Rem: $w = (w_j)$ sono tali $w_j = w_{-j} > 0$

$w_j \nearrow \infty$

(crescenti)

$$h(w) := \left\{ u \in \ell^2(\mathbb{C}) \ : \ |u|_w^2 := \sum_{j \in \mathbb{Z}} |u_j|^2 w_j^2 < \infty \right\}$$

Ora considero lo spazio $\mathcal{H}_r(w)$ delle Hamiltoniane

Majorant-regolari (M-regolari) ~~su~~ $B_r(w)$

$P \in \mathcal{H}_r(w) \to \underline{X}_P : B_r(w) \longrightarrow w$ e per

$$P = \sum P^{(d)} \qquad P^{(d)} = \sum_{d_1 + d_2 = d+2} P_{d_1 d_2}$$

$$\sum_{\substack{|\alpha| = d_1 \\ |\beta| = d_2}} P_{\alpha\beta}\, u^\alpha \bar{u}^\beta \ \rightsquigarrow \ \underline{X}_P^{(j,+)} = \sum |P_{\alpha,\beta}|\, \beta_j\, u^\alpha \bar{u}^{\beta - e_j}$$

$$\left(\text{Rem } \overline{X}_P^{(j,-)} = X_P^{(j,+)} \right)$$

$$|\underline{X}_{P_{d_1 d_2}}|_E = \frac{1}{2} \sup_{|u|_w \le 1} \sum_j w_j^2 \left(\sum |P_{\alpha,\beta}| (\beta_j + \alpha_j)\, u^\alpha \bar{u}^{\beta - e_j} \right)^2$$

pongo $v_j = w_j u_j$ se $u \in h(w) \to v \in \ell^2$

$$\sup_{|v|_{\ell_2} \le 1} \sum_j \left(\frac{w_j^2}{\prod w_i^{\alpha_i + \beta_i}} \right)^2 \left(\sum |P_{\alpha,\beta}| (\beta_j + \alpha_j)\, v^\alpha \bar{v}^{\beta - e_j} \right)^2$$

quindi $\quad |P|_{d_1 d_2} = \sup_{|v|_{\ell_2} \le 1} |Y_{d_1 d_2}(P,w)|_{\ell_2}$

$$Y^{(j)}(P,w) = \frac{1}{2} \sum_{\substack{\alpha,\beta \\ |\alpha| = d_1 \\ |\beta| = d_2}} \frac{w_j^2}{w^{\alpha+\beta}} |P_{\alpha,\beta}| (\beta_j + \alpha_j) v^{\alpha} \bar{v}^{\beta - e_j}$$

$$c_w(\alpha,\beta,j) := \frac{w_j^2}{w^{\alpha+\beta}}$$

ora se ho due pesi w e $\tilde{w}$

se $\displaystyle \sup_{\substack{\alpha,\beta,j \\ |\alpha|+|\beta| = d+2 \\ \alpha_j + \beta_j \neq 0}} \frac{c_w(\alpha,\beta,j)}{c_{w'}(\alpha,\beta,j)} < A \equiv A_d$

allora $\forall\ H \in \mathcal{P}^d(w')$ si ha $H \in \mathcal{P}^d(w)$

con $\quad |H|_w \le A_d |H|_{w'}$.

N.B. Se A_d è uniformemente limitata

$\mathcal{H}_r(w') \subseteq \mathcal{H}_r(w)$ ..

Se $\displaystyle \sup_d A_d \left(\frac{r'}{r}\right)^d =: A < \infty$

allora $\mathcal{H}_{r'}(w') \subset \mathcal{H}_r(w)$

quindi in generale mi trovo a calcolare

$$\sup_{\substack{\alpha,\beta,J \\ \alpha_J+\beta_J\neq 0}} \frac{c_w(\alpha,\beta,J)}{c_{w'}(\alpha,\beta,J)} \left(\frac{r'}{r}\right)^{|\alpha|+|\beta|-2} =$$

$$\sup \left(\frac{w'}{w}\right)^{\alpha+\beta-2e_J} \left(\frac{r'}{r}\right)^{|\alpha|+|\beta|-2}$$

In generale non c'è nessun motivo per cui debba esistere $A =$?

Conservazione del Momento:

Considero l'Hamiltoniana (NON regolare)

$$M = \sum_{J\in\mathbb{Z}} J\,|u_J|^2 \qquad (\text{analitica purché } h_w \subseteq h_{1/2})$$

Ricordare che per le hamiltoniane quadratiche in FORMA NORMALE ellittica vale

$$\{M, u^\alpha \bar{u}^\beta\} = i \sum J\cdot(\alpha-\beta)\; u^\alpha \bar{u}^\beta$$

definisco $\pi(\alpha-\beta) = J\cdot(\alpha-\beta)$

Considero il sottospazio $\mathcal{H}_{0,r}(w) \subset \mathcal{H}_r(w)$ delle hamiltoniane regolari che commutano con $\mathcal{M}$

(È un'algebra di Poisson!)

N.B. anche se $\mathcal{M}$ NON è regolare genera un flusso globalmente ben posto su h_w ($\forall$ peso w) $u_J(\tau) = u_J e^{i\tau j}$

Quindi $\mathcal{H}$ commuta con $\mathcal{M}$ se

$$\mathcal{H}(u_J e^{i\tau j}) = \mathcal{H}(u)$$

$$\left(H = \sum_{\substack{\alpha,\beta \\ \pi(\alpha-\beta)=0}} H_{\alpha\beta} u^\alpha \bar{u}^\beta \quad ! \right)$$

Se $u_J \rightsquigarrow u(x) = \sum_J u_J e^{iJx}$

il flusso di $\mathcal{M}$ è la traslazione in x

$$\phi_{\mathcal{M}}(u,\tau) = \sum u_J e^{iJ(x+\tau)} = u(x+\tau)$$

Su $H_{or}(w)$ ho vere regole di monotonia !

quando $c_w(\alpha,\beta,J) = \dfrac{w_J^{\,2}}{\prod_i w_i^{\alpha_i+\beta_i}}$

con $\alpha_J+\beta_J \neq 0$ pongo $\lambda_i := \dfrac{w_i}{w'_i}$

$$\frac{c_w(\alpha,\beta,J)}{c_{w'}(\alpha,\beta,J)} = \frac{1}{\prod_{i\neq\pm J} \lambda_i^{\alpha_i+\beta_i}} \cdot \frac{1}{\lambda_J^{\alpha_J+\beta_J+\alpha_{-J}+\beta_{-J}-2}}$$

se $\alpha_J+\beta_J+\alpha_{-J}+\beta_{-J} \geq 2$ $c_w(\alpha,\beta,J)$ è decrescente

in w !

(Rem $w' \leq w$ se $w'_J \leq w_J$)
cioè $\lambda_J \geq 1$ $\forall J$

Se $\alpha_J+\beta_J+\alpha_{-J}+\beta_{-J} = 1$ $\dfrac{c_w(\alpha,\beta,J)}{c_{w'}(\alpha,\beta,J)} = \dfrac{\lambda_J}{\prod_{i\neq\pm j} \lambda_i^{\alpha_i+\beta_i}}$

Ora per la conservazione del momento

$$\sum_i (\alpha_i-\beta_i)\, i = 0 \implies J = \pm \sum_{i\neq\pm J} (\alpha_i-\beta_i)\, i$$

$$\frac{c_w(\alpha,\beta,J)}{c_{w'}(\alpha,\beta,J)} = \frac{\lambda_{\sum_{i\in\mathbb{N}}(\alpha_i-\beta_i-\alpha_{-i}+\beta_i)\, i}}{\prod_{i\neq j} \lambda_i^{\alpha_i+\beta_i+\alpha_{-i}+\beta_{-i}}}$$

Esempio: (Sobolev) $w_j = \langle j \rangle^{p}$ $\quad w'_j = \langle j \rangle^{p'}$

$$\frac{c_w}{c_{w'}} = \left(\frac{\langle j \rangle^{2}}{\prod \langle i \rangle^{\alpha_i + \beta_i}} \right)^{p-p'}$$

come detto prima se $\alpha_j + \beta_j \geq 2$

allora se $p \geq p' \Rightarrow \left| \frac{c_w}{c_{w'}} \right| \leq 1$

stesso se $\alpha_j + \beta_j + \alpha_{-j} + \beta_{-j} \geq 2$

se $\alpha_j + \beta_j + \alpha_{-j} + \beta_{-j} = 1$

$$* \left(\frac{\langle \sum i(\alpha_i - \beta_i) \rangle}{\prod_{i \neq \pm j} \langle i \rangle^{\alpha_i + \beta_i}} \right)^{p-p'} \leq \left(\frac{1 + \sum |i| (\alpha_i + \beta_i)}{\prod \langle i \rangle^{\alpha_i + \beta_i}} \right)^{p-p'}$$

Ora fissiamo $k = (\alpha_0 + \beta_0 + \alpha_{-1} + \beta_{-1} + \alpha_1 + \beta_1)$

$$* \leq \left(\frac{1 + k + \sum_{\substack{i \geq 2 \\ (i \neq \pm j)}} i \left(\alpha_i + \beta_i + \alpha_{-i} + \beta_{-i} \right)}{\prod_{\substack{i \geq 2 \\ i \neq j}} \langle i \rangle^{\alpha_i + \beta_i + \alpha_{-i} + \beta_{-i}}} \right)^{p-p'}$$

$$\leq 2^{p-p'}(1+k)^{p-p'}\left(\frac{\sum\limits_{i\geq 2}^{\neq j} i\,(\alpha_i+\beta_i+\alpha_{-i}+\beta_{-i})}{\prod\limits_{\substack{i\geq 2\\ i\neq j}}\langle i\rangle^{\alpha_i+\beta_i+\alpha_{-i}+\beta_{-i}}}\right)$$

ma Rem: $|\alpha|+|\beta| = d+2$ (in una comp. omogenea) e $k \leq |\alpha|+|\beta|$

$$\frac{c_W}{c_{W'}} \leq 2^{p-p'}(d+3)^{p-p'}$$

Note $\dfrac{\sum x_i}{\prod x_i} \leq 1$ se $x_i \geq 2$

Per fare in GENERALE

ecco COSTANCE:

Dato $\alpha, \beta \rightsquigarrow \nu(\alpha+\beta)$

$\nu_i = \alpha_i+\beta_i+\alpha_{-i}+\beta_{-i}$; $\nu_1 = \alpha_0+\beta_0+\alpha_1+\beta_1+\alpha_{-1}+\beta_{-1}$

Data una lista $\alpha, \beta \in \mathbb{N}_f^{\mathbb{Z}}$

costruisco la lista

$\{ h > 1$ ripetuto v_h volte$\} \cup \{1$ ripetuto v_1 volte$\}$

$\hat{m}$ è un riarrangiamento decrescente

Esempio: $\alpha_{-3} = 2$, $\alpha_{-2} = 3$, $\alpha_1 = \alpha_0 = 2$, $\alpha_4 = 1$

$\beta_{-7} = 1$, $\beta_5 = 2 = \beta_4$

tutto il resto $= 0$ $\quad (\# \hat{m} = |\alpha| + |\beta|)$

$$\hat{m} = \{7, 5, 5, 4, 4, 4, 3, 3, 2, 2, 2, 1, 1, 1, 1\}$$

Si può riformulare la monotonia in termini di $\hat{m}$.

Lemma $\quad \pi(\alpha - \beta) = 0 \iff \sum \sigma_k \hat{m}_k = 0$

con $\sigma_k = \pm 1$ se $\hat{m}_k \neq 1$ mentre $\sigma_k = \pm 1, 0$ se $\hat{m}_k = 1$

$$\pi(\alpha - \beta) = \sum_{i>0} i(\alpha_i - \beta_i - \alpha_{-i} + \beta_{-i}) =$$

$$\sum_{i>0} \langle i \rangle (\alpha_i - \beta_i - \alpha_{-i} + \beta_{-i}) = \sum \sigma_k \hat{m}_k$$

per $m \neq 0, 1$

$\hat{m}_k = m \quad \alpha_m + \beta_m + \alpha_{-m} + \beta_{-m}$ volte

$\sigma_k = + \quad \alpha_m + \beta_{-m}$ volte $\quad \sigma_k = -$, $\alpha_{-m} + \beta_m$ volte.

Esempio 2: GEVREY

$$W_j(s) = e^{s\langle j\rangle^\theta} \qquad \text{con } 0 < \theta < 1$$

$$C_{W(s)}(\alpha,\beta,j) = e^{2s\langle j\rangle^\theta - s\sum \alpha_i+\beta_i \langle i\rangle^\theta}$$

$$\frac{C_{s+\delta}}{C_s} = e^{-\delta\left(\sum_{i\in\mathbb{N}}(\alpha_i+\beta_i+\alpha_{-i}+\beta_{-i})\langle i\rangle^\theta - 2\langle j\rangle^\theta\right)}$$

<u>Claim:</u> $\sum_{i\in\mathbb{N}}(\alpha_i+\beta_i+\alpha_{-i}+\beta_{-i})\langle i\rangle^\theta - 2\langle j\rangle^\theta) \geq 0$

tranne se $\alpha+\beta = \underline{e}_0$ in cui viene ≥ -1

(ovviamente l'unico caso NON banale è

se $\alpha_j+\beta_j+\alpha_{-j}+\beta_{-j} = 1$)

Riscrivo in termini degli $\hat{n}$

N.B. dato che $\alpha_j+\beta_j+\alpha_{-j}+\beta_{-j}\neq 0$

$\langle j\rangle \le \hat{m}_1$

Lemma: $\sum_{i\in\mathbb{N}}(\alpha_i+\beta_i+\alpha_{-i}+\beta_{-i})\langle i\rangle^\theta = \sum_{k=1}^{\#\hat{m}}\hat{m}_k^\theta$

(Esercizio)

$\sum_{i\in\mathbb{N}}(\alpha_i+\beta_i+\alpha_{-i}+\beta_{-i})\langle i\rangle^\theta - 2\langle j\rangle^\theta \ge$ $\quad (N=\#\hat{m})$

$\sum_{k=1}^{\#\hat{m}}\hat{m}_k^\theta - 2\hat{m}_1^\theta =$

Rem $\sigma_1\hat{m}_1 = -\sum_{k=2}^{N}\sigma_k\hat{m}_k$

$= \sum_{k=2}^{\#\hat{m}}\hat{m}_k^\theta - \hat{m}_1^\theta$

$\hat{m}_1 \le \sum_{k=2}^{N}\hat{m}_k$

$\ge \sum_{k=2}^{N}\hat{m}_k^\theta - \left(\sum \hat{m}_k\right)^\theta$

Se $N=1$ $\quad \alpha_j+\beta_j=0\ \forall j\neq 0$

$\alpha_0+\beta_0=1$ caso

di prima.

Se $N=2$ viene $=0$ (sarebbe $d=0$)

Se $N\ge 3$

$$\hat{m}_2^\theta + \sum_{k=3}^{N} \hat{m}_k^\theta - \left(\hat{m}_2 + \sum_{k=3}^{N} \hat{m}_k \right)^\theta \geq$$

CLAIM $\geq (2 - 2^\theta) \sum_{k=3}^{N} \hat{m}_k^\theta$

Caso $k = 3$ $\qquad x = \frac{m_2}{m_3} \qquad x \geq 1$

$$x^\theta + 1 - (x+1)^\theta \geq (2 - 2^\theta)$$

poi per induzione (Referenza Yuan)

Cioé:

$$m_2^\theta + (2^\theta - 1) \sum \hat{m}_k^\theta - \left(m_2 + \sum \hat{m}_k \right)^\theta \geq 0$$

$\hat{m}_2 \geq \hat{m}_3 \geq \hat{m}_4 \ \ldots \ \geq 1$

é decrescente in tutte le variabili e

per $\hat{m}_i = \hat{m}$ $\forall i \geq 2$ viene $= 0$.

Proiezioni sugli indici:

Dato $I \subset \mathbb{N}_f^{\mathbb{Z}} \times \mathbb{N}_f^{\mathbb{Z}}$ Definisco la

proiezione sugli indici I:

$$H = \sum H_{\alpha\beta} u^{\alpha} \bar{u}^{\beta} \rightarrow \Pi_I H = \sum_{\alpha,\beta \in I} H_{\alpha\beta} u^{\alpha} \bar{u}^{\beta}$$

per esempio $\mathcal{M} := \{ \alpha, \beta : \pi(\alpha - \beta) = 0 \}$

$$\mathcal{H}_{0r} = \Pi_{\mathcal{M}} \mathcal{H}_r$$

Proposizione: Tutte queste proiezioni sono CONTINUE rispetto alla norma dei maggioranti (poi sono idempotent)

Definisco $\mathcal{U}_N = \{ \alpha, \beta : \sum_{\ell \geq 3} \hat{m}_\ell (\alpha + \beta) > K \}$

$\forall\, H \in \mathcal{H}_{0r}^{\geq 1}$; si ha

Lemma: $| \Pi_{\mathcal{U}} H | \leq e^{-\delta K^{\theta}} | H |$

$\cdots_N \cdots_{s+\delta} = \cdots_s$

Altre definizioni utili

Definisco $\quad B_\delta = \{ \alpha, \beta : \quad \hat{m}_3(\alpha+\beta) \le \hat{m}_1^{\,\delta} \}$

e $\quad \mathcal{U} = \mathcal{M} / B_\delta$