{E})^3 = 0$ by pulling back from the base. For this reason we
suspect that, in the case of linear quadric fibrations different from $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, there aren't any such
bundles.

Proposition 6.1. *Let* C *be a smooth irreducible curve. Let* $X = C \times \mathbb{P}^1 \times \mathbb{P}^1$ *and let* $H = \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes$
$\mathcal{O}_{\mathbb{P}^1}(1)$, *where* $\mathcal{L}$ *is a very ample line bundle on* C. *Then the only rank* r *Ulrich vector bundles* $\mathcal{E}$ *for*
(X, H) *such that* $c_1(\mathcal{E})^3 = 0$ *are:*

   (i) $\pi^*(\mathcal{G}(L))$, *where* $\pi : X \to C \times \mathbb{P}^1$ *is one of the two projections,* $L = \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$ *and* $\mathcal{G}$ *is a rank*
   r *Ulrich vector bundle for* $(C \times \mathbb{P}^1, L)$;
   (ii) $(\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(2))^{\oplus s} \oplus (\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))^{\oplus(r-s)}$ *for* $0 \leq s \leq r$, *when* $C = \mathbb{P}^1$ *and*
   $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(1)$.

In particular there are no rank r *Ulrich vector bundles* $\mathcal{E}$ *for* (X, H) *such that* $c_1(\mathcal{E})^2 = 0$.

Proof. Let $\pi_1 : X \to C, \pi_i : X \to \mathbb{P}^1, i = 2, 3$ and $\pi_{ij} = (\pi_i, \pi_j)$ for $1 \leq i < j \leq 3$ be the projections. For
$i = 2, 3$ set $H_i = \pi_i^*(\mathcal{O}_{\mathbb{P}^1}(1))$ and, given any line bundle N on C, set $N_1 = \pi_1^*N$. By [Ha, Exer. III.12.6]
we know that there is line bundle M on C and there are $a, b \in \mathbb{Z}$ such that $\det \mathcal{E} = M_1 + aH_2 + bH_3$.

Then $(\det \mathcal{E})^3 = 0$ gives $6ab \deg M = 0$, whence either $\deg M = 0$ or $a = 0$ or $b = 0$. Since $\mathcal{E}$ is
globally generated we get that M is globally generated and $a \geq 0, b \geq 0$. Therefore, if $\deg M = 0$ we
have that $M = \mathcal{O}_C$. We deduce that there are three possibilities for $\det \mathcal{E}$:

$$\pi_{12}^*(M \boxtimes \mathcal{O}_{\mathbb{P}^1}(a)), \pi_{13}^*(M \boxtimes \mathcal{O}_{\mathbb{P}^1}(b)) \text{ and } \pi_{23}^*(\mathcal{O}_{\mathbb{P}^1}(a) \boxtimes \mathcal{O}_{\mathbb{P}^1}(b)).$$

By Lemma 5.1 we deduce that there is a rank r vector bundle $\mathcal{H}$ on $C \times \mathbb{P}^1$ (respectively on $\mathbb{P}^1 \times \mathbb{P}^1$)
such that $\mathcal{E} \cong \pi_{1i}^*\mathcal{H}$ for $i = 2, 3$ (respectively $\mathcal{E} \cong \pi_{23}^*\mathcal{H}$).

Suppose that $\mathcal{E} \cong \pi_{1i}^*\mathcal{H}$ for $i = 2, 3$. We have that $X \cong \mathbb{P}(\mathcal{O}_{C \times \mathbb{P}^1}^{\oplus 2})$ and $\xi \cong \pi_3^*(\mathcal{O}_{\mathbb{P}^1}(1))$, so that
$H = \xi + \pi_{1i}^*(\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))$. We can then apply Lemma 4.1(ii) and deduce that $\mathcal{E} \cong \pi_{1i}^*(\mathcal{G}(L))$, where
$L = \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{G} := \mathcal{H}(-L)$ is a rank r Ulrich vector bundle for $(C \times \mathbb{P}^1, L)$.

Suppose instead that $\mathcal{E} \cong \pi_{23}^*\mathcal{H}$. Set, for convenience of notation, $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}_Q(1) = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$. We have $H = \pi_1^*\mathcal{L} + \pi_{23}^*(\mathcal{O}_Q(1))$ and, for every $1 \leq i, p \leq 3$, the Künneth formula
gives

$$(6.1) \quad 0 = H^i(X, \mathcal{E}(-pH)) = \bigoplus_{\alpha+\beta=i} H^\alpha(C, -p\mathcal{L}) \otimes H^\beta(Q, \mathcal{H}(-p)) = H^1(C, -p\mathcal{L}) \otimes H^{i-1}(Q, \mathcal{H}(-p)).$$

If $(C, \mathcal{L}) \neq (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ we have $H^1(C, -p\mathcal{L}) \neq 0$, and therefore we get that

$$H^{i-1}(Q, \mathcal{H}(-p)) = 0 \text{ for every } 1 \leq i, p \leq 3.$$

But then

$$0 = \chi(\mathcal{H}(-p)) = r + \frac{1}{2}c_1(\mathcal{H})^2 - c_2(\mathcal{H}) + (1-p)c_1(\mathcal{H}) \cdot \mathcal{O}_Q(1) + rp^2 - 2rp.$$

Now

$$0 = \chi(\mathcal{H}(-2)) - \chi(\mathcal{H}(-1)) = -c_1(\mathcal{H}) \cdot \mathcal{O}_Q(1) + r$$

and

$$0 = \chi(\mathcal{H}(-3)) - \chi(\mathcal{H}(-2)) = -c_1(\mathcal{H}) \cdot \mathcal{O}_Q(1) + 3r$$

thus giving the contradiction $2r = 0$.

Finally if $(C, \mathcal{L}) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ then $H^1(C, -\mathcal{L}) = 0$ and $H^1(C, -p\mathcal{L}) \neq 0$ for $p = 2, 3$, whence (6.1) gives that $H^{i-1}(Q, \mathcal{H}(-p)) = 0$ for $1 \leq i \leq 3$ and $p = 2, 3$. But then $\mathcal{G} := \mathcal{H}(-1)$ is a rank r Ulrich vector bundle on Q. By [BGS, Rmk. 2.5(4)] (see also [Be, Rmk. 2.6], [AHMPL, Exa. 3.2]) the only indecomposable Ulrich bundles for $(Q, \mathcal{O}_Q(1))$ are $\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}$. This gives that

$$\mathcal{E} \cong (\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(2))^{\oplus s} \oplus (\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))^{\oplus(r-s)}$$

for $0 \leq s \leq r$.

Now in case (i) we have that $c_1(\mathcal{E}) = \pi^*(c_1(\mathcal{G}(L)))$ whence $c_1(\mathcal{E})^2 = \pi^*((c_1(\mathcal{G}) + rL)^2)$. But $\mathcal{G}$, being Ulrich, is globally generated, whence $c_1(\mathcal{G}) + rL$ is ample. Therefore $c_1(\mathcal{E})^2 \neq 0$.

In case (ii) we have that $c_1(\mathcal{E}) = (2r - s)H_2 + (r + s)H_3$ and therefore

$$c_1(\mathcal{E})^2 = 2(2r - s)(r + s)H_2 H_3 \neq 0.$$

$\square$

An interesting example, related to Remark 4.3, is given by the projectivised tangent bundle of projective spaces. In the case of $\mathbb{P}^2$ this is a Del Pezzo threefold covered by lines. We recall (see [Sa, Thm. A]) that $\mathbb{P}(T_{\mathbb{P}^b})$ has two $\mathbb{P}^{b-1}$-bundle structures, $\pi : \mathbb{P}(T_{\mathbb{P}^b}) \to \mathbb{P}^b$ and $\pi' : \mathbb{P}(T_{\mathbb{P}^b}) \to \mathbb{P}^b$, given by the fact that $\mathbb{P}(T_{\mathbb{P}^b})$ is a divisor of type $(1, 1)$ on $\mathbb{P}^b \times \mathbb{P}^b$.

Proposition 6.2. *Let $b \geq 2$ and let $X = \mathbb{P}(T_{\mathbb{P}^b})$ with tautological line bundle ξ and projection $\pi : X \to \mathbb{P}^b$. Then ξ is very ample and $\pi^*(\mathcal{O}_{\mathbb{P}^b}(b))$ is an Ulrich line bundle for (X, ξ). In particular $c_1(\pi^*(\mathcal{O}_{\mathbb{P}^b}(b)))^s = 0$ for $b + 1 \leq s \leq 2b - 1$.*

Moreover the only rank r Ulrich vector bundles $\mathcal{E}$ for (X, ξ) with $c_1(\mathcal{E})^{2b-1} = 0$ are $\pi^*(\mathcal{O}_{\mathbb{P}^b}(b))^{\oplus r}$ and $(\pi')^*(\mathcal{O}_{\mathbb{P}^b}(b))^{\oplus r}$. In particular there are no Ulrich vector bundles $\mathcal{E}$ for (X, ξ) with $c_1(\mathcal{E})^s = 0$ for $1 \leq s \leq b$.

Proof. The Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^b} \to \mathcal{O}_{\mathbb{P}^b}(1)^{\oplus(b+1)} \to T_{\mathbb{P}^b} \to 0$$

gives an inclusion $\mathbb{P}(T_{\mathbb{P}^b}) \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^b}(1)^{\oplus(b+1)}) \cong \mathbb{P}^b \times \mathbb{P}^b$ and ξ is the restriction of the tautological line bundle of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^b}(1)^{\oplus(b+1)})$, that is the restriction of $\mathcal{O}_{\mathbb{P}^b}(1) \boxtimes \mathcal{O}_{\mathbb{P}^b}(1)$. Hence ξ is very ample.

Set $n = 2b - 1, \mathcal{F} = T_{\mathbb{P}^b}, L = 0$ and $\mathcal{G} = \mathcal{O}_{\mathbb{P}^b}(-1)$ in Lemma 4.1. Then

$$\pi^*(\mathcal{O}_{\mathbb{P}^b}(b)) = \pi^*(\mathcal{G}((n - b)L + \det \mathcal{F}))$$

and by Lemma 4.1(ii) all we need are the vanishings

(6.2) $$H^q(S^k \Omega^1_{\mathbb{P}^b}(-1)) = 0 \text{ for } q \geq 0 \text{ and } 0 \leq k \leq b - 1.$$

By [Sc, §1] there is an exact sequence

$$0 \to S^k \Omega^1_{\mathbb{P}^b}(-1) \to \mathcal{O}_{\mathbb{P}^b}(-k-1)^{\oplus \binom{k+b}{b}} \to \mathcal{O}_{\mathbb{P}^b}(-k)^{\oplus \binom{k+b-1}{k-1}} \to 0$$

and (6.2) follows immediately. This proves that $\pi^*(\mathcal{O}_{\mathbb{P}^b}(b))$ is an Ulrich line bundle for (X, ξ).

Now set $R = \pi^* H$, where H is a hyperplane in $\mathbb{P}^b$ and let f be a fiber of π.

We claim that

(6.3) $$\xi^{b-1+j} \cdot R^{b-j} = \binom{b+j}{j} \text{ for every } 0 \leq j \leq b.$$

We prove this by induction on j. If $j = 0$ we have

$$\xi^{b-1} \cdot R^b = \xi^{b-1} \cdot f = 1.$$

If $j \geq 1$ recall that we have the relation

$$\sum_{i=0}^{b} (-1)^i \pi^* c_i(T_{\mathbb{P}^b}) \xi^{b-i} = 0$$

and since $c_i(T_{\mathbb{P}^b}) = \binom{b+1}{i} H^i$, this gives

$$\xi^b = \sum_{i=1}^{b} (-1)^{i+1} \binom{b+1}{i} \xi^{b-i} \cdot R^i$$

and therefore

$$\xi^{b-1+j} \cdot R^{b-j} = \sum_{i=1}^{b} (-1)^{i+1} \binom{b+1}{i} \xi^{b-1+j-i} \cdot R^{b+i-j}.$$

Using the fact that $R^s = 0$ for $s \geq b+1$ and applying the inductive hypothesis, we deduce that

$$\xi^{b-1+j} \cdot R^{b-j} = \sum_{i=1}^{j} (-1)^{i+1} \binom{b+1}{i} \xi^{b-1+j-i} \cdot R^{b+i-j} = \sum_{i=1}^{j} (-1)^{i+1} \binom{b+1}{i} \binom{b+j-i}{j-i}.$$

Using the standard notation $\binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!}$ for any integer $k \geq 1$, we can rewrite the above as

$$\xi^{b-1+j} \cdot R^{b-j} = \sum_{i=1}^{j} (-1)^{i+1} \binom{b+1}{i} \binom{b+j-i}{b} = \sum_{i=1}^{b} (-1)^{i+1} \binom{b+1}{i} \binom{b+j-i}{b}.$$

Now for any numerical polynomial $P(x)$ we have that $\sum_{j=0}^{n} (-1)^j \binom{n}{j} P(j) = 0$ if $\deg P < n$ [GKP94, p.190]. Using this we get that the last sum is $\binom{b+j}{j}$ and (6.3) is proved.

Now $c_1(\mathcal{E}) \equiv a\xi + cR$ for some $a, c \in \mathbb{Z}$. Since $\mathcal{E}$ is globally generated we have that $c_1(\mathcal{E})$ is effective, hence $a \geq 0$. Also, using again that $R^s = 0$ for $s \geq b+1$, we get

$$0 = c_1(\mathcal{E})^{2b-1} = \sum_{k=1}^{2b-1} \binom{2b-1}{k} a^k c^{2b-1-k} \xi^k \cdot R^{2b-1-k} = \sum_{j=0}^{b} \binom{2b-1}{b-1+j} a^{b-1+j} c^{b-j} \xi^{b-1+j} \cdot R^{b-j}$$

and using (6.3) we get that

$$0 = \sum_{j=0}^{b} \binom{2b-1}{b-1+j} \binom{b+j}{j} a^{b-1+j} c^{b-j} = a^{b-1} \sum_{j=0}^{b} \binom{2b-1}{b-1+j} \binom{b+j}{j} a^j c^{b-j}.$$

An easy calculation gives that

$$\sum_{j=0}^{b} \binom{2b-1}{b-1+j} \binom{b+j}{j} a^j c^{b-j} = \binom{2b-1}{b} (a+c)^{b-1}(2a+c)$$

so that we get that

$$\binom{2b-1}{b} a^{b-1}(a+c)^{b-1}(2a+c) = 0$$

that is either $a = 0$ or $a > 0, c = -a, -2a$. But the case $c = -2a$ cannot occur, since then $c_1(\mathcal{E}) \sim a\xi - \pi^*\mathcal{O}_{\mathbb{P}^b}(2a)$. But

$$H^0(a\xi - \pi^*\mathcal{O}_{\mathbb{P}^b}(2a)) \cong H^0(S^a T_{\mathbb{P}^b}(-2a)) \cong H^b(S^a \Omega^1_{\mathbb{P}^b}(2a-b-1))^*$$

and the exact sequence

$$0 \to S^a \Omega^1_{\mathbb{P}^b}(2a-b-1) \to \mathcal{O}_{\mathbb{P}^b}(a-b-1)^{\oplus \binom{a+b}{a}} \to \mathcal{O}_{\mathbb{P}^b}(a-b)^{\oplus \binom{a+b-1}{a-1}} \to 0$$

shows that $H^b(S^a \Omega^1_{\mathbb{P}^b}(2a-b-1)) = 0$, contradicting the fact that $c_1(\mathcal{E})$ is globally generated.

Observe that the line bundle $\xi - \pi^*\mathcal{O}_{\mathbb{P}^b}(1)$ is globally generated and $h^0(\xi - \pi^*\mathcal{O}_{\mathbb{P}^b}(1)) = h^0(T_{\mathbb{P}^b}(-1)) = b+1$, so that it gives a morphism $\pi' : X \to \mathbb{P}^b$. This is just the restriction of the second projection given by the inclusion $\mathbb{P}(T_{\mathbb{P}^b}) \subset \mathbb{P}^b \times \mathbb{P}^b$. Also note that π' is again a $\mathbb{P}^{b-1}$-bundle structure on X.

In any case, we have that either $a = 0$ and $c_1(\mathcal{E}) = \pi^*\mathcal{O}_{\mathbb{P}^2}(c)$ or $a > 0, c = -a$ and $c_1(\mathcal{E}) = (\pi')^*\mathcal{O}_{\mathbb{P}^b}(a)$. We can now apply Lemma 5.1 and deduce that there is a rank r vector bundle $\mathcal{H}$ on $\mathbb{P}^b$ such that $\mathcal{E} = \pi^*\mathcal{H}$ or $(\pi')^*\mathcal{H}$. In the first case $\mathcal{E} = \pi^*\mathcal{H}$, setting $\mathcal{F} = T_{\mathbb{P}^b}, L = 0$ and $\mathcal{G} = \mathcal{H}(-b-1)$ we have, by Lemma 4.1, that

(6.4) $H^q(\mathcal{G} \otimes S^k \Omega^1_{\mathbb{P}^b}) = 0$ for $q \geq 0, 0 \leq k \leq b-1$.

Now the Euler sequence gives, when $k \geq 1$, the exact sequence

$$0 \to \mathcal{G} \otimes S^k \Omega^1_{\mathbb{P}^b} \to \mathcal{G}(-k)^{\oplus \binom{k+b}{k}} \to \mathcal{G}(-k+1)^{\oplus \binom{k+b-1}{k-1}} \to 0$$

and (6.4) implies that $H^q(\mathcal{G}(-k)) = 0$ for $q \geq 0$ and $0 \leq k \leq b - 1$. But then $\mathcal{G}(1)$ is an Ulrich vector bundle for $(\mathbb{P}^b, \mathcal{O}_{\mathbb{P}^b}(1))$ and it follows by [ES, Prop. 2.1] (or [Be, Thm. 2.3]) that $\mathcal{G}(1) \cong \mathcal{O}^{\oplus r}_{\mathbb{P}^b}$ and therefore $\mathcal{E} \cong (\pi^*(\mathcal{O}_{\mathbb{P}^b}(b)))^{\oplus r}$. The proof in the other case $\mathcal{E} = (\pi')^* \mathcal{H}$ is analogous. Finally $c_1(\mathcal{E}) = \pi^*(\mathcal{O}_{\mathbb{P}^b}(br))$ whence for any $1 \leq s \leq b$ we get that $c_1(\mathcal{E})^s = b^s r^s R^s \neq 0$. $\qquad\square$

7. Blow-ups at points not on lines

We now lay out the proof of the main theorem.

The following result is quite standard, but we include it for lack of reference. For the second assertion see also [Se, Cor. 6].

Lemma 7.1. *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 1$ and let H be its hyperplane divisor. Let $x \in X$ be a point and let $\mu_x : \widetilde{X} \to X$ be the blow-up of X at x with exceptional divisor E. Then $\mu_x^* H - E$ is ample if and only if there is no line L such that $x \in L \subseteq X$. Also $\mu_x^*(tH) - E$ is very ample for every $t \geq 2$.*

Proof. If there is a line $L \subseteq X$ passing through x, letting $\widetilde{L}$ be its strict transform, we have

$$(\mu_x^* H - E) \cdot \widetilde{L} = H \cdot L - 1 = 0$$

and therefore $\mu_x^* H - E$ is not ample.

Viceversa assume that $\mu_x^* H - E$ is not ample. Since H is very ample, it follows that $\mu_x^* H - E$ is globally generated, hence, for every irreducible subvariety Z of dimension $d \geq 1$ of $\widetilde{X}$ we have that $(\mu_x^* H - E)^d \cdot Z \geq 0$ by Kleiman's theorem [La, Thm. 1.4.9]. Now $\mu_x^* H - E$ is not ample, hence Nakai-Moishezon-Kleiman's theorem [La, Thm. 1.2.23] implies that there is an irreducible subvariety $\widetilde{V}$ of dimension $d \geq 1$ of $\widetilde{X}$ such that and

$$(7.1) \qquad\qquad (\mu_x^* H - E)^d \cdot \widetilde{V} = 0.$$

Let $V = \mu_x(\widetilde{V})$. It must be that $x \in V$, for otherwise (7.1) gives that $H^d \cdot V = 0$, a contradiction. Note that it cannot be that $\widetilde{V} \subset E$ for otherwise we get the contradiction $V = \{x\}$. Hence μ_x gives an isomorphism $\widetilde{V} \setminus E \cap \widetilde{V} \cong V \setminus \{x\}$ and therefore $\widetilde{V}$ is the strict transform of V. Therefore, by [Ha, Cor. II.7.15], we find that $\widetilde{V}$ is the blow-up of V at x with exceptional divisor $E_{|\widetilde{V}}$. By [La, Lemma 5.1.10] we have that $E_{|\widetilde{V}}^d = (-1)^{d+1} \mathrm{mult}_x(V)$ and (7.1) gives

$$0 = (\mu_x^* H - E)^d \cdot \widetilde{V} = H^d \cdot V + (-1)^d (-1)^{d+1} \mathrm{mult}_x(V)$$

that is

$$H^d \cdot V = \mathrm{mult}_x(V).$$

Then $V \subset \mathbb{P}^N$ is a variety having one point whose multiplicity is the degree of V and it follows that there is a line $L \subseteq V$ passing through x.

Finally the very ampleness of $\mu_x^*(tH) - E$ when $t \geq 2$ follows by [BS1, Thm. 2.1] since $\mathcal{J}_{\{x\}/X}(1)$ is globally generated. $\qquad\square$

Using this we have the following result. The idea comes in part from [K, Thm. 0.1].

Theorem 7.2. *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 1$. Let $\mathcal{E}$ be a rank r vector bundle on X that is 0-regular with respect to $\mathcal{O}_X(t)$ for some $t \geq 1$. Then $c_1(\mathcal{E})$ has the following positivity property: For every $x \in X$ and for every subvariety $Z \subseteq X$ of dimension $k \geq 1$ passing through x, we have that*

$$(7.2) \qquad\qquad c_1(\mathcal{E})^k \cdot Z \geq r^k \mathrm{mult}_x(Z)$$

holds if either $t \geq 2$ or $t = 1$ and x does not belong to a line contained in X.

In particular $c_1(\mathcal{E})$ is ample if either $t \geq 2$ or $t = 1$ and there is no line contained in X.

Moreover assume that either $t \geq 2$ or $t = 1$ and X is not covered by lines. Then $\mathcal{E}$ is big and

$$(7.3) \qquad\qquad c_1(\mathcal{E})^n \geq r^n$$

Proof. Let H be the hyperplane divisor of X. Let $x \in X$ be a point and assume that either $t \geq 2$ or $t = 1$ and x does not belong to a line contained in X. Let $\mu : \widetilde{X} \to X$ be the blow-up of X at x with exceptional divisor E. Since H is very ample, it follows that $\widetilde{H} := \mu^*(tH) - E$ is globally generated and it is also ample by Lemma 7.1. As is well-known (see for example [BEL, Proof of Lemma 1.4]), for every integer s such that $0 \leq s \leq n - 1$ we have that

$$(7.4) \qquad\qquad R^j \mu_* \mathcal{O}_{\widetilde{X}}(sE) = \begin{cases} \mathcal{O}_X & \text{if } j = 0 \\ 0 & \text{if } j > 0 \end{cases}.$$

Then, for every $i > 0$, (7.4) and the Leray spectral sequence give

$$H^i(\widetilde{X}, (\mu^*\mathcal{E})(-E)(-i\widetilde{H})) = H^i(\widetilde{X}, \mu^*(\mathcal{E}(-it))((i-1)E)) \cong H^i(X, \mathcal{E}(-it)) = 0$$

since $\mathcal{E}$ is 0-regular with respect to $\mathcal{O}_X(t)$. Therefore $(\mu^*\mathcal{E})(-E)$ is 0-regular with respect to $\widetilde{H}$ and it follows by [La, Thm. 1.8.5] that $(\mu^*\mathcal{E})(-E)$ is globally generated. But then also $c_1(\mu^*\mathcal{E}(-E)) = \mu^*(c_1(\mathcal{E})) - rE$ is globally generated, whence nef. Now let $Z \subseteq X$ be a subvariety of dimension $k \geq 1$ passing through x and let $\widetilde{Z}$ be its strict transform on $\widetilde{X}$. By Kleiman's theorem [La, Thm. 1.4.9] we deduce that

$$(7.5) \qquad\qquad (\mu^*(c_1(\mathcal{E})) - rE)^k \cdot \widetilde{Z} \geq 0.$$

Now, using [La, Lemma 5.1.10], we see that $(E_{|\widetilde{Z}})^k = (-1)^{k+1}\text{mult}_x(Z)$ and (7.5) implies that

$$0 \leq (\mu^*(c_1(\mathcal{E})) - rE)^k \cdot \widetilde{Z} = c_1(\mathcal{E})^k \cdot Z + (-1)^k r^k (-1)^{k+1} \text{mult}_x(Z) = c_1(\mathcal{E})^k \cdot Z - r^k \text{mult}_x(Z).$$

This proves (7.2). Now if either $t \geq 2$ or $t = 1$ and there is no line contained in X, then $c_1(\mathcal{E})$ is ample by Nakai-Moishezon-Kleiman's criterion [La, Thm. 1.2.23].

Assume that either $t \geq 2$ or $t = 1$ and X is not covered by lines. Choosing $Z = X$ we get that

$$c_1(\mathcal{E})^n \geq r^n.$$

Moreover by [DPS, Thm. 2.5] we have, considering the partition $(1, \ldots, 1)$ of n and the corresponding Schur polynomial [La, §8.3], that

$$(7.6) \qquad\qquad s_{(1,\ldots,1)}((\mu^*\mathcal{E})(-E)) \geq 0.$$

Denoting with s_n the n-th Segre class, we know by [La, Exa. 8.3.5] that

$$s_{(1,\ldots,1)}((\mu^*\mathcal{E})(-E)) = s_n(((\mu^*\mathcal{E})(-E))^*)$$

hence we deduce from (7.6) that

$$(-1)^n s_n((\mu^*\mathcal{E})(-E)) \geq 0$$

and therefore

$$(-1)^n \sum_{j=0}^n \binom{r-1+n}{r-1+j} s_j(\mu^*\mathcal{E}) E^{n-j} \geq 0$$

that is

$$s_n(\mathcal{E}^*) = (-1)^n s_n(\mathcal{E}) \geq \binom{r-1+n}{r-1} > 0$$

and then $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is big, namely $\mathcal{E}$ is big. $\square$

Remark 7.3. With the same proof, one can show that if $\mathcal{E}$ is a rank r vector bundle 0-regular with respect to $\mathcal{O}_X(1)$, $x \in X$ does not belong to a line contained in X, $k \geq 1$ and $Z \subseteq X$ is a k-dimensional subvariety passing through x, then

$$s_k(\mathcal{E}^*) \cdot Z \geq \binom{r-1+k}{r-1} \text{mult}_x(Z).$$

Moreover, if X is not covered by lines, also $s_\lambda((\mu^*\mathcal{E})(-E)) \geq 0$ for any degree n Schur polynomial associated to a partition λ [La, §8.3]. This in turn translates into positivity of polynomials in Chern classes of $\mathcal{E}$. For example if $\lambda = (n, 0, \ldots, 0)$ then one gets that $c_n(\mathcal{E}) \geq \binom{r}{n}$ when $r \geq n$.

Now Theorem 1 follows.

Proof of Theorem 1.

Let $\mathcal{E}$ be a rank r Ulrich vector bundle for $(X, \mathcal{O}_X(t))$. Then $\mathcal{E}$ is 0-regular with respect to $\mathcal{O}_X(t)$, so that we can apply Theorem 7.2 and conclude that (1.1) holds and that $c_1(\mathcal{E})$ is ample if either $t \geq 2$ or $t = 1$ and there is no line contained in X. Now assume that $t \geq 2$ or $t = 1$ and X is not covered by lines. Again Theorem 7.2 implies that (1.2) holds and that $\mathcal{E}$ is big. Moreover, if $r \geq 2$ we can apply [Si, Thm. 1] since $\mathcal{E}$ is globally generated. Note that it cannot be that $\mathcal{E} \cong \mathcal{O}_X^{\oplus(r-1)} \oplus \det \mathcal{E}$, for otherwise $\mathcal{O}_X$ would be an Ulrich line bundle for $(X, \mathcal{O}_X(t))$. But then Lemma 2.1 would imply that $(X, \mathcal{O}_X(t)) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, hence $t = 1$ and X is covered by lines, a contradiction. Therefore [Si, Thm. 1] gives that $c_1(\mathcal{E})^n \geq h^0(\mathcal{E}) - r = r(d-1)$ by [Be, (3.1)]. Thus also (1.3) holds. $\square$

8. The case of surfaces

We are now ready to prove our classification in the case of surfaces.

Proof of Theorem 2.

If $(S, \mathcal{O}_S(1), \mathcal{E})$ is as in (i) or (ii), it follows by [ES, Prop. 2.1] (or [Be, Thm. 2.3]) and Lemma 4.1(i) that $\mathcal{E}$ is a rank r Ulrich vector bundle on S with $c_1(\mathcal{E})^2 = 0$.

Vice versa let $\mathcal{E}$ be a rank r Ulrich vector bundle on S such that $c_1(\mathcal{E})^2 = 0$.

If $(S, \mathcal{O}_S(1)) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ it follows by [ES, Prop. 2.1] (or [Be, Thm. 2.3]) that $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}^{\oplus r}$, so that we are in case (i).

Assume from now on that $(S, \mathcal{O}_S(1)) \neq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. We will prove that $(S, \mathcal{O}_S(1), \mathcal{E})$ is as in (ii).

By Remark 2.1 we have that $c_1(\mathcal{E}) \neq 0$. We now claim that $c_2(\mathcal{E}) = 0$.

Since $\mathcal{E}$ is 0-regular with respect to $\mathcal{O}_S(1)$, it is globally generated by [La, Thm. 1.8.5]. It follows by [DPS, Thm. 2.5] that $0 \leq c_2(\mathcal{E}) \leq c_1(\mathcal{E})^2 = 0$, that is $c_2(\mathcal{E}) = 0$.

Alternatively, to show that $c_2(\mathcal{E}) = 0$, consider the map

$$\lambda_{\mathcal{E}} : \Lambda^r H^0(\mathcal{E}) \to H^0(\det \mathcal{E}).$$

As shown in the proof of Lemma 5.1, the linear system $|\text{Im}\lambda_{\mathcal{E}}|$ is base-point free. Choose now a general subspace $V \subseteq H^0(\mathcal{E})$ of dimension r, so that we get a morphism $\varphi : V \otimes \mathcal{O}_S \to \mathcal{E}$ whose degeneracy locus $D = D_{r-1}(\varphi)$ is a smooth curve by [Ba, Statement(folklore), §4.1]. Since φ is generically an isomorphism we get, as in [GL, Proof of Prop. 1.1], an exact sequence of sheaves

$$(8.1) \qquad\qquad 0 \to V \otimes \mathcal{O}_S \to \mathcal{E} \to \mathcal{L} \to 0$$

and its dual

$$(8.2) \qquad\qquad 0 \to \mathcal{E}^* \to V^* \otimes \mathcal{O}_S \to \mathcal{L}' \to 0$$

where $\mathcal{L}$ and $\mathcal{L}'$ are two line bundles on D with $\mathcal{L} \cong N_{D/S} \otimes (\mathcal{L}')^{-1}$. Note that

$$c_2(\mathcal{E}) = \deg \mathcal{L}'.$$

Now both $\mathcal{L}$ and $\mathcal{L}'$ are globally generated by (8.1) and (8.2). Since $D^2 = 0$ we have that $D = \Gamma_1 \cup \ldots \cup \Gamma_s$ with all Γ_i smooth irreducible, $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$ and $\Gamma_i \equiv \Gamma_j$, so that $D \equiv s\Gamma_i$ for all i and therefore $\Gamma_i^2 = 0$ for all i. Since $\mathcal{O}_S(D)$ is globally generated we get that also $\mathcal{O}_S(\Gamma_i)_{|\Gamma_i} \cong \mathcal{O}_S(D)_{|\Gamma_i}$ is globally generated and has degree 0, whence $\mathcal{O}_S(\Gamma_i)_{|\Gamma_i} \cong \mathcal{O}_{\Gamma_i}$. Also

$$\mathcal{L}_{|\Gamma_i} \cong (N_{D/S} \otimes (\mathcal{L}')^{-1})_{|\Gamma_i} \cong (\mathcal{L}'_{|\Gamma_i})^{-1}$$

and both $\mathcal{L}_{|\Gamma_i}$ and $\mathcal{L}'_{|\Gamma_i}$ are globally generated, whence $\mathcal{L}_{|\Gamma_i} \cong \mathcal{L}'_{|\Gamma_i} \cong \mathcal{O}_{\Gamma_i}$. Therefore

$$\mathcal{L} \cong \mathcal{L}' \cong \bigoplus_{i=0}^{s} \mathcal{O}_{\Gamma_i}$$

and this gives that $c_2(\mathcal{E}) = \deg \mathcal{L}' = 0$.

Thus the claim that $c_2(\mathcal{E}) = 0$ is proved and we proceed with the proof.

Let H be a hyperplane divisor on S.

Since $c_1(\mathcal{E})^2 = 0$ we have that (1.2) does not hold, hence Theorem 1 implies that $S \subseteq \mathbb{P}^N = \mathbb{P}H^0(H)$ is covered by lines. By [LP, Thm. 1.4] it follows that (S, H) is a linear $\mathbb{P}^1$-bundle over a smooth curve (possibly with two $\mathbb{P}^1$-bundle structures in the case of $\mathbb{P}^1 \times \mathbb{P}^1$). Hence there is a smooth curve B, a divisor L on B, a rank 2 vector bundle $\mathcal{F}$ on B (that we can assume to be normalized as in [Ha,

V.2.8.1]) such that $S \cong \mathbb{P}(\mathcal{F})$, $H = \xi + \pi^*L$, where $\pi : S \to B$ is the projection and ξ is the tautological line bundle on $\mathbb{P}(\mathcal{F})$.

We claim that there is a $\mathbb{P}^1$-bundle structure $p : S \to C$ onto a smooth curve C and a line bundle M on C such that $\det \mathcal{E} = p^*M$. As a matter of fact we will have that either $p = \pi$ and $C = B$ or $S \cong \mathbb{P}^1 \times \mathbb{P}^1, C = \mathbb{P}^1$ and $p : S \to \mathbb{P}^1$ is the other projection (besides π).

To this end observe that we can write $\det \mathcal{E} = a\xi + \pi^*M$ for some $a \in \mathbb{Z}$ and some line bundle M on B. Then there is $b \in \mathbb{Z}$ such that $\det \mathcal{E} \equiv a\xi + bf$, where f is a fiber of π. Set $e = -\deg \mathcal{F}, l = \deg L$ and $g = g(B)$, so that $H \equiv \xi + lf$ and $K_S \equiv -2\xi + (2g - 2 - e)f$. Since $(\det \mathcal{E})^2 = 0$ we get that $-a^2e + 2ab = 0$ and therefore either $a = 0$ or $b = \frac{ae}{2}$. But $\det \mathcal{E}$ is globally generated, hence $a = \det \mathcal{E} \cdot f \geq 0$ and if $a > 0$ then $-\frac{ae}{2} = \det \mathcal{E} \cdot \xi \geq 0$, that is $e \leq 0$. Moreover observe that, since $c_1(\mathcal{E}) \neq 0$, we have that $H \cdot \det \mathcal{E} > 0$, and we deduce that either

$$(8.3) \qquad\qquad\qquad\qquad a = 0 \text{ and } b > 0$$

or

$$(8.4) \qquad\qquad\qquad a > 0, e \leq 0, b = \frac{ae}{2} \text{ and } l > \frac{e}{2}.$$

Suppose we are in case (8.4). By [C1, Prop. 2.2] we have that

$$(8.5) \qquad c_1(\mathcal{E}) \cdot H = \frac{r}{2}(3H^2 + H \cdot K_S) \text{ and } c_2(\mathcal{E}) = \frac{1}{2}(c_1(\mathcal{E})^2 - c_1(\mathcal{E}) \cdot K_S) - rH^2 + r\chi(\mathcal{O}_S).$$

Since $H^2 = 2l - e$ and $H \cdot K_S = 2g - 2 + e - 2l$, using $c_1(\mathcal{E})^2 = c_2(\mathcal{E}) = 0$ and replacing in (8.5) we get

$$a(l - \frac{e}{2}) = r(2l - e + g - 1) \text{ and } a(1 - g) = r(2l - e + g - 1).$$

These give

$$a(l - \frac{e}{2} + g - 1) = 0$$

that is $l = \frac{e}{2} - g + 1$. Since $l > \frac{e}{2}$ we get that $g = 0$. As is well-known [Ha, Cor. V.2.13], when $g = 0$, we have that $e \geq 0$. Therefore $e = 0, l = 1$ and $S \cong \mathbb{P}^1 \times \mathbb{P}^1$.

We conclude that either we are in case (8.3) and $\det \mathcal{E} \cong \pi^*M$ for some line bundle M on B (in this case we set $C = B$ and $p = \pi$) or we are in case (8.4) and then $B \cong \mathbb{P}^1$ and $\det \mathcal{E} \cong p^*(\mathcal{O}_{\mathbb{P}^1}(a))$ where $p : S \to \mathbb{P}^1$ is the other projection besides π (in this case we set $C = \mathbb{P}^1, L = \mathcal{O}_{\mathbb{P}^1}(1)$ and $M = \mathcal{O}_{\mathbb{P}^1}(a)$).

Thus the claim is proved and Lemma 5.1 implies that there is a rank r vector bundle $\mathcal{H}$ on C such that $\mathcal{E} \cong p^*\mathcal{H}$. Set $\mathcal{G} = \mathcal{H}(-L - \det \mathcal{F})$, so that $\mathcal{E} \cong p^*(\mathcal{G}(L + \det \mathcal{F}))$. Applying Lemma 4.1 to p we deduce that $H^q(\mathcal{G}(-L)) = 0$ for $q \geq 0$.

Thus $(S, \mathcal{O}_S(1), \mathcal{E})$ is as in (ii). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

9. The case of threefolds

In both proofs we will use a result of Lanteri and Palleschi together with a result of Fujita.

9.1. A quick guide to threefolds covered by lines.

Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible threefold covered by lines, let $x \in X$ be a general point and let $\mathcal{H}_x$ be the Hilbert scheme of lines contained in X and passing through x. All the consequences listed below follow from [LP, Thm. 1.4] and [Fu, Introduction]. To simplify the list, following our purposes, we also assume that X has Picard rank $\rho(X) \geq 2$.

We have that $\dim \mathcal{H}_x \leq 1$ with equality if and only if

   (1) $(X, \mathcal{O}_X(1))$ is a linear $\mathbb{P}^2$-bundle over a smooth curve.

Assume that $(X, \mathcal{O}_X(1))$ is not as in (1). Then $(X, \mathcal{O}_X(1))$ is either one of the following

   (2) $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))$;
   (3) $(\mathbb{P}(T_{\mathbb{P}^2}), \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^2})}(1))$;
   (4) a linear quadric fibration over a smooth curve;
   (5) a linear $\mathbb{P}^1$-bundle over a smooth surface.

Note that the cases (2)-(3) above are listed as (e) in [LP, Thm. 1.4], while all the other cases in [LP, Thm. 1.4(e)] have $\rho(X) = 1$, hence they do not appear in our list, except for the Del Pezzo threefold of degree 7, that is also of type (5).

9.2. **Proof of Theorems.**

Proof of Theorem 3.

If $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (i), (ii) or (iii), it follows by [ES, Prop. 2.1] (or [Be, Thm. 2.3]) and Lemma 4.1 that $\mathcal{E}$ is a rank r Ulrich vector bundle on X with $c_1(\mathcal{E})^2 = 0$.

Now suppose that $\mathcal{E}$ is a rank r Ulrich vector bundle on X with $c_1(\mathcal{E})^2 = 0$.

If $(X, \mathcal{O}_X(1)) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ it follows by [ES, Prop. 2.1] (or [Be, Thm. 2.3]) that $\mathcal{E} = \mathcal{O}_{\mathbb{P}^3}^{\oplus r}$, so that we are in case (i).

Assume from now on that $(X, \mathcal{O}_X(1)) \neq (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$. We will prove that $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (ii) or (iii).

By Remark 2.1 we have that $c_1(\mathcal{E}) \neq 0$. Moreover by Lemma 3.2 we have that $\rho(X) \geq 2$.

Since $c_1(\mathcal{E})^3 = 0$ we have that (1.2) does not hold, hence Theorem 1 implies that X is covered by lines. Therefore $(X, \mathcal{O}_X(1))$ is as in (1)-(5) in 9.1.

In case (1) we know by Lemma 5.2 that $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (ii). In cases (2) and (3) we know by Propositions 6.1 and 6.2 that $c_1(\mathcal{E})^2 \neq 0$, hence these cases do not occur.

As case (4) is excluded by hypothesis, it remains to study case (5) with $\Delta(\mathcal{F}) \neq 0$. Hence we have a $\mathbb{P}^1$-bundle structure $\pi : X \to B$. We let ξ be the tautological line bundle.

We claim that $\det \mathcal{E} = \pi^* M$ for some line bundle M on B.

We can write $\det \mathcal{E} = a\xi + \pi^* M$ for some $a \in \mathbb{Z}$ and some line bundle M on B.

Assume that $a \neq 0$. We have

$$0 = c_1(\mathcal{E})^2 = a^2 \xi^2 + 2a\xi\pi^* M + (\pi^* M)^2.$$

Since $\mathcal{F}$ has rank 2 we know that $\xi^2 = \xi\pi^* c_1(\mathcal{F}) - \pi^* c_2(\mathcal{F})$ and we get

$$(a^2 \pi^* c_1(\mathcal{F}) + 2a\pi^* M)\xi + \pi^*(M^2) - a^2 \pi^* c_2(\mathcal{F}) = 0$$

that is

$$(9.1) \qquad a\pi^*(ac_1(\mathcal{F}) + 2M)\xi + \pi^*(M^2 - a^2 c_2(\mathcal{F})) = 0.$$

If $ac_1(\mathcal{F}) + 2M \not\equiv 0$, we can find an ample divisor A on B such that $A \cdot (ac_1(\mathcal{F}) + 2M) \neq 0$. Intersecting in (9.1) with $\pi^* A$ we get the contradiction

$$aA \cdot (ac_1(\mathcal{F}) + 2M) = 0.$$

Therefore $ac_1(\mathcal{F}) + 2M \equiv 0$ and (9.1) gives that $M^2 = a^2 c_2(\mathcal{F})$. But then

$$a^2 \Delta(\mathcal{F}) = 4a^2 c_2(\mathcal{F}) - a^2 c_1(\mathcal{F})^2 = 0$$

again a contradiction. Therefore $a = 0$ and then $\det \mathcal{E} = \pi^* M$.

Thus the claim is proved and Lemma 5.1 implies that there is a rank r vector bundle $\mathcal{H}$ on B such that $\mathcal{E} \cong \pi^* \mathcal{H}$. Setting $\mathcal{G} = \mathcal{H}(-L - \det \mathcal{F})$, we find that $\mathcal{E} \cong \pi^*(\mathcal{G}(L + \det \mathcal{F}))$ and Lemma 4.1 implies that $H^q(\mathcal{G}(-L)) = H^q(\mathcal{G}(-2L) \otimes \mathcal{F}^*) = 0$ for $q \geq 0$. Hence we are in case (iii). $\qquad\square$

Proof of Theorem 4.

If $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (i) or (ii), it follows by Propositions 6.1 and 6.2 that $\mathcal{E}$ is a rank r Ulrich vector bundle on X with $c_1(\mathcal{E})^3 = 0$ and $c_1(\mathcal{E})^2 \neq 0$.

Now suppose that $\mathcal{E}$ is a rank r Ulrich vector bundle on X with $c_1(\mathcal{E})^3 = 0$ and $c_1(\mathcal{E})^2 \neq 0$.

Since $c_1(\mathcal{E})^3 = 0$ we have by Lemma 3.2 that $\rho(X) \geq 2$. Also (1.2) does not hold, whence Theorem 1 implies that X is covered by lines. Therefore $(X, \mathcal{O}_X(1))$ is as in (1)-(5) in 9.1.

In cases (2) and (3) we can apply Propositions 6.1 and 6.2 and conclude that $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (i) or (ii).

Cases (4) and (5) are excluded by hypothesis unless $(X, \mathcal{O}_X(1))$ is also a Del Pezzo threefold, thus belonging to cases (2) or (3).

In the remaining case (1) we have a $\mathbb{P}^2$-bundle structure $\pi : X \cong \mathbb{P}(\mathcal{F}) \to B$ and $\deg \mathcal{F} < 0$ by hypothesis. We let ξ be the tautological line bundle. We will first prove that $\det \mathcal{E} = \pi^* M$ for some line bundle M on B.

We can write

$$\det \mathcal{E} = a\xi + \pi^* M$$

for some $a \in \mathbb{Z}$ and some line bundle M on B. Then there is $b \in \mathbb{Z}$ such that $\det \mathcal{E} \equiv a\xi + bf$, where f is a fiber of π. Set $e = -\deg \mathcal{F} > 0$ and suppose that $a \neq 0$. We have

$$0 = c_1(\mathcal{E})^3 = a^2(-ae + 3b)$$

and therefore $b = \frac{ae}{3}$ and

$$c_1(\mathcal{E}) = a(\xi + \frac{e}{3}f).$$

Since $\mathcal{F}$ is normalized we have that ξ is effective, hence

$$0 \leq c_1(\mathcal{E})^2 \cdot \xi = -a^2 \frac{e}{3} < 0$$

a contradiction.

Therefore $a = 0$, that is $\det \mathcal{E} = \pi^* M$. By Lemma 5.1 we get that there is a rank r vector bundle $\mathcal{H}$ on B such that $\mathcal{E} \cong \pi^* \mathcal{H}$. But then $c_1(\mathcal{E})^2 = 0$, so that this case does not occur. □

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