Abbiamo discusso integrali di funzioni razionali

$$\int \frac{f(x)}{g(x)}\,dx \qquad \text{quando } g(x) \text{ è}$$

di grado ≤ 2.

Abbiamo trattato esplicitamente $f(x) = 1$

$g(x) = ax^2 + bx + c$ con $a \neq 0$

CASO 1. $\Delta = b^2 - 4ac = 0$ $\left(x_0 = \frac{-b}{2a}\right)$

$\rightsquigarrow$ $ax^2 + bx + c = a(x - x_0)^2$

$$\int \frac{1}{ax^2+bx+c} = \frac{1}{a}\int \frac{1}{(x-x_0)^2} = -\frac{1}{a}\,\frac{1}{(x-x_0)}$$

$$= \frac{-2}{2ax+b}$$

CASO 2. $\Delta = b^2 - 4ac < 0$

completo il quadrato:

$$\frac{1}{ax^2+bx+c} = \frac{1}{a\left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c}$$

$$= \frac{1}{a\left(x+\frac{b}{2a}\right)^2 + \frac{-\Delta}{4a}} =$$

$$\frac{4a}{-\Delta} \cdot \frac{1}{\frac{4a^2}{-\Delta}\left(x+\frac{b}{2a}\right)^2 + 1} =$$

$$= \frac{4a}{-\Delta} \cdot \frac{1}{\left(\frac{2ax+b}{\sqrt{-\Delta}}\right)^2 + 1}$$

$$\int \frac{dx}{ax^2+bx+c} =$$

$$\frac{4a}{-\Delta}\int\frac{dx}{\left(\frac{2ax+b}{\sqrt{-\Delta}}\right)^2+1} =$$

pongo $y=\dfrac{2ax+b}{\sqrt{-\Delta}}$

$$dy=\frac{2a}{\sqrt{-\Delta}}\,dx$$

$$=\frac{2}{\sqrt{-\Delta}}\int\frac{dy}{y^2+1} =$$

$$=\frac{2}{\sqrt{-\Delta}}\,\operatorname{arctg}\left(\frac{2ax+b}{\sqrt{-\Delta}}\right)$$

CASO 3. $\Delta > 0$

$$ax^2+bx+c = a(x-x_1)(x-x_2)$$

con $x_{1,2}=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$\frac{1}{a(x-x_1)(x-x_2)} = \frac{1}{a(x_1-x_2)}\left(\frac{1}{x-x_1}-\frac{1}{x-x_2}\right)$$

$$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a(x_1-x_2)} \ln\left(\frac{|x-x_1|}{|x-x_2|}\right)$$

Esempio

$$\int \frac{dx}{x^2+3x+2} \qquad x_{1,2} = \frac{-3 \pm \sqrt{9-8}}{2} \begin{cases} -1 \\ -2 \end{cases}$$

$$\parallel$$

$$\int \frac{dx}{(x+1)(x+2)}$$

ora $$\frac{1}{(x+1)(x+2)} = \left(\frac{1}{x+1} - \frac{1}{x+2}\right)$$

quindi

$$\int \frac{dx}{x^2+3x+2} = \ln\left(|x+1|\right) - \ln\left(|x+2|\right)$$

$$= \ln\left(\frac{|x+1|}{|x+2|}\right)$$

altro esempio

$$\int \frac{1}{(x-1)(x+3)} = \frac{1}{4} \ln\left(\frac{|x-1|}{|x+3|}\right)$$

$$\frac{1}{(x-1)(x+3)} = \frac{1}{4}\left(\frac{1}{x-1} - \frac{1}{x+3}\right)$$

$\parallel$ infatti

$$\frac{x+3-x+1}{(x-1)(x+3)} = \frac{4}{(x-1)(x+3)}$$

torniamo a $\int \frac{f(x)}{g(x)}$

Caso generale ($g(x)$ di grado 2)

Passo 1. Se il grado di $f(x) \geq 2$

divido $f(x)$ per $g(x)$ $\quad f(x) = g(x) \cdot \underset{\text{quoziente}}{q(x)} + \underset{\text{resto}}{r(x)}$

(con la divisione dei polinomi)

Esempio:

$$\int \frac{x^4+3x-1}{x^2+2x-1}\,dx$$

$$
\begin{array}{rrrrrr|l}
4 & 3 & 2 & 1 & 0 & & \\
x^4 & +\,0 & +\,0 & +\,3x & -1 & & \underline{x^2+2x-1} \\
-(x^4 & +\,2x^3 & -\,x^2) & & & & x^2-2x+5 \\
\hline
/ & -2x^3 & +\,x^2 & +\,3x & -1 & & \uparrow \text{quoziente} \\
 & -(-2x^3 & -\,4x^2 & +\,2x) & & & \\
\hline
 & & 5x^2 & +\,x & -1 & & \\
 & & -(5x^2 & +\,10x & -5) & & \\
\hline
\text{Resto} \to & & & -9x & +4 & &
\end{array}
$$

$$\underset{x^4+3x-1}{f(x)} \;=\; \underset{(x^2+2x-1)}{g(x)} \cdot \underset{(x^2-2x+5)}{q(x)} \; \underset{-9x+4}{+\,r(x)}$$

$$x^4+3x-1 = (x^2+2x-1)(x^2-2x+5)-9x+4$$

verifico:

$$x^4 - \cancel{2x^3} + 5x^2 + \cancel{2x^3} - 4x^2 + 10x - x^2 + 2x - 5 - 9x + 4$$

$$= x^4 + 3x - 1 \qquad \text{oK}$$

$$\int \frac{f(x)}{g(x)}\, dx = \int \frac{g(x)\, q(x) + r(x)}{g(x)} =$$

$$\int q(x)\, dx + \int \frac{r(x)}{g(x)}\, dx$$

$r(x)$ ha grado al più $= 1$.

$\int q(x)\, dx$ è facile

Nell' esempio

$$\int \frac{x^4 + 3x - 1}{x^2 + 2x - 1}\, dx =$$

$$\int (x^2 - 2x + 5)\, dx + \int \frac{-9x + 4}{x^2 + 2x - 1}\, dx$$

$$\frac{x^3}{3} - x^2 + 5x - \int \frac{9x-4}{x^2+2x-1}\,dx$$

se pongo $y = x^2+2x-1$ ottengo $dy = (2x+2)\,dx$

scrivo $9x-4 = 9x+9-13 = \frac{9}{2}(2x+2) - 13$

$$\int \frac{9x-4}{x^2+2x-1} = \int \frac{\frac{9}{2}(2x+2)\,dx}{x^2+2x-1} - \int \frac{13\,dx}{x^2+2x-1}$$

$$= \frac{9}{2}\int \frac{dy}{y}\Bigg|_{y=x^2+2x-1} - 13\int \frac{dx}{x^2+2x-1}$$

$\downarrow$

faccio coi metodi di prima.

Di solito il trucco è combinare sostituzioni e integrali di funzioni razionali.

$$\int \frac{dx}{x + \sqrt{2x+3}} =$$

$$y = \sqrt{2x+3}$$

$$2x + 3 = y^2$$

$$x = \frac{y^2 - 3}{2}$$

$$dx = y\,dy$$

$$\int \frac{y\,dy}{\frac{y^2-3}{2} + y}$$

oppure

$$\int \frac{dx}{\cos x + \sin x + 3}$$

$$y = \operatorname{tg} \frac{x}{2}$$

$$dy = \frac{1}{2}(1+y^2)\,dx$$

$$\sin x = \frac{2y}{1+y^2}$$

$$\cos x = \frac{1-y^2}{1+y^2}$$

$$=$$

$$\int \frac{2\,dy}{(1+y^2)\left(\frac{1-y^2}{1+y^2} + \frac{2y}{1+y^2} + 3\right)}$$

$$=$$

$$2 \int \frac{dy}{1 - y^2 + 2y + 3(1+y^2)}$$