LIMIT POINTS OF NETS OF NON-COMMUTATIVE MEASURES ON DEFORMATIONS OF WEYL ALGEBRAS.

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ABSTRACT. Deformations of classical measures to noncommutative ones play an important role in semiclassical and microlocal analysis, and in quantum physics. In this paper, we characterize limit points of nets of noncommutative measures acting on the tensor product of a deformed Weyl algebra and an arbitrary C*-algebra. The limit points are classical vector cylindrical measures on the predual of the phase space (i.e. on the space of Lagrangian description), with values in the dual of the aforementioned C*-algebra. From a physical standpoint, they are interpreted as the partially classical states of composite systems.

CONTENTS

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1. Introduction.

The purpose of this work is to study the relation – of widespread use in both mathematics and physics – between the noncommutative and commutative theory of integration. We would like to underline some of the general features that have been, despite their interest, overlooked so far. The main novelty is the development of a general framework in which both the usual and the partially classical limit could be taken. The partially classical limit describes situations in which only a part of a given theory modifies its behavior from non-commutative to commutative, while the other part remains unchanged. Despite the variety of possible applications in physics, the partially classical limit of quantum theories has rarely been studied rigorously, and only limited to some specific system [Ginibre, Nironi, and Velo, 2006; Frank and Schlein, 2014; Frank and Gang, 2015]. In this paper we study measures acting on tensor products of a Weyl algebra and an arbitrary C*-algebra $\mathfrak{A}$. In that way we are able to treat on the same ground both the standard and the partial classical limit of quantum states: only the Weyl algebra is deformed while the other – that physically represents the unmodified degrees of freedom – remains constant. The most relevant difference from the usual limit is that the classical measures take values in $\mathfrak{A}^*$, instead of being standard measures. Taking $\mathfrak{A}$ to be the trivial abelian algebra generated by a single element, the standard setting is recovered. In the rest of the section we introduce deformations of probability theories, and their applications in physics and analysis.

The main results of the paper can be found in Section 3. In Section 2, we discuss some aspects of vector measure theory, for measures taking values in pointed and generating convex cones of real vector spaces. The results of Section 2 are heavily inspired by [Neeb, 1998; Glöckner, 2003], and could be interesting in their own right.

1.1. Recovering classical measures from noncommutative ones.

Let $(P_h)_{h>0}$ be a family of noncommutative probability theories [see e.g. Voiculescu, Dykema, and Nica, 1992], indexed by a “semiclassical” parameter h. The parameter h is semiclassical in the sense that it measures to some extent the non-commutativity of the theory. If we complete the family with a classical probability theory P_0, we obtain a new family $(P_h)_{h\geq 0}$ that we may call a deformation. At $h = 0$ we have a classical theory, and as h gets bigger, the theory is deformed becoming more and more noncommutative. This picture has to be taken with a grain of salt, for in most applications all the P_h, $h > 0$, are “equally noncommutative” and only P_0 is distinctively different (being commutative). A question that has often been asked – even if not formulated in these exact terms

– is whether it is possible to introduce a topology such that the map $h \mapsto P_h$ is continuous at zero (the continuity in other points is not important for our purposes, and we may assume that the family $(P_h)_{h>0}$ is very regular with respect to h). If it is the case, we are able to obtain a classical probability theory from a family of noncommutative ones, taking the limit $h \to 0$. Alternatively, we have a prescription on how to deform a given classical probability theory to noncommutative ones in a continuous way.

In order to prove eventual continuity properties of a deformation of probability theories, it is necessary to study both deformed probabilities – or more generally measures – and deformed random variables, i.e. measurable functions. In this paper, we focus on the general study of deformed measures in the Weyl tensor deformation. It is much more difficult to study deformed random variables from a general perspective, even in the aforementioned specific deformation; but we plan to address the question in future works.

1.2. **Bohr's correspondence principle in physics.** Let's now review some concrete situations where the ideas above play an important role. We start with physics, and in particular with the interplay between classical and quantum theories. The correspondence principle in quantum mechanics, customarily attributed to N. Bohr [1923], is a necessary condition for any quantum theory. Essentially, it says that *a quantum theory should reproduce the corresponding classical theory, in the regime where the quantum effects become negligible*. Since from a mathematical perspective the physical quantum theories are noncommutative probability theories – where probabilities are called states, and real random variables are called observables – and the physical classical theories are classical probability theories, the framework described in Section 1.1 fits perfectly. In physics, h is customarily taken to be (a quantity proportional to) Planck's constant $\hbar$; and Bohr's correspondence principle is usually justified by means of heuristic or partially rigorous arguments that date back to the beginning of quantum mechanics [e.g. Wentzel, 1926; Kramers, 1926; Brillouin, 1926; Ehrenfest, 1927; Feynman, 1942]. In mathematics, the correspondence has been extensively studied for quantum mechanical systems (phase spaces with finitely many degrees of freedom) and to some extent also for bosonic quantum field theories (phase spaces with infinitely many degrees of freedom); we defer to the next section for a detailed bibliography.

Physical systems with infinite dimensional phase spaces – the so-called field theories – present many open mathematical problems, especially when it comes to quantization. Without entering too much into details, we would like to emphasize a couple of distinctive features that support the point of view taken in this paper. The first one is about the classical description of fields. Finite dimensional classical systems are defined by a finite dimensional space that is called the phase space. The phase space admits a beautiful geometric characterization as the cotangent bundle $T^*\mathcal{M}$ of a smooth (finite dimensional) manifold $\mathcal{M}$, together with its canonical symplectic form. The dynamics of the system is generated by a smooth function on $T^*\mathcal{M}$, the so-called Hamiltonian function $\mathcal{H}$ by means of Poisson brackets. An alternative description of the system is given in the tangent bundle $T\mathcal{M}$, and the dynamics is generated by the so-called Lagrangian function $\mathcal{L}$ via the variational least action principle. The Hamiltonian and Lagrangian functions are related by means of the so-called Legendre transform, and the two descriptions are equivalent. For infinite dimensional systems, some care has to be taken and it is not always possible to define a manifold with smooth structure. If we restrict to systems set on vector spaces, we may easily consider a similar picture. Without loss of generality, we assume that the phase space (V, σ) is a locally convex real vector space with a symplectic form. On (V, σ), we specify an Hamiltonian functional (maybe only densely defined) that, under suitable assumptions, generates a globally well-posed dynamics on (V, σ). If V has a predual V_*, then we may seek a Lagrangian description of the system on V_*. It is not assured, however, that the two pictures are equivalent. Therefore in general a choice must be made. Motivated by properties of relativistic covariance, the Lagrangian description on V_* is often preferred, especially for quantization. The Weyl algebra of quantum observables is, however, customarily built from the phase space (V, σ). As discussed in Section 3.5, Bohr's correspondence principle holds for regular states of the bosonic Weyl algebra, but *the limit classical states are cylinder measures of* V_*. In other words, the Lagrangian classical picture emerges naturally from the quantum-classical correspondence, by a priori considerations on the deformation of the underlying probability theories.

The other relevant feature is related to representations of Weyl algebras. For finite dimensional phase spaces, there is a unique irreducible representation of the Weyl algebra, and regular states are normal with respect to such representation. Therefore it is sufficient to study the deformation explicitly in the representation, as it has been extensively done for the phase space $\mathbb{R}^{2d}$ and the related Schrödinger representation on $L^2(\mathbb{R}^d)$ (again, refer to the next section for bibliographic details). When the Weyl algebra is built on an infinite dimensional phase space, there are (uncountably) many inequivalent irreducible representations. In addition, representations corresponding to free and interacting relativistic theories should be inequivalent (Haag's theorem). Hence it is desirable to study the correspondence principle independently of the chosen representation of the Weyl algebra. The results obtained in Section 3 are all representation independent.

1.3. **Semiclassical and microlocal measures.** In this part of the introduction, we briefly review the existing literature on the deformations of Weyl algebras in connection with semiclassical and microlocal analysis. The deformation that has been studied the most in analysis is the deformation of standard (Borel) probability theory in $\mathbb{R}^{2d}$ to the corresponding Weyl algebra of canonical commutation relations generated by phase space quantum translations, unitarily represented on $L^2(\mathbb{R}^d)$.

In this context, the picture can be considered essentially complete: the continuous behavior along the deformation of both probabilities and random variables have been thoroughly studied, as well as the convergence of quantum to classical dynamics (one-parameter groups of automorphisms of probabilities or random variables). The reader interested in semiclassical techniques is invited to refer e.g. to [Hörmander, 1994; Martinez, 2002; Zworski, 2012, and references thereof contained]; here we focus explicitly on the deformation of measures. It should also be mentioned that deformations, and especially deformation quantization for Poisson manifolds, has been studied also from an algebraic and geometrical point of view [see e.g. Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer, 1978; Fedosov, 1985; Rieffel, 1993, 1994; Kontsevich, 2003; Mazur, 2004, and references thereof contained].

The Wigner, or semiclassical, or defect measures have a long history. They are the classical measures on $\mathbb{R}^{2d}$ that are a limit point of sequences of non-commutative regular measures of the Weyl deformation $\bigl(\mathrm{Weyl}_h(\mathbb{R}^{2d},\sigma_{\mathrm{c}})\bigr)_{h>0}$, where σ_{c} is the canonical symplectic form on $\mathbb{R}^{2d}$ – for a definition of the Weyl deformation see Section 3.1. They were introduced as microlocal defect measures in order to study variational problems with loss of compactness [see Lions, 1985a,b; Tartar, 1990; Gérard, 1991b]. Around the same time, they have been used in semiclassical analysis to provide a rigorous study of the classical limit of quantum mechanics [see Colin de Verdière, 1985; Helffer, Martinez, and Robert, 1987; Gérard, 1991a; Lions and Paul, 1993; Burq, 1996-1997]. Later, in a series of papers, Ammari and Nier [2008, 2009, 2011, 2015] introduced and studied Wigner measures for systems with infinite dimensional phase spaces. The framework they developed has then been used to study mean field and classical limits of both field theories [Ammari and Falconi, 2014, 2016] and many particle systems [Ammari and Breteaux, 2012; Liard and Pawilowski, 2014]. The present work can be seen as a continuation of the seminal ideas in [Ammari and Nier, 2008, Sections 6 and 7]. Infinite dimensional Wigner measures and cylindrical measures have also been introduced – but from a different point of view – by Lewin, Nam, and Rougerie [2014, 2015a,b].

1.4. **Cylindrical measures and perspectives on quantization.** In this last part of the introduction, we discuss the main results of Sections 3.5 and 3.7, in connection with the problem of quantization in field theories. The quantization of classical relativistic theories with infinitely many degrees of freedom is a very important and largely open problem of mathematical physics. Despite many attempts, there is not a satisfactory way to build interacting quantum field theories quantizing functionals on either the classical phase space or the space of the Lagrangian in $3+1$ dimensions. The interested reader may e.g. refer to [Segal, 1960; Balaban and Raczka, 1975; Balaban, Jezuita, and Raczka, 1976; Krée and Raczka, 1978; Helffer and Sjöstrand, 1992; Helffer, 1994; Amour, Kerdelhue, and Nourrigat, 2001; Ammari and Nier, 2008; Amour, Lascar, and Nourrigat, 2015] for various attempts to build a quantization scheme or pseudodifferential calculus for infinite dimensional systems. In a recent paper [Ammari and Falconi, 2016], we were successful in defining the renormalized quantum dynamics of the Nelson model directly by (Wick) quantization, exploiting a suitable symplectic transformation of the phase space classical Hamiltonian functional.

Here in this work only quantum states (measures), and not observables, are considered; nevertheless, we get some insight on the classical structure emerging in the limit, that may be important for the purpose of quantization and subsequent definition of the quantum dynamics. In particular, the following fact could be relevant: *the actual classical space emerging in the limit might be much larger than the one taken as starting point to build the Weyl algebra*. This might affect the choice of both the quantization procedure and the functionals to be quantized. As an example, consider a scheme like the one just mentioned above for the Nelson model (where the phase space and the space of the Lagrangian are isomorphic). In order to define the quantum dynamics by quantization, and directly get rid of all divergences, it is crucial to perform three steps: make a symplectic change of coordinates in the phase space; evaluate the classical Hamiltonian functional; Wick quantize. Now let (V, σ) be the phase space of some Weyl algebra, $\mathscr{E} : V \to \mathbb{R}$ a corresponding classical Hamiltonian. If the meaningful classical space is some $W \supset V$ where $\mathscr{E}$ can be non-trivially extended, we gain additional freedom in two of the three steps above: choosing the coordinate transformation and defining the quantization procedure. In particular, it may be possible that the needed change of coordinates would not preserve V. This idea is corroborated by the fact that the quantum vacuum state for relativistic interacting theories is not normal with respect to the free vacuum. Nets of the latter usually converge to a measure concentrated on V (e.g. to the point measure concentrated in zero), but nets of the former may converge to a measure concentrated outside of V. If that is the case, manipulating $\mathscr{E}$ on V would not be a sensible strategy. Since we do not know a priori where the limit points of nets of the interacting vacuum would be concentrated, it seems relevant to study the Hamiltonian functional on the whole limit space W.

Now let's discuss in detail why the space defined by the classical limit procedure may be quite large. For the notation, refer to Sections 2.2, 3.1 and 3.3. In the discussion just above, we implicitly assumed that the space of the Lagrangian V_* is continuously and linearly embedded in V; however the following discussion is valid in general. In Theorems 3.15 and 3.20, two important facts are proved: *the limit points of nets of regular quantum states are cylindrical measures on the space of the Lagrangian V_*, and every cylindrical measure is reached by at least one net of quantum states.* Therefore, there are quantum states whose classical counterpart are not measures but "pure" cylindrical measures. Cylindrical measures are rather inconvenient as classical states, because the observables of V_* cannot be evaluated on them, unless roughly speaking they are cylindrical as well. It is however possible to *identify every cylindrical measure* *with a Radon measure on a bigger space, that contains* V_*. This is done exploiting a slight variant of Prokhorov's theorem 2.18 [see Schwartz, 1973, Part I, Chapter I, Theorem 22 for standard measures]. More precisely, there is an injection $\mathfrak{c}:\mathcal{M}_{\mathrm{cyl}}(V_*,\mathfrak{A}^*_+)\to\mathcal{M}_{\mathrm{rad}}\bigl(\prod_{\Phi_*\in F(V_*)}(V_*/\Phi_*)^{\mathfrak{c}},\mathfrak{A}^*_+\bigr)$, where $(V_*/\Phi_*)^{\mathfrak{c}}$ are either Čech or one-point compactifications of V_*/Φ_*, and their product is endowed with the product topology (and the subscript $_{\mathrm{rad}}$ stands for Radon measures). In addition, there is a canonical injection $\mathfrak{i}_{V_*}:V_*\to\prod_{\Phi_*\in F(V_*)}(V_*/\Phi_*)^{\mathfrak{c}}$. An interesting fact is that there are cylindrical measures μ such that $\mathfrak{c}\mu$ *is concentrated outside of* $\mathfrak{i}_{V_*}(V_*)$, and by Theorem 3.20, such measures are reached by suitable nets of quantum states. In the light of the above, it seems reasonable to consider $\prod_{\Phi_*\in F(V_*)}(V_*/\Phi_*)^{\mathfrak{c}}$ as the limit classical space. With respect to $\prod_{\Phi_*\in F(V_*)}(V_*/\Phi_*)^{\mathfrak{c}}$, all limit points of nets of quantum states are Radon probability measures. We remark that $\prod_{\Phi_*\in F(V_*)}(V_*/\Phi_*)^{\mathfrak{c}}$ could be rather big – especially if the Čech compactification is chosen. There is also a bijection $\mathfrak{a}:\mathcal{M}_{\mathrm{cyl}}(V_*,\mathfrak{A}^*_+)\to\mathcal{M}_{\mathrm{cyl}}\bigl(\tilde{V}_{\mathrm{ad}},\mathfrak{A}^*_+\bigr)$, where $\tilde{V}_{\mathrm{ad}}$ is V_{ad} endowed with the $\sigma(V_{\mathrm{ad}},V)$ topology – V_{ad} is the algebraic dual of V. Alternatively, $\tilde{V}_{\mathrm{ad}}$ can be seen as the completion of V_* with respect to the weak topology $\sigma(V_*,V)$. If V is second countable, then $\mathcal{M}_{\mathrm{cyl}}\bigl(\tilde{V}_{\mathrm{ad}},\mathfrak{A}^*_+\bigr)=\mathcal{M}_{\mathrm{rad}}\bigl(\tilde{V}_{\mathrm{ad}},\mathfrak{A}^*_+\bigr)$; therefore it might be sufficient to choose $\tilde{V}_{\mathrm{ad}}$ as the classical space. If V is not second countable, there are $\mu\in\mathcal{M}_{\mathrm{cyl}}(V_*,\mathfrak{A}^*_+)$ such that $\mathfrak{c}\mu$ is concentrated outside of $\tilde{V}_{\mathrm{ad}}$ as well.

We conclude with a couple of explicit examples that should help to put the ideas above into context. The first provides a physically reasonable sequence of quantum states that converges to a classical measure concentrated outside of the starting space V_*. Such states already appeared in Lewin et al. [2015b], and are the so-called free Gibbs states for a second-quantized system. Let $\mathrm{Weyl}_h\bigl(L^2(\mathbb{R}^d)\bigr)$ be the Weyl algebra deformation with phase space $(V,\sigma)=\bigl(L^2(\mathbb{R}^d)_{\mathbb{R}},\mathrm{Im}\langle\cdot,\cdot\rangle_2\bigr)$. In physics, it represents the algebra of canonical commutation relations for time-zero scalar fields. A well-known irreducible representation of this deformation is given by means of Weyl operators in the symmetric Fock space $\Gamma_s\bigl(L^2(\mathbb{R}^d)\bigr)_h$ with h dependent commutation relations [see e.g. Fock, 1932; Cook, 1951]. We do not want to enter too much into details, let's just recall that there exists a positive map $d\Gamma_h$ that associates to any self-adjoint A on $L^2(\mathbb{R}^d)$ a self-adjoint operator $d\Gamma_h(A)$ on $\Gamma_s\bigl(L^2(\mathbb{R}^d)\bigr)_h$. Now let H_0 be a self-adjoint operator on $L^2(\mathbb{R}^d)$ – the so-called free Hamiltonian – such that for any $h>0$, $e^{-\beta_h H_0}$ is a trace-class operator and $\beta_h(H-\mu_h)>0$; where $\beta_h>0,\mu_h\in\mathbb{R}$ are the (possibly deformation-dependent) inverse temperature and chemical potential respectively. The regular Gibbs state Ω_h^{G} in the Fock representation is then defined by

$$\Omega_h^{\mathrm{G}}(\cdot)=\frac{\mathrm{Tr}\bigl(e^{-\beta_h d\Gamma_h(H_0-\mu_h)}\,\cdot\,\bigr)}{\mathrm{Tr}\bigl(e^{-\beta_h d\Gamma_h(H_0-\mu_h)}\bigr)}\;;$$

or equivalently by the generating functional (see Definition 3.6)

$$\mathcal{G}_{\Omega_h^{\mathrm{G}}}(f)=e^{-\frac{1}{2}\langle f,\frac{h}{2}(1+e^{-\beta_h(H_0-\mu_h)})(1-e^{-\beta_h(H_0-\mu_h)})^{-1}f\rangle_2}\;.$$

Now suppose that $\frac{h}{2}(1+e^{-\beta_h(H_0-\mu_h)})(1-e^{-\beta_h(H_0-\mu_h)})^{-1} \to K > 0$ in the weak operator topology as $h \to 0$. Hence by Theorem 3.15 the net of Gibbs states $(\Omega_h^{\mathrm{G}})_{h>0}$ converge to the Gaussian cylindrical measure μ_K on $L^2(\mathbb{R}^d)$ associated to the non-degenerate quadratic form $\langle\,\cdot\,, K\,\cdot\,\rangle_2$. By [Minlos, 1963], μ_K is extended to a Radon gaussian measure $\tilde{\mu}_K$ on the tempered distributions $\mathscr{S}'(\mathbb{R}^d)$ supported outside of $L^2(\mathbb{R}^d)$.

The second example concerns the $P(\varphi)_2$ scalar quantum field theories, and it is related to the comment above about free and interacting vacua. The ground state Ω_h^0 of the free scalar field theory in $1+1$ dimensions is the Fock vacuum; following [Segal, 1956; Nelson, 1973], it is convenient to represent it in the functional space $L^2(\mathscr{S}'(\mathbb{R}), d\nu_h)$, where ν_h is the Gaussian measure associated to the quadratic form $h\langle\cdot,\cdot\rangle_2$. In this space, Ω_h^0 is represented by the projection on the vector $1 \in L^2(\mathscr{S}'(\mathbb{R}), d\nu_h)$ and the time-zero free fields $\varphi_h(f)$ – $f \in \mathscr{S}(\mathbb{R})$ – are represented as the multiplication by the coordinate function f. It then follows that the generating functional for Ω_h^0 is the Fourier transform of ν_h as a cylinder Gaussian measure on $L^2(\mathbb{R})$, namely

$$\mathcal{G}_{\Omega_h^0}(f) = e^{-\frac{h}{2}\|f\|_2^2} .$$

Therefore the limit point $h \to 0$ of $(\Omega_h^0)_{h>0}$ is $\nu_0 = \delta(0)$, the Radon measure concentrated in the point $0 \in L^2(\mathbb{R})$. The ground states $\Omega_h^{P(\varphi)_2}$ of interacting $P(\varphi)_2$ theories are vector states in an inequivalent representation of the Weyl algebra of time-zero fields. It has been proved by [Glimm, Jaffe, and Spencer, 1974], using a result by [Newman, 1973], that – at least for weak couplings – there exists a Radon non-gaussian measure μ_h on $\mathscr{S}'(\mathbb{R})$ such that $\Omega_h^{P(\varphi)_2}$ is represented by the projection on the vector $1 \in L^2(\mathscr{S}'(\mathbb{R}), d\mu_h)$ and the time-zero interacting fields are represented as the multiplication by the coordinate function. The corresponding generating functional $\mathcal{G}_{\Omega_h^{P(\varphi)_2}}$ for the interacting ground state has been characterized by [Fröhlich, 1974, 1977], and by its properties it follows that $\Omega_h^{P(\varphi)_2}$ is a regular state. The limit points of $\Omega_h^{P(\varphi)_2}$ are Radon measures on $\mathscr{S}'(\mathbb{R})$ as well. We conjecture that there are interactions such that the ground state has classical limit points concentrated outside of $L^2(\mathbb{R})$ – and if such property is related to Haag's theorem, it may even hold true for any relativistic invariant interacting theory.

2. Elements of the theory of cone-valued vector measures.

In this section, we outline some results of vector integration that will be needed to characterize the classical limit behavior of nets of noncommutative measures in the Weyl algebra. Even if it is possible to recover – with some adaptation – a part of these results from the literature [Neeb, 1998; Glöckner, 2003], it is important to develop the aspects that are most important for our purposes explicitly. As it will become clear in the following, in our framework vector measures behave essentially as standard measures.

For the convenience of the reader, at the beginning of each subsection the needed notation and definitions are listed. The most important definitions are however singled out in the subsequent text.

2.1. **Definition of cone-valued measures.**

- Given a topological space E, we denote by $\mathcal{B}(E)$ its Borel σ-algebra.

- We will always denote by X a real vector space, and by C a *pointed and generating* convex cone in X containing 0. This means that
$$C \cap -C = \{0\} \; ; \; C - C = X \; .$$

- We denote by X_{ad} the algebraic dual of X, and for any $X' \subset X_{ad}$ we denote by C' the dual cone of C defined by $C' = \{\kappa \in X', \kappa(C) \subseteq \mathbb{R}^+\}$. If X is locally convex, X' its topological dual and C is closed, then the Hahn-Banach separation theorem yields
$$C = C'' = \{x \in X, (\forall \kappa \in C^*)\kappa(x) \geq 0\} \; . \tag{c1}$$
We will consider only triples (X, X', C) satisfying (c1).

- We denote by $\mathbb{R}^+_\infty = [0, \infty]$ the extended real semi-line considered as an additive semigroup with the additional rule
$$(\forall x \in \mathbb{R}^+_\infty) \; \infty + x = x + \infty = \infty \; .$$
We also denote by $\mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty]$ the (compact) complete lattice of extended reals, and by $\mathbb{C} \cup \{\infty\} \cong [-\infty, +\infty] \times [-\infty, +\infty]$ the extended complex numbers (one-point compactification of the complex numbers).

- We denote by $C_\infty = \mathrm{Hom}_{\mathrm{mon}}(C', \mathbb{R}^+_\infty)$ the subset of $(\mathbb{R}^+_\infty)^{C'}$ consisting of monoid homomorphisms. C_∞ is a monoid with respect to pointwise addition.

- We denote by $\mathfrak{i}_C$ the natural monoid morphism
$$\mathfrak{i}_C : C \to C_\infty \; , \; (\forall c \in C)(\forall \kappa \in C')\mathfrak{i}_C(c)(\kappa) = \kappa(c) \; .$$
$\mathfrak{i}_C(c_1) = \mathfrak{i}_C(c_2)$ yields $(\forall \kappa \in C')\kappa(c_1 - c_2) = \kappa(c_2 - c_1) = 0$. Therefore (c1) implies $c_1 - c_2 \in C \cap -C$ and by the pointedness of C we have $c_1 - c_2 = 0$. Thus $\mathfrak{i}_C$ is injective and $C \cong \mathfrak{i}_C(C)$ is a submonoid of C_∞.

- The next condition is important to define cone-valued measures:
$$\mathfrak{i}_C(C) = \mathrm{Hom}_{\mathrm{mon}}(C', \mathbb{R}^+) \; . \tag{c2}$$
We discuss later some explicit example of triples that satisfy (c1) and (c2).

- Finally, we denote by $\mathbb{N}_*$ the set of strictly positive natural numbers, i.e. $\mathbb{N}_* = \mathbb{N} \setminus \{0\}$.

Cone valued measures are vector measures generalizing the concept of positive measures. As we will see, they can essentially be seen as suitable collections of the latter, and therefore they share many interesting properties of usual positive measures.

Definition 2.1 (C-valued measures)**.** *Let (X, X', C) be a triple that satisfies (c1)-(c2), and E a topological space. Then $\mu \in (C_\infty)^{\mathcal{B}(E)}$ is a Borel C-valued measure on E iff it is countably additive and $\mu(\emptyset) = 0$.*

Remark 2.2. In the definition above, 0 is the trivial monoid morphism that maps every $\kappa \in C'$ to $0 \in \mathbb{R}^+_\infty$. In addition, countable additivity is intended in the following sense. Let $\{K_j\}_{j\in\mathbb{N}} \subset C_\infty$ be a subset of C_∞; then the countable combination $\sum_{j\in\mathbb{N}} K_j \in C_\infty$ is defined by pointwise convergence in the topology of extended reals of partial sums, i.e. by convergence of the sequences

$$\mathbb{R}^+_\infty \supset (w^\kappa_n)_{n\in\mathbb{N}} = \Bigl(\sum_{j=0}^{n} K_j(\kappa)\Bigr)_{n\in\mathbb{N}} \ , \ \kappa \in C' \ .$$

Therefore a function $\mu \in (C_\infty)^{\mathcal{B}(E)}$ is countably additive iff for any collection $\{b_j\}_{j\in\mathbb{N}} \subset \mathcal{B}(E)$ of mutually disjoint Borel sets,

$$\mu\Bigl(\bigcup_{j\in\mathbb{N}} b_j\Bigr) = \sum_{j\in\mathbb{N}} \mu(b_j) \ .$$

If $C = \mathbb{R}^+$, Definition 2.1 corresponds to the usual one for positive Borel measures. As it was stated before, a key feature of C-valued measures is that they are in fact families of positive measures, indexed by the dual cone C'. The precise statement is the following, whose proof follows almost directly from Definition 2.1 above.

Theorem 2.3 (Neeb [1998])**.** *There is a bijection between C-valued measures μ on E and families of Borel positive measures $(\mu_\kappa)_{\kappa\in C'}$ on E such that for any $b \in \mathcal{B}(E)$, $\bigl(\kappa \mapsto \mu_\kappa(b)\bigr) \in \mathrm{Hom}_{\mathrm{mon}}(C', \mathbb{R}^+_\infty)$, i.e. the map $\kappa \mapsto \mu_\kappa(b)$ belongs to C_∞.*

In the light of Theorem 2.3, we define a C-valued measure μ *finite* if μ_κ is a finite positive Borel measure for any $\kappa \in C'$.

We turn now to integration of (scalar) functions with respect to cone-valued measures. As usual, it is convenient to start with the integration of non-negative functions. Theorem 2.3 is very convenient in this context, since we can simply define cone-valued integration by means of usual integration. Let $f : E \to \mathbb{R}^+_\infty$ be a non-negative Borel measurable function with values on the extended reals. Let $b \in \mathcal{B}(E)$; then we define for any $\kappa \in C'$,

$$\mathbb{R}^+_\infty \ni I_\kappa = \int_b f(x) d\mu_\kappa(x) \ .$$

The map $\kappa \mapsto I_\kappa$ is a monoid morphism, and therefore an element of C_∞, that we denote by $\mu_b(f)$. This leads to the following natural definition.

Definition 2.4 (μ-integrable functions). *Let (X, X', C) be a triple that satisfies (c1)-(c2); and μ a C-valued measure on a topological space E. The* measure *of a* non-negative *measurable function $f \in (\mathbb{R}^+_\infty)^E$ is defined by*

$$C_\infty \ni \mu_b(f) = \left(\kappa \mapsto \int_b f(x) d\mu_\kappa(x)\right) .$$

A non-negative measurable function $f \in (\mathbb{R}^+_\infty)^E$ is μ-integrable *on the Borel set $b \in \mathcal{B}(E)$ iff $\mu_b(f) \in \mathfrak{i}_C(C) = \mathrm{Hom}_{\mathrm{mon}}(C', \mathbb{R}^+)$. In this case, we denote the* integral *by*

$$C \ni \int_b f(x) d\mu(x) = \mathfrak{i}_C^{-1}\big(\mu_b(f)\big) .$$

A complex function *$f \in \mathbb{C}^E$ is μ-integrable on the Borel set $b \in \mathcal{B}(E)$ iff $|f|$ is μ-integrable, and*

$$X_\mathbb{C} \ni \int_b f(x) d\mu(x) = \int_b (\mathrm{Re} f)_+(x) d\mu(x) - \int_b (\mathrm{Re} f)_-(x) d\mu(x)$$
$$+ i\left(\int_b (\mathrm{Im} f)_+(x) d\mu(x) - \int_b (\mathrm{Im} f)_-(x) d\mu(x)\right) ;$$

where $X_\mathbb{C}$ is the complexification of X, and $f = (\mathrm{Re} f)_+ - (\mathrm{Re} f)_- + i\Big((\mathrm{Im} f)_+ - (\mathrm{Im} f)_-\Big)$ with $\{(\mathrm{Re} f)_+, (\mathrm{Re} f)_-, (\mathrm{Im} f)_+, (\mathrm{Im} f)_-\} \subset (\mathbb{R}^+)^E$.

Remark 2.5. If μ is finite, then any $f \in \bigcap_{\kappa \in C'} L^\infty(E, d\mu_\kappa)$ is integrable. In particular, any continuous and bounded function is integrable.

In the next proposition we state the important linearity property of the integral $\int_b f(x) d\mu(x)$. The proof is trivial.

Proposition 2.6. *The mapping $f \mapsto \int_b f(x) d\mu(x)$ is a linear, cone-homomorphism. In other words, for any complex-valued μ-integrable functions f_1, f_2 and $z \in \mathbb{C}$:*

$$\int_b \big(f_1(x) + z f_2(x)\big) d\mu(x) = \int_b f_1(x) d\mu(x) + z \int_b f_2(x) d\mu(x) .$$

In addition, the cone of non-negative μ-integrable functions is mapped into the cone $(C + i\{0\}) \subset X_\mathbb{C}$.

2.2. **Bochner's theorem.** We are now ready to prove a result that will be crucial in our framework: Bochner's theorem for finite C-valued measures. It will be used to characterize the limit points of nets of noncommutative measures by means of their Fourier transform. To prove the theorem we follow closely [Neeb, 1998; Glöckner, 2003].

2.2.1. *Locally compact abelian groups.* In this subsection – if not specified otherwise – we take as measure space G a locally compact abelian group with character group $\hat{G}$; and (X, X', C) a triple satisfying (c1)-(c2) and some or all of the following additional conditions. Let K be a pointed and generating cone in a real vector space A. Then an involution † on $A_\mathbb{C}$ agrees with K if: $a^\dagger a \in K + i\{0\}$ for any $a \in A_\mathbb{C}$, and for any $k \in K$

there exists an $a_k \in A_{\mathbb{C}}$ such that $k = a_k^{\dagger} a_k$. Then we define the following conditions:

(c3) X and X' locally convex ;

(c4) C' pointed and generating in X' ;

(c5) $(X')_{\mathbb{C}}$ is an involutive algebra with involution agreeing with C' ;

(c6) $X_{\mathbb{C}} = (X')^*_{\mathbb{C}}$; i.e. $X_{\mathbb{C}}$ is the topological dual of $(X')_{\mathbb{C}}$.

Given a locally convex real vector space T, there are (infinitely) many ways to endow $T_{\mathbb{C}}$ with a topology in a "natural" way (i.e. satisfying some suitable properties). Therefore one may ask if it is always possible to endow $X_{\mathbb{C}}$ and $(X')_{\mathbb{C}}$ with suitable topologies such that (c6) is satisfied. If X' is a Banach space and $X = (X')^*$ its dual, the answer is that for any so-called natural complexification of X' there is a so-called reasonable complexification of X such that (c6) is satisfied [see Muñoz, 2000, for additional details].

Definition 2.7 (Completely positive functions)**.** *Let G be an abelian group, (X, X', C) a triple satisfying* (c1) *and* (c3)-(c6)*. A function $f \in (X_{\mathbb{C}})^G$ is* completely positive *iff for any $n \in \mathbb{N}_*$, for any $\{g_i\}_{i=1}^n \subset G$ and $\{\tilde{\kappa}_i\}_{i=1}^n \in (X')_{\mathbb{C}}$:*

$$\sum_{i,j=1}^{n} \tilde{\kappa}_j^{\dagger} \tilde{\kappa}_i \Big(f\big(g_i g_j^{-1}\big) \Big) \geq 0 \ .$$

The definition above is the analogous of positive-definiteness for the cone C. In fact, completely positive functions play the same role for cone-valued measures as positive-definite functions for positive measures. In order to study completely positive functions, it is convenient to introduce the following slight generalization. Let $f \in (X_{\mathbb{C}})^G$, where G is an abelian group. Then there exist an associated kernel $F_f(\cdot, \cdot) : G \times G \to X_{\mathbb{C}}$ defined by $F_f(g_1, g_2) = f(g_1 g_2^{-1})$. Hence it is natural to have the following definition.

Definition 2.8 (Completely positive kernels)**.** *Let A be a set, (X, X', C) a triple satisfying* (c1) *and* (c3)-(c6)*. A kernel $F : A \times A \to X_{\mathbb{C}}$ is completely positive iff for any $n \in \mathbb{N}_*$, for any $\{a_i\}_{i=1}^n \subset A$ and $\{\tilde{\kappa}_i\}_{i=1}^n \in (X')_{\mathbb{C}}$:*

$$\sum_{i,j=1}^{n} \tilde{\kappa}_j^{\dagger} \tilde{\kappa}_i \big(F(a_i, a_j) \big) \geq 0 \ .$$

The equivalence of the two definitions for groups is given by the following trivial result.

Lemma 2.9. *Let G be an abelian group, (X, X', C) a triple satisfying* (c1) *and* (c3)-(c6)*. A function $f \in (X_{\mathbb{C}})^G$ is completely positive iff the associated kernel $F_f \in (X_{\mathbb{C}})^{G \times G}$ is completely positive.*

In order to prove Bochner's theorem, we prove a couple of preliminary results related to complete positivity.

Lemma 2.10. *Let (X, X', C) be a triple satisfying (c1)-(c6), and μ a finite C-valued measure on a topological space E. If we denote by $L^2(E, \mu) \subset \mathbb{C}^E$ the space of μ-square integrable functions, i.e.*

$$L^2(E, \mu) = \bigcap_{\kappa \in C'} L^2(E, \mu_k) ;$$

then the integral map $I_\mu : L^2(E, \mu) \times L^2(E, \mu) \to X_{\mathbb{C}}$, defined by

$$I_\mu(f, g) = \int_E f(x)\bar{g}(x) d\mu(x) ,$$

is well-defined and a completely positive kernel.

Proof. The fact that the kernel I_μ is well-defined is easy to prove using the corresponding property of I_{μ_κ}, $\kappa \in C'$. To prove complete positivity, we proceed as follows. Let $n \in \mathbb{N}_*$, $\{f_i\}_{i=1}^n \subset L^2(E, \mu)$ and $\{\tilde{\kappa}_i\}_{i=1}^n \in (X')_{\mathbb{C}}$. Using the decomposition $X'_{\mathbb{C}} = C' - C' + i(C' - C')$, we see that the map

$$X'_{\mathbb{C}} \ni \tilde{\kappa} \mapsto \mu_{\tilde{\kappa}} = \mu_{\kappa_{\mathrm{R}}^+} - \mu_{\kappa_{\mathrm{R}}^-} + i(\mu_{\kappa_{\mathrm{I}}^+} - \mu_{\kappa_{\mathrm{I}}^-})$$

defines a linear morphism from $X'_{\mathbb{C}}$ to the standard signed measures. Now let $\mu_{\tilde{\kappa}}$ be a signed measure, f an everywhere $\mu_{\tilde{\kappa}}$-integrable function. Then there is a signed measure $\mu_{\tilde{\kappa}(f)}$ defined by $d\mu_{\tilde{\kappa}(f)}(x) = f(x)d\mu_{\tilde{\kappa}}(x)$. If we define in addition

$$\mu_{\tilde{\kappa}(f)^\dagger} = \mu_{\tilde{\kappa}^\dagger(\bar{f})} \; , \; \mu_{\tilde{\kappa}_1(f_1)+\tilde{\kappa}_2(f_2)} = \mu_{\tilde{\kappa}_1(f_1)} + \mu_{\tilde{\kappa}_2(f_2)} \; ,$$
$$\mu_{\tilde{\kappa}_1(f_1)\tilde{\kappa}_2(f_2)} = \mu_{\tilde{\kappa}_1\tilde{\kappa}_2(f_1 f_2)} \; ;$$

then it is easy to see, using property (c5), that for any $b \in \mathcal{B}(E)$, $\tilde{\kappa} \in X'_{\mathbb{C}}$ and everywhere $\mu_{\tilde{\kappa}}$-integrable f:

$$\int_b d\mu_{\tilde{\kappa}(f)^\dagger \tilde{\kappa}(f)} \geq 0 .$$

Then

$$\sum_{i,j=1}^n \tilde{\kappa}_j^\dagger \tilde{\kappa}_i \big(I_\mu(f_i, f_j)\big) = \sum_{i,j=1}^n \int_E f_i(x)\bar{f}_j(x) d\mu_{\tilde{\kappa}_j^\dagger \kappa_i} = \sum_{i,j=1}^n \int_E d\mu_{\tilde{\kappa}_j(f_j)^\dagger \tilde{\kappa}_i(f_i)}$$
$$= \int_E d\mu_{\left(\sum_{i=1}^n \tilde{\kappa}_i(f_i)\right)^\dagger \left(\sum_{i=1}^n \tilde{\kappa}_i(f_i)\right)} \geq 0 .$$

⊣

Corollary 2.11. *Let (X, X', C) be a triple satisfying (c1)-(c6), and μ a finite C-valued measure on a topological space E. If we denote by $L^\infty(E, \mu) \subset \mathbb{C}^E$ the space of μ-bounded functions, i.e.*

$$L^\infty(E, \mu) = \bigcap_{\kappa \in C'} L^\infty(E, \mu_k) ;$$

then the integral $I_\mu : L^\infty(E, \mu) \to X_{\mathbb{C}}$ is a completely positive function – considering $L^\infty(E, \mu)$ as an abelian multiplicative group.

The last ingredient needed to formulate Bochner's theorem is the Fourier transform. The Fourier transform extends quite naturally to cone-valued measures.

Definition 2.12 (Fourier transform of a C-valued measure)**.** *Let G be a locally compact abelian group, (X, X', C) a triple satisfying (c1)-(c2), and $\mathcal{M}(\hat{G}, C)$ the set of finite C-valued Borel measures on the character group $\hat{G}$. The* Fourier transform *is a map $\hat{}: \mathcal{M}(\hat{G}, C) \to (X_{\mathbb{C}})^G$, defined by*

$$(\forall g \in G)\ \hat{\mu}(g) = \int_{\hat{G}} \gamma(g) d\mu(\gamma) .$$

Using the definitions above, Bochner's theorem is written in a rather familiar form.

Theorem 2.13 (Bochner)**.** *Let G be a locally compact abelian group, (X, X', C) a triple satisfying (c1)-(c6). The Fourier transform is a bijection between finite C-valued measures on $\hat{G}$ and completely positive ultraweakly continuous functions from G to $X_{\mathbb{C}}$.*

Proof. Let μ be a finite C-valued measure on $\hat{G}$. Finiteness of the measure implies the integrability of $\gamma(g)$, since $(\forall \gamma \in \hat{G})(\forall g \in G)|\gamma(g)| = 1$. In addition, γ is a representation of the abelian group G on the functions $L^\infty(G, \mu)$. Hence it follows by Corollary 2.11 that $\hat{\mu}(\cdot)$ is completely positive. To prove ultraweak continuity, let $\kappa \in C' + i\{0\}$. By Definition 2.4

$$\kappa\big(\hat{\mu}(\cdot)\big) = \int_{\hat{G}} \gamma(\cdot) d\mu_\kappa(\gamma)$$

is the Fourier transform of the finite measure μ_κ, hence continuous. Now by (c4), $(X')_{\mathbb{C}} = C' - C' + i(C' - C')$ and therefore for any $\tilde{\kappa} \in (X')_{\mathbb{C}}$, $\tilde{\kappa}\big(\hat{\mu}(\cdot)\big) \in \mathbb{C}^G$ is continuous. By (c6), this yields the ultraweak continuity of $\hat{\mu}(\cdot)$.

Now let's consider a completely positive ultraweakly continuous function f from G to $X_{\mathbb{C}}$. Then for any $\kappa \in C' + i\{0\}$, $\kappa\big(f(\cdot)\big)$ is a positive definite continuous $\mathbb{C}$-valued function. Continuity trivially follows from ultraweak continuity (since $\kappa \in (X')_{\mathbb{C}}$). To prove positive-definiteness, we exploit complete positivity. By Definition 2.7, for any $n \in \mathbb{N}_*$, $\{g_i\}_{i=1}^n \subset G$ and $\{\tilde{\kappa}_i\}_{i=1}^n \subset (X')_{\mathbb{C}}$,

$$\sum_{i,j=1}^{n} \tilde{\kappa}_j^\dagger \tilde{\kappa}_i \Big(f\big(g_i g_j^{-1}\big) \Big) \geq 0 .$$

Then by property (c5), there exists $\tilde{\kappa}_\kappa \in (X')_{\mathbb{C}}$ such that $\kappa = \tilde{\kappa}_\kappa^\dagger \tilde{\kappa}_\kappa$. So we can choose $\tilde{\kappa}_i = z_i \tilde{\kappa}_\kappa$ for any $i \in \{1, \dots, n\}$, where $z_i \in \mathbb{C}$. Therefore by linearity we obtain

$$\sum_{i,j=1}^{n} \bar{z}_j z_i \kappa \Big(f\big(g_i g_j^{-1}\big) \Big) \geq 0 ;$$

and hence positive-definiteness of $\kappa\big(f(\cdot)\big)$.

The classical Bochner's theorem for locally compact abelian groups [see e.g. Loomis, 1953] implies the existence of a unique positive, finite measure μ_κ such that $\kappa\big(f(\cdot)\big) = \hat{\mu}_\kappa(\cdot)$. Therefore we have a unique family of positive and finite measures $(\mu_\kappa)_{\kappa \in C'}$. In order for it to define a unique finite C-valued measure, it is necessary that $\kappa \mapsto \mu_\kappa$ is additive. Let

$\kappa_1, \kappa_2 \in C'$. Then $\kappa_1+\kappa_2 \in C'$, and there is a unique measure $\mu_{\kappa_1+\kappa_2}$ such that $\hat{\mu}_{\kappa_1+\kappa_2}(\cdot) = (\kappa_1+\kappa_2)\big(f(\cdot)\big) = \kappa_1\big(f(\cdot)\big) + \kappa_2\big(f(\cdot)\big) = \hat{\mu}_{\kappa_1}(\cdot) + \hat{\mu}_{\kappa_2}(\cdot)$. However since the Fourier transform is a linear bijection, it follows that $\mu_{\kappa_1+\kappa_2} = \mu_{\kappa_1} + \mu_{\kappa_2}$. Hence by Theorem 2.3 we have defined a unique C-valued measure μ. In addition, by Definition 2.4 for any $\kappa \in C'$

$$\kappa\big(f(\cdot)\big) = \int_{\hat{G}} \gamma(\cdot) d\mu_\kappa(\gamma) = \kappa\left(\int_{\hat{G}} \gamma(\cdot) d\mu(\gamma)\right) .$$

Now by (c4), it follows that for any $\tilde{\kappa} \in (X')_{\mathbb{C}}$

$$\tilde{\kappa}\big(f(\cdot)\big) = \tilde{\kappa}\left(\int_{\hat{G}} \gamma(\cdot) d\mu(\gamma)\right) ,$$

and therefore by (c6) it follows that

$$f(\cdot) = \int_{\hat{G}} \gamma(\cdot) d\mu(\gamma) .$$

⊣

2.2.2. *Locally convex spaces.* Bochner's Theorem 2.13 can be applied to finite dimensional real vector spaces (seen as abelian groups under addition). In that context, the Fourier transform takes the following form. Let V be a finite dimensional vector space, V^* its topological dual. Given a C-valued measure on V, then its Fourier transform is a function from V^* to $X_{\mathbb{C}}$ defined by

$$\hat{\mu}(\omega) = \int_V e^{i\omega(v)} d\mu(v) .$$

Using a projective argument, we obtain a variant of Bochner's theorem for cylindrical measures on locally convex real vector spaces. Some basic definitions and notations are in order.

- Let L be a *locally convex real vector space*, L^* its topological dual.
- We denote by $F(L)$ the set of subspaces of L with finite codimension, ordered by inclusion.
- For any $\Lambda \in F(L)$, we denote by $p_\Lambda : L \to L/\Lambda$ the canonical projection.
- For any $\Lambda \supset \Xi \in F(L)$, we denote by $p_{\Lambda\Xi} : L/\Xi \to L/\Lambda$ the canonical map obtained quotienting the identity map of L.
- The family $Q(L) = (L/\Lambda, p_{\Lambda\Xi})_{\Lambda\supset\Xi\in F(L)}$ is a projective system of spaces indexed by $F(L)$ that we call the *projective system of finite dimensional quotients* of L.

Definition 2.14 (C-valued cylindrical measure)**.** *Let L be a locally convex real vector space, and (X, X', C) a triple satisfying* (c1)-(c2)*. A family of measures $M = (\mu_\Lambda)_{\Lambda\in F(L)}$ is a cylindrical measure iff it is a projective system of C-valued measures on $Q(L)$.*

In other words, the family $(\mu_\Lambda)_{\Lambda\in F(L)}$ satisfies:

- $\big(\forall\Lambda\in F(L)\big)$ μ_Λ *is a* C*-valued measure on* L/Λ*;*
- *Define for any* $b\in\mathcal{B}(L/\Lambda)$*;* $p^{-1}_{\Lambda\Xi}(b)=\{\xi\in L/\Xi, p_{\Lambda\Xi}(\xi)\in b\}$*, and* $p_{\Lambda\Xi}(\mu_\Xi)(b)=\mu_\Xi\big(p^{-1}_{\Lambda\Xi}(b)\big)$*. Then*

$$\big(\forall\Lambda\supset\Xi\in F(L)\big)\ \mu_\Lambda=p_{\Lambda\Xi}(\mu_\Xi)$$

Remark 2.15. The compatibility condition of Definition 2.14 implies that for any $\Lambda,\Xi\in F(L)$,

$$\mu_\Lambda(L/\Lambda)=\mu_\Xi(L/\Xi)=m\in C_\infty\ .$$

We call m the *total mass* of the cylindrical measure M. A cylindrical measure $M=(\mu_\Lambda)_{\Lambda\in F(L)}$ is *finite* if for any $\Lambda\in F(L)$, the measure μ_Λ is finite.

Every C-valued measure μ on L induces a cylindrical measure $M_\mu=(\mu_\Lambda)_{\Lambda\in F(L)}$ by

$$\big(\forall\Lambda\in F(L)\big)\ \mu_\Lambda=p_\Lambda(\mu)\ ;$$

where for any $b\in\mathcal{B}(L/\Lambda)$, $p_\Lambda(\mu)(b)=\mu\big(p_\Lambda^{-1}(b)\big)$. The compatibility condition is satisfied, and the total mass of M_μ equals the total mass $\mu(L)$ of μ [see Bourbaki, 1969, I IX.4.2 Théorème 1 for additional details]. On the other hand, for *finite dimensional* L any cylindrical measure $M=(\mu_\Lambda)_{\Lambda\in F(L)}$ induces a measure $\mu^{(M)}=\mu_{\{0\}}$.

We are almost ready to define the Fourier transform of cylindrical measures. In order to do that, we denote by $\Lambda^0\subset L^*$ the subspace orthogonal to Λ, i.e. $\Lambda^0=\{l^*\in L^*,(\forall\lambda\in\Lambda)l^*(\lambda)=0\}$. It is possible to identify $(L/\Lambda)^*$ and Λ^0 by means of $(p_\Lambda)^*$.

Definition 2.16 (Fourier transform of cylindrical measures)**.** *Let L be a locally convex space, (X,X',C) a triple satisfying (c1)-(c2), and $\mathcal{M}_{\mathrm{cyl}}(L,C)$ the set of finite C-valued cylindrical measures on L. The* Fourier transform *is a map* $\hat{\ }:\mathcal{M}_{\mathrm{cyl}}(L,C)\to(X_{\mathbb{C}})^{L^*}$*, defined by*

$$(\forall\lambda^0\in\Lambda^0)\ \hat{\mu}(\lambda^0)=\int_{L/\Lambda}e^{i\lambda^0(l)}d\mu_\Lambda(l)\ .$$

We remark that $L^*=\bigcup_{\Lambda\in F(L)}\Lambda^0$ and the consistency condition of Definition 2.14 ensure the above definition is consistent.

With the aid of Theorem 2.13 and a projective argument, it is possible to prove the following result.

Theorem 2.17 (Bochner for cylindrical measures)**.** *Let L be a locally convex space, (X,X',C) a triple satisfying (c1)-(c6). The Fourier transform is a bijection between finite C-valued cylindrical measures on L and completely positive functions from L^* to $X_{\mathbb{C}}$ that are ultraweakly continuous when restricted to any finite dimensional subspace of L^*.*

2.3. **Tightness.** It is useful to have criteria to check whether a given finite cylindrical measure is in fact a (Radon) measure. We follow Bourbaki [1965, 1967, 1969], and introduce the following definitions.

- Let A be a locally compact space. We denote by $\mathscr{F}^+(A) \subset (\mathbb{R}^+_\infty)^A$ the subset of positive functions, and by $\mathscr{K}^+(A) \subset \mathbb{R}^A$ the subset of continuous functions of compact support.

- Let A be a locally compact topological space, (X, X', C) a triple satisfying (c1)-(c2). For any positive finite C-valued measure μ on A, and any function $f \in \mathscr{F}^+(A)$, we define the induced inner measure by

$$\mu^\bullet(f) = \Bigl(\kappa \mapsto \sup_{\substack{K\subset A\\ K \text{ compact}}} \sup_{g\in\mathscr{K}^+, g\leq f} \int_K g(x) d\mu_\kappa(x)\Bigr) .$$

  Given a set $a \subset A$, we define $\mu^\bullet(a) = \mu^\bullet(\chi_a)$, where χ_a is the characteristic function of a. If $\mu^\bullet(f) \in \mathfrak{i}_C(C)$, we define

$$\int_A^\bullet f(x) d\mu(x) = \mathfrak{i}_C^{-1}\bigl(\mu^\bullet(f)\bigr) .$$

Theorem 2.18 (Prokhorov)**.** *Let L be a locally convex real vector space, and (X, X', C) a triple satisfying (c1)-(c2). Given a cylindrical measure $M = (\mu_\Lambda)_{\Lambda\in F(L)}$ there exists a unique Radon C-valued measure μ on L such that for any $\Lambda \in F(L)$, $\mu_\Lambda = p_\Lambda(\mu)$ iff*

$$\text{(P)} \qquad \bigl(\forall\varepsilon > 0\bigr)\bigl(\exists K \subset L, K \text{ compact}\bigr)\bigl(\forall\Lambda \in F(L)\bigr)(\forall\kappa \in C') \\ \mu^\bullet_{\Lambda,\kappa}\bigl(L/\Lambda \setminus p_\Lambda(K)\bigr) \leq \varepsilon .$$

In addition,

$$\mu^\bullet(K) = \inf_{\Lambda\in F(L)} \mu^\bullet_\Lambda\bigl(p_\Lambda(K)\bigr) .$$

Proof. By [Bourbaki, 1969, I IX.4.2, Théorème 1], (P) holds iff for any $\kappa \in C'$, there exists a unique positive Radon measure μ_κ such that for any $\Lambda \in F(L)$, $\mu_{\Lambda,\kappa} = p_\Lambda(\mu_\kappa)$. Uniqueness of the measure, and linearity of p_Λ also ensure the additivity of $\kappa \mapsto \mu_\kappa$, as in the proof of Theorem 2.13. The last statement also follows from the analogous statements for $\mu^\bullet_\kappa(K)$. ⊣

If L is a real separable Hilbert space, Theorem 2.17 takes a simpler form. We remark that in this context, for any $\Lambda \in F(L)$ we can identify L/Λ with a finite dimensional Hilbert subspace of L, and p_Λ with the orthogonal projector from L onto L/Λ. To this extent, we denote by $\mathbb{F}(\mathscr{H})$ the finite dimensional subspaces of a separable real Hilbert space $\mathscr{H}$, and for any $h \in \mathbb{F}(\mathscr{H})$, by $\mathbb{P}_h$ the corresponding orthogonal projection.

Theorem 2.19. *Let $\mathscr{H}$ be a real separable Hilbert space, and (X, X', C) a triple satisfying (c1)-(c2). In addition, let $B_{\mathscr{H}}(r)$ be the ball of radius r in $\mathscr{H}$. Given a* finite *cylindrical measure $M = (\mu_h)_{h\in\mathbb{F}(\mathscr{H})}$ there exists a unique* finite *Radon C-valued measure μ on $\mathscr{H}$ such that for any $h \in$*

$\mathbb{F}(\mathscr{H})$, $\mu_h = \mathbb{P}_h(\mu)$ *iff*

$$(\mathrm{P}_{\mathscr{H}_s}) \quad \begin{aligned} &\big(\forall\varepsilon > 0\big)\big(\exists r > 0\big)\big(\forall h \in \mathbb{F}(\mathscr{H})\big)\big(\forall\kappa \in C'\big)\;\; r > s \Rightarrow \\ &\qquad\qquad \mu_{h,\kappa}\Big(h \setminus \mathbb{P}_h\big(B_{\mathscr{H}}(r)\big)\Big) \leq \varepsilon \end{aligned}$$

Proof. The proof is analogous to the one of Theorem 2.17, using the corresponding result for positive finite cylindrical measures [see Skorohod, 1974]. ⊣

2.4. **Signed and complex vector measures.** As in the scalar case, it is possible to introduce signed and complex vector measures.

- A *Riesz space* $(V, \leq)$ is a partially ordered real vector space such that:
  - $x \leq y \;\Rightarrow\; (\forall\, z \in B)\, x + z \leq y + z$;
  - $0 \leq x \;\Rightarrow\; (\forall\, 0 < \lambda \in \mathbb{R})\, 0 \leq \lambda x$;
  - $(\forall \{x, y\} \subset B)\, (\exists\, x \vee y \in B)\, x \leq x \vee y\,;\, y \leq x \vee y\,;\, (\forall u \in B)\, (x \leq u\,;\, y \leq u) \Rightarrow x \vee y \leq u$ ($x \vee y$ is called the *supremum* of $\{x, y\}$, and analogously it is possible to define the *infimum* $x \wedge y$).
- Given a convex pointed cone C of a real vector space X, we define the relation $\leq_C$ by
$$x \leq_C y \text{ iff } y - x \in C\ .$$
If for any $\{x, y\} \subset X$, there exist the supremum $x \vee_C y$ with respect to the partial order $\leq_C$, then $(X, \leq_C)$ is a Riesz space. In this case, we say that C is a *lattice cone* of X. Every pointed and generating cone is a lattice cone, for if C is generating there exist for any $x \in X$ the positive and negative part with respect to $\leq_C$.
- The extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$ is not an additive monoid, since $+\infty - \infty$ is not defined. However both $(-\infty, +\infty]$ and $[-\infty, +\infty)$ are additive monoids.
- We define $X_\infty = \mathrm{Hom}_{\mathrm{mon}}(C', \mathbb{R} \cup \{-\infty, +\infty\})$ as the subset of functions $f \in (\mathbb{R} \cup \{-\infty, +\infty\})^{C'}$ satisfying the following properties:
  - If $\pm\infty \in \mathrm{Ran}\, f$, then $\mp\infty \notin \mathrm{Ran}\, f$;
  - $f : C' \to \mathrm{Ran}\, f$ is a monoid homomorphism.

  This definition is justified by the fact that since $+\infty - \infty$ is not defined, signed measures may only take either $+\infty$ or $-\infty$ as a value (in order to be additive). This has also to be the case for signed vector measures, and therefore they will have X_∞ as target space, see Definition 2.20 below.
- $(X_{\mathbb{C}})_\infty = \mathrm{Hom}_{\mathrm{mon}}(C', \mathbb{C} \cup \{\infty\})$.

Let's consider the extension to vector measures of the concept of signed measures. This is easily done by means of X_∞ defined above.

Definition 2.20 (Signed vector measures)**.** *Let (X, X', C) be a triple that satisfies (c1)-(c2), E a topological space. A function $\mu \in (X_\infty)^{\mathcal{B}(E)}$ is a Borel signed vector measure on E iff it is countably additive and $\mu(\emptyset) = 0$.*

The following useful lemma follows directly from the definition of signed measures.

Lemma 2.21. *Every C-valued measure is also a signed measure. Any real linear combination of two C-valued measures is a signed measure, provided at least one of the two measures is finite.*

The important Theorem 2.3 can be easily adapted to hold for signed measures as well.

Theorem 2.22. *There is a bijection between signed vector measures μ on E and families of Borel signed measures $(\mu_\kappa)_{\kappa\in C'}$ on E such that for any $b \in \mathcal{B}(E)$, $\big(\kappa \mapsto \mu_\kappa(b)\big) \in X_\infty$.*

A signed measure μ is *finite* iff for any $\kappa \in C'$, μ_κ is finite. The idea behind signed vector measures is that, as in the case of standard measures, they are the sum of two cone-valued measures. Therefore it is reasonable to define them as a collection indexed only by the dual cone C', in order to prevent possible "sign incongruences" on μ_κ due to the action of a $\kappa \notin C'$. As a matter of fact, with this definition we can indeed prove the existence of a unique Jordan decomposition for signed vector measures. The precise statement is contained in the following result.

Theorem 2.23. *Let (X, X', C) be a triple satisfying (c1)-(c2); and μ a signed vector measure on a topological space E. Then there exist three C-valued measures $\mu^+, \mu^-, |\mu|$ such that:*

- *$\mu = \mu^+ - \mu^-$, and the decomposition is unique;*
- *$|\mu| = \mu^+ + \mu^-$;*
- *At least one between μ^+ and μ^- is finite;*
- *μ is finite iff $|\mu|$ is finite.*

In addition, $\mu^+ = \mu \vee_C 0$, $\mu^- = \mu \wedge_C 0$ and $|\mu| = |\mu|_C$. The operations $+$, $-$, $\vee_C$, $\wedge_C$ and $|\cdot|_C$ on measures are defined pointwise on Borel sets, and 0 is the measure identically zero.

Proof. Let μ be a signed vector measure. Then $(\mu_\kappa)_{\kappa\in C'}$ is the corresponding family of signed measures. By Jordan decomposition of signed measures, for any $\kappa \in C'$, there exist a unique decomposition $\mu_\kappa = \mu_\kappa^+ - \mu_\kappa^-$, with μ_κ^+ and μ_κ^- positive measures with at least one of the two finite, and μ_κ is finite iff $|\mu_\kappa|$ is finite. Hence if $(|\mu_k|)_{\kappa\in C'}$ is the image of a C-valued measure $|\mu|$, μ is finite iff $|\mu|$ is finite. In addition, suppose that there exists a $\tilde{\kappa} \in C'$ such that $\mu_{\tilde{\kappa}}^+$ is not finite. Then $+\infty \in \operatorname{Ran}\mu$, and therefore $-\infty \notin \operatorname{Ran}\mu$, i.e. for any $\kappa \in C'$, μ_κ^- is finite. It follows that if

$(\mu^-_\kappa)_{\kappa\in C'}$ is the image of a C-valued measure, such measure is finite. An analogous statement holds with plus replaced by minus. By Lemma 2.21, to prove the first part of the theorem it remains only to check that the families $(\mu^+_\kappa)_{\kappa\in C'}$ and $(\mu^-_\kappa)_{\kappa\in C'}$ are C-valued measures, i.e. that for any $b\in\mathcal{B}(E)$, the maps $\kappa\mapsto\mu^\pm_\kappa(b)$ are monoid morphisms. On one hand, we have by the fact that $\mu\in X_\infty$ and then Jordan decomposition that

$$\mu_{\kappa_1+\kappa_2}(b)=\mu_{\kappa_1}(b)+\mu_{\kappa_2}(b)=\mu^+_{\kappa_1}(b)+\mu^+_{\kappa_2}(b)-\bigl(\mu^-_{\kappa_1}(b)+\mu^-_{\kappa_2}(b)\bigr)\ ;$$

on the other hand, by Jordan decomposition we have also that

$$\mu_{\kappa_1+\kappa_2}(b)=\mu^+_{\kappa_1+\kappa_2}(b)-\mu^-_{\kappa_1+\kappa_2}(b)\ .$$

Now since the decomposition is unique, it follows that

$$\mu^\pm_{\kappa_1+\kappa_2}(b)=\mu^\pm_{\kappa_1}(b)+\mu^\pm_{\kappa_2}(b)\ ,$$

i.e. the map is a monoid morphism.

To prove the last part, let $\mu=\mu^+-\mu^-$ be a vector signed measure with the respective decomposition. Then for any $b\in\mathcal{B}(E)$, we have that

$$X\ni\mu(b)=\mu^+(b)-\mu^-(b)\ ;\ \mu^+(b),\mu^-(b)\geq_C 0\ .$$

Since C is pointed and generating, $(X,\leq_C)$ is a Riesz space and the decomposition in positive and negative parts is unique. Then it follows that $\mu^+=\mu\vee_C 0$, $\mu^-=\mu\wedge_C 0$ and therefore $|\mu|=|\mu|_C$. ⊣

The complex vector measures are defined in an analogous fashion, and they are the sum of four C-valued measures. We quickly mention the basic definitions and results without proof, for they are equivalent to the ones for signed vector measures.

Definition 2.24 (Complex vector measures)**.** *Let (X,X',C) be a triple that satisfies (c1)-(c2), E a topological space. A function $\mu\in\bigl((X_{\mathbb{C}})_\infty\bigr)^{\mathcal{B}(E)}$ is a Borel complex vector measure on E iff it is countably additive and $\mu(\emptyset)=0$.*

Lemma 2.25. *Under the identifications $\mathbb{R}\ni\alpha\to\alpha+i0$, $+\infty\to\infty$, $-\infty\to\infty$; every signed vector measure is also a complex measure. Any complex linear combination of two signed measures is a complex measure.*

Theorem 2.26. *Let (X,X',C) be a triple satisfying (c1)-(c2); and μ a complex vector measure on a topological space E. Then there exist five C-valued measures $\mu^+_{\mathrm{R}},\mu^-_{\mathrm{R}},\mu^+_{\mathrm{I}},\mu^-_{\mathrm{I}},|\mu|$ such that:*

- *$\mu=\mu^+_{\mathrm{R}}-\mu^-_{\mathrm{R}}+i(\mu^+_{\mathrm{I}}-\mu^-_{\mathrm{I}})$, and the decomposition is unique;*
- *$|\mu|=\mu^+_{\mathrm{R}}+\mu^-_{\mathrm{R}}+\mu^+_{\mathrm{I}}+\mu^-_{\mathrm{I}}$;*
- *At least one between μ^+_{R} and μ^-_{R}, and one between μ^+_{I} and μ^-_{I} are finite;*
- *μ is finite iff $|\mu|$ is finite, or equivalently if $\mu^+_{\mathrm{R}},\mu^-_{\mathrm{R}},\mu^+_{\mathrm{I}},\mu^-_{\mathrm{I}}$ are all finite.*

Corollary 2.27. *The integral with respect to a* finite *complex vector measure* μ *is a map* $\int_{(\cdot)} d\mu : \mathcal{B}(E) \to X_{\mathbb{C}}$ *defined by*

$$\int_b d\mu = \int_b d\mu_{\mathrm{R}}^+ - \int_b d\mu_{\mathrm{R}}^- + i\Big(\int_b d\mu_{\mathrm{I}}^+ - \int_b d\mu_{\mathrm{I}}^-\Big) .$$

2.5. **A concrete realization: duals of C*-algebras.** In this subsection, we discuss a relevant class of triples satisfying the properties (c1)-(c6). The standard results are recalled without proof.

The triples we consider are related to C*-algebras. They will be crucial in the following, for they are related to the commutative limit points of nets of Weyl noncommutative measures.

- Given a C*-algebra $\mathfrak{A}$, we denote by $\mathfrak{A}_+$ the set of elements with positive spectrum, and by $\mathfrak{A}_\dagger$ the set of self-adjoint elements.
- If $\mathfrak{A}_\dagger^*$ is the continuous dual of the set of self-adjoint elements $\mathfrak{A}_\dagger$ of a C*-algebra $\mathfrak{A}$, we denote by $\mathfrak{A}_+^*$ the functionals that are positive when acting on $\mathfrak{A}_+$.

In order to verify conditions (c1)-(c6), we make use of the following classical result [see e.g. Takesaki, 1979].

Theorem 2.28. *Let* $\mathfrak{A}$ *be a* C^**-algebra. Then:*

- $\mathfrak{A}_\dagger$ *is a real Banach subspace of* $\mathfrak{A}$ *and* $\mathfrak{A} = \mathfrak{A}_\dagger + i\mathfrak{A}_\dagger$.
- $\mathfrak{A}_+$ *is a closed, pointed and generating convex cone of* $\mathfrak{A}_\dagger$.
- $\mathfrak{A}_+^*$ *is a pointed and generating convex cone of* $\mathfrak{A}_\dagger^*$*; in particular for any* $\alpha \in \mathfrak{A}_\dagger^*$ *there is a unique decomposition*
$$\alpha = \alpha^+ - \alpha^- \ , \quad \text{with } \alpha^+, \alpha^- \in \mathfrak{A}_+^* \ .$$
- $(\mathfrak{A}_\dagger^*)_{\mathbb{C}} = \mathfrak{A}^*$.

By means of Theorem 2.28, conditions (c1), (c3)-(c6) are immediately proved. Condition (c2) is proved using a remarkable result of Neeb [1998, Lemma I.5]. In fact, if we call C_1' the set of elements of C' with $\mathfrak{A}_\dagger$-norm one, then $C_1' - C_1'$ is a 0-neighbourhood of $\mathfrak{A}_\dagger$. In Section 3, the $\mathfrak{A}_+^*$-valued measures play an important role; from the discussion above it follows that all the results of Section 2 can be freely used there. For later reference, the result is written explicitly as a theorem.

Theorem 2.29. *Let* $\mathfrak{A}$ *be a* C^**-algebra. Then the triple* $(\mathfrak{A}_\dagger^*, \mathfrak{A}_\dagger, \mathfrak{A}_+^*)$ *satisfies* (c1)-(c6)*.*

3. LIMIT POINTS OF NETS OF NONCOMMUTATIVE MEASURES.

In this section we study the limit points of suitable equicontinuous nets of noncommutative measures in deformations of Weyl algebras. As explained in the introduction, such deformations and nets emerge in many branches of analysis and mathematical physics such as microlocal analysis, semiclassical analysis, and the rigorous study of classical and mean field effective behavior of bosonic quantum systems. They can also be seen as families of noncommutative measures that behave as classical measures in the limit; therefore they provide a link between the theories of noncommutative and classical integration.

3.1. **The noncommutative setting.** We begin introducing the Weyl algebra and some of its properties we will exploit the most. To avoid confusion with the operation of taking topological duals, we denote by † the involution of a $*$-algebra. For additional basic definitions and results concerning C*-algebras, refer to Section 2.5 and references thereof contained.

Definition 3.1 (Weyl algebra)**.** *Let (V,σ) be a couple consisting of a real vector space and a non-degenerate antisymmetric bilinear form. Then the* Weyl algebra $\mathrm{Weyl}(V,\sigma)$ *is the C^*-algebra generated by the set of elements*

$$\{W(v), v \in V\}$$

that satisfies the following properties:

- $W(v) \neq 0$ *for any* $v \in V$*;*
- $W(-v) = W(v)^\dagger$ *for any* $v \in V$*;*
- $W(v)W(w) = e^{-i\sigma(v,w)}W(v+w)$ *for any* $v, w \in V$*.*

The definition is well posed because of the following result.

Theorem 3.2 (Slawny [1972])**.** *Given a couple (V,σ) the Weyl algebra* $\mathrm{Weyl}(V,\sigma)$ *is uniquely determined up to $*$-isomorphisms.*

Any Weyl algebra is a unital noncommutative C*-algebra generated by unitary elements. These properties can be easily derived from Definition 3.1.

Let us introduce a real positive deformation parameter $h \geq 0$ measuring the "degree of non-commutativity" of a Weyl algebra.

Definition 3.3 (Weyl deformation)**.** *Let (V,σ) be a couple consisting of a topological real vector space and a non-degenerate antisymmetric bilinear form (symplectic form). Then the* Weyl deformation $\big(\mathrm{Weyl}_h(V,\sigma)\big)_{h\geq 0}$ *is a family of C^*-algebras where* $\mathrm{Weyl}_h(V,\sigma)$ *is generated by the set of elements*

$$\{W_h(v), v \in V\}$$

that has the following properties:

- $W_h(v) \neq 0$ *for any* $v \in V$*;*
- $W_h(-v) = W_h(v)^\dagger$ *for any* $v \in V$*;*
- $W_h(v)W_h(w) = e^{-ih\sigma(v,w)}W_h(v+w)$ *for any* $v, w \in V$*.*

It is clear that for any $h > 0$, the algebra $\mathrm{Weyl}_h(V,\sigma)$ is $*$-isomorphic to $\mathrm{Weyl}(V,\sigma)$ since we can identify the generators in the following way: $W_h(v) = W(h^{1/2}v)$. For $h = 0$, the algebra $\mathrm{Weyl}_0(V,\sigma)$ is the abelian unital C*-algebra of almost periodic functions of V (seen as an additive abelian group) [see H. Bohr 1947; von Neumann 1934; Bochner and von

Neumann 1935, for additional details]. Essentially, the Weyl deformations $\mathrm{Weyl}_h(V,\sigma)$ contain infinitely many identical copies of the Weyl algebras $\mathrm{Weyl}(V,\sigma)$, and a single commutative C*-algebra of almost periodic functions. It is therefore interesting to study the apparent discontinuity following descending nets $h_\beta \to 0$ of the deformation parameter, in a suitable sense. As a starting point, in this work we focus on the study of nets of noncommutative measures. In fact, if we interpret each element $\mathrm{Weyl}_h(V,\sigma)$ of the deformation as an algebra of random variables of a probability theory, it is natural to define the associated noncommutative probabilities as the norm one elements of $\bigl(\mathrm{Weyl}_h(V,\sigma)\bigr)^*_+$, and more generally the (finite) noncommutative positive measures as the elements of $\bigl(\mathrm{Weyl}_h(V,\sigma)\bigr)^*_+$. Since for a C*-algebra $\mathfrak{X}$, it is possible to decompose the dual as $\mathfrak{X}^* = \mathfrak{X}^*_+ - \mathfrak{X}^*_+ + i(\mathfrak{X}^*_+ - \mathfrak{X}^*_+)$, it is sufficient to characterize only positive measures.

As discussed in the introduction, we consider the following more general setting. Let (V,σ) be a real vector space with a symplectic form, and $\bigl(\mathrm{Weyl}_h(V,\sigma)\bigr)_{h\geq 0}$ the corresponding Weyl deformation; and let $\mathfrak{A}$ be a C*-algebra. We consider the tensor product deformation

$$(\mathfrak{W}_h)_{h\geq 0} = \bigl(\mathrm{Weyl}_h(V,\sigma) \otimes_{\gamma_h} \mathfrak{A}\bigr)_{h\geq 0} \; ;$$

where the index γ_h stands for a suitable choice of cross-norm for the tensor product C*-algebra [see e.g. Takesaki, 1979]. For the sake of simplicity, we consider the same choice γ_h for any $h > 0$. We remark that in applications, it is sometimes important to consider the enveloping von Neumann algebra $\mathrm{Weyl}_h(V,\sigma)''$ in place of $\mathrm{Weyl}_h(V,\sigma)$. The majority of our results extend to the deformation $\bigl(\mathrm{Weyl}_h(V,\sigma)'' \otimes_{\gamma_h} \mathfrak{A}\bigr)_{h\geq 0}$ or to any deformation $(\mathfrak{V}_h \otimes_{\gamma_h} \mathfrak{A})_{h\geq 0}$ such that each $\mathrm{Weyl}_h(V,\sigma)$ is a subalgebra of $\mathfrak{V}_h$. It will be pointed out explicitly in the text when a result do not extend to the aforementioned situations. Finally, if one is interested only in the deformation $\bigl(\mathrm{Weyl}_h(V,\sigma)\bigr)_{h\geq 0}$ – in other words in the complete classical limit – it suffices to take $\mathfrak{A}$ to be the trivial C*-algebra generated by a single element.

3.2. **Partial evaluation.** On each tensor algebra $\mathfrak{W}_h$ defined above, there is a natural map that plays an important role, and we call it partial evaluation.

Definition 3.4 (Partial evaluation)**.** *Let (V,σ) be a real vector space with a symplectic form; $\mathfrak{A}$ a C*-algebra; $(\mathfrak{W}_h)_{h\geq 0} = \bigl(\mathrm{Weyl}_h(V,\sigma) \otimes_{\gamma_h} \mathfrak{A}\bigr)_{h\geq 0}$ a corresponding Weyl tensor deformation. For any $h \geq 0$, we define the* partial evaluation map

$$\mathbb{E}^{(\cdot)}_{h,1} : \mathfrak{W}^*_h \to \mathcal{L}\bigl(\mathrm{Weyl}_h(V,\sigma), \mathfrak{A}^*\bigr) \; ,$$

by its action

$$\mathbb{E}^{\Omega_h}_{h,1}(w_h)(a) = \Omega_h(w_h \otimes a) \; ,$$

for any $\Omega_h \in \mathfrak{W}^*_h$, $w_h \in \mathrm{Weyl}_h(V,\sigma)$, and $a \in \mathfrak{A}$. We also define the partial trace *of the (complex) measure $\Omega_h \in \mathfrak{W}^*_h$ as $\mathbb{E}^{\Omega_h}_{h,1}(1)$.*

In the definition above, we have used the notation $\mathcal{L}(X,Y)$ for the continuous linear operators from X to Y; and we have stressed the dependence on the deformation parameter h for it will be important in the following. The partial evaluation does what it is supposed to: given a measure on the tensor algebra, it evaluates any random variable of the first algebra and has as output a (complex) measure on the second algebra. The evaluation of any random variable of the second algebra then gives the same value as evaluating the tensor product of the random variables in the original measure. Of course, we can also define the partial evaluation $\mathbb{E}_{h,2}$ in the same fashion, but since our deformation does not involve $\mathfrak{A}$, we will use $\mathbb{E}_{h,1}$ the most.

The partial evaluation map has some important properties that are summarized in the following proposition [see e.g. Takesaki, 1979, for a proof].

Proposition 3.5. *For any $h \geq 0$, the evaluation map $\mathbb{E}_{h,1}$ is an isometry of $\mathfrak{W}_h^*$ onto $\mathcal{L}\bigl(\mathrm{Weyl}_h(V,\sigma),\mathfrak{A}^*\bigr)$. In addition, an element $\Omega_h \in \mathfrak{W}_h^*$ is a positive measure of total mass m_h – i.e. $\Omega_h \in (\mathfrak{W}_h)_+^*$ and $\lVert\Omega_h\rVert_{\mathfrak{W}_h^*} = m_h$ – iff the resulting evaluation $\mathbb{E}_{h,1}^{\Omega_h} : \mathrm{Weyl}_h(V,\sigma) \to \mathfrak{A}^*$ is completely positive and the partial trace $\mathbb{E}_{h,1}^{\Omega_h}(1) \in \mathfrak{A}_+^*$ satisfies*

$$\lVert\mathbb{E}_{h,1}^{\Omega_h}(1)\rVert_{\mathfrak{A}^*} = m_h \; .$$

3.3. **The generating map and regular measures.** Given a measure on the Weyl algebra, it is possible to define its generating functional [see Segal, 1961]; in our framework it is not a functional, but a map from V to $\mathfrak{A}^*$. Throughout this section, we take $h > 0$ if not specified otherwise.

Definition 3.6 (Generating map)**.** *Let $\Omega_h \in (\mathfrak{W}_h)_+^*$ be a (positive) measure, we define the* generating map $\mathcal{G}_{\Omega_h} : V \to \mathfrak{A}^*$ *by*

$$\mathcal{G}_{\Omega_h}(v) = \mathbb{E}_{h,1}^{\Omega_h}\bigl(W_h(v)\bigr) \; , \; v \in V \; .$$

The generating map is used to define a very important class of measures (and hence its name), the so-called regular measures. As it will become clearer in the following, nets of regular measures are the good choice in order to be sure to have commutative measures as limit points. They are defined as follows.

Definition 3.7 (Regular measures)**.** *Let $\Omega_h \in (\mathfrak{W}_h)_+^*$ be a (positive) measure, $\mathcal{G}_{\Omega_h}$ its generating map. Then Ω_h is* regular *iff for any $v \in V$, the map $\mathcal{G}_{\Omega_h}(\,\cdot\, v) : \mathbb{R} \to \mathfrak{A}^*$ is continuous when $\mathfrak{A}^*$ is endowed with the ultraweak topology (ultraweakly continuous).*

There are many equivalent definitions of regular measures. We will make use also of the following, that can be proved e.g. using the properties of map $\mathbb{E}_{h,2}$ and the equivalent result for trivial $\mathfrak{A}$ [see Bratteli and Robinson, 1997, Section 5.2.3]. Let R be a finite dimensional real vector space. We say that a positive measure ω_h on $\mathrm{Weyl}_h(R,\sigma)\otimes_{\gamma_h}\mathfrak{A}$ is Fock-normal iff for any $a \in \mathfrak{A}_+$, $\mathbb{E}_{h,2}^{\omega_h}(a)$ is represented as a (positive) trace class operator in the unique irreducible representation of $\mathrm{Weyl}_h(R,\sigma)$ (its uniqueness up to unitary equivalence is guaranteed by Stone-von Neumann's theorem).

Proposition 3.8. *Let $\Omega_h \in (\mathfrak{W}_h)^*_+$ be a (positive) measure. Then Ω_h is regular iff for any finite dimensional $R \subset V$ its restriction ω_h to $\mathrm{Weyl}_h(R,\sigma) \otimes_{\gamma_h} \mathfrak{A}$ is a Fock-normal measure.*

In particular, it follows that the generating map of a regular measure is ultraweakly continuous when restricted to finite dimensional subspaces of V.

The following result is an extension to our setting of the main result of the aforementioned paper of Segal [1961]. The idea is that regular measures are uniquely determined by the generating map, and the latter is "almost" completely positive (up to a complex phase factor) and ultraweakly continuous on finite dimensional subspaces.

Proposition 3.9. *Let (V,σ) be a real vector space with a symplectic form, $\mathfrak{A}$ a C^*-algebra and $(\mathfrak{W}_h)_{h\geq 0}$ a corresponding tensor Weyl deformation. Then for any $h>0$, a map $\mathcal{G}_h : V \to \mathfrak{A}^*$ is the generating map of a regular measure $\Omega_h \in (\mathfrak{W}_h)^*_+$ of partial trace $\alpha_h \in \mathfrak{A}^*_+$ iff all the restrictions of $\mathcal{G}_h$ to finite dimensional subspaces of V are ultraweakly continuous, $\mathcal{G}_h(0) = \alpha_h$ and*

$$\sum_{j,k\in F} \mathcal{G}_h(v_j - v_k) e^{ih\sigma(v_j,v_k)} (a_k^\dagger a_j) \geq 0\;;$$

where the $v_j \in V$ are arbitrary as well as the $a_j \in \mathfrak{A}$, and F is any finite index set. The map $\mathcal{G}_h$ uniquely determines Ω_h.

Remark 3.10. If in $(\mathfrak{W}_h)_{h\geq 0}$ we replace $\mathrm{Weyl}_h(V,\sigma)$ by its enveloping von Neumann algebra or any algebra that contains the Weyl algebra as a subalgebra, $\mathcal{G}_h$ does not determine Ω_h uniquely.

Proof. Let's start with the easy "only if" part. Ultraweak continuity follows from Proposition 3.9, the other two properties follow from Proposition 3.5: in fact $W_h(0) = 1$;

$$\sum_{j,k\in F} \mathbb{E}^{\Omega_h}_{h,1}\bigl(W_h(v_k)^\dagger W_h(v_j)\bigr)(a_k^\dagger a_j) \geq 0$$

by complete positivity of $\mathbb{E}^{\Omega_h}_{h,1}$; and $W_h(-v)W_h(w) = e^{ih\sigma(v,w)} W_h(w-v)$ by definition of the Weyl algebra. To prove the "if" part and uniqueness, we act with the generating map on an arbitrary $a \in \mathfrak{A}_+$. Since $\mathfrak{A} = \mathfrak{A}_+ - \mathfrak{A}_+ + i(\mathfrak{A}_+ - \mathfrak{A}_+)$, this suffices to characterize the map $\mathcal{G}_h : V \to \mathfrak{A}^*$ by linearity. Let's denote by $\mathcal{G}^a_h(\cdot) = \mathcal{G}_h(\cdot)(a) : V \to \mathbb{C}$. By Theorem 1 of [Segal, 1961], to $\mathcal{G}^a_h$ corresponds a unique regular measure $\omega^a_h \in \bigl(\mathrm{Weyl}_h(V,\sigma)\bigr)^*_+$ such that $\mathcal{G}^a_h(\cdot) = \omega^a_h\bigl(W_h(\cdot)\bigr)$. By the last property of $\mathcal{G}_h$ this defines a unique completely positive map $\omega^{(\cdot)}_h : \mathfrak{A} \to \bigl(\mathrm{Weyl}_h(V,\sigma)\bigr)^*$. Therefore the analogous of Proposition 3.5 for $\mathbb{E}_{h,2}$ yields that $\Omega_h = \mathbb{E}^{-1}_{h,2}(\omega^{(\cdot)}_h)$ is a positive regular measure of total mass $\mathcal{G}_h(0)$, uniquely determined by $\mathcal{G}_h$. ⊣

3.4. **Compactness.** In this section we discuss compactness properties of nets of generating maps associated to nets of regular measures. Let $(\mathfrak{W}_h)_{h\geq 0}$ be a tensor Weyl deformation. We are interested in nets of measures $(\Omega_{h_\beta})_{\beta\in B}$ such that $h_\beta \neq 0$ for any $\beta \in B$, $h_\beta \to 0$ and $\Omega_{h_\beta} \in (\mathfrak{W}_{h_\beta})^*_+$ regular for any $\beta \in B$. Let's denote by $G_\Omega \subset (\mathfrak{A}^*_{\mathrm{uw}})^V$ and $G_\Omega(v) \subset \mathfrak{A}^*_{\mathrm{uw}}$ the following sets:

$$G_\Omega = \{\mathcal{G}_{\Omega_{h_\beta}}, \beta \in B\} \; ; \; G_\Omega(v) = \{\mathcal{G}_{\Omega_{h_\beta}}(v), \beta \in B\} \; , \; v \in V \; ;$$

where $\mathfrak{A}^*_{\mathrm{uw}}$ is the space $\mathfrak{A}^*$ endowed with the ultraweak topology. The first result is that the family of images of a given point is pointwise compact, provided the total masses of $(\Omega_{h_\beta})_{\beta\in B}$ are bounded.

Lemma 3.11. *Let $(\Omega_{h_\beta})_{\beta\in B}$ be a net of measures in the Weyl tensor deformation. If there exists $m > 0$ such that*

$$\sup_{\beta\in B} \|\Omega_{h_\beta}\|_{\mathfrak{W}^*_{h_\beta}} = m \; ,$$

then $G_\Omega(v)$ is precompact for any $v \in V$. It then follows that G_Ω is precompact as a subset of $\mathcal{F}_{\mathrm{s}}(V, \mathfrak{A}^*_{\mathrm{uw}})$, the space of functions in $(\mathfrak{A}^*_{\mathrm{uw}})^V$ endowed with the uniform structure of simple convergence.

Proof. It follows from Definition 3.6 of the generating map – since the Weyl operators are unitary – that for any $v \in V$, $\beta \in B$ and $a \in \mathfrak{A}$

$$|\mathcal{G}_{\Omega_{h_\beta}}(v)(a)| \leq \|\Omega_{h_\beta}\|_{\mathfrak{W}^*_{h_\beta}} \|a\|_{\mathfrak{A}} \leq m\|a\|_{\mathfrak{A}} \; .$$

Therefore $G_\Omega(v)$ is contained in the ball of radius m of $\mathfrak{A}^*$, and therefore it is precompact in the ultraweak topology by Banach-Alaoglu's theorem. ⊣

Let $R \subseteq V$ be a finite dimensional subspace of V. Then we define the set $G_\Omega\big|_R \subset (\mathfrak{A}^*_{\mathrm{uw}})^R$ by

$$G_\Omega\big|_R = \{\mathcal{G}_{\Omega_{h_\beta}}\big|_R, \beta \in B\} \; .$$

For the next result we make use of a classical result of microlocal and semiclassical analysis for the so-called Wigner measures [see e.g. Lions and Paul, 1993; Tartar, 1990; Helffer et al., 1987; Gérard, 1991b, for additional details]. It is a remarkable consequence of those results that each $G_\Omega\big|_R$ is equicontinuous.

Lemma 3.12. *Let $(\Omega_{h_\beta})_{\beta\in B}$ be a net of regular measures in the Weyl tensor deformation. If there exists $m > 0$ such that*

$$\sup_{\beta\in B} \|\Omega_{h_\beta}\|_{\mathfrak{W}^*_{h_\beta}} = m \; ,$$

then $G_\Omega\big|_R$ is equicontinuous for any finite dimensional $R \subseteq V$.

Proof. Let $a \in \mathfrak{A}$; then $a = a^+_{\mathrm{R}} - a^-_{\mathrm{R}} + i(a^+_{\mathrm{I}} - a^-_{\mathrm{I}})$, with $a^+_{\mathrm{R}}, a^-_{\mathrm{R}}, a^+_{\mathrm{I}}, a^-_{\mathrm{I}} \in \mathfrak{A}_+$. If we define

$$(\mathbb{C})^R \supset G^a_\Omega\big|_R = \{\mathcal{G}_{\Omega_{h_\beta}}(\cdot)(a)\big|_R, \beta \in B\} \; ,$$

then by linearity $G_\Omega^a\big|_R = G_\Omega^{a_\mathrm{R}^+}\big|_R - G_\Omega^{a_\mathrm{R}^-}\big|_R + i(G_\Omega^{a_\mathrm{I}^+}\big|_R - G_\Omega^{a_\mathrm{I}^-}\big|_R)$. Therefore if $G_\Omega^a\big|_R$ is equicontinuous for any $a \in \mathfrak{A}_+$, it follows that $G_\Omega\big|_R \subset (\mathfrak{A}^*_{\mathrm{uw}})^R$ is equicontinuous. The equicontinuity of $G_\Omega^a\big|_R$ is yielded by [Bourbaki, 1972, TG X.19 Corollaire 3], since it is a precompact set in the space of continuous functions – endowed with the uniform structure of compact convergence – from the locally compact space R to the uniform space $\mathbb{C}$. The fact that $G_\Omega^a\big|_R$ is precompact follows from the fact that for any net it is always possible to extract a convergent subnet that is the Fourier transform of a Wigner measure. The last fact is proved using Proposition 3.8 – that ensures that the restricted measures $\mathbb{E}^{\omega_{h_\beta}}_{h_\beta,2}(a)$ are Fock-normal – and then the standard result of microlocal analysis [see e.g. Lions and Paul, 1993, Théorème III.1]. ⊣

An immediate consequence of Lemma 3.11 and 3.12 is that for any finite dimensional $R \subseteq V$, $G_\Omega\big|_R$ is precompact in the space $\mathcal{C}_\mathrm{c}(R, \mathfrak{A}^*_{\mathrm{uw}})$ of continuous functions from the locally compact R to the uniform space $\mathfrak{A}^*_{\mathrm{uw}}$, endowed with the uniform structure of compact convergence – again it suffices to apply [Bourbaki, 1972, TG X.19 Corollaire 3].

Proposition 3.13. *Let $(\Omega_{h_\beta})_{\beta\in B}$ be a net of regular measures in the Weyl tensor deformation. If there exists $m > 0$ such that*

$$\sup_{\beta\in B} \|\Omega_{h_\beta}\|_{\mathfrak{W}^*_{h_\beta}} = m \;,$$

then $G_\Omega\big|_R$ is precompact in $\mathcal{C}_\mathrm{c}(R, \mathfrak{A}^*_{\mathrm{uw}})$ for any finite dimensional $R \subseteq V$.

Combining the results above, we see that the set G_Ω is precompact in $\mathcal{F}_\mathrm{s}(V, \mathfrak{A}^*_{\mathrm{uw}})$, and each of its restrictions to finite dimensional subspaces $R \subseteq V$ is equicontinuous and hence precompact in $\mathcal{C}_\mathrm{c}(R, \mathfrak{A}^*_{\mathrm{uw}})$. However, the uniform structures of compact and simple convergence agree on equicontinuous subsets of $\mathcal{C}(A, B)$ [Bourbaki, 1972, TG X.16 Théorème 1] for any topological space A and uniform space B. Therefore given a net of regular measures that "descends" in the deformation, with uniformly bounded masses, there is always at least one limit point of simple convergence for the corresponding generating map, and every limit point is ultraweakly continuous when restricted to any finite dimensional subset.

Theorem 3.14. *Let (V, σ) be a real vector space with a symplectic form, $\mathfrak{A}$ a C^*-algebra, and $(\mathfrak{W}_h)_{h\geq 0}$ a corresponding Weyl tensor deformation. Then there exist a non-empty set of limit points of the generating map $\mathcal{G}_{\Omega_{h_\beta}}$ in $\mathcal{F}_\mathrm{s}(V, \mathfrak{A}^*_{\mathrm{uw}})$, for any descending net of measures $(\Omega_{h_\beta})_{\beta\in B}$, $h_\beta \to 0$, provided there exists $m > 0$ such that*

$$\sup_{\beta\in B} \|\Omega_{h_\beta}\|_{\mathfrak{W}^*_{h_\beta}} = m \;.$$

If Ω_{h_β} is regular for all $\beta \in B$, every limit point g_Ω belongs to $\mathcal{C}_\mathrm{c}(R, \mathfrak{A}^*_{\mathrm{uw}})$, when restricted to any finite dimensional subspace $R \subseteq V$; in any case g_Ω satisfies:

$$\sum_{j,k\in F} g_\Omega(v_j - v_k)(a_k^\dagger a_j) \geq 0 \;;$$

where the $v_j \in V$ *are arbitrary as well as the* $a_j \in \mathfrak{A}$*, and* F *is any finite index set.*

3.5. **Locally convex spaces and identification of limit measures.** By means of Theorem 3.14, we have provided a characterization of limit points of the generating functional of nets of noncommutative measures descending in the deformation. Our goal is, however, to characterize directly limit points of nets of measures. Inspired by semiclassical analysis in general and [Ammari and Nier, 2008] in particular, this can be done by means of Bochner's theorem 2.17 whenever V has a locally convex predual V_*. There is an important caveat. The limit point of a descending net of noncommutative regular measures of the Weyl tensor algebra is – as we will see – identified with a commutative (cylindrical) measure; such measure however is not on the "phase space" (V, σ), but on its predual V_*. If V is finite dimensional, there is a bijection between the measures on V_* and V. More generally, we can make the following identification for any (V_*, V) such that there is a linear continuous map $u : V_* \to V$. Let $\Phi \in F(V)$ – for the notation refer to Section 2.2.2 – then $\Phi_* = u^{-1}(\Phi) \in F(V_*)$ and the quotient map $u_\Phi : V_*/\Phi_* \to V/\Phi$ is linear. In particular, if we consider $\Phi, \Psi \in F(V)$ with $\Phi \supset \Psi$ we have $\Phi_* \supset \Psi_* \in V_*$ and the commutative diagram

$$\begin{array}{ccccc} V_* & \xrightarrow{p_{\Psi_*}} & V_*/\Psi_* & \xrightarrow{p_{\Phi_*\Psi_*}} & V_*/\Phi_* \\ {\scriptstyle u}\downarrow & & {\scriptstyle u_\Psi}\downarrow & & {\scriptstyle u_\Phi}\downarrow \\ V & \xrightarrow{p_\Psi} & V/\Psi & \xrightarrow{p_{\Phi\Psi}} & V/\Phi \end{array} .$$

Therefore given an $\mathfrak{A}^*_+$-valued cylindrical measure $M_* = (\mu_{\Phi_*})_{\Phi_* \in F(V_*)}$ on V_*, the family $M = (\mu_\Phi)_{\Phi \in F(V)}$ defined by

$$\mu_\Phi = u_\Phi(\mu_{u^{-1}(\Phi)})$$

is an $\mathfrak{A}^*_+$-valued cylindrical measure on V. From the physics perspective, the difference is rather important when considering quantum field theories. In a (classical) field theory, the Lagrangian and Hamiltonian picture are not necessarily equivalent, for the former is set in the tangent bundle $T\mathcal{M}$ of some (convenient) manifold, the latter in the cotangent bundle $T^*\mathcal{M}$ with its canonical symplectic form (phase space). Here V_* plays the role of the tangent bundle, and (V, σ) of the phase space, so on V_* we have Lagrangian description and in (V, σ) Hamiltonian description. The former is often preferred for quantization of relativistic theories, since the Lagrangian map is relativistically covariant, while the Hamiltonian map is not. It is therefore remarkable that even if the Weyl algebra is defined by the phase space (V, σ), the limit classical measures that emerge act naturally on the space of the Lagrangian theory rather than on the phase space itself.

Let's now expand the above comments. Let V be a topological vector space with a locally convex predual V_*, and let $(\mathfrak{R}_h)^*_+ \subseteq (\mathfrak{W}_h)^*_+$ be the set of regular measures and $\mathfrak{R}^*_+ = \bigcup_{h>0} (\mathfrak{R}_h)^*_+$. Consider the set of both noncommutative and commutative (finite) positive measures of the deformation

$$\mathcal{M}_{h\geq 0}(\mathfrak{W}_h) = \mathfrak{R}_+^* \cup \mathcal{M}_{\mathrm{cyl}}(V_*, \mathfrak{A}_+^*) \; .$$

By Proposition 3.9, we can identify each regular measure with a function (the generating map) from V to $\mathfrak{A}^*$, and by Theorems 2.17 and 2.29 we can identify each $\mathfrak{A}_+^*$-valued cylindrical measure on V_* with a function (the Fourier transform) from V to $\mathfrak{A}^*$. Now we denote by $\tilde{\mathcal{C}}_{\mathrm{f}}(V, \mathfrak{A}^*_{\mathrm{uw}})$ the topological space of functions from V to $\mathfrak{A}^*$ – the latter endowed with the ultraweak topology – that are continuous when restricted to any finite dimensional subspace of V, endowed with the uniform structure of simple convergence. By the aforementioned identification, the topology of $\tilde{\mathcal{C}}_{\mathrm{f}}(V, \mathfrak{A}^*_{\mathrm{uw}})$ induces a topology on $\mathcal{M}_{h\geq 0}(\mathfrak{W}_h)$. We denote by $\mathcal{M}_{h\geq 0}(\mathfrak{W}_h)_{\mathcal{C}}$ the set of measures with the induced topology. Theorem 3.14 then translates in the following result.

Theorem 3.15. *Let (V, σ) be a real topological vector space with a symplectic form that has a locally convex predual V_*, $\mathfrak{A}$ a C^*-algebra, and $(\mathfrak{W}_h)_{h\geq 0}$ a corresponding Weyl tensor deformation. Consider a descending net $(\Omega_{h_\beta})_{\beta\in B} \subset \mathfrak{R}_+^*$, $h_\beta \to 0$, such that there exists $m > 0$ such that*

$$\sup_{\beta\in B} \|\Omega_{h_\beta}\|_{\mathfrak{W}^*_{h_\beta}} = m \; .$$

Then there exists a subnet $(\Omega_{h_\gamma})_{\gamma\in C}$ such that

$$\Omega_{h_\gamma} \to M_* \; ,$$

where $M_* \in \mathcal{M}_{\mathrm{cyl}}(V_*, \mathfrak{A}_+^*)$ and the convergence holds in $\mathcal{M}_{h\geq 0}(\mathfrak{W}_h)_{\mathcal{C}}$.

By means of generating functionals and Fourier transforms, we are able to treat noncommutative regular measures and classical $\mathfrak{A}_+^*$-valued cylindrical measures on the same grounds; and even if they are quite different objects, we are able to prove that the latter are limit points of descending nets of the former. This means that regular noncommutative measures behave "nicely", i.e. continuously, at the boundary of the Weyl deformation. The physical interpretation is that as far as states are concerned, Bohr's correspondence principle is satisfied: to a quantum mechanical state for $h > 0$, it corresponds a classical cylindrical state (cylindrical probability distribution) on the space V_* of the Lagrangian theory; and subsequently a cylindrical state on the phase space (V, σ), provided there exists a continuous and linear map $u : V_* \to V$. From the results of the preceding sections, it is also clear that nets of non-regular measures will in general fail to have classical cylinder measures as limit points. Therefore they are not suitable for considering the limiting behavior $h \to 0$. Again, from a physical standpoint this is not unreasonable, since non-regular states appear in physics mostly in relation to typically quantum behaviors, such as infrared divergence [see e.g. Acerbi, Morchio, and Strocchi, 1993a,b].

3.6. **An alternative identification.** In Definition 2.24, we chose the algebra at $h = 0$ to be the commutative unital C*-algebra of almost periodic functions from V to $\mathbb{C}$, with the supremum norm. Let's denote by

AP(V) such algebra. We want to show that the limit points of nets of noncommutative measures could also be seen as positive measures belonging to $\big(\mathrm{AP}(V)\otimes_{\gamma_0}\mathfrak{A}\big)^*$, provided that V has a locally convex predual V_* linearly and continuously embedded in V. Even if the interpretation given in Section 3.5 is more useful for applications, this point of view may also be of interest.

We define the subalgebra SAP(V) of smooth almost periodic functions to be the subalgebra of AP(V) generated by

$$\left\{f_{v^*}(w)=e^{iv^*(w)}, v^*\in V^*\right\}\ .$$

As proved in Theorem 3.15, any limit point of a descending net of noncommutative regular measures with bounded masses is identified with a cylindrical measure M_* belonging to $\mathcal{M}_{\mathrm{cyl}}(V_*,\mathfrak{A}^*_+)$, and therefore with the corresponding cylindrical measure $M\in\mathcal{M}_{\mathrm{cyl}}(V,\mathfrak{A}^*_+)$ – see the construction at the beginning of Section 3.5. Now, it is not difficult to prove that the cylinder integral with respect to M is a continuous completely positive linear map from SAP(V) to $\mathfrak{A}^*$.

Lemma 3.16. *Let V be a locally convex space, $\mathfrak{A}$ a C^*-algebra; and let $M\in\mathcal{M}_{\mathrm{cyl}}(V,\mathfrak{A}^*_+)$. Then $\int_V^{(\mathrm{cyl})}dM\in\mathcal{L}\big(\mathrm{SAP}(V),\mathfrak{A}^*\big)$ and it is completely positive.*

Proof. The action of the cylinder integral on the generators of SAP(V) is given by the Fourier transform:

$$\int_V^{(\mathrm{cyl})}f_{v^*}(w)dM(w)=\int_V^{(\mathrm{cyl})}e^{iv^*(w)}dM(w)=\hat{M}(v^*)\ .$$

Therefore the cylinder integral acts linearly on the generators of SAP(V) and it is completely positive by Bochner's theorem 2.17. Now let

$$f_n(w)=\sum_{j=1}^n z_j f_{v_j^*}(w)$$

be a linear combination of generators. Therefore there exists a $\Phi_n\in F(V)$ such that, for any $j\in\{1,\dots,n\}$, $v_j^*\in(\Phi_n)^0$; and

$$\int_V^{(\mathrm{cyl})}f_n(w)dM(w)=\int_{V/\Phi_n}\Big(\sum_{j=1}^n z_je^{iv_j^*(w)}\Big)d\mu_{\Phi_n}(w)\ ;$$

for the notation refer to Section 2.2.2. Using the corresponding result for the standard measures $\mu_{\Phi_n,\kappa}$, it is not difficult to prove that

$$\Big\|\int_V^{(\mathrm{cyl})}f_n(w)dM(w)\Big\|_{\mathfrak{A}^*}\leq\|f_n\|_\infty\|\hat{M}(0)\|_{\mathfrak{A}^*}\ ;$$

where $\|\cdot\|_\infty$ denotes the supremum norm. Now let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to the supremum norm, that converges to $f\in$ SAP(V). By means of the above, it is possible to define the cylinder integral $\int_V^{(\mathrm{cyl})}f(w)dM(w)$. In addition,

$$\Big\|\int_V^{(\mathrm{cyl})}f(w)dM(w)\Big\|_{\mathfrak{A}^*}\leq\|f\|_\infty\|\hat{M}(0)\|_{\mathfrak{A}^*}\ .$$

Therefore the cylinder integral is a linear and continuous map from $\mathrm{SAP}(V)$ to $\mathfrak{A}^*$, and it is completely positive since it is completely positive on linear combinations of generators. ⊣

Corollary 3.17. *To any cylindrical measure $M \in \mathcal{M}_{\mathrm{cyl}}(V, \mathfrak{A}^*_+)$, it is possible to associate an $\Omega_M \in \big(\mathrm{SAP}(V) \otimes_{\gamma_0} \mathfrak{A}\big)^*_+$, by means of*

$$\Omega_M = (\mathbb{E}_{h,1})^{-1}\Big(\int_V^{(\mathrm{cyl})} dM\Big) \; .$$

Proof. By Lemma 3.16, we can associate to M its cylinder integral, and the latter is a completely positive element of $\mathcal{L}\big(\mathrm{SAP}(V), \mathfrak{A}^*\big)$. Therefore by Proposition 3.5, $\Omega_M = (\mathbb{E}_{h,1})^{-1}\big(\int_V^{(\mathrm{cyl})} dM\big)$ is a positive measure, i.e. it belongs to $\big(\mathrm{SAP}(V) \otimes_{\gamma_0} \mathfrak{A}\big)^*_+$. ⊣

Using Tietze's extension theorem, Ω_M can be continuously extended to $\tilde{\Omega}_M \in \big(\mathrm{AP}(V) \otimes_{\gamma_0} \mathfrak{A}\big)^*_+$.

Theorem 3.18. *Let (V, σ) be a real topological vector space with a symplectic form such that V has a locally convex predual V_* linearly and continuously embedded in it. In addition, let $\mathfrak{A}$ be a C^*-algebra, and $(\mathfrak{W}_h)_{h\geq 0}$ a corresponding Weyl tensor deformation. Consider a descending net $(\Omega_{h_\beta})_{\beta\in B} \subset \mathfrak{R}^*_+$, $h_\beta \to 0$, such that there exists $m > 0$ such that*

$$\sup_{\beta\in B} \|\Omega_{h_\beta}\|_{\mathfrak{W}^*_{h_\beta}} = m \; .$$

Then the set of its limit points with respect to the topology of $\mathcal{M}_{h\geq 0}(\mathfrak{W}_h)_{\mathcal{C}}$ is not empty, and each limit point can be identified with a positive measure $\tilde{\Omega} \in \big(\mathrm{AP}(V) \otimes_{\gamma_0} \mathfrak{A}\big)^*_+$.

3.7. **Every cylindrical measure is a limit point.** In this section we prove that every cylindrical $\mathfrak{A}^*_+$-valued measure on V_* can be reached taking the limit of a suitable net of regular measures. In order to prove the result, we use the following lemma, that can be proved by standard arguments of semiclassical analysis. The proof relies on the fact that squeezed coherent states on $L^2(\mathbb{R}^d)$ converge to measures concentrated on a point of $\mathbb{R}^{2d}$; and that linear combinations of point measures are dense in the space of finite measures $\mathcal{M}(\mathbb{R}^{2d}, \mathbb{C})$, endowed with the weak topology [see e.g. Parthasarathy, 1967]. The extension to the Weyl tensor deformation does not present difficulties.

Lemma 3.19. *Let R be a finite dimensional real vector space with a symplectic form σ and predual $R_* \cong R$. For any $\mu \in \mathcal{M}(R_*, \mathfrak{A}^*_+)$, there exists a net of measures $(\omega_{h_\beta})_{\beta\in B}$, such that for any $\beta \in B$, $\omega_{h_\beta} \in \big(\mathrm{Weyl}_{h_\beta}(R, \sigma) \otimes_{\gamma_{h_\beta}} \mathfrak{A})^*_+$ is Fock-normal, and*

$$\omega_{h_\beta} \to \mu$$

with respect to the topology of $\mathcal{M}_{h\geq 0}\big(\mathrm{Weyl}_h(R, \sigma) \otimes_{\gamma_h} \mathfrak{A}\big)_{\mathcal{C}}$.

With the aid of this result, and of the projective structure of cylindrical measures, we can prove the "surjectivity" of the classical limit, in the sense that every commutative cylindrical measure is reached by some net of regular non-commutative measures.

Theorem 3.20. *Let (V,σ) be a real topological vector space with a symplectic form, such that it has a locally convex predual V_*; and let $\mathfrak{A}$ be a C^*-algebra. Then to any $M \in \mathcal{M}_{\mathrm{cyl}}(V_*,\mathfrak{A}^*_+)$ there corresponds a net of regular measures $(\Omega_{h_\beta})_{\beta\in B} \subset \mathfrak{R}^*_+$ such that $\Omega_{h_\beta} \to M$ in $\mathcal{M}_{h\geq 0}(\mathfrak{W}_h)_{\mathcal{C}}$.*

Proof. Let $\mathcal{M}_{\mathrm{cyl}}(V_*,\mathfrak{A}^*_+) \ni M = (\mu_{\Phi_*})_{\Phi_*\in F(V_*)}$. Combining Lemma 3.19 with the compatibility condition of cylindrical measures – refer to Section 2.2.2 – there exists a family $(\omega_\gamma^{\Phi_*})_{\delta\in D,\Phi_*\in F(V_*)}$ of nets of non-commutative measures such that for any $\delta\in D$ and $\Phi_*\in V_*$,

$$\omega_{h_\delta}^{\Phi_*} \in \bigl(\mathrm{Weyl}_{h_\delta}(V_*/\Phi_*,\sigma)\otimes_{h_\delta}\mathfrak{A}\bigr)^*_+ \text{ is Fock-normal ;}$$

$\omega_{h_\delta}^{\Phi_*} \to \mu_{\Phi_*}$ with respect to the topology of $\mathcal{M}_{h\geq 0}\bigl(\mathrm{Weyl}_{h_\delta}(V_*/\Phi_*,\sigma)\otimes_{\gamma_{h_\delta}}\mathfrak{A}\bigr)_{\mathcal{C}}$; and such that $\mathrm{Weyl}_{h_\delta}(V_*/\Phi_*,\sigma)\otimes_{\gamma_{h_\delta}}\mathfrak{A} \subset \mathrm{Weyl}_{h_\delta}(V_*/\Psi_*,\sigma)\otimes_{\gamma_{h_\delta}}\mathfrak{A}$ yields

$$\omega_{h_\delta}^{\Phi_*} = \omega_{h_\delta}^{\Psi_*}\Big|_{\mathrm{Weyl}_{h_\delta}(V_*/\Phi_*,\sigma)\otimes_{\gamma_{h_\delta}}\mathfrak{A}} .$$

Now since the three topologies

$$\mathcal{M}_{h\geq 0}\bigl(\mathrm{Weyl}_h(V_*/\Phi_*,\sigma)\otimes_{\gamma_h}\mathfrak{A}\bigr)_{\mathcal{C}}\ ,\ \mathcal{M}_{h\geq 0}\bigl(\mathrm{Weyl}_h(V_*/\Psi_*,\sigma)\otimes_{\gamma_h}\mathfrak{A}\bigr)_{\mathcal{C}}\ ,$$

and $\mathcal{M}_{h\geq 0}(\mathfrak{W}_h)_{\mathcal{C}}$ agree on common subspaces for any $\Phi_*\subset\Psi_*\in F(V_*)$, it is possible to extract a subnet $(\Omega_{h_\beta})_{\beta\in B}\subset\mathfrak{R}^*_+$ such that each $\Omega_{h_{\beta_\delta}}$ extends ω_{h_δ} for any $\delta\in D$, and $\Omega_{h_\beta}\to M$ in $\mathcal{M}_{h\geq 0}(\mathfrak{W}_h)_{\mathcal{C}}$ – here we have set $\beta_\delta\in f^{-1}(\delta)\subset B$, where $f: B\to D$ is the monotone final function defining the subnet. ⊣

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