Fano 4-folds having a prime divisor of Picard number 1

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Introduction

Let X be a smooth complex Fano variety of dimension $n \geq 3$, and let $D \subset X$ be a prime divisor. Consider the induced push-forward of 1-cycles $i_*\colon \mathcal{N}_1(D) \to \mathcal{N}_1(X)$, and let $\mathcal{N}_1(D, X) := \mathrm{Im}(i_*)$.
Assume that there exists $D \subset X$ s.t. $\dim \mathcal{N}_1(D, X) = 1$. Then the Picard number ρ_X of X is at most 3 [T, CD], and we have the following results:

- Tsukioka [T] classifies the case where $D \cong \mathbb{P}^{n-1}$;
- Casagrande and Druel [CD] treated the case of $\rho_X = 3$, without any further restriction on D.

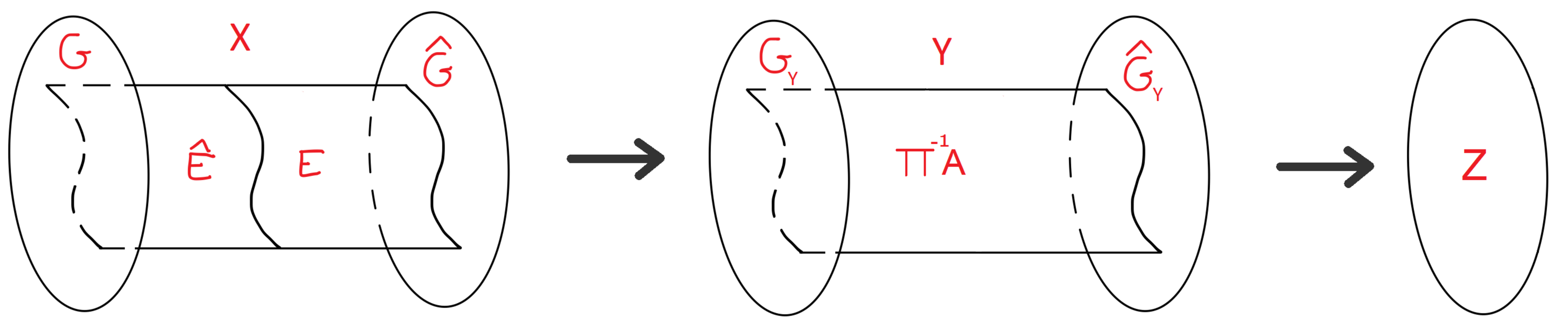
The objective is to study in detail the case $\rho_X = 3$ and $n = 4$, and to give a complete classification.

Construction [CD]

Let X be a smooth Fano variety of dimension $n \geq 3$ and $\rho_X = 3$. Assume that there exists a prime divisor D such that $\dim \mathcal{N}_1(D, X) = 1$. Then X is isomorphic to one of the varieties constructed below.

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[rr, bend right, "\varphi"'] & Y \arrow[r, "\pi"] & Z \end{tikzcd}$$

- Z smooth Fano with $\rho_Z = 1$, $\dim(Z) = n-1$, and index $i_Z \geq 2$;
- Fix integers $a \geq 0$, $d \geq 1$ satisfying $a \leq i_Z - 1$ and $d - a \leq i_Z - 1$, and assume that $\exists$ a smooth hypersurface $A \in |\mathcal{O}_Z(d)|$; $\quad (*)$
- $Y = \mathbb{P}(\mathcal{O}_Z \oplus \mathcal{O}_Z(a))$, G_Y and $\widehat{G}_Y$ are sections corresponding to the projections $\mathcal{O}_Z \oplus \mathcal{O}_Z(a) \twoheadrightarrow \mathcal{O}_Z, \mathcal{O}_Z(a)$;
- $\sigma\colon X \to Y$ is the blow-up along $\widehat{G}_Y \cap \pi^{-1}(A)$, with $E = \mathrm{Exc}(\sigma)$. Moreover $G, \widehat{G}, \widehat{E}$ are the transforms of $G_Y, \widehat{G}_Y$ and $\pi^{-1}(A)$.
- φ is a conic bundle, A is the discriminant divisor, and $E \cup \widehat{E}$ is the union of singular fibres.



Moreover, φ admits a second factorization

$$\begin{tikzcd} X \arrow[r, "\widehat{\sigma}"] \arrow[rr, bend right, "\varphi"'] & \widehat{Y} \arrow[r, "\widehat{\pi}"] & Z \end{tikzcd},$$

where $\widehat{Y} = \mathbb{P}(\mathcal{O}_Z \oplus \mathcal{O}_Z(d-a)) \xrightarrow{\widehat{\pi}} Z$ is a projective bundle, and $\widehat{\sigma}$ is the blow-up along $\widehat{\sigma}(G) \cap \widehat{\pi}^{-1}(A)$ with exceptional divisor $\widehat{E}$; $\widehat{\sigma}(G)$ and $\widehat{\sigma}(\widehat{G})$ are sections of $\widehat{\pi}$, and the role of Y and $\widehat{Y}$ is interchanged.

Case: $\dim X = 4$

We aim to determine and study all the families of Fano varieties as above in the case of dimension 4. In this case, there are 7 families of smooth Fano 3-folds Z with $\rho_Z = 1$ and $i_Z \geq 2$, and we have that condition $(*)$ is satisfied for all admitted pairs (a, d). Moreover, the pairs (a, d) and $(d-a, d)$ give rise to isomorphic Fano 4-folds.

Theorem

There are 28 families of Fano 4-folds X with $\rho_X = 3$, and having a prime divisor D with $\dim \mathcal{N}_1(D, X) = 1$. Among them, 22 families have rational members, and the very general member of 4 other families is not rational. The linear system $|-K_X|$ is free, except for two families where its base locus is either 1 or 2 points. In all cases, a general element of $|-K_X|$ is smooth.

Every Fano 4-fold as above corresponds to a triplet (Z, a, d).

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Numerical invariants

Hodge numbers: use Hodge polynomial $e(X) := \sum_{p,q} h^{p,q}(X) u^p v^q$.
By construction, we have: $e(X) = e(Z) \cdot e(\mathbb{P}^1) + e(A) \cdot \left(e(\mathbb{P}^1) - 1\right)$.
$\boldsymbol{h^0(-K_X)}$, $\boldsymbol{K_X^4}$, $\boldsymbol{K_X^2 \cdot c_2(X)}$: Riemann-Roch and Kodaira vanishing theorems yield $h^0(-K_X) = 1 + \frac{1}{12}(2K_X^4 + K_X^2 \cdot c_2(X))$.

Linear system $|-K_X|$

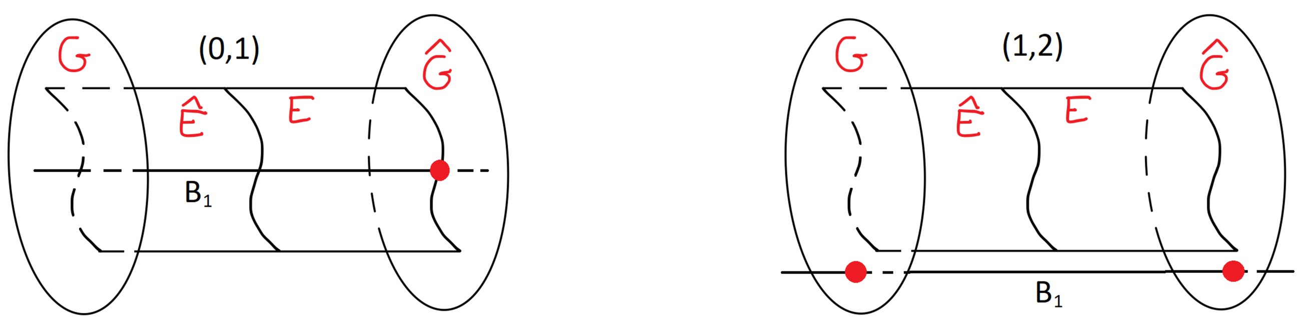
To study the base locus of $|-K_X|$, we find special effective divisors in the linear system. Let $H \in |\mathcal{O}_Z(1)|$ be an effective divisor, then:

$$-K_X \sim (i_Z - a)\varphi^* H + 2\widehat{G} + E \sim (i_Z + a - d)\varphi^* H + 2G + \widehat{E} \sim$$
$$\sim i_Z \varphi^* H + G + \widehat{G}.$$

Let $B_t := \varphi^{-1}\mathrm{Bs}(|\mathcal{O}_Z(t)|)$. Then:

$$\mathrm{Bs}(|-K_X|) \subseteq \left(B_{i_Z} \cup G \cup \widehat{G}\right) \cap \left(B_{i_Z - a} \cup \widehat{G} \cup E\right) \cap \left(B_{i_Z - d + a} \cup G \cup \widehat{E}\right).$$

There are only two families for which $\mathrm{Bs}(|-K_X|) \neq \emptyset$, and they are obtained by taking $Z =$ a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1, 1, 1, 2, 3)$, and $(a, d) = (0, 1), (1, 2)$.



Families of deformation

Let $g\colon \mathcal{Z} \to T$ be a family of smooth Fano 3-folds: we can assume that $\exists \mathcal{H} \in \mathrm{Pic}(\mathcal{Z})$ which restricts to $\mathcal{O}_{Z_t}(1)$ on the fibres. Take the open set $\widetilde{T} \subset \mathbb{P}_T(g_*(\mathcal{H}^{\otimes d})^\vee)$ parametrizing smooth surfaces $A_t \in |\mathcal{O}_{Z_t}(d)|$. Then, if $\widetilde{\mathcal{Z}} := \mathcal{Z} \times_T \widetilde{T}$, we can contruct from $\widetilde{\mathcal{Z}}$ a family $\mathcal{X} \xrightarrow{f} \widetilde{T}$ whose fibers are the Fano 4-folds of the Theorem. Note that $\dim(\widetilde{T}) = \dim(T) + h^0(\mathcal{O}_Z(d)) - 1$.
Proposition 1. Let $\mathcal{X} \to S$ be a smooth family of Fano varieties. If there exists a fibre X_0 which is isomorphic to one of the varieties of the Thm., then all the fibres appear in the same family.
Proposition 2. Fix integers a, d. Let X and X' be varieties of the Thm. constructed from Z and, respectively, $A, A' \in |\mathcal{O}_Z(d)|$. Then $X \cong X'$ if and only if $\exists \psi \in \mathrm{Aut}(Z)$ s.t $\psi(A) = A'$.
Consequence: If T and S are the base of the Kuranishi families of, respectively, Z and X, then $\dim(T) = h^1(\mathcal{T}_Z)$, $\dim(S) = h^1(\mathcal{T}_X)$, and the natural morphism $\widetilde{T} \to S$ is surjective. Therefore:

$$h^1(\mathcal{T}_X) \leq h^1(\mathcal{T}_Z) + h^0(\mathcal{O}_Z(d)) - 1.$$

[T] T. Tsukioka. *Classification of Fano manifolds containing a negative divisor isomorphic to projective space*. Geom. Dedicata 123 (2006).
[CD] C. Casagrande, S. Druel. *Locally unsplit families of rational curves of large anticanonical degree on Fano manifolds*. IMRN, 21 (2015).
[S] S. A. Secci. *Fano 4-folds having a prime divisor of Picard number 1*. Preprint arXiv:2103.16140 (2021).