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Estimates for normal forms of differential equations near an equilibrium point

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1. Introduction

The theory of normal forms, invented by Poincaré, gives simple forms to which a differential equation can be reduced in the neighbourhood of an equilibrium point by a change of variables (see [1], Chap. 5 for a nice introduction to normal form theory and see [2] for a more detailed treatment). Now, while the problem is easily solved in the class of formal vector-valued power series [1], in the class of analytic vector fields convergence problems arise and the power series giving the normalizing transformation are generally divergent [2]. Nevertheless in order to obtain significant information on the behaviour of solutions up to finite times it is often sufficient to normalize only to a finite order. In view of this, in the present paper, we give, making use of the algorithm of Lie transform, a normal form theorem for vector fields around an equilibrium point, providing estimates for convergence radii and remainders, when the normalization is brought up to a finite order r, generalizing a similar theorem recently given for the Hamiltonian case [5]. The method can be extended to $r = \infty$ in the case considered by Poincaré and Dulac, and their classical theorems are recovered. The estimates given here do not instead allow to recover the stronger results of Siegel and Brjuno.

However, looking for stability results over a large, although finite, time interval, and taking into account the classical theorem of Carathéodory-Cartan, the only interesting case is that of a system of weakly coupled harmonic oscillators. By considering the particular case of a reversible system with highly non-resonant frequencies, we are able to show how, by optimizing the order r of normalization, one obtains exponential estimates for the time of stability of the solutions of the type recently obtained by Nekhoroshev [9] for the Hamiltonian case.

We will give the basic definitions of the algorithm of Lie transform and a corresponding existence theorem with the necessary estimates in section 2. Section 3 is devoted to the algebraic part of the main theorem on normal forms, which will be given in section 4. In section 5, as a corollary of the main theorem, we will give a theorem of Poincaré and Dulac referring to the case $r = \infty$. In section 6 we will give the application to the exponential estimate for the case of weakly coupled harmonic oscillators.

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2. The Lie transform

Let V_ε and X_ε be two vector fields on $\boldsymbol{C}^d$, depending on a parameter ε varying in a neighbourhood of zero in $\boldsymbol{C}$, and let F_ε be the flow generated by V_ε according to $\frac{d}{d\varepsilon} F_\varepsilon(z) = V_\varepsilon(F_\varepsilon(z))$. The Lie transform of X_ε by V_ε at "time" ε is the vector field $F_{\varepsilon *} X_\varepsilon$, where the linear operator $F_{\varepsilon *}$ is defined by

$$F_{\varepsilon *} X_\varepsilon(z) := DF_\varepsilon(F_\varepsilon^{-1}(z)) \cdot X_\varepsilon(F_\varepsilon^{-1}(z)) .$$

If $V(\varepsilon, z)$ and $X(\varepsilon, z)$ are analytic in a neighbourhood of 0 in $\boldsymbol{C} \times \boldsymbol{C}^d$, then, for every (ε, z) in a neighbourhood of 0 in $\boldsymbol{C} \times \boldsymbol{C}^d$ we will have $F_{\varepsilon *} X_\varepsilon(z) = \sum_{k \geqq 0} X^{(k)}(z) \varepsilon^k$, with $X^{(k)}(z) := \frac{1}{k!} \frac{\partial^k}{\partial \varepsilon^k}\Big|_{\varepsilon = 0} F_{\varepsilon *} X_\varepsilon(z)$. Usually the Lie transform is defined as $F_\varepsilon^* X_\varepsilon$ with $F_\varepsilon^* := (F_\varepsilon^{-1})_*$, and so $F_{\varepsilon *} X_\varepsilon$ should be called the inverse Lie transform. The advantage of the inverse transform is a simpler expression of the recursive formulas for $X^{(k)}$ (see below and [6]; for an extensive bibliography on Lie series see [12]). The key identity is

$$\frac{\partial}{\partial \varepsilon} F_{\varepsilon *} Y = - L_{V_\varepsilon} F_{\varepsilon *} Y, \tag{2.1}$$

where Y is an autonomous vector field, and L_{V_ε} denotes the the Lie derivative of Y with respect to the vector field V_ε. We recall that, given two vector fields W and Z, the Lie derivative of Z with respect to W is $L_W Z := DZ \cdot W - DW \cdot Z$. Notice that one has $\frac{\partial}{\partial \varepsilon} F_\varepsilon^* Y = F_\varepsilon^* L_{V_\varepsilon} Y$, with $F_\varepsilon^* L_{V_\varepsilon} Y \neq L_{V_\varepsilon} F_\varepsilon^* Y$ since V_ε is non-autonomous. This produces more complicated recursive formulas.

The aim of Lie series theory is to express $X^{(k)}$ in terms of the coefficients of the expansion in ε of V_ε and X_ε. This is given by the following

Theorem 1. *Let* $V_\varepsilon = \sum_{k \geqq 0} V_k \varepsilon^k$, $X_\varepsilon = \sum_{k \geqq 0} X_k \varepsilon^k$ *be two vector fields on* $\boldsymbol{C}^d$ *depending on a parameter* ε, *with* $V(\varepsilon, z)$, $X(\varepsilon, z)$ *analytic in a neighbourhood of* $0 \in \boldsymbol{C} \times \boldsymbol{C}^d$. *Denote by* F_ε *the flow generated by* V_ε, *and let*

$$F_{\varepsilon *} X_\varepsilon = \sum_{k \geqq 0} X^{(k)} \varepsilon^k , \quad F_{\varepsilon *} X_j = \sum_{k \geqq 0} X_j^{(k)} \varepsilon^k .$$

Then one has

$$X^{(n)} = \sum_{j=0}^{n} X_j^{(n-j)} , \tag{2.2}$$

where $X_j^{(n)}$ is recursively defined by

$$X_j^{(0)} := X_j\,, \quad X_j^{(n)} := -\frac{1}{n}\sum_{k=1}^{n} L_{V_{k-1}} X_j^{(n-k)}\,. \tag{2.3}$$

Proof. Taking into account the power expansions of X_ε and V_ε in ε, using (2.1), the bilinearity of Lie derivative, and Newton's rule for derivatives, we have

$$\begin{aligned} X_k^{(n)} &= \frac{1}{n!}\frac{\partial^n}{\partial\varepsilon^n}\bigg|_{\varepsilon=0} F_{\varepsilon *} X_k = -\frac{1}{n!}\frac{\partial^{n-1}}{\partial\varepsilon^{n-1}}\bigg|_{\varepsilon=0} L_{V_\varepsilon} F_{\varepsilon *} X_k \\ &= -\frac{1}{n!}\sum_{j=0}^{n-1}\binom{n-1}{j} L_{j!V_j}\frac{\partial^{n-1-j}}{\partial\varepsilon^{n-1-j}}\bigg|_{\varepsilon=0} F_{\varepsilon *} X_k \\ &= -\frac{1}{n}\sum_{j=1}^{n} L_{V_{j-1}}\frac{1}{(n-j)!}\frac{\partial^{n-j}}{\partial\varepsilon^{n-j}}\bigg|_{\varepsilon=0} F_{\varepsilon *} X_k\,, \end{aligned}$$

and this gives (2.2). Moreover one has

$$\begin{aligned} X^{(n)} &= \frac{1}{n!}\frac{\partial^n}{\partial\varepsilon^n}\bigg|_{\varepsilon=0} F_{\varepsilon *} X_\varepsilon = \frac{1}{n!}\sum_{k\geqq 0}\frac{\partial^n}{\partial\varepsilon^n}\bigg|_{\varepsilon=0} \varepsilon^k F_{\varepsilon *} X_k \\ &= \frac{1}{n!}\sum_{k\geqq 0}\sum_{j=0}^{n}\binom{n}{j}\frac{\partial^j}{\partial\varepsilon^j}\bigg|_{\varepsilon=0}\varepsilon^k\frac{\partial^{n-j}}{\partial\varepsilon^{n-j}}\bigg|_{\varepsilon=0} F_{\varepsilon *} X_k \\ &= \sum_{j=0}^{n}\frac{1}{(n-j)!}\frac{\partial^{n-j}}{\partial\varepsilon^{n-j}}\bigg|_{\varepsilon=0} F_{\varepsilon *} X_j\,, \end{aligned}$$

i.e. (2.3), so that the statement is proven.

Now we give some definitions and notations:
We consider the space H_k of the $\boldsymbol{C}^d$-valued homogeneous polynomials of degree k on $\boldsymbol{C}^d$. In the space H_k we consider the basis $\{h_{j,\alpha}\}_{1\leqq j\leqq d,\,|\alpha|=k}$, with $\alpha\in \boldsymbol{Z}_+^d$ and $|\alpha| := \sum_{j=1}^{d}\alpha_j$, defined by $h_{j,\alpha}(z) := z_1^{\alpha_1}\,\ldots\,z_d^{\alpha_d}\,e_j$, $\{e_j\}_{1\leqq j\leqq d}$ being a basis of $\boldsymbol{C}^d$. Via this basis a generic element $V\in H_k$ will be written as $V = \sum_{1\leqq j\leqq d,\,|\alpha|=k} V^{j,\alpha} h_{j,\alpha}$, with $V^{j,\alpha}\in\boldsymbol{C}$. Moreover, use will be made of the norm on H_k defined by

$$\|V\| := \sum_{j,\alpha} |V^{j,\alpha}|\,.$$

We shall consider domains $\mathscr{D}_\varrho$ of $\boldsymbol{C}^d$ defined by

$$\mathscr{D}_\varrho := \left\{ z = (z_1,\ldots,z_d)\in\boldsymbol{C}^d : \|z\| = \left(\sum_{j=1}^{d} z_j\bar z_j\right)^{1/2} < \varrho \right\}$$

$$\bar{\mathscr{D}}_\varrho := \left\{ z = (z_1,\ldots,z_d)\in\boldsymbol{C}^d : \|z\| = \left(\sum_{j=1}^{d} z_j\bar z_j\right)^{1/2} \leqq \varrho \right\}$$

and $\|\cdot\|_\varrho$ will denote the supremum norm on $\bar{\mathscr{D}}_\varrho$.

We are now interested in giving an estimate on the norm of $L_V X$ when X and V are two homogeneous polynomials. This is given by the following

Lemma 1. *If $X \in H_n$ and $V \in H_m$, then*

$$\| L_V X \| \leqq (m+n) \| V \| \| X \| . \tag{2.4}$$

Proof. From the definition of Lie derivative we have

$$\begin{aligned} L_V X &= \sum_{1 \leqq j,\, k \leqq d,\, |\alpha| = m,\, |\beta| = n} V^{j,\alpha} X^{k,\beta} L h_{j,\alpha} h_{k,\beta} \\ &= \sum V^{j,\alpha} X^{k,\beta} (\partial_j h_{k,\beta} \cdot h_{j,\alpha} - \partial_k h_{j,\alpha} \cdot h_{k,\beta}) \\ &= \sum V^{j,\alpha} X^{k,\beta} (\beta_j h_{k,\alpha+\beta-\delta_j} - \alpha_k h_{j,\alpha+\beta-\delta_k}) \\ &= \sum (\beta_j V^{j,\alpha} X^{k,\beta} - \alpha_j V^{k,\alpha} X^{j,\beta}) h_{k,\alpha+\beta-\delta_j} \end{aligned}$$

($\delta_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ where "1" is at place j). From this, and $\alpha_j \leqq m$, $\beta_j \leqq n$, one gets

$$\| L_V X \| \leqq (m+n) \sum_{j,k,\alpha,\beta} | V^{j,\alpha} | | X^{k,\beta} | = (m+n) \sum_{j,\alpha} | V^{j,\alpha} | \sum_{k,\beta} | X^{k,\beta} | ,$$

and (2.4) follows.

Consider now a nonautonomous vector field $V_\varepsilon = \sum_{k \geqq 0} V_k \varepsilon^k$, with V_k homogeneous polynomials; this induces a flow F_ε on $\boldsymbol{C}^d$, and we are interested in considering the time-one map defined by $\varphi(z) := F_1(z)$ on a neighbourhood of the origin of $\boldsymbol{C}^d$. Given an autonomous vector field $X = \sum_{k \geqq 0} X_k$, with X_k homogeneous polynomials, the transformation φ above transforms it into $\varphi_* X = \sum_{k \geqq 0} X^{(k)}$, with $X^{(k)}$ given by the formulas (2.2) and (2.3) in Theorem 1, with $\varepsilon = 1$. The analyticity properties of such transformation are given in the following

Theorem 2. *Let $X = \sum_{k \geqq 0} X_k$, $V_\varepsilon = \sum_{k \geqq 0} V_k \varepsilon^k$ be two vector fields with $X_k \in H_{k+1}$ and $V_k \in H_{k+2}$; assume $\| X_k \| \leqq M_1 / \varrho_1^k$ and $\| V_k \| \leqq M_2 / \varrho_2^k$ with positive real constants $M_1, M_2, \varrho_1, \varrho_2$. Consider the time-one Lie transform $\varphi_* = F_{1*}$ defined by Theorem 1. Then, denoting*

$$\varphi_* X = \sum_{k \geqq 0} X^{(k)} ,$$

one has $X^{(n)} \in H_{n+1}$, and $\varphi_* X$ is an analytic vector field in the domain $\mathscr{D}_{\varrho *}$, with

$$\varrho_* = \frac{\varrho_1}{1 + \varrho_1 (3 M_2 + 1/\varrho_2)} . \tag{2.5}$$

Moreover, one has, in the closed domain $\bar{\mathscr{D}}_{\varrho'}$, for any positive $\varrho' < \varrho_*$, the bounds

$$\| \varphi_* X \|_{\varrho'} \leqq \varrho' M_1 \left(1 - \frac{\varrho'}{\varrho_*}\right)^{-1}, \tag{2.6}$$

$$\left\| \sum_{k \geqq n} X^{(k)} \right\|_{\varrho'} \leqq \varrho' M_1 (\varrho'/\varrho_*)^n \left(1 - \frac{\varrho'}{\varrho_*}\right)^{-1}. \tag{2.7}$$

Proof. We first consider $X_\varepsilon = \sum_{k \geqq 0} X_k \varepsilon^k$, with X_k given above, and we note that $\varphi_* X$ is nothing but $F_{\varepsilon *} X_\varepsilon$ at $\varepsilon = 1$. It is easy to verify, by (2.2) and (2.3), that if $X_k \in H_{k+1}$, and $V_k \in H_{k+2}$, then $X_j^{(n)} \in H_{n+j+1}$, and therefore $X^{(n)} \in H_{n+1}$. In order to prove the theorem we look for a sequence of positive constants C_j^n such that $\| X_j^{(n)} \| \leqq C_j^n$. Since, by Lemma 1,

$$\| X_j^{(n)} \| \leqq \frac{1}{n} \sum_{k=1}^{n} \| L_{V_{k-1}} X_j^{(n-k)} \| \leqq \frac{n+j+2}{n} \sum_{k=1}^{n} \| V_{k-1} \| \, \| X_j^{(n-k)} \|$$

and since $\| V_k \| \leqq M_2/\varrho_2^k$, we can recursively define

$$C_j^0 := \| X_j \|, \quad C_j^n := \frac{n+j+2}{n} M_2 \varrho_2 \sum_{k=1}^{n} C_j^{n-k} \frac{1}{\varrho_2^k}.$$

We prove now that

$$C_j^n \leqq \frac{n+j}{n} \left(3 M_2 + \frac{1}{\varrho_2}\right) C_j^{n-1}.$$

Indeed, this is trivially true for $n = 1$; for $n > 1$ we isolate the first term in the sum, and write

$$\begin{aligned} C_j^n &= \frac{n+j+2}{n} M_2 C_j^{n-1} + \frac{n+j+2}{n} M_2 \sum_{k=1}^{n-1} C_j^{n-k-1} \frac{1}{\varrho_2^k} \\ &\leqq \left(\frac{n+j+2}{n} M_2 + \frac{1}{\varrho_2}\right) C_j^{n-1} \\ &\leqq \frac{n+j}{n} \left(3 M_2 + \frac{1}{\varrho_2}\right) C_j^{n-1}, \end{aligned}$$

where the definition of C_j^{n-1} and the trivial inequality

$$\frac{n+j+2}{n} \leqq 3 \frac{n+j}{n}$$

have been used. So the stated inequality is proven. Then we have

$$\| X_j^{(n)} \| \leqq \frac{(n+j)!}{n! \, j!} \left(3 M_2 + \frac{1}{\varrho_2}\right)^n \| X_j \|,$$

and therefore

$$\| X^{(n)} \| \leqq \sum_{k=0}^{n} \binom{n}{k} \left(3M_2 + \frac{1}{\varrho_2} \right)^{n-k} \frac{M_1}{\varrho_1^k}$$

$$= M_1 \left(\frac{1}{\varrho_1} + 3M_2 + \frac{1}{\varrho_2} \right)^n .$$

This proves (2.5). Then (2.6) and (2.7) follow by observing that

$$\left\| \sum_{k \geqq n} X^{(k)} \right\|_{\varrho'} \leqq \varrho' M_1 \sum_{k \geqq n} (\varrho'/\varrho_*)^k = \varrho' M_1 (\varrho'/\varrho_*)^n \left(1 - \frac{\varrho'}{\varrho_*} \right)^{-1} ,$$

and that (2.6) is nothing but (2.7) for $n = 0$. This ends the proof.

3. The Normal Form Theorem

Starting now with a vector field $X = \sum_{k \geqq 0} X_k$, with $X_k \in H_{k+1}$, we try to use the Lie transform φ_* of Theorem 2, in order to determine a suitable vector field $V_\varepsilon = \sum_{k \geqq 0} V_k \varepsilon^k$, with $V_k \in H_{k+2}$, such that $\varphi_* X$ has the simplest possible form. So we deduce an equation for both V_ε and the transformed vector field $\varphi_* X$. To this end we use the equation

$$F_{\varepsilon *} X_\varepsilon = Z_\varepsilon = \sum_{k \geqq 0} Z_k \varepsilon^k , \quad 0 \leqq |\varepsilon| \leqq 1 , \tag{3.1}$$

where $F_{\varepsilon *}$ is the Lie transform generated by the unknown vector field V_ε. The vector field $Z = \varphi_* X$ is said to be a normal form for X if $L_S Z = 0$, where S is the diagonalizable part of the linear vector field X_0. We need to put the eq. (3.1) in a more explicit form. Since $X_0 = Z_0$, we have that $\varphi_* X = Z$ is equivalent to

$$\frac{\partial}{\partial \varepsilon} F_{\varepsilon *} X_\varepsilon = \frac{\partial}{\partial \varepsilon} Z_\varepsilon , \quad 0 \leqq |\varepsilon| \leqq 1 .$$

We now use the identity

$$\frac{\partial}{\partial \varepsilon} F_{\varepsilon *} X_\varepsilon = - L_{V_\varepsilon} F_{\varepsilon *} X_\varepsilon + F_{\varepsilon *} \frac{\partial}{\partial \varepsilon} X_\varepsilon ,$$

which, by (2.2) and (2.3), and the relation

$$\frac{\partial}{\partial \varepsilon} X_\varepsilon = \sum_{k \geqq 0} (k+1) X_{k+1} \varepsilon^k ,$$

can be explicitly written as

$$(n+1) X^{(n+1)} = - \sum_{k=0}^{n} L_{V_k} X^{(n-k)} + \sum_{k=0}^{n} (k+1) X_{k+1}^{(n-k)} .$$

By isolating in the first sum the term for $k = n$, since $X^{(j)} = Z_j$, one has

$$L_{X_0} V_n - (n+1) Z_{n+1} = \sum_{k=0}^{n-1} L_{V_k} Z_{n-k} - \sum_{k=0}^{n} (k+1) X_{k+1}^{(n-k)}, \tag{3.2}$$

and this is a recursive equation for V_n and Z_{n+1}, since the r.h.s. is known once $V_0, \ldots, V_{n-1}$ and $Z_0, \ldots, Z_n$ are known.

To solve (3.2) we need to discuss equations of the form $L_X V = W$. To this end we make use of the following

Lemma 2. *Let A be an element of H_1, and let $\sigma(A) = \{\lambda_1, \ldots, \lambda_d\}$ be the spectrum of A. Denoting by $L_{A,k}$ the restriction of L_A to H_k, $L_{A,k}$ is in lower triangular form, and*

$$\sigma(L_{A,k}) = \{(\Lambda|\alpha) - \lambda_j, \quad 1 \leqq j \leqq d, \quad \alpha \in Z_+^d, |\alpha| = k\} \tag{3.3}$$

is the spectrum of $L_{A,k}$ where $\Lambda \in \mathbf{C}^d$ is the vector $(\lambda_1, \ldots, \lambda_d)$, and $(\,|\,)$ is the usual scalar product. Moreover if $V \in H_k$ has the form $V = \sum_{1 \leqq j \leqq d, |\alpha| = k} V^{j,\alpha} h_{j,\alpha}$, then it is

$$(L_A V)^{j,\alpha} = ((\Lambda|\alpha) - \lambda_j) V^{j,\alpha} + \sum_{k=2}^{d} \eta_{0,\alpha_{k-1}} \sigma_k (\alpha_k + 1) V^{j,\alpha+\delta_k-\delta_{k-1}} - \sigma_j V^{j-1,\alpha}, \tag{3.4}$$

where $h_{j,\alpha}(z) = z^\alpha e_j$, $\{e_j\}_{1 \leqq j \leqq d}$ is the basis of $\mathbf{C}^d$ in which A is in canonical Jordan form, $\sigma_k = 0$ if $\lambda_k \neq \lambda_{k-1}$, else $\sigma_k = 1$ or 0 as the case may be, and $\eta_{jk} = 0$ if $j = k$, $\eta_{jk} = 1$ otherwise.

Proof. First we prove (3.4). Write $A(z) = \sum_{k=1}^{d} A^k(z) e_k$, with $A^k(z) = \lambda_k z_k + \sigma_k z_{k-1}$, since A is in canonical Jordan form. Then, by the definition of Lie derivative, one has

$$\begin{aligned}
L_A h_{j,\alpha}(z) &= D h_{j,\alpha}(z) \cdot A(z) - A(h_{j,\alpha}(z)) \\
&= \sum_{k=1}^{d} A^k(z) \frac{\partial}{\partial z_k} h_{j,\alpha}(z) - A^k(z^\alpha e_j) e_k \\
&= \sum_{k=1}^{d} \alpha_k z^{\alpha-\delta_k} (\lambda_k z_k + \sigma_k z_{k-1}) e_j - \lambda_j z^\alpha e_j - \sigma_{j+1} z^\alpha e_{j+1} \\
&= (\Lambda|\alpha) h_{j,\alpha}(z) + \sum_{k=2}^{d} \sigma_k \alpha_k h_{j,\alpha+\delta_{k-1}-\delta_k}(z) - \lambda_j h_{j,\alpha}(z) - \sigma_{j+1} h_{j+1,\alpha}(z) \\
&= ((\Lambda|\alpha) - \lambda_j) h_{j,\alpha}(z) + \sum_{k=2}^{d} \sigma_k \alpha_k h_{j,\alpha+\delta_{k-1}-\delta_k}(z) - \sigma_{j+1} h_{j+1,\alpha}(z).
\end{aligned}$$

From this, equality (3.4) is easily deduced. We now prove (3.3). To this end we order the multi-indices (j, α) by lexicographic order as follows: (j, α) precedes (k, β) if and only if the first non-zero difference $k - j$, $\beta_1 - \alpha_1, \ldots, \beta_d - \alpha_d$ is positive. Since $(j, \alpha) < (j, \alpha - \delta_k + \delta_{k-1})$ and $(j, \alpha) < (j+1, \alpha)$, it turns out that

L_A is a lower triangular matrix with respect to the basis $\{h_{j,\alpha}\}$ and (3.3) follows. So the lemma is proven.

We come now to the solution of eq. (3.2). First note that from (3.4) it follows that $H_k = N_k \oplus R_k$ where

$$\begin{aligned} N_k &= \{V \in H_k \mid L_S V = 0, S \text{ the diagonalizable part of } A\} \\ &\equiv \{V \in H_k \mid V = \sum_{(\Lambda|\alpha) = \lambda_j} V^{j,\alpha} h_{j,\alpha}\} \end{aligned}$$

and

$$\begin{aligned} R_k &= \{V \in H_k \mid L_A W = V, W \in H_k\} \\ &\equiv \{V \in H_k \mid V = \sum_{(\Lambda|\alpha) \neq \lambda_j} V^{j,\alpha} h_{j,\alpha}\}\,. \end{aligned}$$

Now, denote by $W_{n+1} \in H_{n+2}$ the r.h.s. of (3.2). Then, we solve (3.2) by simply equating $(n+1)\, Z_{n+1}$ to the part of W_{n+1} belonging to N_{n+2}, and using the remaining terms to determine V_n. Precisely we use the lower triangular form of the operator L_{X_0} stated in Lemma 2 to recursively define the coefficients of V_n and Z_{n+1} as follows:

if $(\Lambda|\alpha) \neq \lambda_j$, then

$$\begin{aligned} V_n^{j,\alpha} &:= ((\Lambda|\alpha) - \lambda_j)^{-1} \\ &\quad \cdot \left(W_{n+1}^{j,\alpha} + \sigma_j V_n^{j-1,\alpha} - \sum_{k=2}^{d} \eta_{0,\alpha_{k-1}} \sigma_k (\alpha_k + 1) V_n^{j,\alpha+\delta_k-\delta_{k-1}} \right) \\ Z_{n+1}^{j,\alpha} &:= 0\,, \end{aligned} \tag{3.5}$$

if $(\Lambda|\alpha) = \lambda_j$, then

$$\begin{aligned} V_n^{j,\alpha} &:= 0 \\ Z_{n+1}^{j,\alpha} &:= -\frac{W_{n+1}^{j,\alpha}}{n+1}\,, \end{aligned} \tag{3.6}$$

with

$$\begin{aligned} W_1 &:= -X_1 \\ W_{n+1} &:= \sum_{k=0}^{n-1} L_{V_k} Z_{n-k} - \sum_{k=0}^{n} (k+1)\, X_{k+1}^{(n-k)} \\ &\equiv \sum_{k=1}^{n} L_{V_{k-1}} Z_{n+1-k} - \sum_{k=1}^{n+1} k X_k^{(n+1-k)}\,. \end{aligned} \tag{3.7}$$

Now we choose a fixed integer $r \geqq 2$ and define V_n for $n \leqq r-2$ by (3.5), while keeping $V_n = 0$ for $n > r-2$. So we obtain the following

Theorem 3 (The Normal Form Theorem). *Let $X = \sum_{k \geqq 0} X_k$, with $X_k \in H_{k+1}$, be an analytic vector field, and let S be the diagonalizable part of X_0. For every integer $r \geqq 2$ there exists a nonautonomous polynomial vector field V_ε of degree r*

without constant and linear terms such that the transformed vector field $\varphi_* X$ *is in normal form up to terms of degree* r *in its expansion, i.e.*

$$\varphi_* X = X_0 + \sum_{k=1}^{r-1} Z_k + R_r, \tag{3.8}$$

with $Z_k \in H_{k+1}$ *such that* $L_S Z_k = 0$, *and the remainder* R_r *is an analytical vector field with no terms of degree smaller than* $r + 1$ *in its expansion.*

4. The main theorem

Now we will use Theorem 2 to give estimates on the radius of convergence and on the norm of the remainder terms R_r in the Normal Form Theorem.

Theorem 4. *Let* X, V_ε *and* $r \geqq 2$ *be defined as in Theorem 3. Assume that* $\| X_k \| \leqq M/\varrho^k$ *for* $k \geqq 1$, *with positive real constants* M, ϱ, *and that* X_0 *is in Jordan canonical form. Then the remainder* R_r *in* (3.8) *is an analytical vector field in the domain* $\mathscr{D}_{\varrho_r}$, *with*

$$\varrho_r = \frac{\varrho}{1 + M_r(3 + K_r)}, \tag{4.1}$$

and in the closed domain $\bar{\mathscr{D}}_{\varrho'}$, *for any positive* $\varrho' < \varrho_r$, *one has the bound*

$$\| R_r \|_{\varrho'} \leqq \varrho' M' (\varrho'/\varrho_r)^r \left(1 - \frac{\varrho'}{\varrho_r}\right)^{-1} \tag{4.2}$$

where

$$M_r = \begin{cases} \max\{1, 2M A_r\}/2 & \text{if } X_0 \text{ is diagonal} \\ \max\{1, 4\mu_r^{d-1} M A_r\}\, \mu_r^{2(d-1)}/2 & \text{otherwise} \end{cases}$$

$$K_r = \begin{cases} 24 & \text{if } X_0 \text{ is diagonal} \\ 24/\mu_r^{d-1} & \text{otherwise} \end{cases}$$

$$\mu_r = \max\{1{,}6(d - s) A_r\}$$

$$A_r = \max_{2 \leqq k \leqq r} \left\{ \frac{|\alpha| - 1}{|(\Lambda|\alpha) - \lambda_j|}, |\alpha| = k, \lambda_j \in \sigma(X_0), (\Lambda|\alpha) \neq \lambda_j \right\}$$

$$s = \text{number of distinct eigenvalues of } X_0$$

$$M' = \max\{\| X_0 \|, M\}$$

Proof. In order to be able to use Theorem 2 we must estimate $\| V_n \|$ from the recursive formulas (3.5). Let us first consider the special case in which X_0 is diagonal; then, by (3.5), one has

$$\| V_n \| \leqq A_r \frac{\| W_{n+1} \|}{n + 1}.$$

We now come to the estimate of $\| V_n \|$. First we note that, by (3.6), one has

$$\| Z_n \| \leqq \frac{\| W_n \|}{n},$$

and we look for a sequence of suitable positive constants C_n, with $n \geqq 1$, such that $\| V_n \| \leqq C_{n+1}$. Since

$$\| W_1 \| = \| X_1 \| \leqq M/\varrho ,$$

$$\| W_n \| \leqq A_r \left((n+2) \sum_{k=1}^{n-1} \frac{\| W_k \|}{k} \frac{\| W_{n-k} \|}{n-k} + \sum_{k=1}^{n} k \| X_k^{(n-k)} \| \right) \quad n \geqq 2 ,$$

$$\| X_j^{(0)} \| = \| X_j \| \leqq M/\varrho^j , \quad j \geqq 1 ,$$

and

$$\| X_j^{(n)} \| \leqq A_r \frac{n+j+2}{n} \sum_{k=1}^{n} \frac{\| W_k \|}{k} \| X_j^{(n-k)} \| \quad n \geqq 1 ,$$

we can recursively define C_n by

$$C_1 := A_r M/\varrho , \qquad C_n := 2 \sum_{k=1}^{n-1} C_k C_{n-k} + \frac{1}{n} \sum_{k=1}^{n} k D_{k,n-k}$$

$$D_{j,0} := A_r M/\varrho^j , \quad D_{j,n} := 2 \frac{n+j}{n} \sum_{k=1}^{n} C_k D_{j,n-k} .$$

Since, by the definition above, it is

$$\frac{j D_{j,n}}{n+j} = \frac{2}{n} \sum_{k=1}^{n} C_k (n-k+j) \frac{j D_{j,n-k}}{n-k+j} ,$$

then one has

$$C_k \leqq \beta_k / 2 , \tag{4.3}$$

with the sequence $\{\beta_n\}_{n \geqq 1}$ recursively defined by

$$\beta_1 := N_r/\varrho , \qquad \beta_n := \sum_{k=1}^{n-1} \beta_k \beta_{n-k} + \sum_{k=1}^{n} \alpha_{k,n-k} \tag{4.4}$$

$$\alpha_{j,0} := N_r/\varrho^j , \quad \alpha_{j,n} := \frac{n+j}{n} \sum_{k=1}^{n} \beta_k \alpha_{j,n-k} - \frac{1}{n} \sum_{k=1}^{n} k \beta_k \alpha_{j,n-k} \tag{4.5}$$

with $N_r = \max\{1, 2 A_r M\}$. So, we look for an estimate of the sequences (4.4) and (4.5).

We first prove that

$$\sum_{k=1}^{n} \alpha_{k,n-k} \leqq n \alpha_{1,n-1} . \tag{4.6}$$

To this end, defining the formal power series

$$\alpha_j(z) := \sum_{n \geqq 0} \alpha_{j,n} z^n$$

$$\alpha(z_1, z_2) := \sum_{j \geqq 1} \alpha_j(z_2) z_1^j = \sum_{j \geqq 1, n \geqq 0} \alpha_{j,n} z_1^j z_2^n$$

$$\hat{\alpha}(z) := \alpha(z, z) = \sum_{n \geqq 1} \left(\sum_{k=1}^{n} \alpha_{k,n-k} \right) z^n$$

$$\beta(z) := \sum_{k \geqq 1} \beta_k z^k ,$$

one has, by (4.4),

$$\beta(z) = \beta(z)^2 + \hat{\alpha}(z) , \tag{4.7}$$

and, by (4.5),

$$\begin{aligned} z \frac{d}{dz} \alpha_j(z) &= z \frac{d}{dz} (\alpha_j(z) \beta(z)) + j \alpha_j(z) \beta(z) - z \left(\frac{d}{dz} \beta(z) \right) \alpha_j(z) \\ &= z \beta(z) \frac{d}{dz} \alpha_j(z) + j \beta(z) \alpha_j(z) . \end{aligned}$$

This gives the equation

$$\frac{d}{dz} \alpha_j(z) = j \frac{\beta(z)/z}{1 - \beta(z)} \alpha_j(z) , \quad \alpha_j(0) = \alpha_{j,0} = N_r / \varrho^j , \tag{4.8}$$

which can be solved as

$$\alpha_j(z) = N_r \left(\frac{\exp \xi(z)}{\varrho} \right)^j , \quad \xi(z) := \int_0^z \frac{\beta(s)/s}{1 - \beta(s)} \, ds ,$$

and from this, using the definition above for $\hat{\alpha}(z)$, there follows

$$\hat{\alpha}(z) = N_r \sum_{j \geqq 1} ((z/\varrho) \exp \xi(z))^j = \frac{N_r (z/\varrho) \exp \xi(z)}{1 - (z/\varrho) \exp \xi(z)} = \frac{z \alpha_1(z)}{1 - z \alpha_1(z)/N_r} . \tag{4.9}$$

Since, by (4.9) and (4.7), $z \alpha_1(z) \prec \hat{\alpha}(z) \prec \beta(z)$ ($\prec$ denotes Cauchy majorization), one has, by (4.8) with $j = 1$,

$$\begin{aligned} \hat{\alpha}(z) \prec \frac{z \alpha_1(z)}{1 - \beta(z)} &= z \alpha_1(z) \left(1 + \frac{\beta(z)}{1 - \beta(z)} \right) = z \alpha_1(z) + z^2 \frac{d}{dz} \alpha_1(z) \\ &= \sum_{n \geqq 1} \alpha_{1,n-1} z^n + \sum_{n \geqq 1} (n-1) \alpha_{1,n-1} z^n = \sum_{n \geqq 1} n \alpha_{1,n-1} z^n . \end{aligned}$$

So the stated inequality (4.6) is proven.

From (4.5) one has

$$n\alpha_{1,n-1} = \frac{n}{n-1} \sum_{k=1}^{n-1} \beta_k (n-k)\, \alpha_{1,n-k-1}\,,$$

and using the inequality (4.6) and the latter formula we are led to introduce two new sequences $\{\gamma_n\}_{n\geqq 1}$, and $\{\delta_n\}_{n\geqq 1}$ recursively defined by

$$\gamma_1 := N_r/\varrho\,, \quad \gamma_n := \sum_{k=1}^{n-1} \gamma_k\, \gamma_{n-k} + \delta_n$$

$$\delta_1 := N_r/\varrho\,, \quad \delta_n := 2 \sum_{k=1}^{n} \gamma_k\, \delta_{n-k}\,,$$

and one can immediately see that the inequalities

$$\beta_n \leqq \gamma_n, \quad \delta_n \leqq \gamma_n$$

hold. So, only the sequence $\{\gamma_n\}_{n\geqq 1}$ is relevant, and we have

$$\beta_n \leqq \varepsilon_n \tag{4.10}$$

where the sequence $\{\varepsilon_n\}_{n\geqq 1}$ is recursively defined by

$$\varepsilon_1 := N_r/\varrho\,, \quad \varepsilon_n := 3 \sum_{k=1}^{n-1} \varepsilon_k\, \varepsilon_{n-k}\,.$$

The latter definition implies

$$\varepsilon_n = 3^{n-1} (N_r/\varrho)^n\, \sigma_n\,, \tag{4.11}$$

with

$$\sigma_1 := 1\,, \quad \sigma_n := \sum_{k=1}^{n-1} \sigma_k\, \sigma_{n-k}\,,$$

and we only need to estimate the sequence $\{\sigma_n\}_{n\geqq 1}$.

An estimate for σ_n is found as follows.

Let $\sigma(z) = \sum_{k\geqq 1} \sigma_k z^k$. Then we have $\sigma(z) = z + \sigma(z)^2$ or, alternatively,

$$\sigma(z) = (1 - (1 - 4z)^{1/2})/2\,.$$

If we set $f(z) = (1 - 4z)/4$, then we have

$$\frac{1}{n!}\frac{d^n}{dz^n}\sigma(z) = \frac{1}{2(n!)}\left(1 - \frac{1}{2}\right)\left(2 - \frac{1}{2}\right)\dots\left(n - 1 - \frac{1}{2}\right) f(z)^{-n+1/2}\,,$$

so

$$\sigma_n = \frac{1}{n!}\frac{d^n}{dz^n}\sigma(0) \leqq \frac{1}{2n} f(0)^{-n+1/2} = \frac{4^{n-1}}{n}\,. \tag{4.12}$$

This gives the required estimate. Finally we use the inequalities (4.10) and (4.3) to get

$$\| V_n \| \leqq C_{n+1} \leqq \tfrac{1}{2} \varepsilon_{n+1} ,$$

and in conclusion we have, by (4.11) and (4.12),

$$\| V_n \| \leqq \frac{12^n}{2(n+1)} \left(\frac{N_r}{\delta}\right)^{n+1} \leqq \frac{N_r}{2\varrho} \left(\frac{12 N_r}{\varrho}\right)^n \equiv \frac{M_r}{\varrho} \left(\frac{K_r M_r}{\varrho}\right)^n .$$

This gives the estimate if X_0 is diagonal.

The case when X_0 is in Jordan canonical form can be reduced to the previous one as follows.
Define the isomorphism $V \mapsto \hat{V}$ of H_n by

$$\hat{V}^{j,\alpha} := V^{j,\alpha} \mu_r^{(\sum_{k=1}^{d} k\alpha_k - j)}$$

which corresponds to transforming vector fields by the dilatation $z_k \mapsto \mu_r^k z_k$.

Since $\| \hat{V} \| = \sum_{j,\alpha} | V^{j,\alpha} | \mu_r^{(\sum_{k=1}^{d} k\alpha_k - j)}$, one has

$$\mu_r^{n-d} \| V \| \leqq \| \hat{V} \| \leqq \mu_r^{nd-1} \| V \| . \tag{4.13}$$

From (3.5), and the above definition of $\hat{V}^{j,\alpha}$, there follows

$$((\Lambda|\alpha) - \lambda_j) \hat{V}_n^{j,\alpha} = \hat{W}_{n+1}^{j,\alpha} + \frac{\sigma_j}{\mu_r} \hat{V}_n^{j-1,\alpha} - \frac{1}{\mu_r} \sum_{k=2}^{d} \eta_{0,\alpha_{k-1}} \sigma_k (\alpha_k + 1) \hat{V}_n^{j,\alpha_k + \delta_k - \delta_{k-1}} .$$

Taking the modules, and summing over j and α, one has

$$\| \hat{V}_n \| \leqq \frac{3(d-s) A_r}{\mu_r} \| \hat{V}_n \| + A_r \frac{\| \hat{W}_{n+1} \|}{n+1} \leqq \frac{1}{2} \| \hat{V}_n \| + A_r \frac{\| \hat{W}_{n+1} \|}{n+1} ,$$

or equivalently,

$$\| \hat{V}_n \| \leqq 2 A_r \frac{\| \hat{W}_{n+1} \|}{n+1} .$$

Since the isomorphism $V \mapsto \hat{V}$ is induced by an isomorphism of C^d, from the properties of the Lie derivative one has

$$\widehat{L_V W} = L_{\hat{V}} \hat{W},$$

for every pair of vector fields V and W. Therefore one has, by (3.7) and Theorem 1,

$$\hat{W}_n = \sum_{k=1}^{n-1} L_{\hat{V}_{k-1}} \hat{Z}_{n-k} - \sum_{k=1}^{n} k \hat{X}_k^{(n-k)} ,$$

where $\hat{X}_k^{(n-k)}$ denotes the Lie transform of $\hat{X}_k$ by $\hat{V}_\varepsilon = \sum_{k\geq 0} \hat{V}_k \varepsilon^k$ at order $n-k$, i.e.

$$\hat{X}_j^{(0)} := \hat{X}_j \quad \hat{X}_j^{(n)} := -\frac{1}{n}\sum_{k=1}^{n} L_{\hat{V}_{k-1}} \hat{X}_j^{(n-k)} .$$

In conclusion, one has, by (4.13),

$$\| \hat{X}_j \| \leqq \mu_r^{(j+1)d-1} \| X_j \| \leqq (\mu_r^d/\varrho)^j \mu_r^{d-1} M ,$$

and one can apply to $\| \hat{V}_n \|$ the same estimates obtained for $\| V_n \|$, with A_r, M, ϱ substituted by $2A_r$, $\mu_r^{d-1} M$, ϱ/μ_r^d respectively. Therefore one has

$$\| \hat{V}_n \| \leqq \mu_r^d \frac{\hat{N}_r}{2\varrho}\left(\frac{12\mu_r^d \hat{N}_r}{\varrho}\right)^n, \quad \hat{N}_r = \max\{1, 4A_r\mu_r^{d-1} M\} ,$$

and, by (4.13),

$$\| V_n \| \leqq \frac{\| \hat{V}_n \|}{\mu_r^{n+2-d}} \leqq \frac{\hat{N}_r \mu_r^{2(d-1)}}{2\varrho}\left(\frac{12\mu_r^{d-1} \hat{N}_r}{\varrho}\right)^n \equiv \frac{M_r}{\varrho}\left(\frac{K_r M_r}{\varrho}\right)^n .$$

So we have obtained for $\| V_n \|$ an estimate of the form required to apply Theorem 2 with $\varrho_1 = \varrho$, $\varrho_2 = \varrho/K_r M_r$, $M_1 = M'$, $M_2 = M_r/\varrho$. The statement follows by straightforward application of that theorem. This ends the proof.

5. The Poincaré-Dulac Theorem

In the previous section we have given general estimates for normal forms of vector fields up to a finite order r. In this section we will make some considerations about the case $r = \infty$. A look at the constants entering in Theorem 4 shows that it is essential to know the behaviour of A_r as r goes to infinity. To this end we assume that the d-uple $\Lambda = (\lambda_1, \ldots, \lambda_d)$ of eigenvalues of $X_0 \in H_1$ is in the Poincaré domain, i.e. that the convex hull $K(\Lambda)$ of the d points $\lambda_1, \ldots, \lambda_d$ in the complex plane does not contain the origin. In such case one has the following

Lemma 3. *If Λ is in the Poincaré domain then there exists a number $K > 0$ such that*

$$|(\Lambda \,|\, \alpha) - \lambda_j| \geqq K(|\alpha| - 1), \quad 1 \leqq j \leqq d$$

for all $|\alpha| \geqq 2$ such that $(\Lambda \,|\, \alpha) \neq \lambda_j$. *Moreover*

$$\max\{|\alpha| \geqq 2 : (\Lambda \,|\, \alpha) = \lambda_j, \quad 1 \leqq j \leqq d\} < \infty ,$$

and there exists an integer p, $1 \leqq p \leqq d$, and a numbering of the λ_j, such that

$$\begin{aligned} &(\Lambda \,|\, \alpha) \neq \lambda_j \quad \text{for all } j \leqq p , \\ &(\Lambda \,|\, \alpha) = \lambda_j \quad \textit{implies } j > p \quad \textit{and } \alpha_k = 0 \quad \text{for all } k \geqq j . \end{aligned}$$

Proof. For the first statement see [10], for the remaining ones see [3].

The lemma above implies that there exists a number $K > 0$ such that $A_r \leqq K$ for all $r \geqq 2$. So Theorem 4 has the following

Corollary (The Poincaré-Dulac theorem). *Let* $X = \sum_{k \geqq 0} X_k$ *be an analytic vector field in a neighbourhood of zero in* $\mathbf{C}^d$ *with* $X_k \in H_{k+1}$ *and* X_0 *in Jordan canonical form. Let* $\Lambda = (\lambda_1, \dots \lambda_d)$ *be the d-uple of eigenvalues of* X_0. *If* Λ *is in the Poincaré domain then* X *is bianalytically equivalent to a polynomial vector field of degree*

$$s = \max\{|\alpha| \geqq 2 : (\Lambda \mid \alpha) = \lambda_j, \quad 1 \leqq j \leqq d\}$$

of the form

$$X_0 + Z, \quad L_S Z = 0,$$

where S *is the diagonal part of* X_0. *Moreover there exists an integer* p, $1 \leqq p \leqq d$, *such that*

$$Z^k = 0 \quad \textit{for all } k \leqq p,$$

and

$$\frac{\partial}{\partial z_j} Z^k = 0 \quad \textit{for all } j \geqq k > p.$$

The theorem above was first proven by Poincaré [10] when X_0 is diagonal and non resonant, i.e. when $(\Lambda \mid \alpha) \neq \lambda_j$ for all $\alpha \in Z_+^d$ such that $|\alpha| \geqq 2$, and was then generalized to the resonant case by Dulac [3]. After these early results several attempts were made to obtain convergence with less stringent conditions on the spectrum of the linear part X_0 of the vector field. The first success in this direction was obtained by Siegel [11], who proved convergence by assuming, besides non-resonance, the additional condition

$$|(\Lambda \mid \alpha) - \lambda_j| \geqq c |\alpha|^{-v},$$

with c, v positive constants. This condition on Λ holds for almost all vectors in the sense of Lebesgue measure, so that, in the non-resonant case, convergence is a generic fact. On the contrary, when X_0 is resonant, and Λ in not in the Poincaré domain, severe a priory conditions on the normal form are needed to assure convergence. More precisely Brjuno [2] has shown that every analytic vector field with a linear part X_0 such that the interior of $K(\Lambda)$ contains zero (in $\mathbf{C}$), and such that $\sigma(X_0)$ satisfies the condition that the infinite sum

$$\sum_{k \geqq 0} \frac{1}{2^k} \log \frac{1}{\omega_k}$$

is convergent, where

$$\omega_k := \min\{|(\Lambda\,|\,\alpha) - \lambda_j|, \quad 0 < |\alpha| \leqq 2^{k+1}, \quad 1 \leqq j \leqq d, \quad (\Lambda\,|\,\alpha) \neq \lambda_j\},$$

is analytically normalizable provided the normal form is tangent to the foliations $z^\beta = \text{const}$ corresponding to the relations $(\Lambda\,|\,\beta) = 0$, with $\beta \in \boldsymbol{R}^d_+$, $\beta \neq 0$. This geometric interpretation of Brjuno's results is due to Martinet [7]. We remark that the above statement generalizes Siegel's results when X_0 is not resonant.

In spite of their mathematical beauty, the above theorems are generally useless if one is concerned with stability problems. Indeed one has the following

Theorem 5 (Carathéodory-Cartan). (see [8] and references therein) *Let $X = \sum_{k \geqq 0} X_k$ with $X_k \in H_{k+1}$, be an analytical vector field. Necessary and sufficient conditions for the stability of the critical point $z = 0$ for all real times are that*

1. X_0 is diagonalizable with purely imaginary eigenvalues, and

2. X is analytically linearizable

Since for real, i.e. $X = \bar{X}$, vector fields the resonant case is unavoidable (the complex eigenvalues appear in complex conjugate pairs), the above theorem makes Siegel's linearization theorem useless for the stability problem of the solutions of real differential equations. Nevertheless, from the viewpoint of physical applications, it is sufficient to obtain informations on the stability of the solutions for large but finite times, for example times of the order of the age of the universe. To this end the estimates given in Theorem 4 may be very useful, as we will show by an example in the following section.

6. The exponential estimates for reversible systems of coupled harmonic oscillators

Consider now a vector field which satisfies the hypothesis 1. of Carathéodory-Cartan theorem, i.e. that the linear part is a system of harmonic oscillators. Among these systems one can consider two particular classes of interest in physical applications, namely the Hamiltonian systems and the reversible systems [8]. The Hamiltonian case was already discussed in ref. [4], so let's consider here the reversible case.

Precisely, let $Y = \sum_{k \geqq 0} Y_k$, with $Y_k \in H_{k+1}$, be an analytical vector field defined in a neighbourhood of zero in $\boldsymbol{C}^{2d}$ such that $\bar{Y} = Y$, with linear part

$$Y_0^k(z) = \begin{cases} z_{k+d} & 1 \leqq k \leqq d \\ -\omega_{k-d}^2 \, z_{k-d} & d+1 \leqq k \leqq 2d \end{cases}$$

Assume moreover the reversibility condition

$$Y = -RYR,$$
$$R^k(z) := \begin{cases} z_k & 1 \leqq k \leqq d \\ -z_k & d+1 \leqq k \leqq 2d \end{cases}$$

This characterizes Y as the complexification of a real differential equation on $\boldsymbol{R}^{2d}$ describing a reversible system of coupled harmonic oscillators with frequencies $\omega_1, \dots, \omega_d$. If we define, as usual, the complex linear transformation B by

$$B^k(z) := \begin{cases} z_k - (i/\omega_k)\, z_{k+d} & 1 \leqq k \leqq d \\ z_{k-d} + (i/\omega_{k-d})\, z_k & d+1 \leqq k \leqq 2d \end{cases}$$

then the vector field $X := BYB^{-1}$ has the form $X = i \sum_{k \geqq 0} X_k$, with $X_k \in H_{k+1}$. Moreover, one has

$$X_0^k(z) = \begin{cases} \omega_k z_k & 1 \leqq k \leqq d \\ -\omega_{k-d} z_k & d+1 \leqq k \leqq 2d \end{cases} \tag{6.1}$$

and, denoting

$$X_k^j(z) = \sum_{|\alpha|+|\beta|=k+1} X_k^{j,\alpha,\beta} z'^{\alpha} z''^{\beta},$$

where $z' := (z_1, \dots, z_d)$, and $z'' := (z_{d+1}, \dots, z_{2d})$, one has

$$\bar{X}_k = X_k, \tag{6.2}$$
$$X_k^{j,\alpha,\beta} = -X_k^{j+d,\beta,\alpha}. \tag{6.3}$$

Indeed, using the fact that Y is a real vector field, the relation $X_k^{j,\alpha,\beta} = -\bar{X}_k^{j+d,\beta,\alpha}$ easily follows, while the stronger conditions (6.2) and (6.3) can be shown to be equivalent to the reality and reversibility of Y.

Now, by Theorem 3, we can put the vector field X in normal form up to terms of degree r, where r is an integer greater than one. Using

$$(L_{iX_0} V_n)^{j,\alpha,\beta} = i((\Omega \,|\, \alpha - \beta) - \lambda_j)\, V_n^{j,\alpha,\beta},$$

with $\Omega = (\omega_1, \dots, \omega_d)$, $\lambda_j = \omega_j$ if $1 \leqq j \leqq d$, $\lambda_j = -\omega_{j-d}$ otherwise, it is easy to prove, by induction, that the definitions (2.2), (3.6), and (3.7) imply

$$\varphi_* X = i(X_0 + Z + R_r) \quad \text{with } L_{X_0} Z = 0, \tag{6.4}$$

and that the vector fields Z and R_r have the same properties, i.e. (6.2) and (6.3), as X. Moreover, if V_ε is the generator of the flow φ, then V_ε is a real vector field such that $V_k^{j,\alpha,\beta} = V_k^{j+d,\beta,\alpha}$. From this there follows that the transformation $B^{-1} \varphi B$, namely the time-one flow of $B^{-1} V_\varepsilon B$, is a real change of variables, and that the vector field $B^{-1}(\varphi_* X)\, B = (B^{-1} \varphi B)_* Y$ is a real and reversible normal form for Y.

Let us now suppose that the frequencies $\omega_1, \dots, \omega_d$ are nonresonant, i.e.

$$(\Omega \,|\, \alpha) \neq 0 \quad \text{for all } \alpha \in \boldsymbol{Z}^d \setminus \{0\}. \tag{6.5}$$

Since, by (3.6),

$$Z_n^j(z) = \sum_{\substack{|\alpha|+|\beta|=n+1 \\ (\Omega|(\alpha-\beta))=\lambda_j}} Z_n^{j,\alpha,\beta} z'^{\alpha} z''^{\beta},$$

one has, by (6.5),

$$Z_n^j(z) = z_j \sum_{2|\gamma|=n} Z_n^{j,\gamma}(z' z'')^{\gamma}.$$

Along the lines of the Nekhoroshev's like results, we determine now the normalization order r by the condition that the size of the remainder R_r, as estimated by theorem 4, is close to a minimum. Thus, we prove the following

Theorem 6. *Let $X = i \sum_{k \geqq 0} X_k$ be a vector field satisfying the conditions* (6.2) *and* (6.3), *and with linear part as in* (6.1). *Assume moreover*

$$|(\Omega|\alpha)| \geqq c^{-1} |\alpha|^{-(\nu-1)}$$

with $c > 0$, $\nu \geqq 1$, for all $\alpha \in \mathbf{Z}^d \setminus \{0\}$, and $\| X_k \| \leqq M/\varrho^k$ for $k \geqq 1$, with M and ϱ positive real constants. For every positive $\varrho' \leqq K/(2e)^\nu \sqrt{2}$ let $\varphi_* X = i(X_0 + Z + R_{r_{\mathrm{opt}}})$ be the transformed field of X according to theorem 3, *where Z is the normal form up to the optimal order*

$$r_{\mathrm{opt}} = \left[\frac{1}{e}\left(\frac{K}{\sqrt{2}\,\varrho'}\right)^{\frac{1}{\nu}}\right]$$

with

$$K = \frac{\varrho}{15 + 27c2^{\nu-1}M}.$$

Finally, consider a solution $z(t)$ of the differential equation given by the field $\varphi_* X$, with real initial datum $z(0) = z_0$, i.e. $\bar{z}_{0j} = z_{j+d}$, and define $\| z' \|^2 := \sum_{j=1}^{d} z_j \bar{z}_j$. Then, for all initial data satisfying $\| z'_0 \|^2 \leqq \varrho'^2 - \delta^2$, with $0 < \delta \leqq \varrho'$, one has

$$|\, \| z'(t) \|^2 - \| z'_0 \|^2 \,| \leqq \delta^2$$

for all times

$$|t| \leqq T_0 \exp\left(\frac{\nu}{e}\left(\frac{K}{\sqrt{2}\,\varrho'}\right)^{\frac{1}{\nu}}\right)$$

with

$$T_0 = \frac{\delta^2}{4\varrho'^2 M'}$$

$$M' = \max\{\| X_0 \|, M\}.$$

(Here $[\cdot]$ denotes integer part).

Proof. First we show that the function $f(z) := \sum_{j=1}^{d} z_j z_{j+d}$ is a prime integral for the field$X_0 + Z$. Indeed one has

$$\begin{aligned} L_{X_0} f(z) &= \sum_{j=1}^{2d} X_0^j(z) \frac{\partial}{\partial z_j} f(z) \\ &= \sum_{j=1}^{d} \omega_j z_j z_{j+d} - \omega_j z_{j+d} z_j = 0 , \end{aligned}$$

and, by the symmetry properties (6.3) for the coefficients of Z,

$$\begin{aligned} L_{Z_n} f(z) &= \sum_{j=1}^{2d} Z_n^j(z) \frac{\partial}{\partial z_j} f(z) \\ &= \sum_{\substack{1 \leqq j \leqq d \\ 2|\gamma| = n}} (Z_n^{j,\gamma}(z', z'')^\gamma + Z_n^{j+d,\gamma}(z' z'')^\gamma) z_j z_{j+d} = 0 . \end{aligned}$$

The time derivative of $f(z(t))$ is then

$$\begin{aligned} L_{R_r} f(z) &= \sum_{\substack{1 \leqq j \leqq d \\ |\alpha| + |\beta| \geqq r+1}} z_{j+d} R_r^{j,\alpha,\beta} z'^\alpha z''^\beta + z_j R_r^{j+d,\alpha,\beta} z'^\alpha z''^\beta \\ &= - \sum_{\substack{1 \leqq j \leqq 2d \\ |\alpha| + |\beta| \geqq r+1}} z_j R_r^{j,\beta,\alpha} z'^\alpha z''^\beta , \end{aligned}$$

and one has

$$\| L_{R_r} f(z) \|_\varrho \leqq \varrho \| R_r \|_\varrho . \tag{6.6}$$

Now, given $\delta \leqq \varrho'$, we look for a constant t_δ such that $| \| z'(t) \|^2 - \| z'_0 \|^2 | \leqq \delta^2$ for all times $|t| \leqq t_\delta$ and for all initial data z_0 such that $\bar{z}_{0j} = z_{0j+d}$ and $\| z'_0 \|^2 \leqq \varrho'^2 - \delta^2$. Since $\bar{z}_j(t) = z_{j+d}(t)$ for all times, and $f(z(t)) = \| z(t) \|^2/2 = \| z'(t) \|^2$ for $z(t)$ real, one has, by (6.6),

$$\begin{aligned} | \| z'(t) \|^2 - \| z'_0 \|^2 | &\leqq |t| \| L_{R_r} f(z) \|_{\sqrt{2}\varrho'} \\ &\leqq |t| (\sqrt{2} \varrho') \| R_r \|_{\sqrt{2}\varrho'} . \end{aligned}$$

So we can take

$$t_\delta = \frac{\delta^2}{\sqrt{2} \varrho' \| R_r \|_{\sqrt{2}\varrho'}} . \tag{6.7}$$

Now we use the estimates for the remainder given in Theorem 3. From $|(\Omega | \alpha)| \geqq c^{-1} |\alpha|^{-(\nu-1)}$ there follows

$$A_r \leqq c(r-1)(r+1)^{\nu-1} \leqq c 2^{\nu-1} r^\nu ,$$

and therefore, by (4.1) and $\max\{1, 2M A_r\}/2 \leqq 1/2 + M A_r$,

$$\varrho_r \geqq \frac{\varrho}{15 + 27 c 2^{\nu-1} M r^\nu} \geqq \frac{\varrho}{(15 + 27 c 2^{\nu-1} M) r^\nu} \equiv \frac{K}{r^\nu} .$$

With this estimate for ϱ_r, we have, by (4.2), taking $2\sqrt{2}\,\varrho' \leqq \varrho_r$,

$$\| R_r \|_{\sqrt{2}\varrho'} \leqq 2\sqrt{2}\,\varrho' M' \left(\frac{\sqrt{2}\,\varrho'}{\varrho_r}\right)^r \leqq 2\sqrt{2}\,\varrho' M' \left(\frac{\sqrt{2}\,\varrho'}{K} r^\nu\right)^r .$$

To optimize the above estimates we look for the integer r_{opt} for which $(\sqrt{2}\,\varrho' r^\nu / K)^r$ is minimum. It is easy to prove that $r_{\mathrm{opt}} = [(1/e)\,(K/\sqrt{2}\,\varrho')^{1/\nu}]$, $r_{\mathrm{opt}} \geqq 2$ if $\varrho' \leqq K/(2e)^\nu \sqrt{2}$, and therefore one has

$$\| R_{r_{\mathrm{opt}}} \|_{\sqrt{2}\varrho'} \leqq 2\sqrt{2}\,\varrho' M' \left(\frac{\sqrt{2}\,\varrho'}{K} r_{\mathrm{opt}}^\nu\right)^{r_{\mathrm{opt}}} \leqq 2\sqrt{2}\,\varrho' M' \exp\left(-\frac{\nu}{e}\left(\frac{K}{\sqrt{2}\,\varrho'}\right)^{\frac{1}{\nu}}\right).$$

The theorem follows by inserting the above estimates in (6.7).

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Abstract

We consider the problem of finding a normal form for differential equations in the neighbourhood of an equilibrium point, and produce general explicit estimates for both the normal form at a finite order and the remainder, using the method of Lie transforms. With such technique, the classical Poincaré-Dulac theorems are recovered, and the problem of the stability of a reversible system of coupled harmonic oscillators up to exponentially large times is discussed.

Riassunto

Si considera il problema di porre in forma normale un sistema di equazioni differenziali nell'intorno di un punto di equilibrio, e si danno in generale stime esplicite sia per la forma normale troncata ad un ordine finito che per i resti. Si fa uso dell'algoritmo della trasformata di Lie. Con questo metodo si riottengono i teoremi classici di Poincaré-Dulac, e si discute il problema della stabilità per tempi esponenzialmente lunghi di un sistema reversibile di oscillatori armonici accoppiati.

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