Definizione: $\omega \in \mathbb{R}^{\mathbb{Z}}$ è diofanteo se

$$\forall\, \ell \in \mathbb{Z}^{\mathbb{Z}}_f : \quad |\omega \cdot \ell| \geq \frac{\gamma}{\prod_i (1+\ell_i^2 \langle i \rangle^2)}$$

Fisso $Q := \{ \omega \in \mathbb{R}^{\mathbb{Z}} : |\omega_j - j^2| \leq \frac{1}{2} \}$

e chiamo D_γ i Diofantei in Q

where $\mathrm{ad}_S(\cdot) := \{S, \cdot\}$.

Lemma 3.7. *Fix $s \geq 0$ and $\sigma > 0$ and $\omega \in \mathtt{D}_\gamma$. For any $R \in \mathcal{H}^d_{r,s}$ with $d \geq 1$ and such that $\Pi_{\mathcal{K}} R = 0$, the Homological equation $L_\omega S = R$ has a unique solution $S = L_\omega^{-1} R \in \mathcal{H}^d_{r,s+\sigma}$ such that $\Pi_{\mathcal{K}} S = 0$ and moreover*

$$\left\| L_\omega^{-1} R \right\|_{r,s+\sigma} \leq \gamma^{-1} e^{\mathcal{C}_1 \sigma^{-\frac{3}{\theta}}} \|R\|_{r,s} \tag{22}$$

Devo far vedere che

$$\frac{w(s+\sigma)^{2j-(\alpha+\beta)}}{|\omega \cdot (\alpha - \beta)|} \leq \gamma^{-1} e^{C_1 \sigma^{-\frac{3}{\theta}}} w(s)^{2j-(\alpha+\beta)}$$

$\forall\, \alpha, \beta, j$ t.c. $\quad |\alpha| + |\beta| \geq 3 \quad \pi(\alpha - \beta) = 0$

$\alpha_j + \beta_j \neq 0$

$$\frac{e^{-\sigma\left(\sum(\alpha_i+\beta_i)\langle i\rangle^\theta - 2\langle j\rangle\right)}}{|\omega\cdot(\alpha-\beta)|} \leq e^{C_1\sigma^{-3/\theta}}$$

Abbiamo visto che il numeratore ≤ 1

quindi se $|\omega\cdot(\alpha-\beta)| \geq 1$ ovvio

Lemma (Bonde)

se $\sum i^2(\alpha_i-\beta_i) > 2|\alpha-\beta|$ [e $\alpha\neq\beta$]

(Rem $|\ |$ è la norma ℓ_1)

allora $|\omega\cdot(\alpha-\beta)| \geq 1$

$\omega_j = j^2 + \xi_j$ con $|\xi_j| \leq \frac{1}{2}$

$$|\omega\cdot(\alpha-\beta)| \geq \left|\sum i^2(\alpha_i-\beta_i)\right| - |\xi\cdot\alpha-\beta|$$

$$\geq 2|\alpha-\beta| - |\alpha-\beta|\,|\xi|_\infty$$

$$\geq |\alpha-\beta| \geq 1$$

Lemma A.7. *Consider $\alpha, \beta \in \mathcal{M}$ with $\alpha \neq \beta$ and $|\alpha|+|\beta| \geq 3$. If*

$$\left|\sum_i (\alpha_i - \beta_i) i^2\right| \leq 2 \sum_i |\alpha_i - \beta_i|\,, \tag{38}$$

then for all j such that $\alpha_j + \beta_j \neq 0$ one has

$$\sum_i |\alpha_i - \beta_i| \langle i \rangle^{\theta/2} \leq C_* \left(\sum_i (\alpha_i + \beta_i) \langle i \rangle^\theta - 2\langle j \rangle^\theta \right), \qquad C_* = \frac{7}{2 - 2^\theta} \tag{39}$$

Facciamo prima 3 casi banali

Proof. Let us first prove the lemma in the case that $\alpha_i = \beta_i$ for all $i \neq 0$. Then (39) reads

1

$$|\alpha_0 - \beta_0| \leq C_*(|\alpha_0 + \beta_0| - 2)$$

Now

$$|\alpha_0 - \beta_0| \leq |\alpha_0 + \beta_0| \leq \frac{C_*}{3}|\alpha_0 + \beta_0| \leq C_*(|\alpha_0 + \beta_0| - 2\frac{|\alpha_0 + \beta_0|}{3}) \leq C_*(|\alpha_0 + \beta_0| - 2)$$

since $|\alpha| + |\beta| \geq 3$.

2 $\alpha_i = \beta_i \quad \forall\, i \neq 0, j_0 \quad (j_0 \neq 0)$ non è compatibile con $\pi(\alpha-\beta) = 0$.

3 $\alpha_i = \beta_i \quad \forall\, i \neq 0, j_1, j_2 \quad (j_1 \neq j_2)$ e $|\alpha_{j_i} - \beta_{j_i}| = 1 \quad i = 1,2$

allora $\pi(\alpha-\beta) = 0 \Rightarrow j_1 = -j_2$

quindi $\quad (38) \Rightarrow \quad 2 J_1^2 \leq 2(2 + |\alpha_0 - \beta_0|)$

quindi $\quad J_1^{\theta/2} \leq (2 + |\alpha_0 - \beta_0|)^{\theta/4}$

e $\quad \sum |\alpha_i - \beta_i| \langle i \rangle^{\theta/2} = 2|J_1|^{\theta/2} + |\alpha_0 - \beta_0|$

$$\leq 2(2 + |\alpha_0 - \beta_0|)^{\theta/4} + |\alpha_0 - \beta_0|$$

$$\leq 4 + 3(\alpha_0 + \beta_0)$$

Dato che $|\alpha| + |\beta| \geq 3$ ho che $\alpha_0 + \beta_0 \neq 0$

quindi $\quad \sum |\alpha_i - \beta_i| \langle i \rangle^{\theta/2} \leq 7(\alpha_0 + \beta_0)$

ora

$$\sum (\alpha_i + \beta_i) \langle i \rangle^\theta - 2\langle j \rangle^\theta = \alpha_0 + \beta_0 + 2\langle J_1 \rangle^\theta - 2\langle j \rangle^\theta$$

ora se $j = J_1$ viene $\alpha_0 + \beta_0$

mentre se $j = 0$ viene $\geq \alpha_0 + \beta_0$

quindi in conseguenza

$$\sum |\alpha_i - \beta_i| \langle i \rangle^{\theta/2} \leq 7\left(\sum (\alpha_i + \beta_i) \langle i \rangle^\theta - 2\langle j \rangle^\theta \right) \quad \blacksquare$$

Resta α, β t.c. esistono almeno 2 indici

$J_1 \neq J_2 \neq 0$ con $\alpha_{J_i} + \beta_{J_i} \neq 0$ e almeno

un $\underline{J_i}$ soddisfa $\alpha_{\underline{J_i}} + \beta_{\underline{J_i}} \geq 2$

Mi servono NOTAZIONI (Brrrrr)

data una lista $\ell \in \mathbb{Z}^{\mathbb{Z}}_f$

costruisco la lista non ordinata:

$L = \{ h \in \mathbb{Z}/\{0\}$ ripetuto $|\ell_h|$ volte $\}$

e chiamo $\hat{m}(\ell)$ un riordinamento di L

in modo che $|\hat{m}_1| \geq |\hat{m}_2| \geq \ldots \geq |\hat{m}_D|$

Nota in L NON ho traccia di ℓ_0!

Esempio: $\ell_{-7}=1$ $\ell_{-3} = -4$, $\ell_{-2} = 1$, $\ell_0 = 6$, $\ell_5 = -3$

$L = \{-7, -3, -3, -3, -3, -2, 5\}$

$\hat{m} = \{-7, 5, -3, -3, -3, -3, -2\}$ (se ho $7, -7$ metto prima il positivo)

Quindi dati $\alpha, \beta \in \mathcal{M}$ definisco

$$\hat{n} = \hat{n}(\alpha+\beta) \quad e \quad \hat{m} = \hat{m}(\alpha-\beta)$$

Rem. $N = \# \hat{n} = |\alpha| + |\beta| \geq 3$

$D = \# \hat{m}$ se $\exists$ almeno 2 J $(J_1 \neq J_2 \neq 0)$

con $\alpha_{J_i} - \beta_{J_i} \neq 0$ e almeno 1 ha

$$\alpha_{J_i} - \beta_{J_i} \geq 2 \Rightarrow \# \hat{m} \geq 3$$

Osservazione: $D + \alpha_0 + \beta_0 \leq N$

inoltre $\tilde{m} = \hat{m}, 1, \underbrace{1, \ldots, 1}_{N-D}$

soddisfa $\tilde{m}_i \lesssim \hat{n}_i \quad \forall\, i = 1, \ldots, N$

$$\sum (\alpha_i - \beta_i) i^2 = \sum_{\ell=1}^{D} \hat{m}_\ell^2 \sigma_\ell \quad \text{con } \sigma_\ell = \pm 1$$

inoltre $\sum |\alpha_i - \beta_i| = D + |\alpha_0 - \beta_0|$

[Nota Bene se $\hat{m}_a = \hat{m}_b \Rightarrow \sigma_a = \sigma_b$

la conservazione del momento viene

$$\sum i\,(\alpha_i-\beta_i) = \sum \hat{m}_\ell \sigma_\ell$$ con gli stessi σ_ℓ di prima

$\sigma = (\sigma_\ell)_{\ell=1}^{D}$ $\quad \sigma_\ell = +$ se $\alpha_{\hat m_\ell} - \beta_{\hat m_\ell} > 0$

$\sigma_\ell = -$ se $\alpha_{\hat m_\ell} - \beta_{\hat m_\ell} < 0$ (se è $=0$ NON è nella lista $\hat{m}$)

Lemma A.10. *Given $\alpha \neq \beta \in \mathbb{N}_f^{\mathbb{Z}}$, such that $\pi(\alpha-\beta)=0$, $N, D \geq 3$ and satisfying* (38)*, we have*

iare

$$|\widehat{m}_1| \leq 7 \sum_{l\geq 3} \widehat{n}_l^2 \,. \tag{48}$$

Proof. By (38),(45) and (46)

$$\widehat{m}_1^2 + \sigma_1\sigma_2 \widehat{m}_2^2 \;\leq\; 2(D + |\alpha_0-\beta_0|) + \sum_{l=3}^{D} \widehat{m}_l^2 \leq 2N + \sum_{l=3}^{D} \widehat{m}_l^2 \leq 2N + \sum_{l=3}^{N} \widehat{n}_l^2 \leq 7 \sum_{l=3}^{N} \widehat{n}_l^2 \,.$$

since (recall $N \geq 3$) $2N \leq 6(N-2) \leq 6\sum_{l=3}^{N} \widehat{n}_l^2$.

If $\sigma_1\sigma_2 = 1$ then

$$|\widehat{m}_1|, |\widehat{m}_2| \leq \sqrt{7\sum_{l\geq 3} \widehat{n}_l^2}.$$

If $\sigma_1\sigma_2 = -1$

$$(|\widehat{m}_1| + |\widehat{m}_2|)(|\widehat{m}_1| - |\widehat{m}_2|) = m_1^2 - m_2^2 \leq 7\sum_{l\geq 3} \widehat{n}_l^2.$$

Now, if $|m_1| \neq |m_2|$ then

$$|\widehat{m}_1| + |\widehat{m}_2| \leq 7\sum_{l\geq 3} \widehat{n}_l^2.$$

Conversely, if $|m_1| = |m_2|$, by (47), $m_1 \neq m_2$, hence $m_1 = -m_2$. By substituting this relation into (44), we have

$$2|\widehat{m}_1| \leq \sum_{l\geq 3} |\widehat{m}_l| \leq \sum_{l\geq 3} \widehat{n}_l^2 \,,$$

concluding the proof. $\square$

Conclusione della dim:

α, β : $N, \Delta \geq 3$; $\sum J^2 (\alpha_j - \beta_j) \leq 2 \sum |\alpha_i - \beta_i|$

quindi $|\hat{m}_1| \leq 7 \sum_{\ell \geq 3} \hat{m}_\ell$

$$\sum \langle i \rangle^{\frac{\theta}{2}} |\alpha_i - \beta_i| \leq 2 |\hat{m}_1|^{\frac{\theta}{2}} + \sum_{\ell = 3}^{\Delta} |\hat{m}_\ell|^{\frac{\theta}{2}} + |\alpha_0 - \beta_0|$$

$$\leq 2 \cdot 7^{\frac{\theta}{2}} \left(\sum \hat{m}_\ell^2 \right)^{\theta/2} + \sum_{\ell = 3}^{\Delta} \hat{m}_\ell^{\frac{\theta}{2}}$$

$$\leq 2 \cdot 7^{\frac{\theta}{2}} \sum \hat{m}_\ell^{\theta} + \sum \hat{m}_\ell^{\frac{\theta}{2}}$$

$$\leq \left(2 \cdot \sqrt{7} + 1 \right) \sum_{\ell \geq 3} \hat{m}_\ell^{\theta} \leq$$

$$\frac{7}{2 - 2^{\theta}} \left(\sum (\alpha_i + \beta_i) \langle i \rangle^{\theta} - 2 \langle j \rangle^{\theta} \right)$$

$\blacksquare$

REM

$$\left(\sum (\alpha_i + \beta_i) \langle i \rangle^{\theta} - 2 \langle j \rangle^{\theta} \right) \geq (2 - 2^{\theta}) \sum_{\ell \geq 3} \hat{m}_\ell^{\theta}$$

$\forall\ \alpha, \beta, j$ t.c. $\quad |\alpha| + |\beta| \geq 3 \quad \pi(\alpha - \beta) = 0$

$\alpha_j + \beta_j \neq 0 \quad ; \quad \left| \sum j^2 (\alpha_j - \beta_j) \right| \leq 2 \sum |\alpha_i - \beta_i|$

$$\frac{e^{-\sigma\left(\sum(\alpha_i+\beta_i)\langle i\rangle^{\theta} - 2\langle j\rangle\right)}}{|\omega\cdot(\alpha-\beta)|} \leq$$

$$\frac{e^{-\sigma\frac{(2-2^{\theta})}{7}\left(\sum\langle i\rangle^{\theta/2}|\alpha_i-\beta_i|\right)}}{\gamma} \prod_i \left(1 + |\alpha_i - \beta_i| \langle i\rangle\right)^2 = ✱$$

ora pongo $\quad a = \frac{\sigma(2-2^{\theta})}{14}$

$$f_i(x) = -a\langle i\rangle^{\theta/2} x + \ln\left(1 + x\langle i\rangle\right)$$

$$✱ = \gamma^{-1} e^{2\sum_i f_i(|\alpha_i - \beta_i|)}$$

Rem $\quad \ln(1 + x\langle i\rangle) \leq 1 + \ln x + \ln\langle i\rangle$

$$f_i(x) \leq -a\langle i\rangle^{\theta/2}x + \ln x + 1 + \ln\langle i\rangle$$

ora

$$\max_{x\geq 1}\left(-a\langle i\rangle^{\theta/2}x + \ln x\right) = \begin{cases} -a\langle i\rangle^{\theta/2} & \text{if } \langle i\rangle \geq i_0 \\ -1 + \ln\frac{1}{a} - \frac{\theta}{2}\ln\langle i\rangle & \langle i\rangle < i_0 \end{cases}$$

con $i_0 := \left(\frac{1}{a}\right)^{2/\theta}$

infatti derivo

$$-a\langle i\rangle^{\theta/2} + \frac{1}{x} = 0 \quad \Rightarrow \quad x_{max} = \frac{1}{a\langle i\rangle^{\frac{\theta}{2}}}$$

quindi se $\langle i\rangle \geq i_0$ $\quad x_{max} < 1$ e il

massimo è in $x=1$

se $\langle i\rangle < i_0$ calcolo in x_{max}

$$\sum_{i:\, e_i\neq 0} f_i(|e_i|) \leq \sum_{\substack{i\geq i_0\\ e_i\neq 0}} f_i(|e_i|) + \sum_{\langle i\rangle < i_0} f_i(|e_i|)$$

$$\le -a \sum_{\substack{i \ge i_0 \\ i:\, \ell_i \neq 0}} \langle i \rangle^{\frac{\theta}{2}} + 2 i_0 \left(-1 + \ln \frac{1}{a}\right) - \frac{\theta}{2} \sum \ln \langle i \rangle$$

Tolgo tutte le cose NEGATIVE!

$$\le 2 i_0 \ln \frac{1}{a} = 2 \left(\frac{1}{a}\right)^{\frac{2}{\theta}} \ln \frac{1}{a}$$

Quindi $\star \le e^{4 \left(\frac{1}{a}\right)^{\frac{2}{\theta}} \ln \frac{1}{a}}$

Rem: $a = \dfrac{\sigma(2 - 2^{\theta})}{14}$ ∎

<u>Esercizio</u>: FARE IL CASO SOBOLEV.

Peso 0:

(con $\pi = 0$)

$$\sup_{\substack{j, \boldsymbol{\alpha}, \boldsymbol{\beta} \\ \boldsymbol{\alpha}_j + \boldsymbol{\beta}_j \neq 0}} \frac{\langle j \rangle^2}{\prod_i \langle i \rangle^{\alpha_i + \beta_i}} \le \frac{\widehat{n}_1}{\prod_{l \ge 2} \widehat{n}_l} \le \frac{\sum_{l=2}^N \widehat{n}_l + \cancel{|\pi|}}{\prod_{l=2}^N \widehat{n}_l}$$

$$\le \frac{(N-1)\widehat{n}_2 + \cancel{|\pi|}}{\prod_{l=2}^N \widehat{n}_l} \le \frac{N + \cancel{|\pi|}}{\prod_{l=3}^N \widehat{n}_l}.$$

shown that

Peso 1: se $\left| \sum i^2 (\alpha_i - \beta_i) \right| < 2 \left(\sum |\alpha_i + \beta_i| \right)$

e $N, \Delta \ge 3$

sicuro si può far meglio!

$$\prod_i \left(1+\left|\boldsymbol{\alpha}_i-\boldsymbol{\beta}_i\right|\langle i\rangle\right) \le e^{27}(1+|\cancel{\pi}|)^3 N^6 \prod_{l=3}^{N} \widehat{n}_l^{\tau_0}$$

Dimostrare quindi

Thm.

$$\left|L_\omega^{-1}R\right|_{r-\rho,\eta-\sigma,\mathrm{w}(p+\tau,s,a)} \le \gamma^{-1}\mathcal{C}_2(r/\rho,\sigma,\tau)|R|_{r,\eta,\mathrm{w}(p,s,a)}$$

(Su CMP ci sono le costanti)