Teorema

Se f è derivabile in x_0 cioè $\exists$ finito

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

① f è continua in x_0

② la retta tangente approssima il grafico di f

$$\hookrightarrow \quad f(x) - \Big(\overbrace{f(x_0) + f'(x_0)(x - x_0)}^{\text{eq. della retta tg}} \Big) = R(x)$$

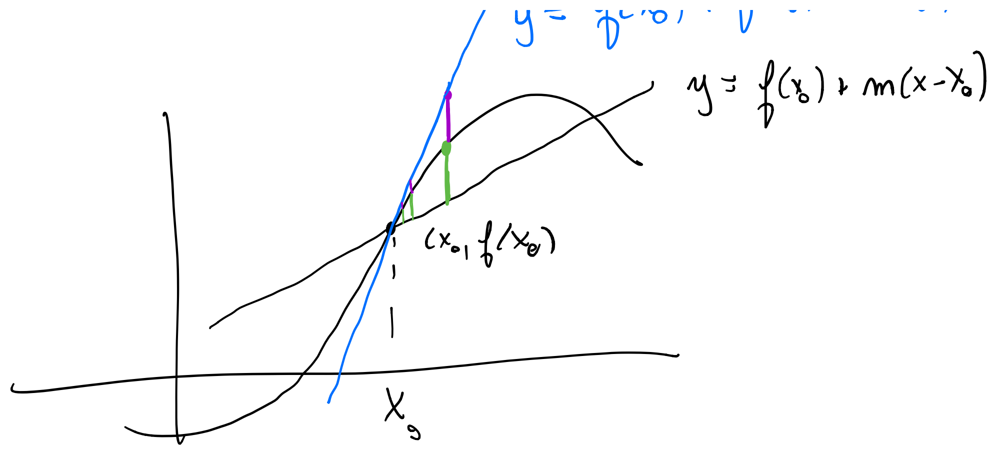
e $|R(x)| << |x - x_0|$ $\qquad R(x) = o(|x - x_0|)$

$\Updownarrow$

$$\lim_{x \to x_0} \frac{f(x) - \big(f(x_0) + f'(x_0)(x - x_0) \big)}{x - x_0} = 0$$

la retta tangente è la MIGLIORE APPROSSIMAZIONE di $f(x)$ tramite una funzione lineare VICINO a x_0.

$\ldots = f(x) + f'(x_0)(x - x_0)$



Esempi di derivate

$x \to 1 \qquad f(x) = 1 \quad \forall x$

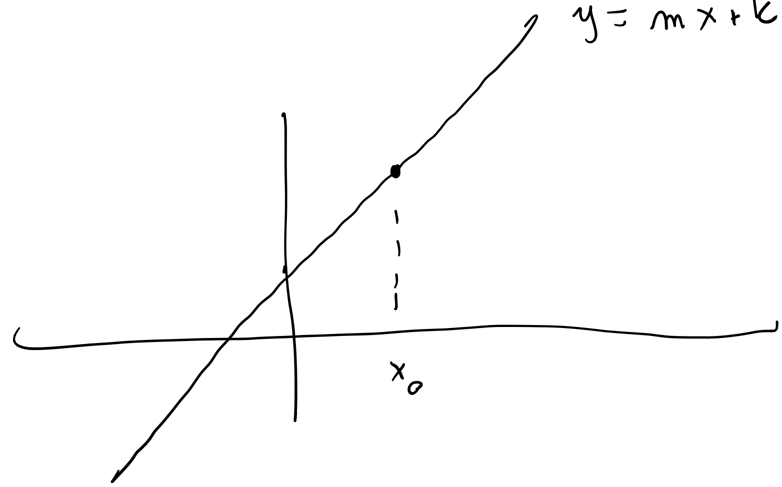
$f'(x) = 0 \quad \forall x \qquad f'(x) = \lim_{h \to 0} \frac{1 - 1}{h} = 0$

$f(x) \to mx + k \qquad m, k$ due numeri

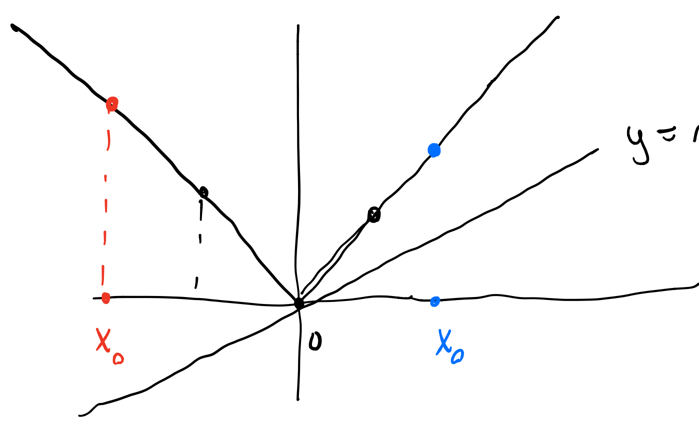
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$\lim_{h \to 0} \frac{(m(x+h) + \cancel{k}) - (mx + \cancel{k})}{h}$$

$$= \lim_{h \to 0} \frac{m(\cancel{x} + h - \cancel{x})}{h} = m$$



$$x \longrightarrow |x| \qquad f(x) = \begin{cases} x & \text{se } x \geq 0 \\ -x & \text{se } x \leq 0 \end{cases}$$



$f'(x_0) = 1 \quad \text{se} \quad x_0 > 0$

$f'(x_0) = -1 \quad \text{se} \quad x_0 < 0$

$$= \lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \frac{f(0+h) - f(0)}{h} =: f'(\ldots$$

$$= \lim_{h \to 0^-} \frac{|0+h| - |0|}{h}$$

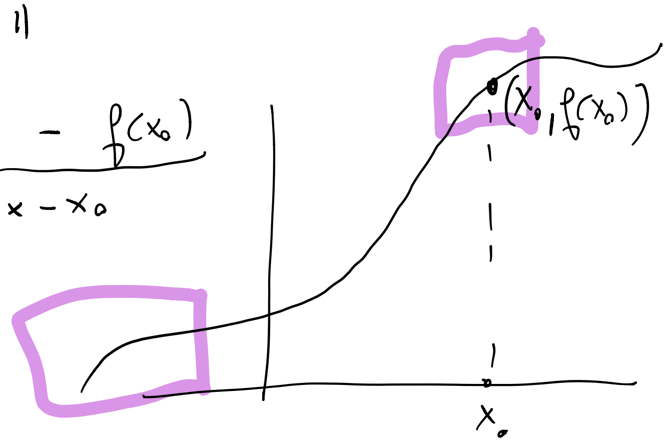
$$\lim_{h \to 0^+} \frac{h}{h} = 1$$

$$\lim_{h \to 0^-} \frac{-h}{h} = -1$$

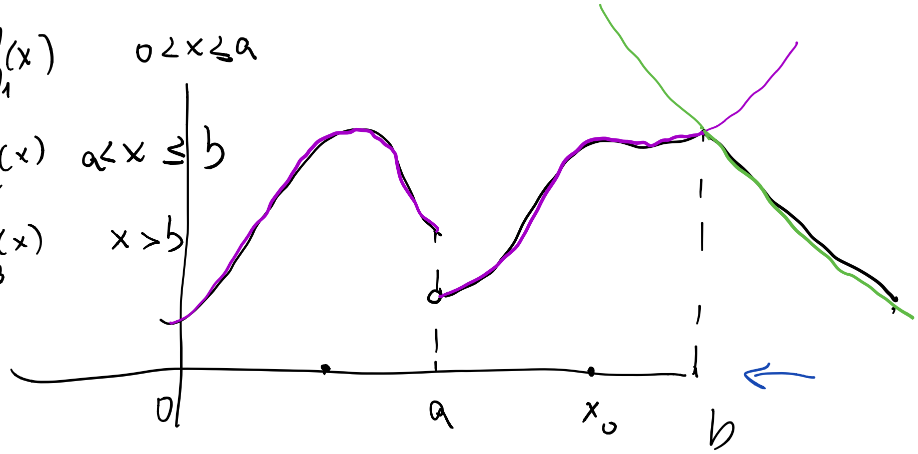
$$\lim_{h \to 0} \frac{f(\overbrace{x_0 + h}^{x}) - f(x_0)}{h}$$

$$\|$$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$



$$f(x) = \begin{cases} f_1(x) & 0 < x \leq a \\ f_2(x) & a < x \leq b \\ f_3(x) & x > b \end{cases}$$

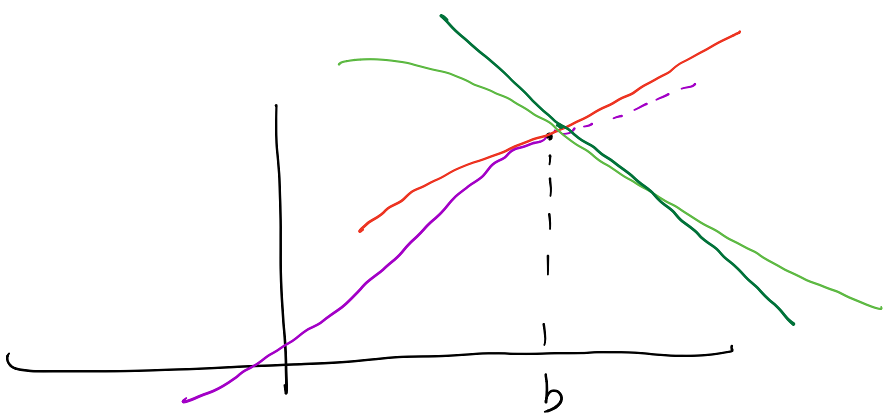


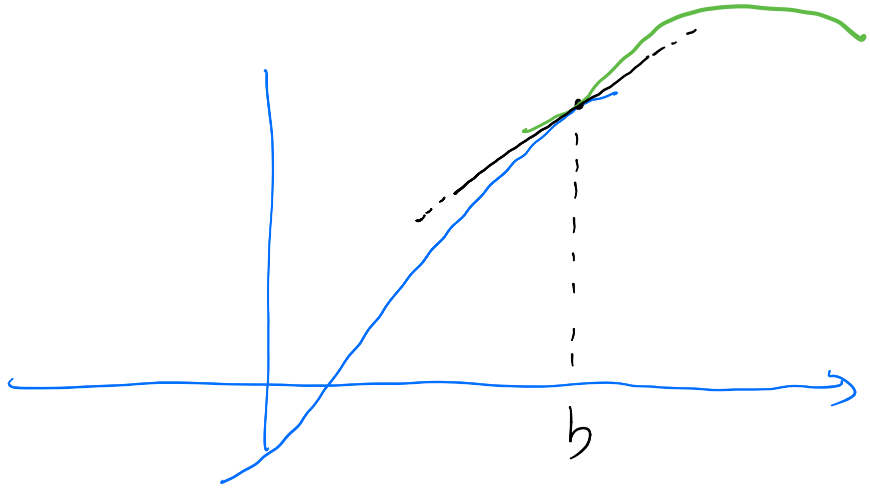
$x_0 = a$ $\qquad\qquad$ $x_0 = b$

$$\lim_{x \to b^-} \frac{f_2(x) - f_2(b)}{x - b} = f_2'(b)$$

$$\parallel$$

$$\lim_{x \to b^+} \frac{f_3(x) - f_3(b)}{x - b} = f_3'(b)$$





$f(x) : D \to \mathbb{R} \qquad x_0 \in D$

$\left[f(x) \text{ è continua in } x_0 \right]$ e

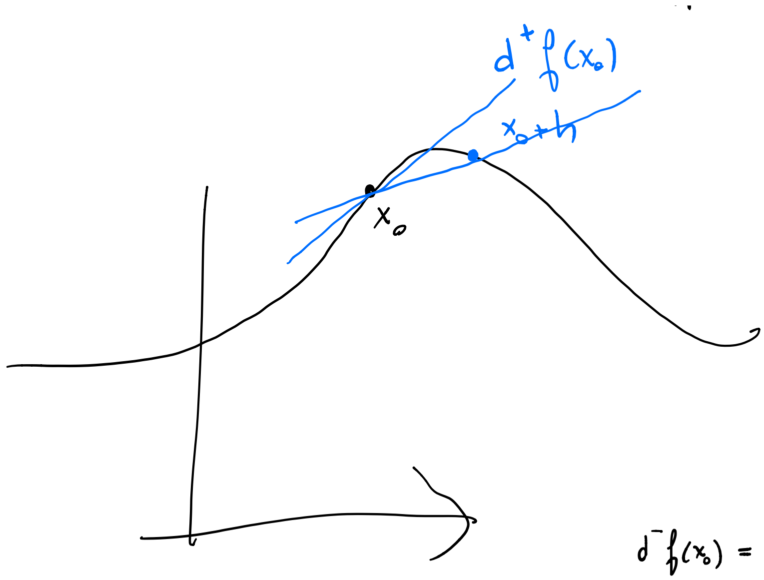
DEF.

$$\Rightarrow \frac{d}{dx^+} f(x)\Big|_{x=x_0} = \lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h}$$

$$\parallel$$

$$\Rightarrow d^+ f(x_0) = \lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h}$$

$$d^- f(x_0) = \lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h}$$

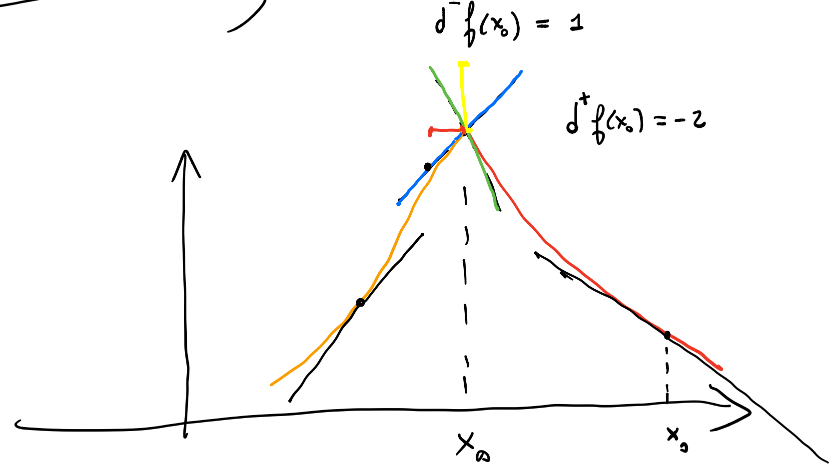


$f'(x_0)$

$df(x_0)$

$\frac{df}{dx}(x_0)$

$\left.\frac{df}{dx}\right|_{x=x_0}$



se f è derivabile en x_0

$\lambda f \quad (\lambda \in \mathbb{R})$ è derivabile

$$\frac{d}{dx}(\lambda f) = \lambda \frac{df}{dx}$$

$$\frac{d}{dx}\lambda f = \lim_{h \to 0} \frac{\lambda f(x+h) - \lambda f(x)}{h}$$

$$= \lambda \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lambda \frac{df}{dx}$$

$$(f \pm g)' = f' \pm g'$$

$$\Rightarrow (fg)' = f'g + fg'$$

$$(fg)'(x) = f'(x)\, g(x) + f(x)\, g'(x)$$

$$\parallel$$

$$\lim_{h \to 0} \frac{(fg)(x+h) - (fg)(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x)\, g(x)}{h}$$

$$(x^2)' = (x \cdot x)' = 1 \cdot x + x \cdot 1 = 2x$$

$$(x^3)' = (x^2 \cdot x)' = (x^2)' \cdot x + x^2 \cdot (x)' = 2x \cdot x + x^2$$

$$(fg)' = f'g + fg'$$

$$= 3x^2$$

$$\frac{d}{dx} x^n = n\, x^{n-1}$$ (si dimostra per induzione)

$$f(x) = 7x^3 - 4x^2 + 2x + 1$$

$$\frac{d}{dx} f(x) = 7 \frac{d}{dx} x^3 - 4 \frac{d}{dx} x^2 + 2 \frac{d}{dx} x + \frac{d}{dx} 1$$

$$= 7 \cdot (3x^2) - 4 \cdot (2x) + 2 \cdot 1 + 0$$

$$1 = x \cdot \frac{1}{x} \qquad \text{vero} \quad \forall x \neq 0$$

$$\frac{d}{dx} 1 = \frac{d}{dx}\left(\underset{f}{x} \cdot \underset{g}{\frac{1}{x}} \right) \qquad \text{per} \quad x \neq 0$$

$$\|$$

$$0 = \left(\frac{d}{dx} x\right) \cdot \frac{1}{x} + x \cdot \left(\frac{d}{dx} \frac{1}{x}\right)$$

$$0 = 1 \cdot \frac{1}{x} + x \cdot \left(\frac{1}{x}\right)'$$

$$x \cdot \left(\frac{1}{x}\right)' = -\frac{1}{x} \implies \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$\frac{1}{x} = x^{-1} = \qquad\qquad (x^n)' = n\, x^{n-1}$$

$$\left(\frac{1}{x}\right)' = (-1)\, x^{-1-1}$$

Lemme $\quad \forall \alpha \in \mathbb{R} \quad (x^\alpha)' = \alpha\, x^{\alpha-1}$

$$\left(x^{\frac{3}{2}}\right)' = \frac{3}{2}\,x^{\frac{3}{2}-1} = \frac{3}{2}\,x^{\frac{1}{2}}$$

$$(\alpha = \frac{3}{2}$$

$$\left(x^{-\pi}\right)' = -\pi\,x^{-\pi-1}$$

$$1 = g \cdot \frac{1}{g} \qquad \forall x : g(x) \neq 0$$

$$(1)' = \left(g \cdot \frac{1}{g}\right)'$$

$$\left(\frac{1}{g}\right)' \qquad \overset{\parallel}{0} = g' \cdot \frac{1}{g} + g \cdot \left(\frac{1}{g}\right)'$$

$$g \cdot \left(\frac{1}{g}\right)' = -\frac{g'}{g}$$

$$\boxed{\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}}$$

$$\left(\frac{1}{x^2+1}\right)' = -\frac{2x}{(x^2+1)^2}$$

$g(x) = x^2+1$ $\qquad$ $g'(x) = 2x$

① $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$; ② $(f \cdot h)' = f'h + f h'$

Applico ② con $h = \frac{1}{g}$

$$\frac{f}{g} = f \cdot \frac{1}{g}$$

$$\left(\frac{f}{g}\right)' = \left(f \cdot \frac{1}{g}\right)' = f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)'$$

$$= \frac{f'}{g} + f \cdot \left(-\frac{g'}{g^2}\right) = \frac{f'}{g} - \frac{fg'}{g^2}$$

$$= \frac{gf' - fg'}{g^2}$$

f, g entrambe derivabili

$$f \circ g\,(x) = f(g(x))$$

(A) $d\, f(g(x)) = f'(g(x)) \cdot g'(x)$

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(B) $(f \circ g)' = (f' \circ g) \cdot g'$

$$h(x) = (3x^2 + 2)^3 = f(g(x))$$

$$g(x) = 3x^2 + 2 \qquad f(y) = y^3$$

$$g(x) = 3x^2 + 2 \qquad g'(x) = 6x$$

$$f(y) = y^3 \qquad f'(y) = 3y^2 \qquad f'(g(x)) = 3(g(x))^2$$

$$h'(x) = 3(3x^2+2)^2 \cdot 6x$$

$$=$$

$$3\,(g(x))^2 \cdot g'(x)$$

$$D\, f(g(x)) = f'(g(x)) \cdot g'(x)$$

$$\frac{1}{6x^4 - 2x^2} = f(g(x))$$

$$f(y) = \frac{1}{y}$$

$$g(x) = 6x^4 - 2x^2$$

$$f'(y) = -\frac{1}{y^2}$$

$$\boxed{f'(g(x)) = -\frac{1}{(g(x))^2} = -\frac{1}{(6x^4 - 2x^2)^2}}$$

$$g'(x) = 6 \cdot 4\, x^3 - 2 \cdot 2\, x = 24\, x^3 - 4x$$

$$d\left(\frac{1}{6x^4 - 2x^2}\right) = -\frac{1}{(6x^4 - 2x^2)^2} \cdot (24\, x^3 - 4x)$$

$$\frac{d}{dx}\left(x^2 + 4x\right)^{\alpha} = \alpha\,(x^2 + 4x)^{\alpha - 1} \cdot (2x + 4)$$

$$f(y) = y^{\alpha} \qquad f'(y) = \alpha\, y^{\alpha - 1}$$

$$g(x) = x^2 + 4x$$

$$(e^x)' := \lim_{h \to 0} \frac{e^{(x+h)} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$\Rightarrow (e^x)' = e^x$$

$$e^{4x} = f(g(x))$$

$$f(y) = e^y$$

$$g(x) = 4x$$

$$\frac{d}{dx} e^{4x}$$

$$f(y) = e^y \qquad f'(y) = e^y \qquad g'(x) = 4$$

$$\frac{d}{dx} e^{4x} = e^{4x} \cdot 4$$

$$\Rightarrow \quad a^x = e^{\ln a \cdot x}$$

a numero positivo

$$a = e^{\ln a}$$

$$\frac{d}{dx} a^x = a^x \cdot \ln a$$

$$\Rightarrow \quad x = e^{\ln x}$$

vero $\forall \; x > 0$

$$(x)' = \left(e^{\ln x}\right)'$$

$$\downarrow$$

$$1 = e^{\ln x} \cdot (\ln x)'$$

$$1 = x \cdot (\ln x)'$$

$$f(y) = e^y$$

$$g(x) = \ln x$$

$$f'(y) = e^y$$

$$g'(x) = (\ln x)'$$

$$(\ln x)' = \frac{1}{x}$$

(!) $$\frac{d}{dx} x^\alpha = \alpha x^{\alpha - 1}$$

$$\frac{d}{dx} e^x = e^x \qquad \frac{d}{dx} \ln x = \frac{1}{x}$$

$(x^2+3x-2)^{\frac{3}{2}}$; $\sqrt{(\ln x + 2)}$

e^{x^2+5} ; $\ln(2x^4+\sqrt{x})$

$\ln(\ln(x))$; $\dfrac{1}{\ln 3x + 2}$

$$(\operatorname{sen} x)' = \lim_{h \to 0} \frac{\operatorname{sen}(x+h) - \operatorname{sen}(x)}{h}$$

$$= \lim_{h \to 0} \frac{\operatorname{sen} x \cos h + \cos x \operatorname{sen} h - \operatorname{sen} x}{h}$$

$$= \lim_{h \to 0} \left[\operatorname{sen} x \left(\frac{\cos h - 1}{h} \right) + \cos x \frac{\operatorname{sen} h}{h} \right]$$

$$= \operatorname{sen} x \lim_{h \to 0} \left(\frac{\cos h - 1}{h} \right) + \cos x \lim_{h \to 0} \frac{\operatorname{sen} h}{h}$$

$$\parallel \quad (\cos h - 1) \sim -\frac{h^2}{2}$$

$$\sin x \lim_{h \to 0} \frac{-h^{\cancel{2}}}{2\cancel{h}} + \cos x \lim_{h \to 0} 1$$

$$0 + \cos x$$

$$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x$$