Stable cohomology of the moduli space of trigonal curves

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Abstract

We prove that the rational cohomology $H^i(\mathcal{T}_g;\mathbf{Q})$ of the moduli space of trigonal curves of genus g is independent of g in degree $i < \lfloor g/4 \rfloor$. This makes possible to define the stable cohomology ring as $H^\bullet(\mathcal{T}_g;\mathbf{Q})$ for a sufficiently large g, which turns out to be isomorphic to the tautological ring.

Introduction

We work over the field of complex numbers $\mathbf{C}$. Let C be a smooth algebraic curve. We will say that C is *trigonal* if it has *gonality* 3, i.e. it is a smooth non-hyperelliptic curve which is a (ramified) triple cover of $\mathbf{P}^1$.

Motivation and previous works

Let $\mathcal{T}_g$ be the moduli space of trigonal curves of genus g. There is a natural inclusion

$$\mathcal{T}_g \subseteq \mathcal{M}_g$$

into the moduli space of smooth curves of genus g. Thus $\mathcal{T}_g$ is a stratum of the stratification of $\mathcal{M}_g$ by gonality.

The rational cohomology ring of $\mathcal{T}_g$ is completely known for low genera. It has been computed for $g = 2, 3, 4$ by Mumford [4], Looijenga [3] and Tommasi [7], resp. and for $g = 5$, [8].

Main results

Theorem. *For $i < \lfloor g/4 \rfloor$, the rational cohomology of $\mathcal{T}_g$ is*

$$H^i(\mathcal{T}_g;\mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0, \\ \mathbf{Q}(-1), & i = 2, \\ \mathbf{Q}(-2), & i = 4, \\ 0, & \text{otherwise}. \end{cases} \qquad (1)$$

The above result, together with a description of the rational Chow ring of $\mathcal{T}_g$ by Patel and Vakil [5] and by Canning and Larson [1], also yield the following

Corollary. *For i, g as above,*

$$H^i(\mathcal{T}_g;\mathbf{Q}) = \begin{cases} R^{i/2}(\mathcal{T}_g), & i \text{ even}, \\ 0, & i \text{ odd}. \end{cases} \qquad (2)$$

Setting

We consider the natural embedding of trigonal curves in *Hirzebruch surfaces*: a trigonal curve of genus g can be embedded in $\mathbb{F}_n := \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}^1_{\mathbf{P}}(n))$ as a divisor of class

$$C \sim 3E_n + dF_n,$$

where $d = \frac{g+3n+2}{2}$, $n \equiv g \bmod 2$ and $0 \le n \le \frac{g+2}{3}$.

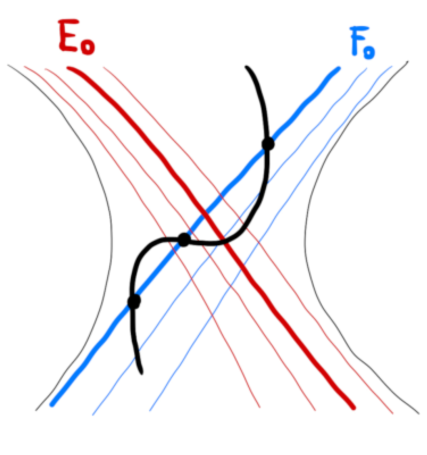

Figure 1: The curve C in $\mathbb{F}_0 \cong \mathbf{P}^1 \times \mathbf{P}^1$.

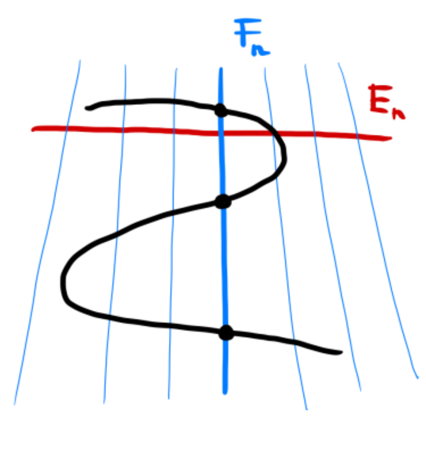

Figure 2: The curve C in $\mathbb{F}_n$, for any $n \ge 1$.

The integer n is called the *Maroni invariant* of the curve C and it defines a stratification of $\mathcal{T}_g$.

Maroni stratification

$$\begin{cases} \mathcal{N}_s \subset \cdots \subset \mathcal{N}_0 = \mathcal{T}_g & g \text{ even}, \\ \mathcal{N}_s \subset \cdots \subset \mathcal{N}_1 = \mathcal{T}_g & g \text{ odd}; \end{cases}$$

where s is the largest integer s.t. $s \equiv g \bmod 2$ and $s \le \frac{g+2}{3}$. For any $n;\ 0 \le n \le s$,

$$\mathcal{N}_n := \{[C] \in \mathcal{T}_g | C \text{ has Maroni invariant } \ge n\}.$$

Proof

We compute first the stable cohomology of the Maroni strata $N_n := \mathcal{N}_n \setminus \mathcal{N}_{n+2}$.

Maroni strata as quotients of complements of discriminants

Let $V_{d,n} := H^0(\mathbb{F}_n; \mathcal{O}_{\mathbb{F}_n}(3E_n + dF_n))$, $X_{d,n} \subset V_{d,n}$ the locus of smooth sections and $\Sigma_{d,n} := V_{d,n} \setminus X_{d,n}$ the *discriminant*. Let $G_n := Aut(\mathbb{F}_n)$.

- For $n = 0$, G_0 is reductive and isogenous to $\mathbf{C}^* \times SL_2 \times SL_2$,

$$H^\bullet([X_{d,0}/(\mathbf{C}^* \times SL_2 \times SL_2)];\mathbf{Q}) \cong H^\bullet(N_0;\mathbf{Q}).$$

- For $n > 0$, G_n is not reductive, but it is homotopy equivalent to its reductive part $\mathbf{C}^* \times GL_2$,

$$H^\bullet([X_{d,n}/(\mathbf{C}^* \times GL_2)];\mathbf{Q}) \cong H^\bullet(N_n;\mathbf{Q}).$$

Gorinov-Vassiliev's method

The method computes the Borel-Moore homology of the discriminant, which is equivalent to the cohomology of its complement due to *Alexander duality*.

It is based on a classification of the *singular configurations* of the elements of $\Sigma_{d,n}$,

$$X_1, \ldots, X_M \subseteq \mathbb{F}_n.$$

From [2], there exists a spectral sequence $E^\bullet_{p,q} \Rightarrow \bar{H}_{p+q}(\Sigma_{d,n})$ whose p-th column in the first page is given by the Borel-Moore homology of X_p.

Fix $N > 1$ and set $X_p := B(\mathbb{F}_n, p)$ the space of unordered configurations of p points on $\mathbb{F}_n$. Then

$$E^1_{p,q} = \bar{H}_{q-2(\dim V_{d,n}-3p)-p+1}(B(\mathbb{F}_n,p);\pm\mathbf{Q}) \otimes \mathbf{Q}(v_{d,n} - 3p),$$

provided that $d \ge 2N + 3n - 1$.

Under this assumption, we can also bound the dimension of the stratum corresponding to the remaining configurations $X_N, \ldots, X_M$.

Precisely, we find that the Borel-Moore homology of $\bar{H}_i(\Sigma_{d,n})$ is defined only by $X_1, \ldots, X_4$ in degree $i > 2\dim V_{d,n} - N - 1$, and by Alexander duality, $H^i(X_{d,n})$ is defined only by the first four columns in degree $i < N \Leftrightarrow i \le \frac{d-3n+1}{2}$.

Generalized Leray-Hirsch theorem

It is a criterion to determine if

$$H^\bullet(X_{d,n}) \cong H^\bullet(X_{d,n}/G) \otimes H^\bullet(G), \qquad (3)$$

for some reductive group G acting on $X_{d,n}$ with finite stabilizers.

From a theorem of Peters and Steenbrink [6], a sufficient condition for (3) to hold is given by the surjectivity of the orbit map in cohomology $\rho^* : H^\bullet(X_{d,n}) \to H^\bullet(G)$. This map is surjective for both $\mathbf{C}^* \times SL_2 \times SL_2$ and GL_2. From this we deduce the stable cohomology of N_0 and of $X_{d,n}/GL_2$. By studying the Leray spectral sequence associated to the fibration

$$X_{d,n}/GL_2 \xrightarrow{\mathbf{C}^*} X_{d,n}/(\mathbf{C}^* \times GL_2)$$

we also deduce the stable cohomology of N_n, for $n > 0$.

Gysin spectral sequence

Table 1: Spectral sequence converging to $\bar{H}_\bullet(\mathcal{T}_g;\mathbf{Q}) \otimes \mathbf{Q}(-\dim \mathcal{T}_g)$ with g even.

$\ldots$	-4	N_6 -3	N_4 -2	N_2 -1	N_0 0	
					$\mathbf{Q}$	0
				$\mathbf{Q}(-1)$		-1
						-2
				$\mathbf{Q}(-2)$		-3
			$\mathbf{Q}(-3)$ ←			-4
					$\mathbf{Q}(-3)$	-5
			$\mathbf{Q}(-4)$ ←	$\mathbf{Q}(-4)$		-6
		$\mathbf{Q}(-5)$ ←				-7
				$\mathbf{Q}(-5)$		-8
		$\mathbf{Q}(-6)$ ←	$\mathbf{Q}(-6)$			-9
	$\mathbf{Q}(-7)$ ←					-10
			$\mathbf{Q}(-7)$			-11
	$\ldots$ ←	$\mathbf{Q}(-8)$				-12
	$\ldots$	$\mathbf{Q}(-9)$				-13

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