$$= \lim_{x\to 1} \frac{(8e^6-1)}{x-1} + 35e^6 + \frac{R_2(x)}{(x-1)^2}$$

$$= \lim_{x\to 1} \frac{(8e^6-1)}{x-1} + \lim_{x\to 1} 35e^6 + \lim_{x\to 1} \frac{R_2(x)}{(x-1)^2}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\frac{8e^6-1}{0} \qquad\qquad 35e^6 \qquad\qquad 0$$

$$|R_2(x)| \ll |x-1|^2$$

il limite NON esiste ma

$$\lim_{x\to 1^-} \frac{8e^6-1}{x-1} = -\infty \qquad \lim_{x\to 1^+} \frac{8e^6-1}{x-1} = +\infty$$

— . — . —

Formula di approx di Taylor a ordine $n \geq 1$

sia $f : \mathbb{R} \to \mathbb{R}$ che sia C^n (f continua, f' continua, f'' continua, $f^{(3)}$ continua, ..., $f^{(n)}$ continua)

$f^{(3)}$ "derivata terza" è la derivata di f''

$f^{(n)}$ "derivata n'ma" è la derivata di $f^{(n-1)}$

allora posso scrivere fissato x_0

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + R_2$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \frac{1}{3!} f^{(3)}(x_0)(x-x_0)^3$$
$$+ \frac{1}{4!} f^{(4)}(x_0)(x-x_0)^4 + \cdots$$
$$+ \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n + R_n$$

$$|R_n| \ll |x-x_0|^n$$

$$f(x) = e^x \; ; \; f'(x) = e^x ; \; f''(x) = e^x , \ldots , f^{(n)}(x) = e^x$$

$x_0 = 0$ (proviamo la formula di Taylor $n=6$)

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \frac{1}{3!} f^{(3)}(x_0)(x-x_0)^3$$
$$+ \frac{1}{4!} f^{(4)}(x_0)(x-x_0)^4 + \frac{1}{5!} f^{(5)}(x_0)(x-x_0)^5 + \frac{1}{6!} f^{(6)}(x_0)(x-x_0)^6 + R_6$$

① $f(x) = 1 + 1 \cdot x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4!} + \frac{x^5}{5!} + \underbrace{\frac{x^6}{6!} + R_6}_{=}$

② $f(x) = 1 + 1 \cdot x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4!} + \frac{x^5}{5!} + R_5$

③ $f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4!} + \boxed{R_4}$

$$\Rightarrow \lim_{x \to 0} \frac{e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{6}}{x^4} =$$

$$\lim_{x \to 0} \frac{\cancel{1} + \cancel{x} + \cancel{\frac{x^2}{2}} + \cancel{\frac{x^3}{6}} + \frac{x^4}{4!} + \boxed{R_4} \cancel{-1} \cancel{-x} \cancel{-\frac{x^2}{2}} \cancel{-\frac{x^3}{6}}}{x^4}$$

$$= \lim_{x \to 0} \frac{\frac{x^4}{4!}}{x^4} + \lim_{x \to 0} \frac{R_4}{x^4}$$

$$\qquad \| \qquad\qquad \|$$

$$\frac{1}{4!} \quad + \quad 0$$

$$\lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$$

$f(x) = \operatorname{sen} x \quad ; \qquad f'(x) = \cos x \quad ; \quad f''(x) = -\operatorname{sen} x \qquad f^{(3)} = -\cos x$

$f^{(4)}(x) = \operatorname{sen} x \qquad f^{(5)}(x) = \cos x \qquad f^{(6)} = -\operatorname{sen} x \qquad f^{(7)} = -\cos x$

$$\operatorname{sen} x = 0 + 1 \cdot x + \frac{0}{2} x^2 + -\frac{1}{6} x^3 + \frac{1}{4!} 0 \cdot x^4 + 1 \cdot \frac{1}{5!} x^5 + R_5$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \frac{1}{3!} f^{(3)}(x_0)(x-x_0)^3$$

$$+ \frac{1}{4!} f^{(4)}(x_0)(x-x_0)^4 + \frac{1}{5!} f^{(5)}(x_0)(x-x_0)^5 + \cancel{\frac{1}{6!} f^{(6)}(x_0)(x-x_0)^6} + R$$

R_5

$$\operatorname{sen} x = x - \frac{x^3}{6} + \frac{x^5}{5!} + R_5 = x - \frac{x^3}{6} - \frac{x^5}{5!} + R_6$$

$$\lim_{x \to 0} \frac{\operatorname{sen} x - x + \frac{x^3}{6}}{x^5} =$$

$$\lim_{x \to 0} \frac{\cancel{x} - \cancel{\frac{x^3}{6}} + \frac{x^5}{5!} + R_5 \cancel{- x} + \cancel{\frac{x^3}{6}}}{x^5} =$$

$$\lim_{x \to 0} \frac{\frac{x^5}{5!}}{x^5} + \lim_{x \to 0} \frac{R_5}{x^5} = \frac{1}{5!} + 0$$

$$\lim_{x \to 0} \frac{\sin x - tg\, x}{x^3} \qquad \sin x = x - \frac{x^3}{6} + R_3$$

$$f(x) = tg\, x \;;\; f'(x) = 1 + (tg\, x)^2 \;;\; f''(x) = 2\, tg\, x \left(1 + (tg\, x)^2\right)$$

$$f^{(3)}(x) = 2\underbrace{(1+tg^2 x)(1+tg^2 x)}_{=2} + 2\, tg\, x \cdot \underbrace{2\, tg\, x\,(1+tg^2 x)}_{=0}$$

$$tg(x) = 0 + 1 \cdot x + \frac{1}{2}\, 0 \cdot x^2 + \frac{1}{3!} \cdot 2x^3 + R_3$$

$$tg(x) = x + \frac{x^3}{3} + R_3$$

$$\sin x - tg\, x = \cancel{x} - \frac{x^3}{6} - \left(\cancel{x} + \frac{x^3}{3}\right) + R_3$$

chiamo R_3 qualsiasi cosa che sia $<< |x - x_0|^3$

$$\lim_{x \to 0} \frac{\sin x - tg\, x}{x^3} = \lim_{x \to 0} \underbrace{\frac{-\frac{x^3}{6} - \frac{x^3}{3}}{x^3}}_{= -\frac{1}{6} - \frac{1}{3}} + \lim_{x \to 0} \underbrace{\frac{R_3}{x^3}}_{=0}$$

① Se $f(x)$ è pari $f(x) = f(-x)$

allora

$$f(x) = f(0) + f'(0)\,x + \frac{1}{2} f''(0)\,x^2 + \frac{1}{3!} f^{(3)}(0)\,x^3 + R$$

$$\shortparallel \qquad \shortparallel \qquad \shortparallel \qquad \shortparallel \qquad \shortparallel \qquad \shortparallel$$

$$f(-x) = f(0) - f'(0)\,x + \frac{1}{2} f''(0)\,x^2 - \frac{1}{3!} f^{(3)}(0)\,x^3 + R$$

l'unico modo possibile è $f'(0) = 0$; $f^{(3)}(0) = 0$

LEMMA Se f è pari nello sviluppo in zero tutti i termini $f^{(k)}(0)$ con k DISPARI valgono 0

$f^{(k)}(0) = 0 \quad \forall\, k$ DISPARI

LEMMA Se f è dispari nello sviluppo in zero tutti i termini $f^{(k)}(0)$ con k PARI valgono 0

$f^{(k)}(0) = 0 \quad \forall\, k$ PARI

Polinomio di Taylor $f(g(x))$

e^{4x} in $x_0 = 0$ al grado 5

$\shortparallel$

$f(g(x))$ $f(y) = e^y$; $g(x) = 4x$

$y = 4x$ se $x = 0$ $\Rightarrow$ $y = 0$

sviluppo $f(y)$ vicino $y_0 = 0$ al grado 5

$\to e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \frac{y^4}{4!} + \frac{\boxed{y^5}}{5!} + o(y^5)$

per $y = 4x$

$e^{4x} = 1 + 4x + \frac{(4x)^2}{2} + \frac{(4x)^3}{3!} + \frac{(4x)^4}{4!} + \frac{(4x)^5}{5!} + \underline{\underline{o(x)^5}}$

$o(4x)^5$ $\lim_{x \to 0} \frac{o(4x)^5}{(4x)^5} = 0$

$\Downarrow$

$\lim_{x \to 0} \frac{o(4x)^5}{x^5} = 0$

e^{x^2} sviluppo all'ordine 6

$x_0 = 0$

$$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + o(y^3)$$ sviluppo per $k=3$

sostituite $y = x^2$

$$\Rightarrow e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + o((x^2)^3)$$

$$= 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + o(x^6)$$

e^{x^2+2x} vicino a $x_0 = 0$

$y = x^2 + 2x \Rightarrow y_0 = 0$

sviluppo all'ordine 4

$$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \frac{y^4}{4!} + o(y^4)$$

$\Downarrow$ sostituisco $y = x^2 + 2x$

$$e^{x^2+2x} = 1 + (2x + x^2) + \frac{(2x+x^2)^2}{2} + \frac{(2x+x^2)^3}{3!}$$
$$+ \frac{(2x+x^2)^4}{4!} + o(2x+x^2)^4$$

$\Downarrow$

$$o(2x+x^2)^4 \underset{x\to 0}{<<} (2x+\cancel{x^2})^4 \sim (2x)^4$$

$$2x+x^2 \sim 2x \quad x \to 0$$

$$o(2x+x^2)^4 = o(2x)^4 = o(x^4)$$

$$e^{x^2+2x} = 1 + (2x+x^2) + \frac{(2x+x^2)^2}{2} + \frac{(\overset{1}{2x}+\overset{2}{x^2})^3}{3!}$$

$$+ \frac{(2x+x^2)^4}{4!} + o(x^4)$$

$$= 1 + 2x + x^2 + \frac{1}{2}\left(4x^2 + x^4 + 4x^3\right)$$

$$+ \frac{1}{3!}\left[(2x)^3 + x^6 + 3(2x)^2 \cdot x^2 + 3 \cdot (2x) \cdot (x^2)^2\right]$$

$$+ \frac{1}{4!}\left[(2x)^4 + x^8 + 4(2x)^3 \cdot x^5 + 6(2x)^2 \cdot (x^2)^2 + 4(2x) \cdot (x^2)^3\right.$$

$$= 1 + 2x + x^2 + 2x^2 + \frac{x^4}{2} + 2x^3$$

$$+ \frac{1}{6}\left(8x^3 + 12x^4\right) + \frac{1}{8 \cdot 3 \cdot} 16x^4 + o(x^4)$$

$$= 1 + 2x + 3x^2 + \left(2 + \frac{4}{3}\right)x^3 + \left(\underbrace{\frac{1}{2} + 2 + \frac{2}{3}}\right)x^4 + o(x^4)$$

$$e^{x^2+2x} = 1 + 2x + 3x^2 + \frac{10}{3}x^3 + \frac{19}{6}x^4 + o(x^4)$$

$$\parallel \qquad \parallel$$

$$h(x) = h(0) + h'(0)\,x + \frac{1}{2}h''(0)\,x^2 + \frac{1}{6}h^{(3)}(0)\,x^3 + \frac{1}{4!}h^{(4)}(0)\,x^4 + o(x^4)$$

$h(x) = e^{x^2+2x}$ $\qquad$ $h'(0) = 2$

$h(0) = 1$ $\qquad$ $\frac{1}{2} h''(0) = 3$

$h''(0) = 6$

$\frac{1}{6} h^{(3)}(0) = \frac{10}{3}$ $\qquad$ $h^{(3)}(0) = \frac{10 \cdot 6}{3} = 20$

$\frac{1}{24} h^{(4)}(0) = \frac{19}{6}$ $\qquad$ $h^{(4)}(0) = \frac{24 \cdot 19}{6} = 76$