Dalla puntata precedente

$$w_j(s) = \langle j \rangle^{p_0} e^{s\langle j\rangle^{\theta} + a|j|}$$

$\mathcal{H}_{r,s}$ hamiltoniane regolari su $B_r(w(s))$

che conservano il momento e t_c.

$$H = \sum_{\substack{|\alpha|+|\beta|\geq 2 \\ \pi(\alpha-\beta)=0}} H_{\alpha\beta}\, u^{\alpha}\bar{u}^{\beta} \qquad (\text{di grado} \geq 0)$$

Oss. $\pi(\alpha-\beta) = \sum i(\alpha_i - \beta_i) = 0$

implica che $P^{(-1)} = Re(H_{0,e_0} u_0)$ la condizione

$d \geq 0$ esclude questi termini.

N.B. se $d = 0$ $\quad H = \sum \lambda_j |u_j|^2 + P(u_0)$

Vario il parametro s.

Lemma 3.5. *If $H \in \mathcal{H}_{r,s}$, then for all $s_1 \geq s$ one has*

$$\|H\|_{r,s_1} \leq \|H\|_{r,s}\,.$$

devo dimostrare che

$$\sup_{\substack{\alpha,\beta:\\ |\alpha|+|\beta|\geq 2 \\ \alpha+\beta \neq \underline{e}_0}} \quad \sup_{s:\, \alpha_j+\beta_j\neq 0} \left(\frac{W(s)}{W(s+\delta)}\right)^{\alpha+\beta-2\underline{e}_j} \leq 1$$

$$\left(\frac{W(s+\delta)}{W(s)}\right)_J := \frac{w_j(s+\delta)}{w_j(s)} = e^{\delta\langle j\rangle^\theta}$$ (tutto il resto si cancella)

quindi voglio mostrare che

$$\sup_{\substack{\alpha,\beta:\\ |\alpha|+|\beta|\geq 2 \\ \alpha+\beta \neq \underline{e}_0}} \quad \sup_{s:\, \alpha_j+\beta_j\neq 0} -\sum \langle i\rangle^\theta(\alpha_i+\beta_i)+2\langle j\rangle^\theta \leq 0$$

cioé $\forall\ \alpha,\beta:\ \pi(\alpha-\beta)=0;\ |\alpha|+|\beta|\geq 2\quad \alpha_j+\beta_j\neq 0$

$$-2\langle j\rangle^\theta + \sum \langle i\rangle^\theta(\alpha_i+\beta_i) \geq 0$$

Rappresentazione utile

(gli $\hat{n}$)

:ar **Definition A.2.** Given a vector $v = (v_i)_{i\in\mathbb{Z}} \in \mathbb{N}^{\mathbb{Z}}_f$ with $|v| \geq 2$ we denote by $\widehat{n} = \widehat{n}(v)$ the vector $(\widehat{n}_l)_{l\in I}$ (where $I \subset \mathbb{N}$ is finite) which is the decreasing rearrangement of

$$\{\mathbb{N} \ni h > 1 \quad \text{repeated } v_h + v_{-h} \text{ times}\} \cup \{1 \quad \text{repeated } v_1 + v_{-1} + v_0 \text{ times}\}$$

Remark A.3. *A good way of envisioning this list is as follows. Given an infinite set of variables $(x_i)_{i\in\mathbb{Z}}$ and a vector $v = (v_i)_{i\in\mathbb{Z}} \in \mathbb{N}^{\mathbb{Z}}_f$ consider the monomial $x^v := \prod_i x_i^{v_i}$. We can write*

$$x^v = \prod_i x_i^{v_i} = x_{j_1}x_{j_2}\cdots x_{j_{|v|}}, \qquad \textit{with} \quad j_k \in \mathbb{Z}$$

then $\widehat{n}(v)$ is the decreasing rearrangement of the list $(\langle j_1\rangle, \ldots, \langle j_{|v|}\rangle)$.

Example A.4. *Let us set*

$$v_{-1} = 2, v_0 = 3, v_1 = 1, v_3 = 1, v_4 = 2.$$

Hence, 1 is repeated 6 times, 3 is repeated 1 time, and 4 is repeated 2 times :

$$\hat{n}_1 = 4, \hat{n}_2 = 4, \hat{n}_3 = 3, \hat{n}_4 = \cdots = \hat{n}_9 = 1$$

Given $\alpha, \beta \in \mathbb{N}^{\mathbb{Z}}_f$ with $|\alpha| + |\beta| \geq 2$ from now on we define

$$\widehat{n} = \widehat{n}(\alpha + \beta) \qquad \text{and set} \quad N := |\alpha| + |\beta|$$

which is the cardinality of $\widehat{n}$. We observe that, $N \geq 2$ and since

ent (30)
$$0 = \sum_{i\in\mathbb{Z}} i(\alpha_i - \beta_i) = \sum_{h>0} h(\alpha_h - \beta_h - \alpha_{-h} + \beta_{-h}),$$

there exists a choice of $\sigma_i = \pm 1, 0$ such that[3]

:ci (31)
$$\sum_l \sigma_l \widehat{n}_l = 0.$$

with $\sigma_l \neq 0$ if $\widehat{n}_l \neq 1$. Hence,

1za (32)
$$\widehat{n}_1 \leq \sum_{l\geq 2} \widehat{n}_l.$$

$$0 = \sum_{h>0} h(\alpha_h + \beta_{-h}) - \sum_{h>0} h(\alpha_{-h} + \beta_h)$$

$$\sum \hat{n} = \sum h\left[(\alpha_h + \beta_{-h}) + (\alpha_{-h} + \beta_h)\right] + \text{cond. di } 0$$

$\hat{n} = h$ compare $(\alpha_h + \beta_{-h} + \alpha_{-h} + \beta_h)$ volte

$\sigma_e : \hat{n}_e = h$ è $+$ $\alpha_h + \beta_{-h}$ volte

$-$ $\alpha_{-h} + \beta_h$ volte

per $h > 1$

per $h = 1$ e $+$ $\alpha_1 + \beta_{-1}$ volte $-$ $\alpha_{-1} + \beta_1$ volte

e 0 $\alpha_0 + \beta_0$ volte.

Ora se $\hat{n} = (\hat{n}_1, \hat{n}_2, \ldots)$

con $N \geq 2$ soddisfa $\sum \sigma_e \hat{n}_e = 0$

con $\sigma_e = 0$ solo se $\hat{n}_e = 1$

allora $\sum_{i=1}^{N} \hat{n}_i^{\theta} - 2\hat{n}_1^{\theta} \geq (2 - 2^{\theta}) \sum_{i \geq 3} \hat{n}_i^{\theta}$

Dim. se $\hat{n}_1 = \cancel{2}\ 1$ allora tutti gli $\hat{n}_i = 1$

quindi mi viene

$$N - 2 \geq (2-2^{\theta}) N - 2 \qquad \text{(Vero!)}$$

Se $\hat{n}_1 > 1$ allora $\sum \sigma_\ell \hat{n}_\ell = 0$

implica $\hat{n}_1 \leq \sum_{\ell \geq 2} \hat{n}_\ell$

$$\sum_{i=1}^{N} \hat{n}_i^{\theta} - 2\hat{n}_1^{\theta} = \sum_{i=2}^{N} \hat{n}_i^{\theta} - \hat{n}_1^{\theta} \geq$$

$$\sum_{i=2}^{N} \hat{n}_i^{\theta} - \left(\sum_{i=2}^{N} \hat{n}_i \right)^{\theta}$$

Se $N=2$ per conservazione del momento

$\hat{n}_1 = \hat{n}_2 \Rightarrow$ vale

Se $N \geq 3$

$$\hat{n}_2^{\theta} + \sum_{i=3}^{N} \hat{n}_i^{\theta} - \left(\hat{n}_2 + \sum_{i=3}^{N} \hat{n}_i \right)^{\theta} \geq (2-2^{\theta}) \sum_{i=3}^{N} \hat{n}_i^{\theta}$$

$$f(x) = x_2^\theta - \left(\sum_{i=2}^{N} x_i\right)^\theta + (2^\theta - 1) \sum_{i=3} x_i^\theta$$

$f(x)$ cresce in $x_2 \geq x_3$

$f(x_3, x_3, x_4 \dots)$ cresce in $x_3 \geq x_4$

$$x_3^\theta - \left(2x_3 + \sum_{i=4}^{N} x_i\right)^\theta + (2^\theta - 1)\left(x_3^\theta + \sum_{i=4}^{N} \dots\right)$$

$$2^\theta x_3^\theta - \left(2x_3 + \sum_{i=4}^{N} x_i\right)^\theta + (2^\theta - 1) \sum_{i=4}^{N} x_i^\theta\Bigg)$$

$f(x_4, x_4, x_4, x_5 \dots)$ cresce in $x_4 \geq x_5$

iterando

$$x_{k+1}^\theta - (k x_{k+1} + \text{Resto})^\theta + (2^\theta - 1)(k-1)\, x_{k+1}^\theta$$

cresce in x_{k+1} (ok così!)

restano tutti = e viene!

Ora dimostro 3.5.

$$\sum \langle l \rangle^{\theta} (\alpha_l + \beta_l) = \sum_{\ell=1}^{N} \hat{m}_\ell^{\theta} \quad \text{(per costruzione)}$$

dato che $\alpha_j + \beta_j \neq 0$ $\langle j \rangle \leq \hat{m}_1$

quindi

$$\sum \langle l \rangle^{\theta} (\alpha_l + \beta_l) - 2\langle j \rangle^{\theta} \geq \sum \hat{m}_\ell^{\theta} - 2\hat{m}_1^{\theta}$$

e il risultato segue.

in particolare se

$$H = \sum H_{\alpha\beta} u^{\alpha} \bar{u}^{\beta} \qquad \text{t.c.} \quad \forall \alpha, \beta$$

$$\sum_{\ell=3}^{|\alpha|+|\beta|} \hat{m}_\ell^{\theta}(\alpha+\beta) > K \qquad \text{allora}$$

$$|H|_{r, s+\delta} < e^{-\delta K} |H|_s$$

I piccoli divisori:

La chiave degli algoritmi BNF è

risolvere le equazioni omologiche

Vediamo lo schema che voglio seguire

Parto da $H = \sum \omega_j |u_j|^2 + P$

con $P \in \mathcal{P}^{\geq 1}$

cerco $S \in \mathcal{P}^{\geq 1}$ t.c.

$$e^{\{S,\cdot\}} H = \sum \omega_j |u_j|^2 + Z + P_+ \quad \text{con} \quad P_+ \in \mathcal{P}^{\geq 2}$$

e $\{\sum \omega_j |u_j|, Z\} = 0$

uso la formula dell'exp

Nota $e^{\{S,\cdot\}} - 1 - \{S,\cdot\}$ mappa $\mathcal{P}^{\geq 0} \to \mathcal{P}^{\geq 2}$

(ci sono almeno 2 parentesi di Poisson)

con S che ha grado ≥ 1 ,

allo stesso modo $e^{\{S,\cdot\}}-1 \quad \mathcal{P}^{\geq 1} \longrightarrow \mathcal{P}^{\geq 2}$

quindi

$$\Pi^{(d\leq 2)}\left(e^{\{S,\cdot\}}H\right) = \sum \omega_j |u_j|^2 + P + \{S, \sum \omega_j |u_j|^2\}$$

REM:

Definizione: dato $I \subset \{(\alpha,\beta) : \pi(\alpha-\beta)=0\}$

(sottoinsieme di indici)

$$\Pi_I H = \sum_{\alpha,\beta\in I} H_{\alpha\beta}\, u^\alpha \bar{u}^\beta$$

$\Pi^{(d\leq d_0)}$ è la proiezione $I := \{\alpha,\beta : |\alpha|+|\beta|\leq d_0+2\}$

<u>Quindi</u>: Bisogna (SE POSSIBILE)

$$P + \{S, \sum \omega_j |u_j|^2\} = Z$$

in modo che Z commuta con $\sum \omega_j |u_j|^2$

Definisco $L_\omega : H_{r,w} \to H_{r,w}$ (?)

non è detto!

$$L_\omega S = \{ \sum \omega_j |u_j|^2 , S \}$$

se $S = \sum S_{\alpha\beta} u^\alpha \bar{u}^\beta$

$$L_\omega S \overset{\sim}{=} \sum S_{\alpha\beta} \, i\omega \cdot (\alpha-\beta) \, u^\alpha \bar{u}^\beta$$

(e poi se $\omega \notin \ell^\infty$ non è detto che sia convergente!)

definisco quindi

$$\Pi_R H = \sum_{\omega(\alpha-\beta)\neq 0} H_{\alpha,\beta} u^\alpha \bar{u}^\beta$$

$$\Pi_{Ker} H = \sum_{\omega\cdot(\alpha-\beta)=0} H_{\alpha\beta} u^\alpha \bar{u}^\beta$$

$\Rightarrow \quad \Pi_R H = H \quad (H \in \mathcal{H}^R_{r,w})$

$L_\omega S = H$ ammette formalmente

una soluzione

$$S_{\alpha,\beta} := \frac{H_{\alpha,\beta}}{i(\omega\cdot(\alpha-\beta))}$$

D_γ: ω: $|\omega_j - j^2| \le \frac{1}{2}$ e

$\forall\, \ell \in \mathbb{Z}^{\mathbb{Z}}_f$

$$|\omega\cdot\ell| \ge \frac{\gamma}{\prod_i (1+\ell_i^2\langle i\rangle^2)}$$

where $\mathrm{ad}_S(\cdot) := \{S,\cdot\}$.

Lemma 3.7. *Fix $s \ge 0$ and $\sigma > 0$ and $\omega \in \mathtt{D}_\gamma$. For any $R \in \mathcal{H}^d_{r,s}$ with $d \ge 1$ and such that $\Pi_{\mathcal{K}} R = 0$, the Homological equation $L_\omega S = R$ has a unique solution $S = L_\omega^{-1} R \in \mathcal{H}^d_{r,s+\sigma}$ such that $\Pi_{\mathcal{K}} S = 0$ and moreover*

$$\left\|L_\omega^{-1} R\right\|_{r,s+\sigma} \le \gamma^{-1} e^{\mathcal{C}_1 \sigma^{-\frac{3}{\theta}}} \|R\|_{r,s} \tag{22}$$

Questo è doloroso

Devo far vedere che

$$\frac{W(s+\sigma)^{2j-(\alpha+\beta)}}{|\omega\cdot(\alpha-\beta)|} \le \gamma^{-1} e^{C_1\sigma^{-\frac{3}{\theta}}} W(s)^{2j-(\alpha+\beta)}$$

$\forall\ \alpha,\beta,j$ t.c. $|\alpha|+|\beta|\ge 3$ $\quad \pi(\alpha-\beta)=0$

$\alpha_j+\beta_j\neq 0$

$$\frac{e^{-\sigma\left(\sum(\alpha_i+\beta_i)\langle i\rangle^{\theta}-2\langle j\rangle\right)}}{|\omega\cdot(\alpha-\beta)|} \le e^{C_1\sigma^{-\frac{3}{\theta}}}$$

Abbiamo visto che il numeratore ≤ 1

quindi se $|\omega\cdot(\alpha-\beta)|\ge 1$ ovvio

Lemma (Banale)

se $\sum i^2(\alpha_i-\beta_i) > 2|\alpha-\beta|$ [e $\alpha\neq\beta$]

(Rem $|\ |$ è la norma ℓ_1)

allora $|\omega\cdot(\alpha-\beta)| \geq 1$

$\omega_j = j^2 + \xi_j$ con $|\xi_j| \leq \frac{1}{2}$

$$|\omega\cdot(\alpha-\beta)| \geq \left|\sum i^2(\alpha_i-\beta_i)\right| - |\xi\cdot\alpha-\beta|$$

$$\geq 2|\alpha-\beta| - |\alpha-\beta|\,|\xi|_\infty$$

$$\geq |\alpha-\beta| \geq 1$$

LEMMA (Bourgain)

Lemma A.7. *Consider $\alpha,\beta\in\mathcal{M}$ with $\alpha\neq\beta$ and $|\alpha|+|\beta|\geq 3$. If*

$$\left|\sum_i(\alpha_i-\beta_i)i^2\right| \leq 2\sum_i|\alpha_i-\beta_i|, \tag{38}$$

then for all j such that $\alpha_j+\beta_j\neq 0$ one has

$$\sum_i|\alpha_i-\beta_i|\langle i\rangle^{\theta/2} \leq C_*\left(\sum_i(\alpha_i+\beta_i)\langle i\rangle^\theta - 2\langle j\rangle^\theta\right), \qquad C_* = \frac{7}{2-2^\theta} \tag{39}$$