Geometry of Elliptic Normal Curves of Degree 6

Anatoli Shatsila [1]

[1] Jagiellonian University in Kraków, Poland

Introduction

By the Riemann-Roch Theorem, one can embed any elliptic curve $C = \mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ as an elliptic normal curve of degree $n \geq 3$ in $\mathbb{P}^{n-1}$ via the complete linear system $|\mathcal{O}_C(n \cdot O)|$, where O is the origin in C. Assume the embedding is given by

$$\varphi : C \ni z \mapsto \varphi(z) = (X_0(z) : X_1(z) : \ldots : X_{n-1}(z)) \in \mathbb{P}^{n-1}.$$

By C_n we denote the image $\varphi(C)$.

We consider embeddings of C_n invariant under the action of the Heisenberg group so that for any $z \in C$ both $\sigma(\varphi(z)) = (X_{n-1}(z) : X_0(z) : \ldots : X_{n-3}(z) : X_{n-2}(z))$ and $\tau(\varphi(z)) = (X_0(z) : \varepsilon X_1(z) : \ldots : \varepsilon^{n-1} X_{n-1}(z))$ with $\varepsilon := \frac{2\pi i}{n}$, are in C_n. In addition, we will assume the existence of the point $c \in C$ that satisfies $X_i(c) = 0 \Leftrightarrow i = 0$.

The functions $\{x_m\}_{m \in \mathbb{Z}/n\mathbb{Z}}$ considered in [1, Section I.2] and defined using Weierstrass sigma-functions induce an immersion satisfying the above conditions. In this case we can take $c := \frac{\omega_1}{2} + \frac{\omega_2}{2n}$. For convenience, we will assume that C is embedded by x_m - all results and proofs are valid for any immersion with the mentioned properties.

The cases $n = 3, 4$ are classical. The case $n = 5$ has been considered in the monograph [1], where many beautiful geometric construction were used to study the interrelation between the Horrocks-Mumford vector bundle and the normal bundle of elliptic curves of degree 5. In the work we study the geometry in the case $n = 6$.

Main result

By C_p and C_{pq} we denote the images of C_6 under the projection from a general point and a general line respectively. Then

- the ideal $I(C_p)$ of the curve C_p is generated by three polynomials of degree 2 and two polynomials of degree 3.
- the ideal $I(C_{pq})$ of the curve C_{pq} is generated by two polynomials of degree 3 and three polynomials of degree 4.

Main steps of the proof

- The curve C_p (resp. C_{pq}) is k-normal for all $k \geq 2$ (resp. $k \geq 3$) and
$$h^0(\mathcal{I}_{C_p}(2)) = 3, h^0(\mathcal{I}_{C_p}(3)) = 17,$$
$$h^0(\mathcal{I}_{C_{pq}}(3)) = 2, h^0(\mathcal{I}_{C_{pq}}(4)) = 11.$$
- The ideal of the curve C_p (resp. C_{pq}) is generated in degree 3 (resp. 4) - follows from Castelnuovo-Mumford regularity theory.
- For a general point $P \in \mathbb{P}^5$ there are at least two distinct triples of points (denoted by R_1, R_2, R_3 and T_1, T_2, T_3) on C_6 spanning linear subspaces which contain P.
- There exist 4 points $A_1, A_2, A_3, A_4 \in C_6$ different from R_i, T_i such that their span contains P.
- Let $Q_i = V(q_i)$ with $\{q_i\}_{i=1}^3$ being a basis of $H^0(\mathcal{I}_{C_6}(2))$. The intersection $Q_1 \cap Q_2 \cap Q_3$ is a curve. This is the most technical part of the proof which uses certain geometric relations between the points on C_6 defined above.
- There are no linear syzygies between q_i's hence $17 - 3 \cdot 5 = 2$ yields the result.
- The second part of the proof for C_{pq} is similar and even easier due to the fact that $h^0(\mathcal{I}_{C_{pq}}(3)) = 2$.

Elliptic normal curves and quadric hypersurfaces

Theorem. ([2], Theorem 3.1) Let $C = \mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ be an elliptic curve and $C_6 \subset \mathbb{P}^5$ be its embedding as a normal elliptic sextic. Then there exists a 9-dimensional space of quadric hypersurfaces containing C_6. A basis of this space is given by

$$\begin{aligned}
Q_0 &= x_0^2 + x_3^2 + \alpha(x_2x_4 + x_5x_1),\\
Q_1 &= x_1^2 + x_4^2 + \alpha(x_3x_5 + x_0x_2),\\
Q_2 &= x_2^2 + x_5^2 + \alpha(x_4x_0 + x_1x_3),\\
Q_0' &= x_0^2 - x_3^2 + \beta(x_2x_4 - x_5x_1),\\
Q_1' &= x_1^2 - x_4^2 + \beta(x_3x_5 - x_0x_2),\\
Q_2' &= x_2^2 - x_5^2 + \beta(x_4x_0 - x_1x_3),\\
Q_0'' &= x_0x_1 + x_3x_4 + \gamma x_2x_5,\\
Q_1'' &= x_1x_2 + x_4x_5 + \gamma x_3x_0,\\
Q_2'' &= x_2x_3 + x_5x_0 + \gamma x_4x_1
\end{aligned}$$

with

$$\begin{aligned}
\alpha &= -\frac{x_3^2(\omega)}{x_2(\omega)x_4(\omega)+x_5(\omega)x_1(\omega)},\\
\beta &= \frac{x_3^2(\omega)}{x_2(\omega)x_4(\omega)-x_5(\omega)x_1(\omega)},\\
\gamma &= -\frac{x_3(\omega)x_4(\omega)}{x_2(\omega)x_5(\omega)}
\end{aligned}$$

where $\omega = \frac{\omega_1}{2} + \frac{\omega_2}{12}$.

Lemma. ([2], Lemma 3.3) The following relations hold:

$$\alpha\beta(\alpha + \beta) = -2,$$
$$\gamma = \alpha\beta.$$

Theorem. ([2], Theorem 3.6) The ideal $I(\mathrm{Sec}(C_6))$ is generated by two cubic surfaces F_1 and F_2 given by

$$F_1 = 2(\alpha^2\beta^2 - \alpha - \beta)x_0x_2x_4 + \sum_{i=0}^{2} \sigma^i(-2x_0^3 + 2(\beta-\alpha)x_1x_2x_3 + \alpha\beta(\beta-\alpha)x_0x_3^2),$$

$$F_2 = 2(\alpha^2\beta^2 - \alpha - \beta)x_1x_3x_5 + \sum_{i=0}^{2} \sigma^i(-2x_1^3 + 2(\beta-\alpha)x_2x_3x_4 + \alpha\beta(\beta-\alpha)x_1x_4^2).$$

References

[1] K. Hulek. Projective geometry of elliptic curves. Soc. Math. de France, Asterisque, 137, 1986.

[2] A. Shatsila. Geometry of elliptic normal curves of degree 6. arXiv:2203.11672, 2022.

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 anatoli.shatsila@student.uj.edu.pl