Quasi-periodic solutions with Sobolev regularity of NLS on $\mathbb{T}^d$ with a multiplicative potential

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Abstract: We prove the existence of quasi-periodic solutions for Schrödinger equations with a *multiplicative* potential on $\mathbb{T}^d$, $d \geq 1$, finitely *differentiable* nonlinearities, and tangential frequencies constrained along a *pre-assigned* direction. The solutions have only Sobolev regularity both in time and space. If the nonlinearity and the potential are C^∞ then the solutions are C^∞. The proofs are based on an improved Nash-Moser iterative scheme, which assumes the weakest tame estimates for the inverse linearized operators ("Green functions") along scales of Sobolev spaces. The key off-diagonal decay estimates of the Green functions are proved via a new multiscale inductive analysis. The main novelty concerns the measure and "complexity" estimates.

Keywords: Nonlinear Schrödinger equation, Nash-Moser Theory, KAM for PDE, Quasi-Periodic Solutions, Small Divisors, Infinite Dimensional Hamiltonian Systems.

2000AMS subject classification: 35Q55, 37K55, 37K50.

1 Introduction

The first existence results of quasi-periodic solutions of Hamiltonian PDEs have been proved by Kuksin [28] and Wayne [38] for one dimensional, analytic, nonlinear perturbations of linear wave and Schrödinger equations. The main difficulty, namely the presence of arbitrarily "small divisors" in the expansion series of the solutions, is handled via KAM theory. These pioneering results were limited to Dirichlet boundary conditions because the eigenvalues of the Laplacian had to be simple. In this case one can impose the so-called "second order Melnikov" non-resonance conditions to solve the linear homological equations which arise at each KAM step, see also Pöschel [35]. Such equations are linear PDEs with constant coefficients and can be solved using Fourier series. Already for periodic boundary conditions, where two consecutive eigenvalues are possibly equal, the second order Melnikov non-resonance conditions are violated.

Later on, another more direct bifurcation approach has been proposed by Craig and Wayne [17], who introduced the Lyapunov-Schmidt decomposition method for PDEs and solved the small divisors problem, for periodic solutions, with an analytic Newton iterative scheme. The advantage of this approach is to require only the "first order Melnikov" non-resonance conditions, which are essentially the minimal assumptions. On the other hand, the main difficulty of this strategy lies in the inversion of the linearized operators obtained at each step of the iteration, and in achieving suitable estimates for their inverses in high (analytic) norms. Indeed these operators come from linear PDEs with *non-constant* coefficients and are small perturbations of a diagonal operator having arbitrarily small eigenvalues.

In order to get estimates in analytic norms for the inverses, called Green functions by the analogy with Anderson localization theory, Craig and Wayne developed a coupling technique inspired by the methods of Fröhlich-Spencer [24]. The key properties are:

(i) "separations" between singular sites, namely the Fourier indexes of the small divisors,

(ii) "localization" of the eigenfunctions of $-\partial_{xx} + V(x)$ with respect to the exponentials.

Property (ii) implies that the matrix which represents, in the eigenfunction basis, the multiplication operator for an analytic function has an exponentially fast decay off the diagonal. Then the "separation properties" (i) imply a very "weak interaction" between the singular sites. Property (ii) holds in dimension 1, i.e. $x \in \mathbb{T}^1$, but, for $x \in \mathbb{T}^d$, $d \geq 2$, some counterexamples are known, see [23].

The "separation properties" (i) are quite different for periodic or quasi-periodic solutions. In the first case the singular sites are "separated at infinity", namely the distance between distinct singular sites increases when the Fourier indexes tend to infinity. This property is exploited in [17]. On the contrary, it never holds for quasi-periodic solutions, even for finite dimensional systems. For example, in the ODE case where the small divisors are $\omega \cdot k$, $k \in \mathbb{Z}^\nu$, if the frequency vector $\omega \in \mathbb{R}^\nu$ is diophantine, then the singular sites k where $|\omega \cdot k| \leq \rho$ are "uniformly distributed" in a neighborhood of the hyperplane $\omega \cdot k = 0$, with nearby indices at distance $O(\rho^{-\alpha})$ for some $\alpha > 0$.

This difficulty has been overcome by Bourgain [6], who extended the approach of Craig-Wayne in [17] via a multiscale inductive argument, proving the existence of quasi-periodic solutions of 1-dimensional wave and Schrödinger equations with polynomial nonlinearities. In order to get estimates of the Green functions, Bourgain imposed lower bounds for the determinants of most "singular sub-matrices" along the diagonal. This implies, by a repeated use of the "resolvent identity" (see [24], [10]), a sub-exponentially fast decay of the Green functions. As a consequence, at the end of the iteration, the quasi-periodic solutions are Gevrey regular.

At present, KAM theory for 1-dimensional semilinear PDEs has been sufficiently understood, see e.g. [29], [30], [16], but much work remains for PDEs in higher space dimensions, due to the more complex properties of the eigenfunctions and eigenvalues of

$$(-\Delta + V(x))\, \psi_j(x) = \mu_j\, \psi_j(x)\,.$$

The main difficulties for PDEs in higher dimensions are:

1. the eigenvalues μ_j appear in clusters of unbounded sizes.
2. the eigenfunctions $\psi_j(x)$ are (in general) "not localized" with respect to the exponentials.

Problem 2 has been often bypassed considering pseudo-differential PDEs substituting the multiplicative potential $V(x)$ by a "convolution potential"

$$V * (e^{\mathrm{i} j \cdot x}) = m_j e^{\mathrm{i} j \cdot x}\,, \ m_j \in \mathbb{R}\,, \ j \in \mathbb{Z}^d\,,$$

which, by definition, is diagonal on the exponentials. The scalars m_j are called the "Fourier multipliers".

Concerning problem 1, since the approach of Craig-Wayne and Bourgain requires only the first order Melnikov non-resonance conditions, it works well, in principle, in case of multiple eigenvalues, in particular for PDEs in higher spatial dimensions.

Actually the first existence results of periodic solutions for NLW and NLS on $\mathbb{T}^d$, $d \geq 2$, have been established by Bourgain in [7]-[10]. Here the singular sites form huge clusters (not only points as in $d = 1$) but are still "separated at infinity". The nonlinearities are polynomial and the solutions have Gevrey regularity in space and time.

Recently these results were extended in [2]-[5] to prove the existence of periodic solutions, with only Sobolev regularity, for NLS and NLW in any dimension and with merely differentiable nonlinearities. Actually in [4], [5] the PDEs are defined not only on tori, but on any compact Zoll manifold, Lie group and homogeneous space. These results are proved via an abstract Nash-Moser implicit function theorem (a simple Newton method is not sufficient). Clearly, a difficulty when working with functions having only Sobolev regularity is that the Green functions will exhibit only a polynomial decay off the diagonal, and not exponential (or sub-exponential). A key concept that one must exploit are the interpolation/tame estimates. For PDEs on Lie groups only weak properties of "localization" (ii) of the eigenfunctions hold, see [5]. Nevertheless these properties imply a block diagonal decay, for the matrix which represents the multiplication operator in the eigenfunctions basis, sufficient to achieve the tame estimates.

We also mention that existence of periodic solutions for NLS on $\mathbb{T}^d$ has been proved, for analytic nonlinearities, by Gentile-Procesi [26] via the Lindstedt series techniques, and, in the differentiable case, by Delort [18] using paradifferential calculus.

Regarding quasi-periodic solutions, Bourgain [10] was the first to prove their existence for PDEs in higher dimension, actually for nonlinear Schrödinger equations with Fourier multipliers and polynomial nonlinearities on $\mathbb{T}^d$ with $d = 2$. The Fourier multipliers, in number equal to the tangential frequencies of the quasi-periodic solution, play the role of external parameters. The main difficulty arises in the multi-scale argument to estimate the decay of the Green functions. Due to the degeneracy of the eigenvalues of the Laplacian the singular sub-matrices that one has to control are huge. If $d = 2$, careful estimates on the number of integer vectors on a sphere, allowed anyway Bourgain to show that the required non-resonance conditions are fulfilled for "most" Fourier multipliers.

More recently Bourgain [13] improved the techniques in [10] proving the existence of quasi-periodic solutions for nonlinear wave and Schrödinger equations with Fourier multipliers on any $\mathbb{T}^d$, $d > 2$, still for polynomial nonlinearities. The improvement in [13] comes from the use of sophisticated techniques developed in the context of Anderson localization theory in Bourgain-Goldstein-Schlag [14], Bourgain [11], see also Bourgain-Wang [15]. These techniques (sub-harmonic functions, Cartan theorem, semi-algebraic sets) mainly concern fine properties of rational and analytic functions, especially measure estimates of sublevels. Actually the nonlinearities in [13] are taken to be polynomials in order to use semialgebraic techniques. Very recently, Wang [37] has generalized the results in [13] for NLS with no Fourier multipliers and with supercritical nonlinearities. The main step is a Lyapunov-Schmidt reduction in order to introduce parameters and then be able to apply the results of [13].

We also remark that, in the last years, the KAM approach has been extended by Eliasson-Kuksin [21] for nonlinear Schrödinger equations on $\mathbb{T}^d$ with a convolution potential and analytic nonlinearities. The potential plays the role of "external parameters". The quasi-periodic solutions are C^∞ in space. Clearly an advantage of the KAM approach is to provide also a stability result: the linearized equations on the perturbed invariant tori are reducible to constant coefficients, see also [22].

For the cubic NLS in $d = 2$ the existence of quasi-periodic solutions has been recently proved by Geng-Xu-You [25] via a Birkhoff normal form and a modification of the KAM approach in [21], see also Procesi-Procesi [36], valid in any dimension.

In the present paper we prove -see Theorem 1.1- the existence of quasi-periodic solutions for nonlinear Schrödinger equations on $\mathbb{T}^d$, $d \geq 1$, with:

1. merely *differentiable* nonlinearities, see (1.2),
2. a *multiplicative* (merely differentiable) potential $V(x)$, see (1.3),
3. a *pre-assigned* (Diophantine) direction of the tangential frequencies, see (1.4)-(1.5) .

The quasi-periodic solutions in Theorem 1.1 have the same Sobolev regularity both in time and space, see remark 5.3. Moreover, we prove that, if the potential and the nonlinearity are of class C^∞, then the quasi-periodic solutions are C^∞-functions of (t, x).

Let us make some comments on the results.

1. Theorem 1.1 confirms the natural conjecture about the persistence of quasi-periodic solutions for Hamiltonian PDEs into a setting of finitely many derivatives (as in the classical KAM theory [33], [34], [39]), stated for example by Bourgain [9], page 97. The nonlinearities in Theorem 1.1, as well as the potential, are sufficiently many times differentiable, depending on the dimension and the number of the frequencies. Of course we can not expect the existence of quasi-periodic solutions of the Schrödinger equation under too weak regularity assumptions on the nonlinearities. Actually, for finite dimensional Hamiltonian systems, it has been rigorously proved that, if the vector field is not sufficiently smooth, then all the invariant tori could be destroyed and only discontinuous Aubry-Mather invariant sets survive, see e.g. [27]. We have not tried to estimate the minimal smoothness exponents, see however remark 1.2. This could be interesting for comparing Theorem 1.1 with the well posedness results of the Cauchy problem.

2. Theorem 1.1 is the first existence result of quasi-periodic solutions with a multiplicative potential $V(x)$ on $\mathbb{T}^d$, $d \geq 2$. We never exploit properties of "localizations" of the eigenfunctions of $-\Delta + V(x)$ with respect to the exponentials, that actually might *not* be true, see [23]. Along the multiscale analysis we use the exponential basis which diagonalizes $-\Delta + m$ where m is the average of $V(x)$, see (2.5), and not the eigenfunctions of $-\Delta + V(x)$. In [10] Bourgain considered analytic multiplicative periodic potentials of the special form $V_1(x_1) + \ldots + V_d(x_d)$ to ensure localization properties of the eigenfunctions, leaving open the natural problem for a general multiplicative potential $V(x)$.

We also underline that Theorem 1.1 holds for any *fixed* potential $V(x)$: we do not extract parameters from V, the role of external parameters being played by the frequency $\omega = \lambda\bar{\omega}$.

3. For finite dimensional systems, the existence of quasi-periodic solutions with tangential frequencies constrained along a fixed direction has been proved by Eliasson [19] (with KAM theory) and Bourgain [8] (with a multiscale approach). The main difficulty clearly relies in satisfying the Melnikov non-resonance conditions, required at each step of the iterative process, using only *one* parameter. Bourgain raised in [8] the question if a similar result holds true also for infinite dimensional Hamiltonian systems. This has been recently proved in [1] for 1-dimensional PDEs, verifying the second order Melnikov non-resonance conditions of KAM theory. Theorem 1.1 (and its method of proof) answers positively to Bourgain's conjecture also for PDEs in higher space dimension. The non-resonance conditions that we have to fulfill are of first order Melnikov type, see the end of section 1.2.

The proof of Theorem 1.1 is based on a Nash-Moser iterative scheme and a multiscale analysis of the linearized operators as in [13]. However, our approach presents many differences with respect to Bourgain's one [13], about:

1. the iterative scheme,
2. the multiscale proof of the Green's functions polynomial decay estimates.

Referring to section 1.2 for a detailed exposition of our approach, we outline here the main differences.

1. Since we deal with merely differentiable nonlinearities we need all the power of the Nash-Moser theory in scales of Sobolev functions spaces. A Newton method valid in analytic Banach scales is not sufficient. This means that the superexponential smallness of the error terms due to finite dimensional truncations, see (7.60), can not be obtained, in Sobolev scales, decreasing the analyticity strips, but using the structure of the iteration and the interpolation estimates of the Green functions, see lemmas 7.8, 7.9, 7.12. This is a key idea when dealing with matrices with a merely polynomial off-diagonal decay.

Actually, the Nash-Moser scheme developed in section 7 also improves the one in [2]-[4], requiring the minimal tame properties (7.62) for the inverse linearized operators, see comments after (1.14).

Another comment is in order: we do not follow the "analytic smoothing technique" suggested by Moser in [33] of approximating the differentiable Hamiltonian PDE by analytic ones. This technique is very efficient for finite dimensional Hamiltonian systems, see [34], [39], but it seems quite delicate for PDEs (especially in dimensions $d \geq 2$) because of the presence of large clusters of small divisors. So we prefer a more direct Nash-Moser iterative procedure more similar, in spirit, to [32].

2. The main difference between our multiscale approach, which is developed to prove the Green functions estimates (7.62), and the one in [13], [14], [11], [15], concerns the way we prove inductively the existence of "large sets" of N_n-good parameters, see Definition 5.2. Quoting Bourgain [12] "*...the results in [13] make essential use of the general perturbative technology (based on subharmonicity and semi-algebraic set theory) [...]. This technique enables us to deal with large sets of 'singular sites' [...], something difficult to achieve with conventional eigenvalue methods.*". Actually, exploiting that $-\Delta + V(x)$ is positive definite, we are able to prove the necessary measure and "complexity" estimates by using only elementary eigenvalue variation arguments, see section 6.

Another deep difference is required for dealing with a multiplicative potential $V(x)$: we define "very regular" sites (see Definition 4.2) depending on the potential V.

We hope that this novel approach will be useful also for extending the results of [11], [13], [14], [15].

We tried to present the steps of proof in an abstract setting (as much as possible) in order to develop a systematic procedure, alternative to KAM theory, for the search of quasi-periodic solutions of PDEs. The proof of Theorem 1.1 is completely self-contained. All the techniques employed are elementary and based on abstract arguments valid for many PDEs. Only the "separation properties" of the bad sites (section 5) will change, of course, for different PDEs.

Since the aim of the present paper is to focus on the small divisors problem for quasi-periodic solutions with Sobolev regularity of NLS with a multiplicative potential on $\mathbb{T}^d$ and differentiable nonlinearities, we have considered, among many possible variations, quasi-periodically forced nonlinear perturbations of linear Schrödinger equations. In this way, we avoid the Lyapunov-Schmidt decomposition. Clearly the small divisors difficulty for quasi-periodically forced NLS is the same as for autonomous NLS.

We now state precisely our results.

1.1 Main result

We consider d-dimensional nonlinear Schrödinger equations with a potential V, like

$$\mathrm{i}u_t - \Delta u + V(x)u = \varepsilon f(\omega t, x, |u|^2)u + \varepsilon g(\omega t, x)\,, \quad x \in \mathbb{T}^d\,, \tag{1.1}$$

where $V \in C^q(\mathbb{T}^d;\mathbb{R})$ for some q large enough, $\varepsilon > 0$ is a small parameter, the frequency vector $\omega \in \mathbb{R}^\nu$ is non resonant (see (1.5)), the nonlinearity is quasi-periodic in time and only finitely many times differentiable, more precisely

$$f \in C^q(\mathbb{T}^\nu \times \mathbb{T}^d \times \mathbb{R};\mathbb{R})\,, \quad g \in C^q(\mathbb{T}^\nu \times \mathbb{T}^d;\mathbb{C}) \tag{1.2}$$

for some $q \in \mathbb{N}$ large enough. Moreover we suppose

$$-\Delta + V(x) \geq \beta_0 I\,,\ \beta_0 > 0\,. \tag{1.3}$$

Remark 1.1. *Condition* (1.3) *is used for the measure estimates of section 6. Actually for autonomous NLS it can be always verified after a gauge-transformation $u \mapsto e^{-\mathrm{i}\sigma t}u$ for σ large enough.*

We assume that the frequency vector ω is a small dilatation of a fixed Diophantine vector $\bar\omega \in \mathbb{R}^\nu$, namely

$$\omega = \lambda\bar\omega\,, \quad \lambda \in \Lambda := [1/2, 3/2]\,, \quad |\bar\omega| \leq 1\,, \tag{1.4}$$

where, for some $\gamma_0 \in (0,1)$, $\tau_0 > \nu - 1$,

$$|\bar\omega \cdot l| \geq \frac{2\gamma_0}{|l|^{\tau_0}}\,, \quad \forall l \in \mathbb{Z}^\nu \setminus \{0\}\,, \tag{1.5}$$

and $|l| := \max\{|l_1|, \ldots, |l_\nu|\}$. For definiteness we fix $\tau_0 := \nu$.

If $g(\omega t, x) \not\equiv 0$ then $u = 0$ is not a solution of (1.1) for $\varepsilon \neq 0$.

- Question: do there exist quasi-periodic solutions of (1.1) for sets of parameters (ε, λ) of positive measure?

This means looking for $(2\pi)^{\nu+d}$-periodic solutions $u(\varphi, x)$ of

$$\mathrm{i}\omega \cdot \partial_\varphi u - \Delta u + V(x)u = \varepsilon f(\varphi, x, |u|^2)u + \varepsilon g(\varphi, x)\,. \tag{1.6}$$

These solutions will be, for some $(\nu + d)/2 < s \leq q$, in the Sobolev space

$$H^s := H^s(\mathbb{T}^\nu \times \mathbb{T}^d;\mathbb{C}) \ := \ \Big\{u(\varphi, x) = \sum_{(l,j)\in\mathbb{Z}^\nu\times\mathbb{Z}^d} u_{l,j}e^{\mathrm{i}(l\cdot\varphi + j\cdot x)} \quad \text{such that} \ \ \|u\|_s^2 := K_0 \sum_{i\in\mathbb{Z}^{\nu+d}} |u_i|^2\langle i\rangle^{2s} < +\infty\Big\} \tag{1.7}$$

where

$$i := (l, j)\,, \quad \langle i\rangle := \max(|l|, |j|, 1)\,, \quad |j| := \max\{|j_1|, \ldots, |j_d|\}.$$

For the sequel we fix $s_0 > (d + \nu)/2$ so that there is the continuous embedding

$$H^s(\mathbb{T}^{\nu+d}) \hookrightarrow L^\infty(\mathbb{T}^{\nu+d})\,, \quad \forall s \geq s_0\,, \tag{1.8}$$

and H^s is a Banach algebra with respect to the multiplication of functions. The constant $K_0 > 0$ in the definition (1.7) of the Sobolev norm $\| \ \|_s$ is independent of s. The value of K_0 is fixed (large enough) so that $|u|_{L^\infty} \leq \|u\|_{s_0}$ and the interpolation inequality

$$\|u_1 u_2\|_s \leq \frac{1}{2}\|u_1\|_{s_0}\|u_2\|_s + \frac{C(s)}{2}\|u_1\|_s\|u_2\|_{s_0}\,, \quad \forall s \geq s_0\,,\ u_1, u_2 \in H^s\,, \tag{1.9}$$

holds with $C(s) \geq 1$ and $C(s) = 1, \forall s \in [s_0, s_1]$; the constant s_1 is defined in (7.16) and depends only on $d, \nu, \tau_0 := \nu$. With respect to the standard Moser-Nirenberg interpolation estimate in Sobolev spaces, see e.g. [31], the additional property in (1.9) is that one of the constants is independent of s. The proof of (1.9) is given for example in Appendix of [4], see also [31].

The main result of this paper is:

Theorem 1.1. *Assume (1.5). There are $s := s(d,\nu)$, $q := q(d,\nu) \in \mathbb{N}$, such that: $\forall\, V \in C^q$ satisfying (1.3), $\forall f, g \in C^q$, there exist $\varepsilon_0 > 0$, a map*

$$u \in C^1([0,\varepsilon_0] \times \Lambda; H^s) \quad \text{with} \quad u(0,\lambda) = 0\,,$$

and a Cantor like set $\mathcal{C}_\infty \subset [0,\varepsilon_0] \times \Lambda$ of asymptotically full Lebesgue measure, i.e.

$$|\mathcal{C}_\infty|/\varepsilon_0 \to 1 \quad \text{as} \quad \varepsilon_0 \to 0\,, \tag{1.10}$$

such that, $\forall (\varepsilon,\lambda) \in \mathcal{C}_\infty$, $u(\varepsilon,\lambda)$ is a solution of (1.6) with $\omega = \lambda\bar{\omega}$.
Moreover, if V, f, g are of class C^∞ then $u(\varepsilon,\lambda) \in C^\infty(\mathbb{T}^d \times \mathbb{T}^\nu; \mathbb{C})$.

We have not tried to optimize the estimates for $q := q(d,\nu)$ and $s := s(d,\nu)$.

Remark 1.2. *In [2] we proved the existence of periodic solutions in $H^s_t(\mathbb{T}; H^1_x(\mathbb{T}^d))$ with $s > 1/2$, for one dimensional NLW equations with nonlinearities of class C^6, see the bounds (1.9), (4.28) in [2].*

1.2 Ideas of the proof

Vector NLS. We prove Theorem 1.1 finding solutions of the "vector" NLS equation

$$\begin{cases} \ \ \mathrm{i}\omega \cdot \partial_\varphi u^+ - \Delta u^+ + V(x)u^+ = \varepsilon f(\varphi, x, u^- u^+)u^+ + \varepsilon g(\varphi, x) \\ -\mathrm{i}\omega \cdot \partial_\varphi u^- - \Delta u^- + V(x)u^- = \varepsilon f(\varphi, x, u^- u^+)u^- + \varepsilon \bar{g}(\varphi, x) \end{cases} \tag{1.11}$$

where[1]

$$\mathbf{u} := (u^+, u^-) \in \mathbf{H}^s := H^s \times H^s \tag{1.12}$$

(the second equation is obtained by formal complex conjugation of the first one). In the system (1.11) the variables u^+, u^- are independent. However, note that (1.11) reduces to the scalar NLS equation (1.1) in the set

$$\mathcal{U} := \Big\{ \mathbf{u} := (u^+, u^-) \ : \ \overline{u^+} = u^- \Big\} \tag{1.13}$$

in which u^- is the complex conjugate of u^+ (and viceversa).

Linearized equations. We look for solutions of the vector NLS equation (1.11) in $\mathbf{H}^s \cap \mathcal{U}$ by a Nash-Moser iterative scheme. The main step concerns the invertibility of (any finite dimensional restriction of) the linearized operators at any $\mathbf{u} \in \mathbf{H}^s \cap \mathcal{U}$, namely

$$\mathcal{L}(\mathbf{u}) := L_\omega - \varepsilon T_1 = D_\omega + T$$

[1] In order to give a sense to (1.11) we need to define a smooth extension of $f(\varphi, x, \cdot)$ to $\mathbb{C}$, although $f(\varphi, x, \cdot)$ was not assumed real analytic. Since we look for solutions satisfying $\overline{u^+} = u^-$ we only need to linearize (1.11) at $\mathbf{u} \in \mathcal{U}$, and we require that the differential of (the extended) $f(\varphi, x, \cdot)$ at any $s \in \mathbb{R}$ is $\mathbb{C}$-linear. For instance we can choose, for $z \in \mathbb{C}\backslash\mathbb{R}$,

$$f(\varphi, x, z) := (1 - \mathrm{i})f(\varphi, x, \mathrm{Re}(z)) + \mathrm{i}f(\varphi, x, \mathrm{Re}(z) + \mathrm{Im}(z))\,.$$

described in (2.1)-(2.8), with suitable estimates of the inverse in high Sobolev norm.

An advantage of the vector NLS formulation, with respect to the scalar NLS equation (1.6), is that the operators $\mathcal{L}(\mathbf{u})$ are $\mathbb{C}$-linear and selfadjoint. This is convenient for proving the measure estimates via eigenvalue variation arguments. Moreover the matrix T is Töplitz, see (2.13), and its entries on the lines parallel to the diagonal decay to zero at a polynomial rate.

Matrices with off-diagonal decay. In section 3 we develop an abstract setting for dealing with matrices with polynomial off-diagonal decay. In Definition 3.2 we introduce the s-norm of a matrix and we prove the algebra and interpolation properties (3.16), (3.15). The s-norms are inspired to mimic the behavior of matrices representing the multiplication operator by a function of H^s. This intrinsic setting is very convenient (in particular for the multiscale Proposition 4.1) to estimate the decay of inverse matrices via Neumann series, because product, and then powers, of matrices with finite s-norm will exhibit the same off-diagonal decay.

Improved Nash-Moser iteration. We construct inductively better and better approximate solutions $\mathbf{u}_n$ of the NLS equation (1.11) by a Nash-Moser iterative scheme, see the "truncated" equations (P_n) in Theorem 7.1. The $\mathbf{u}_n \in H_n$, see (7.1), are trigonometric polynomials with a super-exponential number N_n of harmonics, see (7.2).

At each step we impose that, for "most" parameters $(\varepsilon, \lambda) \in [0, \varepsilon_0) \times [1/2, 3/2]$, the eigenvalues of the restricted linearized operators $\mathcal{L}_n := P_n \mathcal{L}(\mathbf{u}_n)_{|H_n}$ are in modulus bounded from below by $O(N_n^{-\tau})$, see Lemma 6.7. The proof exploits that $-\Delta + V$ is positive definite, see (1.3) and remark 1.1. Then the L^2-norm of the inverse satisfies $\|\mathcal{L}_n^{-1}\|_0 = O(N_n^{\tau})$. By Lemma 3.6 this implies that the s-norm (see Definition 3.2) satisfies

$$|\mathcal{L}_n^{-1}|_s \leq N_n^{s+d+\nu} \|\mathcal{L}_n^{-1}\|_0 = O(N_n^{s+d+\nu+\tau}) \, , \ \forall s > 0 \, .$$

Such an estimate is *not* sufficient for the convergence of the Nash-Moser scheme. We need sharper estimates for the Green functions (sublinear decay), of the form

$$|\mathcal{L}_n^{-1}|_s = O(N_n^{\tau' + \delta s}) \, , \quad \delta \in (0,1) \, , \ \tau' > 0 \, , \ \forall s > 0 \, , \tag{1.14}$$

which imply an off-diagonal decay of the inverse matrix coefficients like

$$|(\mathcal{L}_n^{-1})_{i'}^{i}| \leq C \frac{N_n^{\tau' + \delta s}}{\langle i - i' \rangle^s} \, , \quad |i|, |i'| \leq N_n \, ,$$

see Definition 3.10. Actually the conditions (1.14) are optimal for the convergence of the Nash-Moser iterative scheme, as a famous counter-example of Lojasiewicz-Zehnder [32] shows: if $\delta = 1$ the scheme does not converge. By Lemma 3.5 the bound (1.14) implies the interpolation estimate in Sobolev norms

$$\|\mathcal{L}_n^{-1} h\|_s \leq C(s) (N_n^{\tau' + \delta s} \|h\|_{s_1} + N_n^{\tau' + \delta s_1} \|h\|_s) \, , \quad \forall s \geq s_1 \, ,$$

which is sufficient for the Nash-Moser convergence. Note that the exponent $\tau' + \delta s$ in (1.14) grows with s, unlike the usual Nash-Moser theory, see e.g. [39], where the "tame" exponents are s-independent. Actually it is easier to prove these weaker tame estimates, see, in particular, Step II of Lemma 4.3.

In order to prove (1.14) we have to exploit (mild) "separation properties" of the small divisors: several eigenvalues of $\mathcal{L}_n$ are actually much bigger (in modulus) than $N_n^{-\tau}$.

Estimates of Green functions. The core of the paper is to establish the Green functions estimates (1.14) at each step of the iteration, see Lemma 7.7. These follow by an inductive application of the multiscale Proposition 4.1, once verified the "separation property" (H3), see Lemma 7.5.

The "separation properties" of the N_n-bad and singular sites are obtained by Proposition 5.1 for all the parameters (ε, λ) which are N_n-good, see Definition 5.2 and assumption (i). We first use the covariance property (2.20) and the "complexity" information (5.3) on the set $B_N(j_0; \varepsilon, \lambda)$ in (5.2) (the set of the "bad" θ) to bound the number of "bad" time-Fourier components, see Lemma 5.1 (this idea goes back to [20]). Next we use also the information that the sites are "singular" to bound the length of a "chain" of N_n-bad and singular sites (with ideas similar to [13]), see Lemma 5.2.

In order to conclude the inductive proof we have to verify that "most" parameters (ε, λ) are N_n-good. For this, we do not invoke sub-harmonic functions theory, Cartan theorem as in [13], [14], [11].

Measure and "complexity" estimates. Using Proposition 6.1 we prove first that most parameters (ε, λ) are N_n-good in a weak sense. The proof of Proposition 6.1 is based on simple eigenvalue variation arguments and Fubini theorem. The main novelty is to use that $-\Delta + V(x)$ is positive definite, see (1.3) and remark 1.1, and to perform the measure estimates in the new set of variables (6.19). In this way we prove that for "most" parameters (ε, λ) the set $B^0_N(j_0; \varepsilon, \lambda)$ in (6.1) (of "strongly" bad θ) has a small measure. This fact and the Lipschitz dependence of the eigenvalues with respect to parameters imply also the complexity bound (6.3), see Lemma 6.3. Finally, using again the multiscale Proposition 4.1 and the separation Proposition 5.1 we conclude inductively that most of these parameters (ε, λ) are actually N_n-good (in the strong sense), see Lemma 7.6.

Definition of regular sites. In order to deal with a multiplicative potential the key idea is to define "very regular" sites, i.e. in Definition 4.2 the constant Θ will be taken large with respect to the potential V, so that the diagonal terms (2.21) dominate also the off diagonal part $V_0(x)$ of the potential, see Lemma 4.1. Taking a large value for the constant Θ does not affect the qualitative properties of the chains of singular sites proved in Lemma 5.2. Then we achieve in section 5 the separation properties for the clusters of small divisors, and the multiscale Proposition 4.1 applies. We refer also to Lemmas 7.3 and 7.4 for a similar construction at the initial step of the iteration.

Melnikov non-resonance conditions. An advantage of the Nash-Moser iterative scheme is to require weaker non-resonance conditions than for the KAM approach. For clarity we collect all the non-resonance conditions that we make along the paper below:

- $\omega = \lambda\bar{\omega}$ is diophantine, see (1.5), (5.6). It is used only in Lemma 5.1 to get separation properties of the bad sites in the time Fourier components.

- $\omega = \lambda\bar{\omega}$ satisfies the non-resonance condition (7.19) of first order Melnikov type. Physically, this assumption means that the forcing frequencies ω do not enter in resonance with the first N_0 normal mode frequencies of the linearized Schrödinger equation at the origin. This is used for the initialization of the Nash-Moser scheme, see subsection 7.1.

- $(\lambda\bar{\omega}, \varepsilon)$ satisfy the "first order Melnikov" non-resonance conditions at each step of the Nash-Moser iteration: the eigenvalues of $A_{N_n}(\lambda\bar{\omega}, \varepsilon)$ have to be $\geq 2N_n^{-\tau}$, see also Lemma 6.7.

- We also verify that most frequencies are N-good (see Definition 5.2) imposing conditions on the eigenvalues of the matrices $A_{N,j_0}(\lambda\bar{\omega}, \varepsilon, \theta)$ as in Lemma 6.6. These requirements can then be seen as other "first order Melnikov" non-resonance conditions.

Sobolev regularities. Along the proof we make use of three different Sobolev regularity thresholds

$$s_0 < s_1 < S\,.$$

The scale $s_0 > (d+\nu)/2$ is simply required to establish the algebra and interpolation estimates, see e.g. (1.9). The Sobolev index s_1 is large enough to have a sufficiently strong decay when proving the multiscale Proposition 4.1, see (4.5). Finally the Sobolev regularity S is large enough (see (7.16)) for proving the convergence of the Nash-Moser iterative scheme in section 7.

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2 The linearized equation

We look for solutions of the vector NLS equation (1.11) in $\mathbf{H}^s \cap \mathcal{U}$ (see (1.13)) by a Nash-Moser iterative scheme. The main step concerns the invertibility of (any finite dimensional restriction of) the family of linearized operators

$$\mathcal{L}(\mathbf{u}) := \mathcal{L}(\omega, \varepsilon, \mathbf{u}) := L_\omega - \varepsilon T_1 \tag{2.1}$$

acting on $\mathbf{H}^s$, where $\mathbf{u} = (u^+, u^-) \in C^1([0, \varepsilon_0] \times \Lambda, \mathbf{H}^s \cap \mathcal{U})$,

$$L_\omega := \begin{pmatrix} \mathrm{i}\omega \cdot \partial_\varphi - \Delta + V(x) & 0 \\ 0 & -\mathrm{i}\omega \cdot \partial_\varphi - \Delta + V(x) \end{pmatrix} \tag{2.2}$$

and

$$T_1 := \begin{pmatrix} p(\varphi, x) & q(\varphi, x) \\ \bar{q}(\varphi, x) & p(\varphi, x) \end{pmatrix} \tag{2.3}$$

with

$$p(\varphi, x) := f(\varphi, x, |u^+|^2) + f'(\varphi, x, |u^+|^2)|u^+|^2\,,\ q(\varphi, x) := f'(\varphi, x, |u^+|^2)(u^+)^2\,. \tag{2.4}$$

Above f' denotes the derivative of $f(\varphi, x, s)$ with respect to s. The functions p, q depend also on ε, λ through $\mathbf{u}$. Note that $u^+u^- = |u^+|^2 \in \mathbb{R}$ since $\mathbf{u} \in \mathcal{U}$, see (1.13).

Decomposing the multiplicative potential

$$V(x) = m + V_0(x) \tag{2.5}$$

where m is the average of $V(x)$ and $V_0(x)$ has zero mean value, we also write

$$L_\omega = D_\omega + T_2 \tag{2.6}$$

where D_ω is the constant coefficient differential operator

$$D_\omega := \begin{pmatrix} \mathrm{i}\omega \cdot \partial_\varphi - \Delta + m & 0 \\ 0 & -\mathrm{i}\omega \cdot \partial_\varphi - \Delta + m \end{pmatrix} \quad \text{and} \quad T_2 := \begin{pmatrix} V_0(x) & 0 \\ 0 & V_0(x) \end{pmatrix}. \tag{2.7}$$

Hence the operator $\mathcal{L}(\mathbf{u})$ in (2.1) can also be written as

$$\mathcal{L}(\mathbf{u}) = D_\omega + T\,, \qquad T := T_2 - \varepsilon T_1\,. \tag{2.8}$$

Lemma 2.1. *$\mathcal{L}(\mathbf{u})$ is symmetric in $\mathbf{H}^0$, i.e. $(\mathcal{L}(\mathbf{u})h, k)_{L^2} = (h, \mathcal{L}(\mathbf{u})k)_{L^2}$ for all h, k in the domain of $\mathcal{L}(\mathbf{u})$.*

Proof. The operator L_ω is symmetric with respect to the L^2-scalar product in $\mathbf{H}^0$, because each $\pm\mathrm{i}\omega \cdot \partial_\varphi - \Delta + V(x)$ is symmetric in $H^0(\mathbb{T}^\nu \times \mathbb{T}^d; \mathbb{C})$. Moreover T_2, T_1 are selfadjoint in $\mathbf{H}^0$ because $V(x)$ and $p(\varphi, x)$ are real valued, being $|u^+|^2 \in \mathbb{R}$ and f real by (1.2), see [5]. ∎

The Fourier basis diagonalizes the differential operator D_ω. In what follows we sometimes identify an operator with the associated (infinite dimensional) matrix in the Fourier basis. The operator $\mathcal{L}(\omega, \varepsilon, \mathbf{u})$ is represented by the infinite dimensional Hermitian matrix

$$A(\omega) := A(\omega, \varepsilon, \mathbf{u}) := D_\omega + T\,, \tag{2.9}$$

where

$$D_\omega := \mathrm{diag}_{i \in \mathbb{Z}^b} \begin{pmatrix} -\omega \cdot l + \|j\|^2 + m & 0 \\ 0 & \omega \cdot l + \|j\|^2 + m \end{pmatrix}, \tag{2.10}$$

$$i := (l, j) \in \mathbb{Z}^b := \mathbb{Z}^\nu \times \mathbb{Z}^d\,, \qquad \|j\|^2 := j_1^2 + \ldots + j_d^2\,, \tag{2.11}$$

and

$$T := (T_i^{i'})_{i \in \mathbb{Z}^b, i' \in \mathbb{Z}^b}\,, \ \ T_i^{i'} := -\varepsilon (T_1)_i^{i'} + (T_2)_i^{i'}\,, \tag{2.12}$$

$$(T_1)_i^{i'} = \begin{pmatrix} p_{i-i'} & q_{i-i'} \\ (\overline{q})_{i-i'} & p_{i-i'} \end{pmatrix}, \quad (T_2)_i^{i'} = \begin{pmatrix} (V_0)_{j-j'} & 0 \\ 0 & (V_0)_{j-j'} \end{pmatrix}, \tag{2.13}$$

where $p_i, q_i, (V_0)_j$ denote the Fourier coefficients of $p(\varphi, x), q(\varphi, x), V_0(x)$.

Note that $(T_i^{i'})^\dagger = T_{i'}^i$ (the symbol $\dagger$ denotes the conjugate transpose) because $(\overline{q})_{i-i'} = \overline{q_{i'-i}}$ and $\overline{p}_i = p_{-i}$, since p is real-valued. The matrix T is *Töplitz*, namely $T_i^{i'}$ depends only on the difference of the indices $i - i'$. Moreover, since the functions p, q in (2.4), as well as the potential V, are in H^s, then $T_i^{i'} \to 0$ as $|i - i'| \to \infty$ at a polynomial rate. In the next section we introduce precise norms to measure such off-diagonal decay.

Moreover we shall introduce a further index $a \in \{0, 1\}$ to distinguish the two eigenvalues $\pm\omega \cdot l + \|j\|^2 + m$ (see (2.21)) and the four elements of each of these 2×2 matrices, see Definition 3.1 and (3.2).

We introduce the one-parameter family of infinite dimensional matrices

$$A(\omega,\theta) := A(\omega) + \theta Y := D_\omega + T + \theta Y \tag{2.14}$$

where

$$Y := \mathrm{diag}_{i\in\mathbb{Z}^b} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{2.15}$$

The reason for adding θY is that, translating the time Fourier indices

$$(l, j) \mapsto (l + l_0, j)$$

in $A(\omega)$, gives $A(\omega,\theta)$ with $\theta = \omega \cdot l_0$, see (2.20): the matrix T remains unchanged under translation because it is Töplitz.

Remark 2.1. *The covariance property* (2.20) *will be exploited in section 5 to prove "separation properties" of the "singular sites".*

We shall study properties of the linearized systems $A(\omega,\varepsilon,\mathbf{u})v = h$ in sections $3-6$. To apply the results of these sections to the Nash-Moser scheme of section 7, we have to keep in mind that $\mathbf{u}$ itself depends on the parameters (ω,ε) (in a C^1 way, with some bound on $\|\mathbf{u}\|_{s_1} + \|\partial_{(\omega,\varepsilon)}\mathbf{u}\|_{s_1}$). Therefore the frame of sections $3-6$ will be the following: we study parametrized families of (infinite dimensional) matrices

$$A(\varepsilon,\lambda,\theta) = D(\lambda) + T(\varepsilon,\lambda) + \theta Y, \tag{2.16}$$

where $D(\lambda)$ is defined by (2.10) with $\omega = \lambda\bar\omega$, and T is a Töplitz matrix such that $|T|_{s_1} + |\partial_{(\lambda,\varepsilon)}T|_{s_1} \leq C$ (C depending on V).

The main goal of the following sections is to prove polynomial off-diagonal decay for the inverse of the $2(2N+1)^b$-dimensional sub-matrices of $A(\varepsilon,\lambda,\theta)$ centered at (l_0, j_0) denoted by

$$A_{N,l_0,j_0}(\varepsilon,\lambda,\theta) := A_{|l-l_0|\leq N, |j-j_0|\leq N}(\varepsilon,\lambda,\theta) \tag{2.17}$$

where

$$|l| := \max\{|l_1|,\ldots,|l_\nu|\}\,, \quad |j| := \max\{|j_1|,\ldots,|j_d|\}\,, \quad |j| \leq \|j\| \leq \sqrt{d}|j|\ ,. \tag{2.18}$$

If $l_0 = 0$ we use the simpler notation

$$A_{N,j_0}(\varepsilon,\lambda,\theta) := A_{N,0,j_0}(\varepsilon,\lambda,\theta)\,. \tag{2.19}$$

If also $j_0 = 0$, we simply write

$$A_N(\varepsilon,\lambda,\theta) := A_{N,0}(\varepsilon,\lambda,\theta)\,,$$

and, for $\theta = 0$, we denote

$$A_{N,j_0}(\varepsilon,\lambda) := A_{N,j_0}(\varepsilon,\lambda,0)\,.$$

We have the following crucial *covariance* property

$$A_{N,l_1,j_1}(\varepsilon,\lambda,\theta) = A_{N,j_1}(\varepsilon,\lambda,\theta + \lambda\bar\omega\cdot l_1)\,, \tag{2.20}$$

which will be exploited in Lemma 5.1.

A major role is played by the eigenvalues of $D(\lambda) + \theta Y$,

$$d_i^\pm := d_i^\pm(\lambda,\theta) := \pm\lambda\bar\omega\cdot l + \|j\|^2 + m \pm \theta\,.$$

In order to distinguish between the $\pm$ sites we introduce an index

$$a \in \{0, 1\}$$

and we denote

$$d_{i,a}(\lambda,\theta) = \begin{cases} \lambda\bar\omega\cdot l + \|j\|^2 + m + \theta & \text{if } a = 0 \\ -\lambda\bar\omega\cdot l + \|j\|^2 + m - \theta & \text{if } a = 1\,. \end{cases} \tag{2.21}$$

3 Matrices with off-diagonal decay

Let us consider the basis of the vector-space $\mathbf{H}^s := H^s \times H^s$ made up by

$$e_{i,0} := (e^{\mathrm{i}(l\cdot\varphi+j\cdot x)}, 0), \ e_{i,1} := (0, e^{\mathrm{i}(l\cdot\varphi+j\cdot x)}), \quad i := (l,j) \in \mathbb{Z}^b := \mathbb{Z}^\nu \times \mathbb{Z}^d \,. \tag{3.1}$$

Then we write any $\mathbf{u} = (u^+, u^-) \in H^s \times H^s$ as

$$\mathbf{u} = \sum_{k\in\mathbb{Z}^b\times\{0,1\}} u_k e_k \,, \quad k := (i,a) \in \mathbb{Z}^b \times \{0,1\} \,,$$

where $\mathbf{u}_{l,j,0} := u^+_{l,j}$, resp. $\mathbf{u}_{l,j,1} := u^-_{l,j}$, denote the Fourier indices of u^+, resp. u^-, see (1.7).

For $B \subset \mathbb{Z}^b \times \{0,1\}$, we introduce the subspace

$$\mathbf{H}^s_B := \Big\{ \mathbf{u} \in H^s \times H^s \ : \ u_k = 0 \text{ if } k \notin B \Big\} .$$

When B is finite, the space $\mathbf{H}^s_B$ does not depend on s and will be denoted $\mathbf{H}_B$. We define

$$\Pi_B : \mathbf{H}^s \to \mathbf{H}_B$$

the L^2-orthogonal projector onto $\mathbf{H}_B$.

In what follows B, C, D, E are finite subsets of $\mathbb{Z}^b \times \{0,1\}$.

We identify the space $\mathcal{L}^B_C$ of the linear maps $L : \mathbf{H}_B \to \mathbf{H}_C$ with the space of matrices

$$\mathcal{M}^B_C := \Big\{ M = (M^{k'}_k)_{k'\in B, k\in C} \,, \ M^{k'}_k \in \mathbb{C} \Big\}$$

according to the following usual definition.

Definition 3.1. *The matrix $M \in \mathcal{M}^B_C$ represents the linear operator $L \in \mathcal{L}^B_C$, if*

$$\forall k' = (i', a') \in B, \, k = (i,a) \in C \,, \quad \Pi_k L e_{k'} = M^{k'}_k e_k \,,$$

where $e_{i,0}$, $e_{i,1}$ are defined in (3.1) *and $M^{k'}_k \in \mathbb{C}$.*

For example, with the above notation, the matrix elements of the matrix $(T_1)^{i'}_i$ in (2.13) are

$$(T_1)^{i',0}_{i,0} = p_{i-i'} \,, \ (T_1)^{i',1}_{i,0} = q_{i-i'} \,, \ (T_1)^{i',0}_{i,1} = (\overline{q})_{i-i'} = \overline{q_{i'-i}} \,, \ (T_1)^{i',1}_{i,1} = p_{i-i'} \,. \tag{3.2}$$

Notations. For any subset B of $\mathbb{Z}^b \times \{0,1\}$, we denote by

$$\overline{B} := \mathrm{proj}_{\mathbb{Z}^b} B \tag{3.3}$$

the projection of B in $\mathbb{Z}^b$.

Given $B \subset B'$, $C \subset C' \subset \mathbb{Z}^b \times \{0,1\}$ and $M \in \mathcal{M}^{B'}_{C'}$ we can introduce the restricted matrices

$$M^B_C := \Pi_C M_{|\mathbf{H}_B} \,, \quad M_C := \Pi_C M \,, \quad M^B := M_{|\mathbf{H}_B} \,. \tag{3.4}$$

If $D \subset \mathrm{proj}_{\mathbb{Z}^b} B'$, $E \subset \mathrm{proj}_{\mathbb{Z}^b} C'$, then we define

$$M^D_E \ \text{ as } \ M^B_C \quad \text{where} \quad B := (D \times \{0,1\}) \cap B', \ C := (E \times \{0,1\}) \cap C' \,. \tag{3.5}$$

In the particular case $D = \{i'\}$, $E := \{i\}$, $i, i' \in \mathbb{Z}^b$, we use the simpler notations

$$M_i := M_{\{i\}} \quad \text{(it is either a line or a group of two lines of } M), \tag{3.6}$$

$$M^{i'} := M^{\{i'\}} \quad \text{(it is either a column or a group of two columns of } M), \tag{3.7}$$

and

$$M_i^{i'} := M_{\{i\}}^{\{i'\}}\,, \tag{3.8}$$

it is a $m \times m'$-complex matrix, where $m \in \{1,2\}$ (resp. $m' \in \{1,2\}$) is the cardinality of C (resp. of B) defined in (3.5) with $E := \{i\}$ (resp. $D = \{i'\}$).

We endow the vector-space of the 2×2 (resp. 2×1, 1×2, 1×1) complex matrices with a norm $|\ |$ such that

$$|UW| \le |U||W|\,,$$

whenever the dimensions of the matrices make their multiplication possible, and $|U| \le |V|$ if U is a submatrix of V.

Remark 3.1. *The notations in* (3.5), (3.6), (3.7), (3.8), *may be not very specific, but it is deliberate: it is convenient not to distinguish the index $a \in \{0,1\}$, which is irrelevant in the definition of the s-norms, in Definition 3.2.*

We also set the L^2-operatorial norm

$$\|M_C^B\|_0 := \sup_{h \in \mathbf{H}_B, h \neq 0} \frac{\|M_C^B h\|_0}{\|h\|_0} \tag{3.9}$$

where $\|\ \|_0 := \|\ \|_{L^2}$.

Definition 3.2. **(s-norm)** *The s-norm of a matrix $M \in \mathcal{M}_C^B$ is defined by*

$$|M|_s^2 := K_0 \sum_{n \in \mathbb{Z}^b} [M(n)]^2 \langle n \rangle^{2s} \tag{3.10}$$

where $\langle n \rangle := \max(|n|, 1)$,

$$[M(n)] := \begin{cases} \max\limits_{i-i'=n, i \in \overline{C}, i' \in \overline{B}} |M_i^{i'}| & \text{if } \ n \in \overline{C} - \overline{B} \\ 0 & \text{if } \ n \notin \overline{C} - \overline{B} \end{cases} \tag{3.11}$$

with $\overline{B} := \mathrm{proj}_{\mathbb{Z}^b} B$, $\overline{C} := \mathrm{proj}_{\mathbb{Z}^b} C$ *(see* (3.3)*), and the constant $K_0 > 0$ is introduced in* (1.7).

It is easy to check that $|\ |_s$ is a norm on $\mathcal{M}_C^B$. It verifies $|\ |_s \le |\ |_{s'}$, $\forall s \le s'$, and

$$\forall M \in \mathcal{M}_C^B\,, \quad \forall B' \subseteq B\,,\ C' \subseteq C\,, \quad |M_{C'}^{B'}|_s \le |M|_s\,.$$

The s-norm is designed to estimate the off-diagonal decay of matrices like T in (2.12) with $p, q, V \in H^s$.

Lemma 3.1. *The matrices T_1, T_2 in* (2.3), (2.7) *with $p, q, V \in H^s$, satisfy*

$$|T_1|_s \le K \|(q,p)\|_s\,, \quad |T_2|_s \le K \|V\|_s\,. \tag{3.12}$$

Proof. By (3.11), (2.13) we get

$$[T_1(n)] := \max_{i-i'=n} \left| \begin{pmatrix} p_{i-i'} & q_{i-i'} \\ \overline{q_{i-i'}} & p_{i-i'} \end{pmatrix} \right| \le K(|p_n| + |q_n|)\,.$$

Hence, the definition in (3.10) implies

$$|T_1|_s^2 = K_0 \sum_{n \in \mathbb{Z}^b} [T_1(n)]^2 \langle n \rangle^{2s} \le K_1 \sum_{n \in \mathbb{Z}^b} (|p_n| + |q_n|)^2 \langle n \rangle^{2s} \le K_2 \|(p,q)\|_s^2$$

and (3.12) follows. The estimate for $|T_2|_s$ is similar. ■

In order to prove that the matrices with finite s-norm satisfy the interpolation inequalities (3.15), and then the algebra property (3.16), the guiding principle is the analogy between these matrices and the operators of the form (2.3), i.e. the multiplication operators for functions. We introduce the subset $\mathcal{H}_+$ of $\cap_{s\geq 0} H^s$ formed by the trigonometric polynomials with positive Fourier coefficients

$$\mathcal{H}_+ := \Big\{ h = \sum h_{l,j} e^{\mathrm{i}(l\cdot\varphi + j\cdot x)} \text{ with } h_{l,j} \neq 0 \text{ for a finite number of } (l,j) \text{ only and } h_{l,j} \in \mathbb{R}_+ \Big\} .$$

Note that the sum and the product of two functions in $\mathcal{H}_+$ remain in $\mathcal{H}_+$.

Definition 3.3. *Given $M \in \mathcal{M}_C^B$, $h \in \mathcal{H}_+$, we say that M is dominated by h, and we write $M \prec h$, if*

$$[M(n)] \leq h_n , \quad \forall n \in \mathbb{Z}^b , \tag{3.13}$$

in other words if $|M_i^{i'}| \leq h_{i-i'}$, $\forall i' \in \mathrm{proj}_{\mathbb{Z}^b} B$, $i \in \mathrm{proj}_{\mathbb{Z}^b} C$.

It is easy to check (B and C being finite) that

$$|M|_s = \min \Big\{ \|h\|_s \ : \ h \in \mathcal{H}_+ \ , \ M \prec h \Big\} \quad \text{and} \quad \exists h \in \mathcal{H}_+ \ , \ \forall s \geq 0 \, , \ |M|_s = \|h\|_s \, . \tag{3.14}$$

Lemma 3.2. *For $M_1 \in \mathcal{M}_D^C$, $M_2 \in \mathcal{M}_C^B$, $M_3 \in \mathcal{M}_D^C$, we have*

$$M_1 \prec h_1 \, , \ M_2 \prec h_2 \, , \ M_3 \prec h_3 \quad \Longrightarrow \quad M_1 + M_3 \prec h_1 + h_3 \quad \text{and} \quad M_1 M_2 \prec h_1 h_2 \, .$$

Proof. Property $M_1 + M_3 \prec h_1 + h_3$ is straightforward. For $i \in \mathrm{proj}_{\mathbb{Z}^b} D$, $i' \in \mathrm{proj}_{\mathbb{Z}^b} B$, we have

$$\begin{aligned} |(M_1M_2)_i^{i'}| = \Big| \sum_{q \in \overline{C} := \mathrm{proj}_{\mathbb{Z}^b} C} (M_1)_i^q (M_2)_q^{i'} \Big| \ &\leq \ \sum_{q\in\overline{C}} |(M_1)_i^q| |(M_2)_q^{i'}| \leq \sum_{q\in\overline{C}} (h_1)_{i-q} (h_2)_{q-i'} \\ &\leq \ \sum_{q \in \mathbb{Z}^b} (h_1)_{i-q}(h_2)_{q-i'} = (h_1h_2)_{i-i'} \end{aligned}$$

implying $M_1M_2 \prec h_1h_2$ by Definition 3.3. ■

We immediately deduce from (1.9) and (3.14) the following interpolation estimates.

Lemma 3.3. (Interpolation) *$\forall s \geq s_0 > (d+\nu)/2$ there is $C(s) \geq 1$, with $C(s_0) = 1$, such that, for any finite subset $B, C, D \subset \mathbb{Z}^b \times \{0,1\}$, $\forall M_1 \in \mathcal{M}_D^C$, $M_2 \in \mathcal{M}_C^B$,*

$$|M_1M_2|_s \leq (1/2)|M_1|_{s_0}|M_2|_s + (C(s)/2)|M_1|_s|M_2|_{s_0} \, , \tag{3.15}$$

in particular,

$$|M_1M_2|_s \leq C(s)|M_1|_s|M_2|_s \, . \tag{3.16}$$

Note that the constant $C(s)$ in Lemma 3.3 is independent of B, C, D. By (3.16) with $s = s_0$, we get (recall that $C(s_0) = 1$)

Lemma 3.4. *For any finite subset $B, C, D \subset \mathbb{Z}^b \times \{0,1\}$, for all $M_1 \in \mathcal{M}_D^C$, $M_2 \in \mathcal{M}_C^B$, we have*

$$|M_1M_2|_{s_0} \leq |M_1|_{s_0}|M_2|_{s_0} \, , \tag{3.17}$$

and, $\forall M \in \mathcal{M}_B^B$, $\forall n \geq 1$,

$$|M^n|_{s_0} \leq |M|_{s_0}^n \qquad \text{and} \qquad |M^n|_s \leq C(s)|M|_{s_0}^{n-1}|M|_s \, , \ \forall s \geq s_0 \, . \tag{3.18}$$

Proof. The second estimate in (3.18) is obtained from (3.15), using $C(s) \geq 1$. ■

The s-norm of a matrix $M \in \mathcal{M}_C^B$ controls also the Sobolev H^s-norm. Indeed, we identify $\mathbf{H}_B$ with the space $\mathcal{M}_B^{\{0\}}$ of column matrices and the Sobolev norm $\| \ \|_s$ is equal to the s-norm $| \ |_s$, i.e.

$$\forall w \in \mathbf{H}_B \, , \quad \|w\|_s = |w|_s \, , \quad \forall s \geq 0 \, . \tag{3.19}$$

Then $Mw \in \mathbf{H}_C$ and the next lemma is a particular case of Lemma 3.3.

Lemma 3.5. (Sobolev norm) *$\forall s \geq s_0$ there is $C(s) \geq 1$ such that, for any finite subset $B, C \subset \mathbb{Z}^b \times \{0,1\}$,*

$$\|Mw\|_s \leq (1/2)|M|_{s_0}\|w\|_s + (C(s)/2)|M|_s\|w\|_{s_0}\,, \quad \forall M \in \mathcal{M}_C^B\,,\ w \in \mathbf{H}_B\,. \tag{3.20}$$

The following lemma is the analogue of the smoothing properties (7.4)-(7.5) of the projection operators.

Lemma 3.6. (Smoothing) *Let $M \in \mathcal{M}_C^B$. Then, $\forall s' \geq s \geq 0$,*

$$M_i^{i'} = 0\,,\ \forall |i - i'| < N \quad \Longrightarrow \quad |M|_s \leq N^{-(s'-s)}|M|_{s'}\,, \tag{3.21}$$

and, for $N \geq N(K_0)$,

$$M_i^{i'} = 0\,,\ \forall |i - i'| > N \quad \Longrightarrow \quad \begin{cases} |M|_{s'} \leq N^{s'-s}|M|_s \\ |M|_s \leq N^{s+b}\|M\|_0\,. \end{cases} \tag{3.22}$$

Proof. Estimate (3.21) and the first bound of (3.22) follow from the definition of the norms $|\ |_s$. The second bound of (3.22) follows by the first bound in (3.22), noting that $|M_i^{i'}| \leq \|M\|_0$, $\forall i, i'$,

$$|M|_s \leq N^s|M|_0 \leq K_0^{1/2}N^s\sqrt{(2N+1)^b}\|M\|_0 \leq N^{s+b}\|M\|_0$$

for $N \geq N(K_0)$. ■

In the next lemma we bound the s-norm of a matrix in terms of the $(s+b)$-norms of its lines.

Lemma 3.7. (Decay along lines) *Let $M \in \mathcal{M}_C^B$. Then, $\forall s \geq 0$,*

$$|M|_s \leq K_1 \max_{i \in \mathrm{proj}_{\mathbb{Z}^b} C} |M_{\{i\}}|_{s+b} \tag{3.23}$$

(we could replace the index b with any $\alpha > b/2$).

Proof. For all $i \in \overline{C} := \mathrm{proj}_{\mathbb{Z}^b} C$, $i' \in \overline{B} := \mathrm{proj}_{\mathbb{Z}^b} B$, $\forall s \geq 0$,

$$|M_i^{i'}| \leq \frac{|M_{\{i\}}|_{s+b}}{\langle i - i' \rangle^{s+b}} \leq \frac{m(s+b)}{\langle i - i' \rangle^{s+b}}$$

where $m(s+b) := \max_{i \in \overline{C}} |M_{\{i\}}|_{s+b}$. As a consequence

$$|M|_s = \Big(\sum_{n \in \overline{C} - \overline{B}} (M[n])^2 \langle n \rangle^{2s} \Big)^{1/2} \leq m(s+b) \Big(\sum_{n \in \mathbb{Z}^b} \langle n \rangle^{-2b} \Big)^{1/2} = m(s+b)K(b)$$

implying (3.23). ■

The L^2-norm and s_0-norm of a matrix are related.

Lemma 3.8. *Let $M \in \mathcal{M}_B^C$. Then, for $s_0 > (d+\nu)/2$,*

$$\|M\|_0 \leq |M|_{s_0}\,. \tag{3.24}$$

Proof. Let $m \in \mathcal{H}_+$ be such that $M \prec m$ and $|M|_s = \|m\|_s$ for all $s \geq 0$, see (3.14). Also for $H \in \mathcal{M}_C^{\{0\}}$, there is $h \in \mathcal{H}_+$ such that $H \prec h$ and $|H|_s = \|h\|_s$, $\forall s \geq 0$. Lemma 3.2 implies that $MH \prec mh$ and so

$$|MH|_0 \leq \|mh\|_0 \leq |m|_{L^\infty}\|h\|_0 \overset{(1.8)}{\leq} \|m\|_{s_0}\|h\|_0 = |M|_{s_0}|H|_0\,, \quad \forall H \in \mathcal{M}_C^{\{0\}}\,.$$

Then (3.24) follows (recall (3.19)). ■

It will be convenient to use the notion of left invertible operators.

Definition 3.4. (Left Inverse) *A matrix $M \in \mathcal{M}_C^B$ is left invertible if there exists $N \in \mathcal{M}_B^C$ such that $NM = I_B$. Then N is called a left inverse of M.*

Note that M is left invertible if and only if M (considered as a linear map) is injective (then $\dim \mathbf{H}_C \geq \dim \mathbf{H}_B$). The left inverses of M are not unique if $\dim \mathbf{H}_C > \dim \mathbf{H}_B$: they are uniquely defined only on the range of M.

We shall often use the following perturbation lemma for left invertible operators. Note that the bound (3.25) for the perturbation in s_0-norm only, allows to estimate the inverse (3.28) also in $s \geq s_0$ norm.

Lemma 3.9. (Perturbation of left invertible matrices) *If $M \in \mathcal{M}_C^B$ has a left inverse $N \in \mathcal{M}_B^C$, then*

$$\forall P \in \mathcal{M}_C^B \qquad \text{with} \qquad |N|_{s_0} |P|_{s_0} \leq 1/2 \,, \tag{3.25}$$

the matrix $M + P$ has a left inverse N_P that satisfies

$$|N_P|_{s_0} \leq 2 |N|_{s_0} \,, \tag{3.26}$$

and, $\forall s \geq s_0$,

$$|N_P|_s \leq \Big(1 + C(s) |N|_{s_0} |P|_{s_0}\Big) |N|_s + C(s) |N|_{s_0}^2 |P|_s \tag{3.27}$$

$$\leq C(s) \Big(|N|_s + |N|_{s_0}^2 |P|_s\Big) \,. \tag{3.28}$$

Moreover,

$$\forall P \in \mathcal{M}_C^B \qquad \text{with} \qquad \|N\|_0 \|P\|_0 \leq 1/2 \,, \tag{3.29}$$

the matrix $M + P$ has a left inverse N_P that satisfies

$$\|N_P\|_0 \leq 2 \|N\|_0 \,. \tag{3.30}$$

Proof. We simplify notations denoting $C(s)$ any constant that depends on s only.

Step I. *Proof of (3.26).*

The matrix $N_P = AN$ with $A \in \mathcal{M}_B^B$ is a left inverse of $M + P$ if and only if

$$I_B = AN(M + P) = A(I_B + NP) \,,$$

i.e. if and only if A is the inverse of $I_B + NP \in \mathcal{M}_B^B$. By (3.25) $|NP|_{s_0} \leq 1/2$, hence the matrix $I_B + NP$ is invertible and

$$N_P = AN = (I_B + NP)^{-1} N = \sum_{p=0}^{\infty} (-1)^p (NP)^p N \tag{3.31}$$

is a left inverse of $M + P$. Estimate (3.26) is an immediate consequence of (3.31), (3.17) and (3.25).

Step II. *Proof of (3.27).*

For all $s \geq s_0$

$$\begin{aligned} \forall p \geq 1, \ |(NP)^p N|_s &\overset{(3.15)}{\leq} C(s) |N|_{s_0} |(NP)^p|_s + C(s) |N|_s |(NP)^p|_{s_0} \\ &\overset{(3.18)}{\leq} C(s) |N|_{s_0} |NP|_{s_0}^{p-1} |NP|_s + C(s) |N|_s |NP|_{s_0}^p \\ &\overset{(3.25),(3.15)}{\leq} C(s) 2^{-p} (|N|_{s_0} |P|_{s_0} |N|_s + |N|_{s_0}^2 |P|_s) \,. \end{aligned} \tag{3.32}$$

We derive (3.27) by

$$|N_P|_s \overset{(3.31)}{\leq} |N|_s + \sum_{p=1}^{\infty} |(NP)^p N|_s \overset{(3.32)}{\leq} |N|_s + C(s) (|N|_{s_0} |P|_{s_0} |N|_s + |N|_{s_0}^2 |P|_s) \,.$$

Finally (3.30) follows from (3.29) as in Step I because the operatorial L^2-norm (see (3.9)) satisfies the algebra property as the s_0-norm in (3.17). ■

4 The multiscale analysis: estimates of Green functions

The main result of this section is the multiscale Proposition 4.1. In the whole section $\delta \in (0,1)$ is fixed and $\tau' > 0$, $\Theta \geq 1$ are real parameters, on which we shall impose some condition in Proposition 4.1.

Given $\Omega, \Omega' \subset E \subset \mathbb{Z}^b \times \{0,1\}$ we define

$$\mathrm{diam}(E) := \sup_{k,k' \in E} |k - k'| \,, \qquad \mathrm{d}(\Omega, \Omega') := \inf_{k \in \Omega, k' \in \Omega'} |k - k'| \,,$$

where, for $k = (i,a)$, $k' := (i', a')$ we set

$$|k - k'| := \max\{|i - i'|, |a - a'|\} \,.$$

Definition 4.1. (N-good/bad matrix) *The matrix $A \in \mathcal{M}_E^E$, with $E \subset \mathbb{Z}^b \times \{0,1\}$, $\mathrm{diam}(E) \leq 4N$, is N-good if A is invertible and*

$$\forall s \in [s_0, s_1] \ , \quad |A^{-1}|_s \leq N^{\tau' + \delta s}. \tag{4.1}$$

Otherwise A is N-bad.

We first define the regular and singular sites of a matrix.

Definition 4.2. (Regular/Singular sites) *The index $k := (i,a) \in \mathbb{Z}^b \times \{0,1\}$ is* REGULAR *for A if $|A_k^k| \geq \Theta$. Otherwise k is* SINGULAR.

Now we need a more precise notion adapted to the induction process.

Definition 4.3. ((A,N)-good/bad site) *For $A \in \mathcal{M}_E^E$, we say that $k \in E \subset \mathbb{Z}^b \times \{0,1\}$ is*

- *(A,N)-*REGULAR *if there is $F \subset E$ such that $\mathrm{diam}(F) \leq 4N$, $\mathrm{d}(k, E \backslash F) \geq N$ and A_F^F is N-good.*
- *(A,N)-*GOOD *if it is regular for A or (A,N)-regular. Otherwise we say that k is (A,N)-*BAD.

Let us consider the new larger scale

$$N' = N^\chi \tag{4.2}$$

with $\chi > 1$.

For a matrix $A \in \mathcal{M}_E^E$ we define $\mathrm{Diag}(A) := (\delta_{kk'} A_k^{k'})_{k,k' \in E}$.

Proposition 4.1. (Multiscale step) *Assume*

$$\delta \in (0, 1/2) \,, \ \tau' > 2\tau + b + 1 \,, \ C_1 \geq 2 \,, \tag{4.3}$$

and, setting $\kappa := \tau' + b + s_0$,

$$\chi(\tau' - 2\tau - b) > 3(\kappa + (s_0 + b) C_1) \,, \ \chi\delta > C_1 \,, \tag{4.4}$$

$$S \geq s_1 > 3\kappa + \chi(\tau + b) + C_1 s_0 \,. \tag{4.5}$$

For any given $\Upsilon > 0$, there exist $\Theta := \Theta(\Upsilon, s_1) > 0$ large enough (appearing in Definition 4.2), and $N_0(\Upsilon, \Theta, S) \in \mathbb{N}$ such that:
$\forall N \geq N_0(\Upsilon, \Theta, S)$, $\forall E \subset \mathbb{Z}^b \times \{0,1\}$ with $\mathrm{diam}(E) \leq 4N' = 4N^\chi$ (see (4.2)), if $A \in \mathcal{M}_E^E$ satisfies

- **(H1)** $|A - \mathrm{Diag}(A)|_{s_1} \leq \Upsilon$
- **(H2)** $\|A^{-1}\|_0 \leq (N')^\tau$
- **(H3)** *There is a partition of the (A,N)-bad sites $B = \cup_\alpha \Omega_\alpha$ with*

$$\mathrm{diam}(\Omega_\alpha) \leq N^{C_1} \,, \quad \mathrm{d}(\Omega_\alpha, \Omega_\beta) \geq N^2 \ , \ \forall \alpha \neq \beta \,, \tag{4.6}$$

then A is N'-good. More precisely

$$\forall s \in [s_0, S] \ , \quad |A^{-1}|_s \leq \frac{1}{4}(N')^{\tau'}\Big((N')^{\delta s} + |A - \mathrm{Diag}(A)|_s\Big) \,. \tag{4.7}$$

The above proposition says, roughly, the following. If A has a sufficient off-diagonal decay (assumption (H1) and (4.5)), and if the sites that can not be inserted in good "small" submatrices (of size $O(N)$) along the diagonal of A are sufficiently separated (assumption (H3)), then the L^2-bound $(H2)$ for A^{-1} implies that the "large" matrix A (of size $N' = N^{\chi}$ with χ as in (4.4)) is good, and A^{-1} satisfies also the bounds (4.7) in s-norm for $s > s_1$. It is remarkable that the bounds for $s > s_1$ follow only by informations on the N-good submatrices in s_1-norm (see Definition 4.1) plus, of course, the s-decay of A.

According to (4.4) the exponent χ, which measures the new scale $N' >> N$, is large with respect to the size of the bad clusters Ω_α, i.e. with respect to C_1. The intuitive meaning is that, for χ large enough, the "resonance effects" due to the bad clusters are "negligible" at the new larger scale.

The constant $\Theta \geq 1$ which defines the regular sites (see Definition 4.2) must be large enough with respect to Υ, i.e. with respect to the off diagonal part $\mathcal{T} := A - \mathrm{Diag}(A)$, see (H1) and Lemma 4.1. In the application to matrices like A in (2.9) the constant Υ is proportional to $\|V\|_{s_1} + \varepsilon\|(p,q)\|_{s_1}$.

The exponent $\tau \geq \tau(b)$ shall be taken large in order to verify condition (H2), imposing lower bounds on the modulus of the eigenvalues of A. Note that χ in (4.4) can be taken large independently of τ, choosing, for example, $\tau' := 3\tau + 2b$ (see remark 7.2).

Finally, the Sobolev index s_1 has to be large with respect to χ and τ, according to (4.5). This is also natural: if the decay is sufficiently strong, then the "interaction" between different clusters of N-bad sites is weak enough.

Remark 4.1. *In (4.6) we have fixed the separation N^2 between the bad clusters just for definiteness: any separation N^{μ}, $\mu > 0$, would be sufficient. Of course, the smaller $\mu > 0$ is, the larger the Sobolev exponent s_1 has to be. See remark 5.2 for other comments on assumption (H3).*

Remark 4.2. *An advantage of the multiscale Proposition 4.1 with respect to analogous lemmata in [13] (see for example Lemma 14.31-[13]) is to require only an L^2-bound for the inverse of A, and not for submatrices. For this we use the notion of left inverse matrix in the proof.*

The proof of Proposition 4.1 is divided in several lemmas. In each of them we shall assume that the hypotheses of Proposition 4.1 are satisfied. We set

$$\mathcal{T} := A - \mathrm{Diag}(A)\,, \qquad |\mathcal{T}|_{s_1} \stackrel{(H1)}{\leq} \Upsilon \,. \tag{4.8}$$

Call G (resp. B) the set of the (A,N)-good (resp. bad) sites. The partition $E = B \cup G$ induces the orthogonal decomposition $\mathbf{H}_E = \mathbf{H}_B \oplus \mathbf{H}_G$ and we write

$$u = u_B + u_G \qquad \text{where} \qquad u_B := \Pi_B u\,, \ u_G := \Pi_G u\,.$$

The next Lemmas 4.1 and 4.2 say that the Cramer system $Au = h$ can be nicely reduced along the good sites G, giving rise to a (non-square) system $A'u_B = Zh$, with a good control of the s-norms of the matrices A' and Z. Moreover A^{-1} is a left inverse of A'.

Lemma 4.1. (Semi-reduction on the good sites) *Let $\Theta^{-1}\Upsilon \leq c_0(s_1)$ be small enough. There exist $\mathcal{M} \in \mathcal{M}_G^E$, $\mathcal{N} \in \mathcal{M}_G^B$ satisfying, if $N \geq N_1(\Upsilon)$ is large enough,*

$$|\mathcal{M}|_{s_0} \leq cN^{\kappa}\,, \quad |\mathcal{N}|_{s_0} \leq c\,\Theta^{-1}\Upsilon\,, \tag{4.9}$$

for some $c := c(s_1) > 0$, and, $\forall s \geq s_0$,

$$|\mathcal{M}|_s \leq C(s)N^{2\kappa}(N^{s-s_0} + N^{-b}|\mathcal{T}|_{s+b})\,, \quad |\mathcal{N}|_s \leq C(s)N^{\kappa}(N^{s-s_0} + N^{-b}|\mathcal{T}|_{s+b})\,, \tag{4.10}$$

such that

$$Au = h \quad \Longrightarrow \quad u_G = \mathcal{N}u_B + \mathcal{M}h\,.$$

Moreover

$$u_G = \mathcal{N}u_B + \mathcal{M}h \quad \Longrightarrow \quad \forall k \text{ regular}\,, \ (Au)_k = h_k\,. \tag{4.11}$$

Proof. It is based on "resolvent identity" arguments like in [13]. The use of the s-norms introduced in section 3 makes the proof very neat.

Step I. *There exist* $\Gamma, L \in \mathcal{M}_G^E$ *satisfying*

$$|\Gamma|_{s_0} \leq C_0(s_1)\Theta^{-1}\Upsilon\,, \quad |L|_{s_0} \leq N^{\kappa}\,, \tag{4.12}$$

and, $\forall s \geq s_0$,

$$|\Gamma|_s \leq C(s)N^{\kappa}(N^{s-s_0} + N^{-b}|\mathcal{T}|_{s+b})\,, \quad |L|_s \leq C(s)N^{\kappa+s-s_0}\,, \tag{4.13}$$

such that

$$Au = h \quad \Longrightarrow \quad u_G + \Gamma u = Lh\,. \tag{4.14}$$

Fix any $k \in G$ (see Definition 4.3). If k is regular, let $F := \{k\}$, and, if k is not regular but (A,N)-regular, let $F \subset E$ such that $\mathrm{d}(k, E\backslash F) \geq N$, $\mathrm{diam}(F) \leq 4N$, A_F^F is N-good. We have

$$Au = h \quad \Longrightarrow \quad A_F^F u_F + A_F^{E\backslash F} u_{E\backslash F} = h_F \quad \Longrightarrow \quad u_F + Qu_{E\backslash F} = (A_F^F)^{-1}h_F \tag{4.15}$$

where

$$Q := (A_F^F)^{-1}A_F^{E\backslash F} = (A_F^F)^{-1}\mathcal{T}_F^{E\backslash F} \in \mathcal{M}_F^{E\backslash F}\,. \tag{4.16}$$

The matrix Q satisfies

$$|Q|_{s_1} \overset{(3.16)}{\leq} C(s_1)|(A_F^F)^{-1}|_{s_1}|\mathcal{T}|_{s_1} \overset{(4.1),(4.8)}{\leq} C(s_1)N^{\tau'+\delta s_1}\Upsilon \tag{4.17}$$

(the matrix A_F^F is N-good). Moreover, $\forall s \geq s_0$, using the interpolation Lemma 3.3, and $\mathrm{diam}(F) \leq 4N$,

$$\begin{aligned} |Q|_{s+b} &\overset{(3.15)}{\leq} C(s)(|(A_F^F)^{-1}|_{s+b}|\mathcal{T}|_{s_0} + |(A_F^F)^{-1}|_{s_0}|\mathcal{T}|_{s+b}) \\ &\overset{(3.22)}{\leq} C(s)(N^{s+b-s_0}|(A_F^F)^{-1}|_{s_0}|\mathcal{T}|_{s_0} + |(A_F^F)^{-1}|_{s_0}|\mathcal{T}|_{s+b}) \\ &\overset{(4.1),(4.8)}{\leq} C(s)N^{(\delta-1)s_0}(N^{s+b+\tau'}\Upsilon + N^{\tau'+s_0}|\mathcal{T}|_{s+b})\,. \end{aligned} \tag{4.18}$$

Applying the projector $\Pi_{\{k\}}$ in (4.15), we obtain

$$Au = h \quad \Longrightarrow \quad u_k + \sum_{k' \in E} \Gamma_k^{k'} u_{k'} = \sum_{k' \in E} L_k^{k'} h_{k'} \tag{4.19}$$

that is (4.14) with

$$\Gamma_k^{k'} := \begin{cases} 0 & \text{if } \ k' \in F \\ Q_k^{k'} & \text{if } \ k' \in E \setminus F \end{cases} \qquad \text{and} \qquad L_k^{k'} := \begin{cases} [(A_F^F)^{-1}]_k^{k'} & \text{if } \ k' \in F \\ 0 & \text{if } \ k' \in E \setminus F. \end{cases} \tag{4.20}$$

If k is regular then $F = \{k\}$, and, by Definition 4.2,

$$|A_k^k| \geq \Theta\,. \tag{4.21}$$

Therefore, by (4.20) and (4.16), the k-line of Γ satisfies

$$|\Gamma_k|_{s_0+b} \leq |(A_k^k)^{-1}\mathcal{T}_k|_{s_0+b} \overset{(4.21),(4.8)}{\leq} C(s_0)\Theta^{-1}\Upsilon\,. \tag{4.22}$$

If k is not regular but (A,N)-regular, since $\mathrm{d}(k, E\backslash F) \geq N$ we have, by (4.20), that $\Gamma_k^{k'} = 0$ for $|k - k'| \leq N$. Hence, by Lemma 3.6,

$$\begin{aligned} |\Gamma_k|_{s_0+b} &\overset{(3.21)}{\leq} N^{-(s_1-s_0-b)}|\Gamma_k|_{s_1} \overset{(4.20)}{\leq} N^{-(s_1-s_0-b)}|Q|_{s_1} \overset{(4.17)}{\leq} C(s_1)\Upsilon N^{\tau'+s_0+b-(1-\delta)s_1} \\ &\leq C(s_1)\Theta^{-1}\Upsilon \end{aligned} \tag{4.23}$$

for $N \geq N_0(\Theta)$ large enough. Indeed the exponent $\tau' + s_0 + b - (1-\delta)s_1 < 0$ because s_1 is large enough according to (4.5) and $\delta \in (0, 1/2)$ (recall $\kappa := \tau' + s_0 + b$). In both cases (4.22)-(4.23) imply that each line Γ_k decays like

$$|\Gamma_k|_{s_0+b} \leq C(s_1)\Theta^{-1}\Upsilon\,, \quad \forall k \in G\,.$$

Hence, by Lemma 3.7, $|\Gamma|_{s_0} \leq C'(s_1)\Theta^{-1}\Upsilon$, which is the first inequality in (4.12). Likewise we prove the second estimate in (4.12). Moreover, $\forall s \geq s_0$, still by Lemma 3.7,

$$|\Gamma|_s \leq K \sup_{k\in G} |\Gamma_k|_{s+b} \overset{(4.20)}{\leq} K|Q|_{s+b} \overset{(4.18)}{\leq} C(s)N^{\kappa}(N^{s-s_0} + N^{-b}|\mathcal{T}|_{s+b})$$

where $\kappa := \tau' + s_0 + b$ and for $N \geq N_0(\Upsilon)$.

The second estimate in (4.13) follows by $|L|_{s_0} \leq N^{\kappa}$ (see (4.12)) and (3.22) (note that by (4.20), since $\mathrm{diam}F \leq 4N$, we have $L_k^{k'} = 0$ for all $|k-k'| > 4N$).

Step II. By (4.14) we have

$$Au = h \quad \Longrightarrow \quad (I_G + \Gamma^G)u_G = Lh - \Gamma^B u_B\,. \tag{4.24}$$

By (4.12), if Θ is large enough (depending on Υ, namely on the potential V_0), we have $|\Gamma^G|_{s_0} \leq 1/2$. Hence, by Lemma 3.9, $I_G + \Gamma^G$ is invertible and

$$|(I_G + \Gamma^G)^{-1}|_{s_0} \overset{(3.26)}{\leq} 2\,, \tag{4.25}$$

$$\forall s \geq s_0\,,\quad |(I_G + \Gamma^G)^{-1}|_s \overset{(3.28)}{\leq} C(s)(1 + |\Gamma^G|_s) \overset{(4.13)}{\leq} C(s)N^{\kappa}(N^{s-s_0} + N^{-b}|\mathcal{T}|_{s+b})\,. \tag{4.26}$$

By (4.24), $Au = h \Longrightarrow u_G = \mathcal{M}h + \mathcal{N}u_B$, with

$$\mathcal{M} := (I_G + \Gamma^G)^{-1}L \quad \text{and} \quad \mathcal{N} := -(I_G + \Gamma^G)^{-1}\Gamma^B \tag{4.27}$$

and estimates (4.9)-(4.10) follow by Lemma 3.3, (4.25)-(4.26) and (4.12)-(4.13).

Note that

$$u_G + \Gamma u = Lh \quad \Longleftrightarrow \quad u_G = \mathcal{M}h + \mathcal{N}u_B\,. \tag{4.28}$$

As a consequence, if $u_G = \mathcal{M}h + \mathcal{N}u_B$ then, by (4.20), for k regular,

$$u_k + (A_k^k)^{-1}\sum_{k'\neq k} A_k^{k'}u_{k'} = (A_k^k)^{-1}h_k\,,$$

hence $(Au)_k = h_k$, proving (4.11). ■

Lemma 4.2. (Reduction on the bad sites) *We have*

$$Au = h \quad \Longrightarrow \quad A'u_B = Zh$$

where

$$A' := A^B + A^G\mathcal{N} \ \in \mathcal{M}_E^B\,, \qquad Z := I_E - A^G\mathcal{M} \ \in \mathcal{M}_E^E\,, \tag{4.29}$$

satisfy

$$|A'|_{s_0} \leq c(\Theta)\,, \qquad |A'|_s \leq C(s,\Theta)N^{\kappa}(N^{s-s_0} + N^{-b}|\mathcal{T}|_{s+b})\,, \tag{4.30}$$

$$|Z|_{s_0} \leq cN^{\kappa}\,, \quad |Z|_s \leq C(s,\Theta)N^{2\kappa}(N^{s-s_0} + N^{-b}|\mathcal{T}|_{s+b})\,. \tag{4.31}$$

Moreover $(A^{-1})_B$ is a left inverse of A'.

Proof. By Lemma 4.1,

$$Au = h \quad \Longrightarrow \quad \begin{cases} A^G u_G + A^B u_B = h \\ u_G = \mathcal{N} u_B + \mathcal{M} h \end{cases} \quad \Longrightarrow \quad (A^G \mathcal{N} + A^B) u_B = h - A^G \mathcal{M} h \,,$$

i.e. $A' u_B = Zh$. Let us prove estimates (4.30)-(4.31) for A' and Z.

Step I. *$\forall\, k$ regular we have $A'_k = 0$, $Z_k = 0$.*

By (4.11), for all k regular,

$$\forall h\,,\ \forall u_B \in \mathbf{H}_B\ , \quad \Big(A^G (\mathcal{N} u_B + \mathcal{M} h) + A^B u_B \Big)_k = h_k \ , \quad i.e. \quad (A' u_B)_k = (Zh)_k \,,$$

which implies $A'_k = 0$ and $Z_k = 0$.

Step II. *Proof of (4.30)-(4.31).*

Call $R \subset E$ the regular sites in E. For all $k \in E \backslash R$, we have $|A_k^k| < \Theta$ (see Definition 4.2). Then (4.8) implies

$$|A_{E\backslash R}|_{s_0} \le \Theta + |\mathcal{T}|_{s_0} \le c(\Theta)\,, \quad |A_{E\backslash R}|_s \le \Theta + |\mathcal{T}|_s\,,\ \forall s \ge s_0\,. \tag{4.32}$$

By Step I and the definition of A' in (4.29) we get

$$|A'|_s = |A'_{E\backslash R}|_s \le |A^B_{E\backslash R}|_s + |A^G_{E\backslash R} \mathcal{N}|_s\,.$$

Therefore, Lemma 3.3, (4.32), (4.9), (4.10), imply

$$|A'|_s \le C(s,\Theta) N^\kappa (N^{s-s_0} + N^{-b} |\mathcal{T}|_{s+b}) \qquad \text{and} \qquad |A'|_{s_0} \le c(\Theta)\,,$$

proving (4.30). The bound (4.31) follows similarly.

Step III. *$(A^{-1})_B$ is a left inverse of A'.*

By $A^{-1} A' = A^{-1}(A^B + A^G \mathcal{N}) = I_E^B + I_E^G \mathcal{N}$ we get

$$(A^{-1})_B A' = (A^{-1} A')_B = I_B^B - 0 = I_B^B$$

proving that $(A^{-1})_B$ is a left inverse of A'. ∎

Now $A' \in \mathcal{M}_E^B$, and the set B is partitioned in clusters Ω_α of size $O(N^{C_1})$, far enough one from another, see (H3). Then, up to a remainder of very small s_0-norm (see (4.35)), A' is defined by the submatrices $(A')_{\Omega'_\alpha}^{\Omega_\alpha}$ where Ω'_α is some neighborhood of Ω_α (the distance between two distinct Ω'_α and Ω'_β remains large). Since A' has a left inverse with L^2-norm $O(N'^\tau)$, so have the submatrices $(A')_{\Omega'_\alpha}^{\Omega_\alpha}$. Since these submatrices are of size $O(N^{C_1})$, the s-norms of their inverse will be estimated as $O(N^{C_1 s} N'^\tau) = O(N'^{\tau + \chi^{-1} C_1 s})$, see (4.41). By Lemma 3.9, provided χ is chosen large enough, A' has a left inverse V with s-norms satisfying (4.33). The details are given in the following lemma.

Lemma 4.3. (Left inverse with decay) *The matrix A' defined in Lemma 4.2 has a left inverse V which satisfies*

$$\forall s \ge s_0\ , \quad |V|_s \le C(s) N^{2\chi\tau + \kappa + 2(s_0+b)C_1} (N^{C_1 s} + |\mathcal{T}|_{s+b})\,. \tag{4.33}$$

Proof. Define $\mathcal{D} \in \mathcal{M}_E^B$ by

$$\mathcal{D}_{k'}^k := \begin{cases} (A')_{k'}^k & \text{if} \ \ (k,k') \in \cup_\alpha (\Omega_\alpha \times \Omega'_\alpha) \\ 0 & \text{if} \ \ (k,k') \notin \cup_\alpha (\Omega_\alpha \times \Omega'_\alpha) \end{cases} \qquad \text{where} \quad \Omega'_\alpha := \{k \in E \ : \ \mathrm{d}(k, \Omega_\alpha) \le N^2/4\}\,. \tag{4.34}$$

Step I. *$\mathcal{D}$ has a left inverse $W \in \mathcal{M}_B^E$ with $\|W\|_0 \le 2(N')^\tau$.*

We define $\mathcal{R} := A' - \mathcal{D}$. By the definition (4.34), if $\mathrm{d}(k',k) < N^2/4$ then $\mathcal{R}_{k'}^k = 0$ and so

$$\begin{aligned} |\mathcal{R}|_{s_0} &\overset{(3.21)}{\le} 4^{s_1} N^{-2(s_1-b-s_0)} |\mathcal{R}|_{s_1-b} \le 4^{s_1} N^{-2(s_1-b-s_0)} |A'|_{s_1-b} \\ &\overset{(4.30),(4.8)}{\le} C(s_1) N^{-2(s_1-b-s_0)} N^\kappa (N^{s_1-b-s_0} + N^{-b}\Upsilon) \le C(s_1) N^{2\kappa - s_1} \end{aligned} \tag{4.35}$$

for $N \geq N_0(\Upsilon)$ large enough. Therefore

$$\begin{aligned}
\|\mathcal{R}\|_0 \|(A^{-1})_B\|_0 &\overset{(3.24)}{\leq} |\mathcal{R}|_{s_0} \|A^{-1}\|_0 \overset{(4.35),(H2)}{\leq} C(s_1) N^{2\kappa - s_1} (N')^\tau \\
&\overset{(4.2)}{=} C(s_1) N^{2\kappa - s_1 + \chi\tau} \overset{(4.5)}{\leq} 1/2 \qquad (4.36)
\end{aligned}$$

for $N \geq N(s_1)$. Since $(A^{-1})_B \in \mathcal{M}_B^E$ is a left inverse of A' (see Lemma 4.2), Lemma 3.9 and (4.36) imply that $\mathcal{D} = A' - R$ has a left inverse $W \in \mathcal{M}_B^E$, and

$$\|W\|_0 \overset{(3.30)}{\leq} 2\|(A^{-1})_B\|_0 \leq 2\|A^{-1}\|_0 \overset{(H2)}{\leq} 2(N')^\tau \,. \qquad (4.37)$$

Step II. *$W_0 \in \mathcal{M}_B^E$ defined by*

$$(W_0)_k^{k'} := \begin{cases} W_k^{k'} & \text{if} \quad (k, k') \in \cup_\alpha (\Omega_\alpha \times \Omega'_\alpha) \\ 0 & \text{if} \quad (k, k') \notin \cup_\alpha (\Omega_\alpha \times \Omega'_\alpha) \end{cases} \qquad (4.38)$$

is a left inverse of $\mathcal{D}$ and $|W_0|_s \leq C(s) N^{(s+b)C_1 + \chi\tau}$, $\forall s \geq s_0$.
Since $W\mathcal{D} = I_B$, we prove that W_0 is a left inverse of $\mathcal{D}$ showing that

$$(W - W_0)\mathcal{D} = 0 \,. \qquad (4.39)$$

Let us prove (4.39). For $k \in B = \cup_\alpha \Omega_\alpha$, there is α such that $k \in \Omega_\alpha$, and

$$\forall k' \in B \,, \ ((W - W_0)\mathcal{D})_k^{k'} = \sum_{q \notin \Omega'_\alpha} (W - W_0)_k^q \mathcal{D}_q^{k'} \qquad (4.40)$$

since $(W - W_0)_k^q = 0$ if $q \in \Omega'_\alpha$, see the Definition (4.38).
Case I: $k' \in \Omega_\alpha$. Then $\mathcal{D}_q^{k'} = 0$ in (4.40) and so $((W - W_0)\mathcal{D})_k^{k'} = 0$.
Case II: $k' \in \Omega_\beta$ for some $\beta \neq \alpha$. Then, since $\mathcal{D}_q^{k'} = 0$ if $q \notin \Omega'_\beta$, we obtain by (4.40) that

$$((W - W_0)\mathcal{D})_k^{k'} = \sum_{q \in \Omega'_\beta} (W - W_0)_k^q \mathcal{D}_q^{k'} \overset{(4.38)}{=} \sum_{q \in \Omega'_\beta} W_k^q \mathcal{D}_q^{k'} \overset{(4.34)}{=} \sum_{k \in E} W_k^q \mathcal{D}_q^{k'} = (W\mathcal{D})_k^{k'} = (I_B)_k^{k'} = 0 \,.$$

Since $\mathrm{diam}(\Omega'_\alpha) \leq 2N^{C_1}$, definition (4.38) implies $(W_0)_k^{k'} = 0$ for all $|k - k'| \geq 2N^{C_1}$. Hence, $\forall s \geq 0$,

$$|W_0|_s \overset{(3.22)}{\leq} C(s) N^{(s+b)C_1} \|W_0\|_0 \overset{(4.37)}{\leq} C(s) N^{(s+b)C_1 + \chi\tau} \,. \qquad (4.41)$$

Step III. *A' has a left inverse V satisfying (4.33).*
Now $A' = \mathcal{D} + \mathcal{R}$, W_0 is a left inverse of $\mathcal{D}$, and

$$|W_0|_{s_0} |\mathcal{R}|_{s_0} \overset{(4.41),(4.35)}{\leq} C(s_1) N^{(s_0+b)C_1 + \chi\tau + 2\kappa - s_1} \overset{(4.5)}{\leq} 1/2$$

(we use also that $\chi > C_1$ by (4.4)) for $N \geq N(s_1)$ large enough. Hence, by Lemma 3.9, A' has a left inverse V with

$$|V|_{s_0} \overset{(3.26)}{\leq} 2|W_0|_{s_0} \overset{(4.41)}{\leq} C N^{(s_0+b)C_1 + \chi\tau} \qquad (4.42)$$

and, $\forall s \geq s_0$,

$$\begin{aligned}
|V|_s &\overset{(3.28)}{\leq} C(s)(|W_0|_s + |W_0|_{s_0}^2 |\mathcal{R}|_s) \leq C(s)(|W_0|_s + |W_0|_{s_0}^2 |A'|_s) \\
&\overset{(4.41),(4.30)}{\leq} C(s) N^{2\chi\tau + \kappa + 2(s_0+b)C_1} (N^{C_1 s} + |\mathcal{T}|_{s+b})
\end{aligned}$$

proving (4.33). $\blacksquare$

PROOF OF PROPOSITION 4.1 COMPLETED. Lemmata 4.1, 4.2, 4.3 imply

$$Au = h \quad \Longrightarrow \quad \begin{cases} u_G = \mathcal{M}h + \mathcal{N}u_B \\ u_B = VZh \end{cases}$$

whence

$$(A^{-1})_B = VZ \qquad \text{and} \qquad (A^{-1})_G = \mathcal{M} + \mathcal{N}VZ = \mathcal{M} + \mathcal{N}(A^{-1})_B \,. \tag{4.43}$$

Therefore, $\forall s \geq s_0$,

$$\begin{aligned} |(A^{-1})_B|_s \quad & \stackrel{(4.43),(3.15)}{\leq} \quad C(s)(|V|_s|Z|_{s_0} + |V|_{s_0}|Z|_s) \\ & \stackrel{(4.33),(4.31),(4.8),(4.42)}{\leq} \quad C(s)N^{2\kappa+2\chi\tau+2(s_0+b)C_1}(N^{C_1 s} + |\mathcal{T}|_{s+b}) \\ & \leq \quad C(s)(N')^{\alpha_1}((N')^{\alpha_2 s} + |\mathcal{T}|_s) \end{aligned}$$

using $|\mathcal{T}|_{s+b} \leq C(s)(N')^b|\mathcal{T}|_s$ (by (3.22)) and defining

$$\alpha_1 := 2\tau + b + 2\chi^{-1}(\kappa + C_1(s_0 + b)) \,, \quad \alpha_2 := \chi^{-1}C_1 \,.$$

We obtain the same bound for $|(A^{-1})_G|_s$. Hence, for $s \in [s_0, S]$,

$$\begin{aligned} |A^{-1}|_s \quad & \leq \quad |(A^{-1})_B|_s + |(A^{-1})_G|_s \leq C(s)(N')^{\alpha_1}((N')^{\alpha_2 s} + |\mathcal{T}|_s) \\ & \stackrel{(4.4)}{\leq} \quad \frac{1}{4}(N')^{\tau'}((N')^{\delta s} + |\mathcal{T}|_s) \end{aligned}$$

for $N \geq N(S)$ large enough, proving (4.7).

5 Separation properties of the bad sites

The aim of this section is to verify the separation properties of the bad sites required in the multiscale Proposition 4.1.

Let $A := A(\varepsilon, \lambda, \theta)$ be the infinite dimensional matrix defined in (2.16). Given $N \in \mathbb{N}$ and $i = (l_0, j_0)$, recall that the submatrix $A_{N,i}$ is defined in (2.17).

Definition 5.1. **(N-good/bad site)** *A site $k := (i, a) \in \mathbb{Z}^b \times \{0, 1\}$ is:*

- *N-REGULAR if $A_{N,i}$ is N-good (Definition 4.1). Otherwise we say that k is N-SINGULAR.*
- *N-GOOD if*

$$k \textit{ is regular } (Definition\, 4.2) \quad \text{or} \quad \textit{all the sites } k' \textit{ with } \mathrm{d}(k', k) \leq N \textit{ are } N-regular \,. \tag{5.1}$$

Otherwise, we say that k is N-BAD.

Remark 5.1. *It is easy to see that a site k which is N-good according to Definition 5.1, is (A_E^E, N)-good according to Definition 4.3, for any set $E = E_0 \times \{0, 1\}$ containing k where $E_0 \subset \mathbb{Z}^b$ is a product of intervals of length $\geq N$. We introduce these different definitions for merely technical reasons: it is more convenient to prove separation properties of N-bad sites for infinite dimensional matrices. On the other hand, for a finite matrix A_E^E, we need the notion of (A_E^E, N)-good sites in order to perform the "resolvent identity" also near the boundary ∂E, see Step I of Lemma 4.1.*

We define

$$B_N(j_0; \varepsilon, \lambda) := \Big\{\theta \in \mathbb{R} \,:\, A_{N,j_0}(\varepsilon, \lambda, \theta) \text{ is } N - bad\Big\} \,. \tag{5.2}$$

Definition 5.2. **(N-good/bad parameters)** *A couple $(\varepsilon, \lambda) \in \mathbb{R}^2$ is N-good for A if*

$$\forall\, j_0 \in \mathbb{Z}^d\,, \quad B_N(j_0; \varepsilon, \lambda) \subset \bigcup_{q=1,\ldots,N^{2d+\nu+4}} I_q \tag{5.3}$$

where I_q are intervals with measure $|I_q| \leq N^{-\tau}$. Otherwise, we say (ε, λ) is N-bad. We define

$$\mathcal{G}_N := \mathcal{G}_N(\mathbf{u}) := \Big\{ (\varepsilon, \lambda) \in [0, \varepsilon_0] \times \Lambda \,:\, (\varepsilon, \lambda) \ \text{ is } \ N - \text{good for } A \Big\}\,. \tag{5.4}$$

The main result of this section is the following proposition. It will enable to verify the assumption (H3) of Proposition 4.1 for the submatrices $A_{N',j_0}(\varepsilon, \lambda, \theta)$, see Lemmata 7.5 and 7.6.

Proposition 5.1. **(Separation properties of N-bad sites)** *There exist $C_1 := C_1(d, \nu) \geq 2$ and $N_1 := N_1(\nu, d, \gamma_0, \tau_0, m, \Theta)$ such that if $N \geq N_1$ and*

- **(i)** *(ε, λ) is N-good for A*
- **(ii)** *$\tau > \chi \tau_0$ (τ_0 is the diophantine exponent of $\bar{\omega}$ in (1.5)),*

then $\forall \theta \in \mathbb{R}$, the N-bad sites $k := (l, j, a) \in \mathbb{Z}^\nu \times \mathbb{Z}^d \times \{0, 1\}$ of $A(\varepsilon, \lambda, \theta)$ with $|l| \leq N'$ admit a partition $\cup_\alpha \Omega_\alpha$ in disjoint clusters satisfying

$$\operatorname{diam}(\Omega_\alpha) \leq N^{C_1(d,\nu)}\,, \quad \mathrm{d}(\Omega_\alpha, \Omega_\beta) > N^2\,, \ \forall \alpha \neq \beta\,. \tag{5.5}$$

We underline that the estimates (5.5) are *uniform* in θ.

Remark 5.2. *The N-bad sites appear necessarily in clusters with increasing size $O(N^{C_1})$, due to the multiplicity of the eigenvalues of the Laplacian; this happens already for the singular sites of periodic solutions, i.e. for $\nu = 1$, see [3]. It is also natural that the separation between clusters of N-bad sites increases with N, because, roughly speaking, the N-bad sites correspond small divisors of size $O(N^{-\alpha})$.*

Remark 5.3. *The geometric structure of the bad and singular sites, determines the regularity of the solutions of Theorem 1.1. Actually, the solutions of Theorem 1.1 have the same Sobolev regularity in time and space because the N-bad clusters are separated in the space-time Fourier indices, see (5.5).*

We first estimate the time Fourier components of the N-singular sites. We use that, by (1.5), the frequency vectors $\omega = \lambda \bar{\omega}$, $\forall \lambda \in [1/2, 3/2]$, are diophantine, namely

$$|\omega \cdot l| \geq \frac{\gamma_0}{|l|^{\tau_0}}\,, \quad \forall l \in \mathbb{Z}^\nu \setminus \{0\}\,, \tag{5.6}$$

and we use the "complexity" information (5.3) on the set $B_N(j_0; \varepsilon, \lambda)$. This kind of argument was used in [20] and [13].

Lemma 5.1. *Assume (i)-(ii) of Proposition 5.1. Then, $\forall j_1 \in \mathbb{Z}^d$, the number of N-singular sites $(l_1, j_1, a_1) \in \mathbb{Z}^\nu \times \mathbb{Z}^d \times \{0, 1\}$ with $|l_1| \leq 2N'$ does not exceed $2N^{2d+\nu+4}$.*

Proof. If (l_1, j_1, a_1) is N-singular then $A_{N,l_1,j_1}(\varepsilon, \lambda, \theta)$ is N-bad (see Definitions 5.1 and 4.1). By (2.20), we get that $A_{N,j_1}(\varepsilon, \lambda, \theta + \lambda \bar{\omega} \cdot l_1)$ is N-bad, namely $\theta + \lambda \bar{\omega} \cdot l_1 \in B_N(j_1; \varepsilon, \lambda)$ (see (5.2)). By assumption, (ε, λ) is N-good, and, therefore, (5.3) holds.

We claim that in each interval I_q there is at most one element $\theta + \omega \cdot l_1$ with $\omega = \lambda \bar{\omega}$, $|l_1| \leq 2N'$. Then, since there are at most $N^{2d+\nu+4}$ intervals I_q (see (5.3)) and $a \in \{0, 1\}$, the lemma follows.

We prove the previous claim by contradiction. Suppose that there exist $l_1 \neq l_1'$ with $|l_1|, |l_1'| \leq 2N'$, such that $\omega \cdot l_1 + \theta$, $\omega \cdot l_1' + \theta \in I_q$. Then

$$|\omega \cdot (l_1 - l_1')| = |(\omega \cdot l_1 + \theta) - (\omega \cdot l_1' + \theta)| \leq |I_q| \leq N^{-\tau}\,. \tag{5.7}$$

By (5.6) we also have

$$|\omega \cdot (l_1 - l_1')| \geq \frac{\gamma_0}{|l_1 - l_1'|^{\tau_0}} \geq \frac{\gamma_0}{(4N')^{\tau_0}} = 4^{-\tau_0} \gamma_0 N^{-\chi \tau_0}\,. \tag{5.8}$$

By assumption (ii) of Proposition 5.1 the inequalities (5.7) and (5.8) are in contradiction, for $N \geq N_0(\gamma_0, \tau_0)$ large enough. ∎

Corollary 5.1. *Assume (i)-(ii) of Proposition 5.1. Then, $\forall j_1 \in \mathbb{Z}^d$, the number of N-bad sites $(l_1, j_1, a_1) \in \mathbb{Z}^\nu \times \mathbb{Z}^d \times \{0,1\}$ with $|l_1| \leq N'$ does not exceed $CN^{3d+2\nu+4}$ for some positive constant C depending only on d and ν.*

PROOF. By Lemma 5.1, the set S of N-singular sites (l, j, a) with $|l| \leq N' + N$, $|j - j_1| \leq N$ has cardinality at most $CN^{2d+\nu+4} \times N^d$. Each N-bad site (l_1, j_1, a_1) with $|l_1| \leq N'$ is included in some N-ball centered at an element of S. Each of these balls contains at most CN^ν sites of the form (l, j_1, a). Hence there are at most $CN^{2d+\nu+4} \times N^d \times N^\nu$ such N-bad sites. ■

We now estimate also the spatial components of the N-bad sites. In order to achieve a partition in clusters we use the notion of "chain" of N-bad sites already used for the search of periodic solutions of NLS and NLW in higher dimension in [7], [3].

Definition 5.3. **(M-chain)** *A sequence $k_0, \ldots, k_L \in \mathbb{Z}^{d+\nu} \times \{0,1\}$ of distinct integer vectors satisfying, for some $M \geq 2$, $|k_{q+1} - k_q| \leq M$, $\forall q = 0, \ldots, L-1$, is called a M-chain of length L.*

Proposition 5.1 will be a consequence of the following lemma. Here we exploit that the N-bad sites $k = (i, a)$ are singular, see (5.1) and Definition 4.2.

Lemma 5.2. *There is $C(d, \nu) > 0$ such that, $\forall \theta \in \mathbb{R}$, $\forall N$, any M-chain of N-bad sites, with $|l_q| \leq N'$, has length*

$$L \leq (MN)^{C(d,\nu)} . \tag{5.9}$$

PROOF. Let $k_q = (l_q, j_q, a_q)$, $q = 0, \ldots, L$, be a M-chain of N-bad sites with $|l_q| \leq N'$. Then

$$\max\{|l_{q+1} - l_q|, |j_{q+1} - j_q|\} \leq M , \quad \forall q \in [0, L] , \tag{5.10}$$

and, by Definitions 5.1 and 4.2, and (2.21),

$$|-\omega \cdot l_q + \|j_q\|^2 + m - \theta| < \Theta \ \ (\text{if } a_q = 1) \quad \text{or} \quad |\omega \cdot l_q + \|j_q\|^2 + m + \theta| < \Theta \ \ (\text{if } a_q = 0) .$$

We deduce one of the following θ-independent inequalities

$$|\pm \omega \cdot (l_{q+1} - l_q) + (\|j_{q+1}\|^2 \pm \|j_q\|^2)| \leq 2(\Theta + m) .$$

By (5.10) we get $|\|j_{q+1}\|^2 \pm \|j_q\|^2| \leq 2(\Theta + m) + |\omega|M \leq K_1 M$ for some $K_1 := K_1(\Theta, m)$. Since $|\|j_{q+1}\|^2 - \|j_q\|^2| \leq \|j_{q+1}\|^2 + \|j_q\|^2$, in any case $|\|j_{q+1}\|^2 - \|j_q\|^2| \leq K_1 M$. Therefore

$$\forall q, q_0 \in [0, L] \ , \ |\|j_q\|^2 - \|j_{q_0}\|^2| \leq |q - q_0| K_1 M \tag{5.11}$$

and, using also (5.10),

$$|j_{q_0} \cdot (j_q - j_{q_0})| = \frac{1}{2} \Big| \|j_q\|^2 - \|j_{q_0}\|^2 - \|j_q - j_{q_0}\|^2 \Big| \leq K_2 |q - q_0|^2 M^2 . \tag{5.12}$$

Let us introduce the subspace of $\mathbb{R}^d$

$$G = \mathrm{Span}_{\mathbb{R}}\{j_q - j_{q'} \ : \ 0 \leq q, q' \leq L \} = \mathrm{Span}_{\mathbb{R}}\{j_q - j_0 \ : \ 0 \leq q \leq L \}$$

and let us call g ($1 \leq g \leq d$) the dimension of G. Define $\delta := (2d+1)^{-2}$. The constants C below (may) depend on Θ, m, d, ν.

Case I. $\forall q_0 \in [0, L]$, $\mathrm{Span}_{\mathbb{R}}\{j_q - j_{q_0} \ : \ |q - q_0| \leq L^\delta \ , \ q \in [0, L] \} = G$.

We select a basis of G from $j_q - j_{q_0}$ ($|q - q_0| \leq L^\delta$), say $f_1, f_2, \ldots, f_g \in G$. By (5.10) we have

$$|f_i| \leq M L^\delta , \qquad \forall i = 1, \ldots, g . \tag{5.13}$$

Decomposing in this basis the orthogonal projection of j_{q_0} on G,

$$P_G j_{q_0} = \sum_{i=1}^{g} x_i f_i \tag{5.14}$$

and taking the scalar products with f_p, $p = 1, \ldots, g$, we get the linear system

$$Fx = b \quad \text{with} \quad F_p^i := f_i \cdot f_p \,, \ b_p := P_G j_{q_0} \cdot f_p = j_{q_0} \cdot f_p \,.$$

Since $\{f_i\}_{i=1,\ldots,g}$ is a basis of G the matrix F is invertible. Since the coefficients of F are integers, $|\det(F)| \geq 1$. By Cramer rule, using that (5.13) implies $|F_p^i| \leq C|f_i||f_p| \leq (ML^\delta)^2$, we deduce that

$$|(F^{-1})_i^{i'}| \leq C(ML^\delta)^{2(g-1)} \,, \quad \forall i, i' = 1, \ldots, g \,. \tag{5.15}$$

By (5.12), we have $|b_i| \leq K_2(ML^\delta)^2$, $\forall i = 1, \ldots, g$, and (5.15) implies

$$|x_{i'}| \leq C(ML^\delta)^{2g} \,, \quad \forall i' = 1, \ldots, g \,. \tag{5.16}$$

From (5.14), (5.13), (5.16), we deduce $|P_G j_{q_0}| \leq C(ML^\delta)^{2g+1}$, $\forall q_0 \in [0, L]$, and

$$|j_{q_1} - j_{q_2}| = |P_G j_{q_1} - P_G j_{q_2}| \leq C(ML^\delta)^{2g+1} \leq C(ML^\delta)^{2d+1}, \quad \forall (q_1, q_2) \in [0, L]^2 \,.$$

Since all the j_q are in $\mathbb{Z}^d$, their number (counted without multiplicity) does not exceed $C(ML^\delta)^{(2d+1)d}$. Thus we have obtained the bound

$$\sharp\{j_q \ ; \ 0 \leq q \leq L\} \leq C(ML^\delta)^{(2d+1)d} \,. \tag{5.17}$$

Now by Corollary 5.1, for each $q_0 \in [0, L]$, the number of $q \in [0, L]$ such that $j_q = j_{q_0}$ is at most $2N^{3d+2\nu+4}$, and so

$$L \leq C(ML^\delta)^{(2d+1)d} 2N^{3d+2\nu+4} \,.$$

Since $\delta(2d + 1)d < 1/2$, we get

$$L \leq C' M^{2d(d+1)} N^{2(3d+2\nu+4)} \tag{5.18}$$

proving (5.9) for N large enough.

Case II. There is $q_0 \in [0, L]$ such that

$$\mu := \dim \mathrm{Span}\{j_q - j_{q_0} \ : \ |q - q_0| \leq L^\delta \,, \ q \in [0, L]\} \leq g - 1 \,,$$

namely all the vectors j_q stay in a affine subspace of dimension $\mu \leq g - 1$. Then we repeat on the sub-chain j_q, $|q - q_0| \leq L^\delta$, the argument of case I, to obtain a bound for L^δ (and hence for L).

Applying at most d-times the above procedure, we obtain a bound for L of the form $L \leq (MN)^{C(d,\nu)}$, proving the lemma. ■

Proof of Proposition 5.1 completed. Set $M := N^2$. We introduce the following equivalence relation in the set

$$\mathcal{S}_N := \Big\{ k = (l, j, a) \in \mathbb{Z}^{\nu+d} \times \{0, 1\} \ : \ |l| \leq N', \, k \text{ is} \, N\text{- bad \ for} \, A(\varepsilon, \lambda, \theta) \Big\}.$$

Definition 5.4. *We say that $x \equiv y$ if there is a N^2-chain $\{k_q\}_{q=0,\ldots,L}$ in $\mathcal{S}_N$ connecting x to y, namely $k_0 = x$, $k_L = y$.*

This equivalence relation induces a partition of the N-bad sites of $A(\varepsilon, \lambda, \theta)$ with $|l| \leq N'$, in disjoint equivalent classes $\cup_\alpha \Omega_\alpha$, satisfying, by Lemma 5.2,

$$\mathrm{d}(\Omega_\alpha, \Omega_\beta) > N^2 \,, \quad \mathrm{diam}(\Omega_\alpha) \overset{(5.9)}{\leq} N^2 (N^3)^{C(d,\nu)} \,. \tag{5.19}$$

This proves (5.5).

6 Measure and "complexity" estimates

We define

$$B^0_N(j_0;\varepsilon,\lambda) \;:=\; \Big\{\theta\in\mathbb{R} \,:\, \|A^{-1}_{N,j_0}(\varepsilon,\lambda,\theta)\|_0 > N^{\tau}\Big\} \tag{6.1}$$

$$\qquad\qquad = \;\Big\{\theta\in\mathbb{R} \,:\, \exists \text{ an eigenvalue of } A_{N,j_0}(\varepsilon,\lambda,\theta) \text{ with modulus less than } N^{-\tau}\Big\} \tag{6.2}$$

where $\| \ \|_0$ is the operatorial L^2-norm defined in (3.9). The equivalence between (6.1) and (6.2) is a consequence of the self-adjointness of $A_{N,j_0}(\varepsilon,\lambda,\theta)$. We also define

$$\mathcal{G}^0_N := \mathcal{G}^0_N(\mathbf{u}) \;:=\; \Big\{(\varepsilon,\lambda)\in[0,\varepsilon_0]\times\Lambda \,:\, \forall\, j_0\in\mathbb{Z}^d\,,\quad B^0_N(j_0;\varepsilon,\lambda)\subset\bigcup_{q=1,\ldots,N^{2d+\nu+4}} I_q \tag{6.3}$$
$$\text{where } I_q \text{ are disjoint intervals with measure } |I_q|\le N^{-\tau}\Big\}\,.$$

Remark 6.1. *The difference between the sets $\mathcal{G}^0_N$ defined in (6.3) and $\mathcal{G}_N$ defined in (5.4) relies in the different definition of $B^0_N(j_0;\varepsilon,\lambda)$ in (6.1) and $B_N(j_0;\varepsilon,\lambda)$ in (5.2). For all $\theta\notin B_N(j_0;\varepsilon,\lambda)$ the matrices $A_{N,j_0}(\varepsilon,\lambda,\theta)$ are N-good, i.e. satisfy bounds on $|A^{-1}_{N,j_0}(\varepsilon,\lambda,\theta)|_s\le N^{\delta s+\tau'}$ for $s\in[s_0,s_1]$, while for all $\theta\notin B^0_N(j_0;\varepsilon,\lambda)$ we only have the L^2- bound $\|A^{-1}_{N,j_0}(\varepsilon,\lambda,\theta)\|_0\le N^{\tau}$. Using the multiscale Proposition 4.1 and the separation Proposition 5.1 (which holds for any θ) we shall prove inductively that the parameters that stay in $\mathcal{G}^0_{N_k}(u_k)$ along the Nash-Moser scheme are in fact also in $\mathcal{G}_{N_k}(u_k)$.*

The aim of this section is to prove the following proposition.

Proposition 6.1. *There is a constant $C>0$ such that, for $N\ge N_0(V,d,\nu)$ large enough and*

$$\varepsilon_0\beta_0^{-1}(\|T_1\|_0+\|\partial_\lambda T_1\|_0)\le c \tag{6.4}$$

small enough (β_0 is defined in (1.3) *and T_1 in* (2.3)*), the set $\mathcal{B}^0_N := (\mathcal{G}^0_N)^c\cap([0,\varepsilon_0]\times\Lambda)$ has measure*

$$|\mathcal{B}^0_N|\le C\,\varepsilon_0 N^{-1}\,. \tag{6.5}$$

Proposition 6.1 is derived from several lemmas based on basic properties of eigenvalues of self-adjoint matrices, which are a consequence of their variational characterization.

Lemma 6.1. *i) Let $A(\xi)$ be a family of square matrices in $\mathcal{M}^E_E$, C^1 in the real parameter $\xi\in\mathbb{R}$. Assume that there is an invertible matrix U such that the matrices $\widetilde{A}(\xi):=A(\xi)U$ are self-adjoint and $\partial_\xi\widetilde{A}(\xi)\ge\beta I$, $\beta>0$. Then, for any $\alpha>0$, the measure*

$$\Big|\Big\{\xi\in\mathbb{R} \,:\, \|A^{-1}(\xi)\|_0\ge\alpha^{-1}\Big\}\Big|\le 2|E|\alpha\beta^{-1}\|U\|_0 \tag{6.6}$$

where $|E|$ denotes the cardinality of the set E.
ii) In particular, if $A=Z+\xi W$ with Z,W selfadjoint, W invertible and $\beta_1 I\le Z\le\beta_2 I$, $\beta_1>0$, then

$$\Big|\Big\{\xi\in\mathbb{R} \,:\, \|A^{-1}(\xi)\|_0\ge\alpha^{-1}\Big\}\Big|\le 2|E|\alpha\beta_2\beta_1^{-1}\|W^{-1}\|_0\,. \tag{6.7}$$

PROOF. i) The eigenvalues of the self-adjoint matrices $\widetilde{A}(\xi)$ can be listed as C^1 functions $\mu_k(\xi)$, $1\le k\le|E|$. Now

$$\Big\{\xi\in\mathbb{R} \,:\, \|A^{-1}(\xi)\|_0\ge\alpha^{-1}\Big\} \;\subset\; \Big\{\xi\in\mathbb{R} \,:\, \|\widetilde{A}^{-1}(\xi)\|_0\ge(\alpha\|U\|_0)^{-1}\Big\}$$
$$\qquad = \;\Big\{\xi\in\mathbb{R} \,:\, \exists k\in[1,|E|]\,,\, |\mu_k(\xi)|\le\alpha\|U\|_0\Big\}$$

because $\widetilde{A}(\xi)$ is selfadjoint. Since $\partial_\xi \widetilde{A}(\xi) \geq \beta I$, we have $\partial_\xi \mu_k(\xi) \geq \beta > 0$ and the measure estimate (6.6) follows readily.

ii) Applying i) with $U = W^{-1}Z$ and self-adjoint matrices $\widetilde{A}(\xi) = ZW^{-1}Z + \xi Z$, we get

$$\left| \left\{ \xi \in \mathbb{R} \,:\, \|A^{-1}(\xi)\|_0 \geq \alpha^{-1} \right\} \right| \leq 2|E|\alpha\beta_1^{-1}\|W^{-1}\|_0\|Z\|_0 \leq 2|E|\alpha\beta_2\beta_1^{-1}\|W^{-1}\|_0,$$

which is (6.7). ■

From the variational characterization of the eigenvalues of selfadjoint matrices we can derive :

Lemma 6.2. *Let A, A_1 be self adjoint matrices. Then their eigenvalues (ranked in nondecreasing order) satisfy the Lipschitz property*

$$|\mu_k(A) - \mu_k(A_1)| \leq \|A - A_1\|_0 \,. \tag{6.8}$$

The continuity property (6.8) of the eigenvalues allows to derive a "complexity estimate" for $B_N^0(j_0;\varepsilon,\lambda)$ knowing its measure, more precisely the measure of

$$B_{2,N}^0(j_0;\varepsilon,\lambda) := \left\{ \theta \in \mathbb{R} \,:\, \|A_{N,j_0}^{-1}(\varepsilon,\lambda,\theta)\|_0 > N^\tau/2 \right\}. \tag{6.9}$$

Lemma 6.3. *$\forall j_0 \in \mathbb{Z}^d$, $\forall(\varepsilon,\lambda) \in [0,\varepsilon_0] \times \Lambda$, we have $B_N^0(j_0;\varepsilon,\lambda) \subset \cup_{q=1,\ldots,2\mathtt{M}N^\tau} I_q$ where I_q are intervals with $|I_q| \leq N^{-\tau}$ and $\mathtt{M} := |B_{2,N}^0(j_0;\varepsilon,\lambda)|$.*

Proof. If $\theta \in B_N^0(j_0;\varepsilon,\lambda)$, by (6.8) and since $\|Y\|_0 = 1$ (see (2.15)), we deduce that

$$\begin{aligned} \left[\theta - N^{-\tau}, \theta + N^{-\tau}\right] \quad &\subset \quad B_{2,N}^0(j_0;\varepsilon,\lambda) \\ &= \quad \left\{ \theta \in \mathbb{R} \,:\, \exists \text{ an eigenvalue of } A_{N,j_0}(\varepsilon,\lambda,\theta) \text{ with modulus less than } 2N^{-\tau} \right\}. \end{aligned}$$

Hence $B_N^0(j_0;\varepsilon,\lambda)$ is included in an union of intervals J_m of disjoint interiors,

$$B_N^0(j_0;\varepsilon,\lambda) \subset \bigcup_m J_m \subset B_{2,N}^0(j_0;\varepsilon,\lambda), \quad \text{with length} \quad |J_m| \geq 2N^{-\tau} \tag{6.10}$$

(if some of the intervals $[\theta - N^{-\tau}, \theta + N^{-\tau}]$ overlap, then we glue them together). We decompose each J_m as an union of (non overlapping) intervals I_q of length between $N^{-\tau}/2$ and $N^{-\tau}$. Then, by (6.10), we get a new covering

$$B_N^0(j_0;\varepsilon,\lambda) \subset \bigcup_{q=1,\ldots,Q} I_q \subset B_{2,N}^0(j_0;\varepsilon,\lambda) \quad \text{with} \quad N^{-\tau}/2 \leq |I_q| \leq N^{-\tau}$$

and, since the intervals I_q do not overlap,

$$QN^{-\tau}/2 \leq \sum_{q=1}^{Q} |I_q| \leq |B_{2,N}^0(j_0;\varepsilon,\lambda)| =: \mathtt{M}.$$

As a consequence $Q \leq 2\,\mathtt{M}\,N^\tau$, which proves the lemma. ■

We estimate the measure $|B_{2,N}^0(j_0;\varepsilon,\lambda)|$ differently for $|j_0| \geq 2N$ or $|j_0| < 2N$. In the next lemmas we assume

$$N \geq N_0(V,\nu,d) > 0 \text{ large enough} \quad \text{and} \quad \varepsilon\|T_1\|_0 \leq 1\,. \tag{6.11}$$

Lemma 6.4. *$\forall |j_0| \geq 2N$, $\forall(\varepsilon,\lambda) \in [0,\varepsilon_0] \times \Lambda$, we have $|B_{2,N}^0(j_0;\varepsilon,\lambda)| \leq CN^{-\tau+d+\nu}$.*

Proof. Recalling (2.19) and (2.16), we have

$$A_{N,j_0}(\varepsilon,\lambda,\theta) = A_{N,j_0}(\varepsilon,\lambda) + \theta Y_{N,j_0} = D_{N,j_0}(\lambda) + T_{N,j_0}(\varepsilon,\lambda) + \theta Y_{N,j_0}\,. \tag{6.12}$$

We claim that, if $|j_0| \geq 2N$ and $N \geq N_0(V, d, \nu)$, see (6.11), then

$$4d|j_0|^2 I \geq A_{N,j_0}(\varepsilon, \lambda) \geq \frac{|j_0|^2}{8} I \,. \tag{6.13}$$

Indeed by (6.12) and (6.8), the eigenvalues $\lambda_{l,j}$ of $A_{N,j_0}(\varepsilon, \lambda)$ satisfy

$$\lambda_{l,j} = \delta^{\pm}_{l,j} + O(\varepsilon \|T_1\|_0 + \|V\|_0) \quad \text{where} \quad \delta^{\pm}_{l,j} := \|j\|^2 \pm \omega \cdot l \,. \tag{6.14}$$

Since $|\omega| = |\lambda||\bar{\omega}| \leq 3/2$ (see (1.4)), $\|j\| \geq |j|$ (see (2.18)), $|j - j_0| \leq N$, $|l| \leq N$, we have

$$\delta^{\pm}_{l,j} \geq (|j_0| - |j - j_0|)^2 - \nu |\omega||l| \geq (|j_0| - N)^2 - \frac{3}{2} \nu N \geq \frac{|j_0|^2}{6} \tag{6.15}$$

for $|j_0| \geq 2N$ and $N \geq N_0(\nu)$ large enough. Moreover, since $\|j\|^2 \leq d|j|^2$,

$$\delta^{\pm}_{l,j} \leq d(|j_0| + |j - j_0|)^2 + \nu|\omega||l| \leq d(|j_0| + N)^2 + 2\nu N \leq 3d|j_0|^2 \tag{6.16}$$

for $N \geq N_0(\nu)$ large enough. Hence (6.14), (6.15), (6.16), (6.11) imply (6.13). As a consequence, by Lemma 6.1-ii) with $W = Y_{N,j_0}$, $\|W^{-1}\|_0 = 1$, we deduce $|B^0_{2,N}(j_0; \varepsilon, \lambda)| \leq C N^{-\tau + d + \nu}$. ∎

Lemmas 6.3 and 6.4 imply that:

Corollary 6.1. *$\forall |j_0| \geq 2N$, $\forall (\varepsilon, \lambda) \in [0, \varepsilon_0] \times \Lambda$, we have*

$$B^0_N(j_0; \varepsilon, \lambda) \subset \bigcup_{q = 1, \ldots, N^{d+\nu+2}} I_q$$

where I_q are intervals satisfying $|I_q| \leq N^{-\tau}$.

We now consider the cases $|j_0| < 2N$.

Lemma 6.5. *$\forall |j_0| < 2N$, $\forall (\varepsilon, \lambda) \in [0, \varepsilon_0] \times \Lambda$, we have*

$$B^0_{2,N}(j_0; \varepsilon, \lambda) \subset I_N := (-11dN^2, 11dN^2) \,.$$

PROOF. The eigenvalues of θY are $\pm\theta$ and (2.18) implies $\|j\|^2 \leq d(|j_0| + |j - j_0|)^2 \leq 9dN^2$. Hence, by (6.12), (6.14), $|l| \leq N$, (1.4), (6.11),

$$\|A_{N,j_0}(\varepsilon, \lambda)\|_0 \leq \|D_{N,j_0}(\lambda)\|_0 + \|T_{N,j_0}(\varepsilon, \lambda)\|_0 \leq 2\nu N + 9dN^2 + C(1 + \|V\|_0) \leq 10dN^2$$

for $N \geq N(V, d, \nu)$ large enough. By Lemma 6.2, if $\theta \notin I_N$ all the eigenvalues of $A_{N,j_0}(\varepsilon, \lambda, \theta) = A_{N,j_0}(\varepsilon, \lambda) + \theta Y_{N,j_0}$ are greater than 1 (actually dN^2). ∎

Lemma 6.6. *$\forall |j_0| < 2N$, the set*

$$\mathbf{B}^0_{2,N}(j_0) := \Big\{ (\varepsilon, \lambda, \theta) \in [0, \varepsilon_0] \times \Lambda \times \mathbb{R} \, : \, \Big\| A^{-1}_{N,j_0}(\varepsilon, \lambda, \theta) \Big\|_0 > N^\tau / 2 \Big\} \tag{6.17}$$

has measure

$$|\mathbf{B}^0_{2,N}(j_0)| \leq \varepsilon_0 N^{-\tau + d + \nu + 3} \,. \tag{6.18}$$

PROOF. By Lemma 6.5, $\mathbf{B}^0_{2,N}(j_0) \subset [0, \varepsilon_0] \times \Lambda \times I_N$. In order to estimate the "bad" $(\varepsilon, \lambda, \theta)$ where at least one eigenvalue of $A_{N,j_0}(\varepsilon, \lambda, \theta)$ is less than $N^{-\tau}$, we introduce the variables

$$\xi := \frac{1}{\lambda} \,, \quad \eta := \frac{\theta}{\lambda} \quad \text{where} \quad (\xi, \eta) \in [2/3, 2] \times 2I_N \tag{6.19}$$

and we consider the self adjoint matrix

$$\frac{1}{\lambda} A_{N,j_0}(\varepsilon, \lambda, \theta) \overset{(6.12)}{=} \mathrm{diag}_{|l| \leq N, |j - j_0| \leq N} \begin{pmatrix} -\bar{\omega} \cdot l & 0 \\ 0 & \bar{\omega} \cdot l \end{pmatrix} + \xi P_{N,j_0} - \varepsilon \xi T_1(\varepsilon, 1/\xi) + \eta Y \tag{6.20}$$

where

$$P := \begin{pmatrix} -\Delta + V(x) & 0 \\ 0 & -\Delta + V(x) \end{pmatrix} \quad \text{satisfies} \quad P \overset{(1.3)}{\geq} \beta_0 I \, .$$

The derivative with respect to ξ of the matrix in (6.20) is

$$P_{N,j_0} - \varepsilon T_1(\varepsilon, 1/\xi) + \frac{\varepsilon}{\xi} \partial_\lambda T_1(\varepsilon, 1/\xi) \overset{(6.4)}{\geq} \frac{\beta_0}{2} I \, ,$$

i.e. positive definite (for ε_0 small enough). By Lemma 6.1, for each fixed η, the set of $\xi \in [2/3, 2]$ such that at least one eigenvalue is $\leq N^{-\tau}$ has measure at most $O(N^{-\tau+d+\nu})$. Then, integrating on $\eta \in I_N$, whose length is $|I_N| = O(N^2)$, on $\varepsilon \in [0, \varepsilon_0]$, and since the change of variables (6.19) has a Jacobian of modulus $\geq 1/8$, we deduce (6.18). ■

By the same arguments (see also the proof of Lemma 7.13) we also get the following measure estimate that will be used in section 7, see $(S4)_n$.

Lemma 6.7. *The complementary of the set*

$$\mathtt{G}_N := \mathtt{G}_N(\mathbf{u}) := \Big\{ (\varepsilon, \lambda) \in [0, \varepsilon_0] \times \Lambda \, : \, \| A_N^{-1}(\varepsilon, \lambda) \|_0 \leq N^{\tau} \Big\} \tag{6.21}$$

has measure

$$|\mathtt{G}_N^c \cap ([0, \varepsilon_0] \times \Lambda)| \leq \varepsilon_0 N^{-\tau+d+\nu+1} \, . \tag{6.22}$$

Remark 6.2. *For periodic solutions (i.e. $\nu = 1$), a similar eigenvalue variation argument which exploits $-\Delta \geq 0$ was used in the Appendix of [10] and in [5].*

As a consequence of Lemma 6.6, for "most" (ε, λ) the measure of $B^0_{2,N}(j_0; \varepsilon, \lambda)$ is "small".

Lemma 6.8. $\forall |j_0| < 2N$, *the set*

$$\mathcal{F}_N(j_0) := \Big\{ (\varepsilon, \lambda) \in [0, \varepsilon_0] \times \Lambda \, : \, |B^0_{2,N}(j_0; \varepsilon, \lambda)| \geq \frac{1}{2} N^{-\tau+2d+\nu+4} \Big\}$$

has measure

$$|\mathcal{F}_N(j_0)| \leq 2\varepsilon_0 N^{-d-1} \, . \tag{6.23}$$

Proof. By Fubini theorem (see (6.17) and (6.9))

$$|\mathbf{B}^0_{2,N}(j_0)| = \int_{[0,\varepsilon_0] \times \Lambda} |B^0_{2,N}(j_0; \varepsilon, \lambda)| d\varepsilon \, d\lambda \, . \tag{6.24}$$

Let $\mu := \tau - 2d - \nu - 4$. By (6.24) and (6.18),

$$\begin{aligned} \varepsilon_0 N^{-\tau+d+\nu+3} &\geq \int_{[0,\varepsilon_0] \times \Lambda} |B^0_{2,N}(j_0; \varepsilon, \lambda)| d\varepsilon \, d\lambda \\ &\geq \frac{1}{2} N^{-\mu} \Big| \Big\{ (\varepsilon, \lambda) \in [0, \varepsilon_0] \times \Lambda \, : \, |B^0_{2,N}(j_0; \varepsilon, \lambda)| \geq \frac{1}{2} N^{-\mu} \Big\} \Big| := \frac{1}{2} N^{-\mu} |\mathcal{F}_N(j_0)| \end{aligned}$$

whence (6.23). ■

By Lemma 6.8, for all $(\varepsilon, \lambda) \notin \mathcal{F}_N(j_0)$ we have the measure estimate $|B^0_{2,N}(j_0; \varepsilon, \lambda)| < N^{-\tau+2d+\nu+4}/2$. Then, Lemma 6.3 implies

Corollary 6.2. $\forall |j_0| < 2N$, $\forall (\varepsilon, \lambda) \notin \mathcal{F}_N(j_0)$, *we have* $B^0_N(j_0; \varepsilon, \lambda) \subset \bigcup_{q=1,\ldots,N^{2d+\nu+4}} I_q$ *with* I_q *intervals satisfying* $|I_q| \leq N^{-\tau}$.

Proposition 6.1 is a direct consequence of the following lemma.

Lemma 6.9. $\mathcal{B}_N^0 \subseteq \bigcup_{|j_0|<2N} \mathcal{F}_N(j_0)$.

Proof. Corollaries 6.1 and 6.2 imply that

$$(\varepsilon, \lambda) \notin \bigcup_{|j_0|<2N} \mathcal{F}_N(j_0) \quad \Longrightarrow \quad (\varepsilon, \lambda) \in \mathcal{G}_N^0$$

(see the definition in (6.3)). The lemma follows. ■

Proof of Proposition 6.1 completed. By Lemma 6.9 and (6.23) we get

$$|\mathcal{B}_N^0| \leq \sum_{|j_0|<2N} |\mathcal{F}_N(j_0)| < (2N+1)^d |\mathcal{F}_N(j_0)| \leq (2N+1)^d 2\varepsilon_0 N^{-d-1} \leq C\varepsilon_0 N^{-1} \,. \tag{6.25}$$

7 Nash Moser iterative scheme

Consider the orthogonal splitting

$$\mathbf{H}^s = H_n \oplus H_n^\perp$$

where $\mathbf{H}^s$ is defined in (1.12) and

$$\begin{aligned} H_n &:= \Big\{ u := \mathbf{u} = (u^+, u^-) \in \mathbf{H}^s \,:\, u = \sum_{|(l,j)| \leq N_n} u_{l,j}\, e^{\mathrm{i}(l\cdot\varphi + j\cdot x)} \Big\} \\ H_n^\perp &:= \Big\{ u := \mathbf{u} = (u^+, u^-) \in \mathbf{H}^s \,:\, u = \sum_{|(l,j)| > N_n} u_{l,j}\, e^{\mathrm{i}(l\cdot\varphi + j\cdot x)} \Big\}, \end{aligned} \tag{7.1}$$

with $u_{l,j} := (u_{l,j}^+, u_{l,j}^-) \in \mathbb{C}^2$, and

$$N_n := N_0^{2^n} \,, \qquad \text{namely} \quad N_{n+1} = N_n^2 \,, \ \forall n \geq 0 \,. \tag{7.2}$$

In the proof we shall take $N_0 \in \mathbb{N}$ large enough depending on ε_0 and V, d, ν, see (7.95). We denote by

$$P_n : \mathbf{H}^s \to H_n \qquad \text{and} \qquad P_n^\perp : \mathbf{H}^s \to H_n^\perp \tag{7.3}$$

the orthogonal projectors onto H_n and $H_n^\perp$. The following "smoothing" properties hold, $\forall n \in \mathbb{N}$, $s \geq 0$, $r \geq 0$,

$$\|P_n u\|_{s+r} \leq N_n^r \|u\|_s \,, \qquad \forall u \in \mathbf{H}^s \tag{7.4}$$

$$\|P_n^\perp u\|_s \leq N_n^{-r} \|u\|_{s+r} \,, \quad \forall u \in \mathbf{H}^{s+r} \,. \tag{7.5}$$

More generally, for $j_0 \in \mathbb{Z}^d$, we denote P_{N,j_0} the orthogonal projector from $\mathbf{H}^s$ onto the subspace

$$H_{N,j_0} := \Big\{ u \in \mathbf{H}^s \,:\, u = \sum_{|(l,j-j_0)| \leq N} u_{l,j}\, e^{\mathrm{i}(l\cdot\varphi + j\cdot x)} \Big\} \,. \tag{7.6}$$

With the above notation $H_n = H_{N_n,0}$, see (7.1), and $P_n := P_{N_n,0}$, see (7.3). Moreover we also denote Π_{N,j_0} the orthogonal projector from $H^{s_0}(\mathbb{T}^d)$ (functions only of the x-variable) onto the space

$$E_{N,j_0} := \Big\{ u(x) := \sum_{|j-j_0| \leq N} u_j e^{\mathrm{i} j\cdot x} \,,\ u_j \in \mathbb{C} \Big\} \,. \tag{7.7}$$

The composition operator on Sobolev spaces

$$f : \mathbf{H}^s \to \mathbf{H}^s \,, \qquad f(u)(t,x) := \begin{pmatrix} f(\varphi, x, u^- u^+) u^+ \\ f(\varphi, x, u^- u^+) u^- \end{pmatrix} ,$$

where $f \in C^q(\mathbb{T}^\nu \times \mathbb{T}^d \times \mathbb{R}; \mathbb{R})$ with

$$q \geq S + 2 \tag{7.8}$$

satisfies the following standard properties (see e.g. [31]): $\forall s \in [s_1, S]$, $s_1 > (d+\nu)/2$,

- **(F1)** (**Regularity**) $f \in C^2(\mathbf{H}^s; \mathbf{H}^s)$.
- **(F2)** (**Tame estimates**) $\forall u, h \in \mathbf{H}^s$ with $\|u\|_{s_1} \le 1$,

$$\|f(u)\|_s \le C(s)\|u\|_s \,, \quad \|(Df)(u)h\|_s \le C(s)(\|h\|_s + \|u\|_s\|h\|_{s_1}) \,. \tag{7.9}$$

$$\|D^2 f(u)[h,v]\|_s \le C(s)\Big(\|u\|_s\|h\|_{s_1}\|v\|_{s_1} + \|v\|_s\|h\|_{s_1} + \|v\|_{s_1}\|h\|_s\Big) \,. \tag{7.10}$$

As a consequence we get

- **(F3)** (**Taylor Tame estimate**) $\forall u \in \mathbf{H}^s$ with $\|u\|_{s_1} \le 1$, $\forall h \in \mathbf{H}^s$ with $\|h\|_{s_1} \le 1$,

$$\|f(u+h) - f(u) - (Df)(u)\, h\|_s \le C(s)(\|u\|_s\|h\|_{s_1}^2 + \|h\|_{s_1}\|h\|_s) \,. \tag{7.11}$$

In particular, for $s = s_1$,

$$\|f(u+h) - f(u) - (Df)(u)\, h\|_{s_1} \le C(s_1)\|h\|_{s_1}^2 \,. \tag{7.12}$$

The values of the constants s_1 and S are fixed in (7.16) below.

Remark 7.1. *The differential $(Df)(u)$ is the operator T_1 defined in* (2.3) *with (p,q) as in* (2.4).

By Lemma 3.1 and the first inequality in (7.9) applied to the composition operators in (2.4), the Töplitz matrix T_1 which represents $Df(u)$ satisfies, $\forall s \in [s_1, S]$,

$$|T_1|_s = |(Df)(u)|_s \le C(s)(1 + \|u\|_s) \,. \tag{7.13}$$

For simplicity of notation we denote $(g, \bar g)$ simply by g. We shall use that g and the potential V satisfy

$$\|g\|_{C^q} \le C \,, \quad \|V\|_{C^q} \le C \,, \tag{7.14}$$

for some fixed constant C.

With the above more concise notations, the vector NLS-equation (1.11) becomes

$$L_\omega u = \varepsilon(f(u) + g) \,. \tag{7.15}$$

For definiteness we fix the Sobolev indices $s_0 < s_1 < S$ as

$$s_0 := b = d + \nu \,, \qquad s_1 := 10(\tau + b)C_2 \,, \qquad S := 12\tau' + 8(s_1 + 1) \,, \tag{7.16}$$

where

$$C_2 := 6(C_1 + 2) \,, \ \tau := \max\{d + \nu + 2, 2\, C_2\, \tau_0 + 1\} \,, \ \tau' := 3\tau + 2b \,, \ \tau_0 := \nu \tag{7.17}$$

(the constant τ_0 is introduced in (1.5)) and $C_1 := C_1(d,\nu) \ge 2$ is defined in Proposition 5.1. Note that s_0, s_1, S defined in (7.16) depend only on d and ν.

We also fix the constant δ in Definition 4.1 as

$$\delta := 1/4 \,. \tag{7.18}$$

Remark 7.2. *By (7.16)-(7.18) the hypotheses (4.3)-(4.5) of Proposition 4.1 are satisfied for any $\chi \in [C_2, 2C_2)$, as well as assumption (ii) of Proposition 5.1. We assume $\tau \ge d + \nu + 2$ in view of (6.22). The strongest condition for S appears in the proof of Lemma 7.10.*

Setting

$$\tau_1 := d + \nu$$

and $\gamma > 0$, we shall implement the first steps of the Nash-Moser iteration restricting λ to the set

$$\begin{aligned} \bar{\mathcal{G}} \ &:= \ \Big\{\lambda \in \Lambda \, : \, \Big\| \Big(\pm \lambda \bar\omega \cdot l + \Pi_0(-\Delta + V(x))_{|E_0} \Big)^{-1} \Big\|_{L^2_x} \le \frac{N_0^{\tau_1}}{\gamma}, \, \forall \, |l| \le N_0 \Big\} \\ &= \ \Big\{\lambda \in \Lambda \, : \, |\pm \lambda \bar\omega \cdot l + \mu_j| \ge \gamma N_0^{-\tau_1}, \, \forall \, |j| \le N_0, \, |l| \le N_0 \Big\} \end{aligned} \tag{7.19}$$

where μ_j are the eigenvalues of $\Pi_0(-\Delta + V(x))_{|E_0}$ where $\Pi_0 := \Pi_{N_0,0}$, $E_0 := E_{N_0,0}$ are defined in (7.7). We shall prove in Lemma 7.13 the measure bound $|\bar{\mathcal{G}}| = 1 - O(\gamma)$ (since $\tau_1 \geq d + \nu$). The constant γ will be fixed in (7.95).

We also define

$$\sigma := \tau' + \delta s_1 + 2 \,. \tag{7.20}$$

Given a set A we denote $\mathcal{N}(A, \eta)$ the open neighborhood of A of width η (which is empty if A is empty).

Theorem 7.1. (Nash-Moser) *There exist $\bar{c}, \bar{\gamma} > 0$ (depending on $d, \nu, V, \gamma_0, \beta_0$) such that, if*

$$N_0 \geq 2\gamma^{-1} \,, \ \gamma \in (0, \bar{\gamma}) \,, \qquad \text{and} \qquad \varepsilon_0 N_0^S \leq \bar{c} \,, \tag{7.21}$$

then there is a sequence $(u_n)_{n\geq 0}$ of C^1 maps $u_n : [0, \varepsilon_0) \times \Lambda \to \mathbf{H}^{s_1} \cap \mathcal{U}$ (see (1.13)) satisfying

$(\mathbf{S1})_n$ $u_n(\varepsilon, \lambda) \in H_n \cap \mathcal{U}$, $u_n(0, \lambda) = 0$, $\|u_n\|_{s_1} \leq 1$, $\|\partial_{(\varepsilon,\lambda)} u_n\|_{s_1} \leq C(s_1) N_0^{\tau_1 + s_1 + 1} \gamma^{-1}$.

$(\mathbf{S2})_n$ $(n \geq 1)$ *For all $1 \leq k \leq n$, $\|u_k - u_{k-1}\|_{s_1} \leq N_k^{-\sigma-1}$, $\|\partial_{(\varepsilon,\lambda)}(u_k - u_{k-1})\|_{s_1} \leq N_k^{-1/2}$.*

$(\mathbf{S3})_n$ $(n \geq 1)$

$$\|u - u_{n-1}\|_{s_1} \leq N_n^{-\sigma} \quad \Longrightarrow \quad \bigcap_{k=1}^{n} \mathcal{G}^0_{N_k}(u_{k-1}) \subseteq \mathcal{G}_{N_n}(u) \tag{7.22}$$

where $\mathcal{G}^0_N(u)$ (resp. $\mathcal{G}_N(u)$) is defined in (6.3) (resp. in (5.4)) .

$(\mathbf{S4})_n$ *Define the set*

$$\mathcal{C}_n := \bigcap_{k=1}^{n} \mathtt{G}_{N_k}(u_{k-1}) \bigcap_{k=1}^{n} \mathcal{G}^0_{N_k}(u_{k-1}) \bigcap \Big([0, \varepsilon_0] \times \bar{\mathcal{G}} \Big) \,, \tag{7.23}$$

where $\mathtt{G}_{N_k}(u_{k-1})$ is defined in (6.21), $\bar{\mathcal{G}}$ in (7.19), $\mathcal{G}^0_{N_k}(u_{k-1})$ in (6.3).

If $(\varepsilon, \lambda) \in \mathcal{N}(\mathcal{C}_n, N_n^{-\sigma})$ then $u_n(\varepsilon, \lambda)$ solves the equation

$$(P_n) \qquad P_n\Big(L_\omega u - \varepsilon(f(u) + g)\Big) = 0 \,.$$

$(\mathbf{S5})_n$ $U_n := \|u_n\|_S$, $U'_n := \|\partial_{(\varepsilon,\lambda)} u_n\|_S$ *(where S is defined in (7.16)) satisfy*

$$(i) \ \ U_n \leq N_n^{2(\tau' + \delta s_1 + 1)} \,, \qquad (ii) \ \ U'_n \leq N_n^{4\tau' + 2s_1 + 4} \,.$$

The sequence $(u_n)_{n\geq 0}$ converges in C^1 norm to a map

$$u \in C^1([0, \varepsilon_0) \times \Lambda, \mathbf{H}^{s_1}) \quad \text{with} \quad u(0, \lambda) = 0 \tag{7.24}$$

and, if (ε, λ) belongs to the Cantor like set

$$\mathcal{C}_\infty := \bigcap_{n\geq 0} \mathcal{C}_n \tag{7.25}$$

then $u(\varepsilon, \lambda)$ is a solution of (1.11), i.e. (7.15), *with $\omega = \lambda\bar{\omega}$.*

The sets of parameters $\mathcal{C}_n$ in $(S4)_n$ are decreasing, i.e.

$$\ldots \subseteq \mathcal{C}_n \subseteq \mathcal{C}_{n-1} \subseteq \ldots \subseteq \mathcal{C}_0 \subset [0, \varepsilon_0] \times \bar{\mathcal{G}} \subset [0, \varepsilon_0] \times \Lambda \,,$$

and it could happen that $\mathcal{C}_{n_0} = \emptyset$ for some $n_0 \geq 1$. In such a case $u_n = u_{n_0}$, $\forall n \geq n_0$ (however the map u in (7.24) is always defined), and $\mathcal{C}_\infty = \emptyset$. Later, in (7.95), we shall specify the values of $\gamma, \varepsilon_0, N_0$, in order to verify that $\mathcal{C}_\infty$ has asymptotically full measure, i.e. (1.10) holds.

The proof of Theorem 7.1 is based on an improvement of the Nash-Moser theorems in [2], [3], [4]. The main difference is that the "tame exponent" $\tau' + \delta s$ in (7.64) depends on the Sobolev index s. We have chosen $\delta = 1/4$ in (7.18) for definiteness. The Nash-Moser iteration would converge for any $\delta < 1$, see section 1.2.

Another difference with respect to the scheme in [2], [3], [4], is that we perform, at the same time, the Nash-Moser iteration and the multiscale argument for proving the invertibility of the linearized operators, see Lemma 7.7. This is more convenient for proving measure estimates.

7.1 Initialization of the Nash-Moser scheme

We perform the first step of the Nash-Moser iteration restricting $\lambda \in \mathcal{N}(\bar{\mathcal{G}}, 2N_0^{-\sigma})$ (the set $\bar{\mathcal{G}}$ is defined in (7.19)).

Lemma 7.1. *For all $\lambda \in \mathcal{N}(\bar{\mathcal{G}}, 2N_0^{-\sigma})$, the operator*

$$\mathcal{L}_0 := P_0(L_{\lambda\bar{\omega}})_{|H_0} \tag{7.26}$$

(where L_ω is defined in (2.2)) is invertible and

$$\|\mathcal{L}_0^{-1}\|_{s_1} \leq 2N_0^{\tau_1+s_1}\gamma^{-1}\,. \tag{7.27}$$

Proof. With the notations of (7.19), for all $\lambda \in \mathcal{N}(\bar{\mathcal{G}}, 2N_0^{-\sigma})$,

$$\forall |(l,j)| \leq N_0\,, \quad |\pm \lambda\bar{\omega}\cdot l + \mu_j| \geq \gamma N_0^{-\tau_1} - 2|\bar{\omega}|N_0^{1-\sigma} \geq \frac{\gamma}{2}N_0^{-\tau_1}\,, \tag{7.28}$$

provided $N_0 \geq 4\gamma^{-1}|\bar{\omega}|$ (recall (7.20), (7.17) and $\tau_1 := d+\nu$). Then $\|\mathcal{L}_0^{-1}\|_0 \leq 2\gamma^{-1}N_0^{\tau_1}$ and (7.27) follows by the smoothing property (7.4). ■

A fixed point of

$$F_0 : H_0 \to H_0\,, \quad F_0(u) := \varepsilon\mathcal{L}_0^{-1}P_0(f(u)+g)\,, \tag{7.29}$$

is a solution of equation (P_0).

Lemma 7.2. *For $\varepsilon\gamma^{-1}N_0^{\tau_1+s_1+\sigma} \leq c(s_1)$ small, $\forall\lambda \in \mathcal{N}(\bar{\mathcal{G}}, 2N_0^{-\sigma})$, the map F_0 is a contraction in $\mathtt{B}_0(s_1) := \{u \in H_0 \,:\, \|u\|_{s_1} \leq \rho_0 := N_0^{-\sigma}\}$.*

Proof. The map F_0 maps $\mathtt{B}_0(s_1)$ into itself, because, $\forall \|u\|_{s_1} \leq \rho_0$,

$$\|F_0(u)\|_{s_1} \overset{(7.27)}{\leq} 2\varepsilon\gamma^{-1}N_0^{\tau_1+s_1}(\|f(u)\|_{s_1} + \|g\|_{s_1}) \overset{(F2),(7.14)}{\leq} \varepsilon\gamma^{-1}N_0^{\tau_1+s_1}C(s_1) \leq \rho_0$$

for $\varepsilon\gamma^{-1}N_0^{\tau_1+s_1+\sigma}$ is small enough. Moreover, $\forall\|u\|_{s_1} \leq \rho_0$,

$$\|(DF_0)(u)\|_{s_1} = \varepsilon\|\mathcal{L}_0^{-1}P_0(Df)(u)_{|H_0}\|_{s_1} \overset{(7.27),(F2)}{\leq} \varepsilon N_0^{\tau_1+s_1}\gamma^{-1}C(s_1) \leq 1/2\,, \tag{7.30}$$

implying that the map F_0 is a contraction in $\mathtt{B}_0(s_1)$. ■

Let $\widetilde{u}_0(\varepsilon,\lambda)$ denote the unique solution of (P_0) in $\mathtt{B}_0(s_1)$ defined for all $(\varepsilon,\lambda) \in [0,\varepsilon_0]\times\mathcal{N}(\bar{\mathcal{G}}, 2N_0^{-\sigma})$. For $\varepsilon = 0$ the map F_0 in (7.29) has $u = 0$ as a fixed point. By uniqueness we deduce $\widetilde{u}_0(0,\lambda) = 0$. Since the contracting map F_0 leaves $B_0(s_1)\cap\mathcal{U}$ invariant (see (1.13)), we deduce that $\widetilde{u}_0(\varepsilon,\lambda) \in \mathcal{U}$. Moreover, by (7.30), the operator

$$\mathcal{L}_0(\varepsilon) := P_0\Big(L_\omega - \varepsilon(Df)(\widetilde{u}_0)\Big)_{|H_0} = \mathcal{L}_0 - \varepsilon P_0(Df)(\widetilde{u}_0)_{|H_0} = \mathcal{L}_0\Big(I - (DF_0)(\widetilde{u}_0)\Big) \tag{7.31}$$

is invertible and

$$\|\mathcal{L}_0^{-1}(\varepsilon)\|_{s_1} \leq 2\|\mathcal{L}_0^{-1}\|_{s_1} \overset{(7.27)}{\leq} 4N_0^{\tau_1+s_1}\gamma^{-1}\,. \tag{7.32}$$

The implicit function theorem implies that $\widetilde{u}_0 \in C^1([0,\varepsilon_0]\times\mathcal{N}(\bar{\mathcal{G}}, 2N_0^{-\sigma}); H_0)$ and

$$\partial_\varepsilon\widetilde{u}_0 = \mathcal{L}_0^{-1}(\varepsilon)P_0(f(\widetilde{u}_0)+g)\,, \quad \partial_\lambda\widetilde{u}_0 = -\mathcal{L}_0^{-1}(\varepsilon)(\partial_\lambda\mathcal{L}_0)\widetilde{u}_0\,. \tag{7.33}$$

Then, by (7.33), (7.32) and $\partial_\lambda L_\omega = \mathrm{diag}(\pm \mathrm{i}\bar{\omega}\cdot\partial_\varphi)$, we get

$$\|\partial_\varepsilon\widetilde{u}_0\|_{s_1} \leq N_0^{\tau_1+s_1}\gamma^{-1}C(s_1)\,, \quad \|\partial_\lambda\widetilde{u}_0\|_{s_1} \leq 4|\bar{\omega}|N_0^{\tau_1+s_1}\gamma^{-1}\|\widetilde{u}_0\|_{s_1+1} \leq CN_0^{\tau_1+s_1+1-\sigma}\gamma^{-1} \tag{7.34}$$

using that $\|\widetilde{u}_0\|_{s_1+1} \leq N_0\|\widetilde{u}_0\|_{s_1} \leq N_0N_0^{-\sigma}$.

Finally we define the C^1 map $u_0 := \psi_0 \widetilde{u}_0 : [0, \varepsilon_0] \times \Lambda \to H_0$ with cut-off function $\psi_0 : \Lambda \to [0,1]$,

$$\psi_0 := \begin{cases} 1 & \text{if } \lambda \in \mathcal{N}(\bar{\mathcal{G}}, N_0^{-\sigma}) \\ 0 & \text{if } \lambda \notin \mathcal{N}(\bar{\mathcal{G}}, 2N_0^{-\sigma}) \end{cases} \qquad \text{and} \qquad |D_\lambda \psi_0| \leq N_0^\sigma C \,. \tag{7.35}$$

Then (7.35), $\|\widetilde{u}_0\|_{s_1} \leq N_0^{-\sigma}$ and (7.34) imply (we have $\partial_\varepsilon \psi_0 \equiv 0$)

$$\|u_0\|_{s_1} \leq N_0^{-\sigma} \,, \quad \|\partial_{(\varepsilon,\lambda)} u_0\|_{s_1} \leq C(s_1) N_0^{\tau_1 + s_1 + 1} \gamma^{-1} \,. \tag{7.36}$$

The statement $(S1)_0$ is proved. Note that $(S2)_0$, $(S3)_0$ are empty. Finally, also property $(S4)_0$ is proved because, by (7.35) the function $u_0(\varepsilon, \lambda)$ solves the equation (P_0) for all $(\varepsilon, \lambda) \in \mathcal{N}(\mathcal{C}_0, N_0^{-\sigma})$, since $\mathcal{C}_0 = [0, \varepsilon_0] \times \bar{\mathcal{G}}$.

For the next steps of the induction we need the following lemma which establishes a property which replaces $(S3)_n$ for the first steps of the induction.

Lemma 7.3. *There exists $N_0 := N_0(S, V) \in \mathbb{N}$ and $c(s_1) > 0$ such that, if*

$$\varepsilon_0 N_0^{\tau' + \delta s_1} \leq c(s_1) \,, \tag{7.37}$$

then $\forall N_0^{1/C_2} \leq N \leq N_0$, $\forall \|u\|_{s_1} \leq 1$, $\mathcal{G}_N(u) = [0, \varepsilon_0] \times \Lambda$.

In order to prove Lemma 7.3 we prefix the following Lemma.

Lemma 7.4. *For $N \geq \tilde{N}(S, V)$ large enough, if*

$$\Big\| \Big(\vartheta \, \mathrm{I} + \Pi_{N,j_0} (-\Delta + V(x))_{|E_{N,j_0}} \Big)^{-1} \Big\|_{L^2_x} \leq N^\tau \,, \quad \vartheta \in \mathbb{R} \,, \tag{7.38}$$

(see the definition of E_{N,j_0} in (7.7)) then, $\forall s \in [s_0, S]$,

$$\Big| \Big(\vartheta \mathrm{I} + \Pi_{N,j_0} (-\Delta + V(x))_{|E_{N,j_0}} \Big)^{-1} \Big|_s \leq \frac{1}{2} N^{\tau' + \delta s} \,. \tag{7.39}$$

Proof. We apply a simplified version of Proposition 4.1 to $\vartheta \mathrm{I} + \Pi_{N,j_0}(-\Delta + V(x))_{|E_{N,j_0}}$. We sketch the main modifications only. The scale N' in Proposition 4.1 is here replaced by N. Assumption (H1) follows from the regularity of the potential $V(x)$ (see Lemma 3.1) and (H2) is (7.38). With respect to Proposition 4.1, we use a stronger version of assumption (H3), calling "good sites" the regular sites only, namely the $j \in \mathbb{Z}^d$, $|j - j_0| \leq N$, such that

$$|d_j| \geq \Theta \qquad \text{where} \qquad d_j := \vartheta + \|j\|^2 + m$$

and m denotes the average of the potential $V(x)$, see (2.5). This is enough because here the singular sites satisfy separation properties. For $\Theta^{-1} \|V\|_{s_1}$ small enough we have the analogue of Lemma 4.1 (the proof is simpler because all the good sites satisfy $|d_j| \geq \Theta$). The separation properties of the singular sites $j \in \mathbb{Z}^d$, $|j - j_0| \leq N$, such that $|d_j| < \Theta$, is proved as in section 5: a M-chain of singular sites has length at most $L \leq M^{C_3(d)}$, see Lemma 5.2 and (5.17). Then, taking $M := N^{\delta/2(1+C_3(d))}$ we get a partition of the singular sites in clusters Ω_α satisfying

$$\mathrm{d}(\Omega_\alpha, \Omega_\beta) > N^{\delta/2(1+C_3(d))} \quad \text{and} \quad \mathrm{diam}(\Omega_\alpha) \leq ML \leq M^{1+C_3(d)} = N^{\delta/2} \,.$$

Estimate (7.39) follows by the arguments of Lemmas 4.2, 4.3 in section 4. ■

Proof of Lemma 7.3. We claim that, $\forall (\varepsilon, \lambda) \in [0, \varepsilon_0] \times \Lambda$, $\forall j_0 \in \mathbb{Z}^d$,

$$B_N(j_0; \varepsilon, \lambda) \subset \bigcup_{|(l, j - j_0)| \leq N} \Big\{ \theta \in \mathbb{R} \, : \, |\delta^{\pm}_{l,j}(\theta)| \leq N^{-\tau} \Big\} \tag{7.40}$$

where

$$\delta^{\pm}_{l,j}(\theta) := \pm(\omega \cdot l + \theta) + \tilde{\mu}_j \,, \ \omega = \lambda\bar{\omega} \,, \ \tilde{\mu}_j := \text{ eigenvalues of } \Pi_{N,j_0}(-\Delta + V(x))_{|E_{N,j_0}}$$

(which depend on N) and the subspace E_{N,j_0} is defined in (7.7). Actually (7.40) is equivalent to

$$|\delta^{\pm}_{l,j}(\theta)| > N^{-\tau} \,, \ \forall \, |(l, j - j_0)| \leq N \quad \Longrightarrow \quad A_{N,j_0}(\varepsilon, \lambda, \theta) \text{ is } N - \text{good} \tag{7.41}$$

with $A = \mathcal{L}(u) = L_\omega + \theta Y - \varepsilon(Df)(u)$. We first prove that the left hand side condition in (7.41) implies

$$Q_{N,j_0} := P_{N,j_0}(L_\omega + \theta Y)_{|H_{N,j_0}} \quad \text{satisfies} \quad |Q^{-1}_{N,j_0}|_s \leq \frac{1}{2} N^{\tau' + \delta s} \,, \ \forall s \in [s_0, S] \,, \tag{7.42}$$

(the subspace H_{N,j_0} is defined in (7.6)). Indeed, the operator L_ω is diagonal in time Fourier basis. The left hand side condition in (7.41) is equivalent to

$$\Big\| \Big(\pm (\lambda\bar{\omega} \cdot l + \theta)\mathrm{I} + \Pi_{N,j_0}(-\Delta + V(x))_{|E_{N,j_0}} \Big)^{-1} \Big\|_{L^2_x} < N^{\tau} \,, \ \ \forall |l| \leq N \,.$$

Lemma 7.4 implies, for $N \geq N_0^{1/C_2} \geq \tilde{N}(V, S)$, that

$$\Big| \Big(\pm (\lambda\bar{\omega} \cdot l + \theta)\mathrm{I} + \Pi_{N,j_0}(-\Delta + V(x))_{|E_{N,j_0}} \Big)^{-1} \Big|_s \leq \frac{1}{2} N^{\tau' + \delta s} \,, \ \ \forall |l| \leq N \,,$$

and (7.42) follows because Q_{N,j_0} is diagonal in time Fourier basis.

We now prove (7.41) by a perturbative argument. By (7.13) and $\|u\|_{s_1} \leq 1$ we have $|(Df)(u)|_{s_1} \leq C(s_1)$. Hence

$$\varepsilon |Q_{N,j_0}|_{s_1} |(Df)(u)|_{s_1} \overset{(7.42)}{\leq} \varepsilon N^{\tau' + \delta s_1} C(s_1) \leq \varepsilon_0 N_0^{\tau' + \delta s_1} C(s_1) \overset{(7.37)}{\leq} 1/2 \,. \tag{7.43}$$

Then, by Lemma 3.9, the matrix $A_{N,j_0}(\varepsilon, \lambda, \theta) = P_{N,j_0}(L_\omega + \theta Y - \varepsilon(Df)(u))_{|H_{N,j_0}}$ is invertible and

$$\forall s \in [s_0, s_1] \,, \ \ |A^{-1}_{N,j_0}(\varepsilon, \lambda, \theta)|_s \overset{(3.26)}{\leq} 2 |Q^{-1}_{N,j_0}|_s \overset{(7.42)}{\leq} N^{\tau' + \delta s} \,, \tag{7.44}$$

namely it is N-good.

Finally, by (7.40), $B_N(j_0; \varepsilon, \lambda)$ is included in an union of $2(2N+1)^b$ intervals of measure $\leq 2N^{-\tau}$, hence of $4(2N+1)^b \leq N^{2d+\nu+4}$ intervals I_q of measure $|I_q| \leq N^{-\tau}$. This proves that any $(\varepsilon, \lambda) \in [0, \varepsilon_0] \times \Lambda$ is N-good (see Definition 5.2) for $A = \mathcal{L}(u)$, namely that (ε, λ) is in $\mathcal{G}_N(u)$, see (5.4). ■

Finally we prove $(S5)_0$. With estimates similar to the proof of $(S1)_0$ using the smallness condition on ε_0 in (7.21), we deduce $(S5)_0$-(i). In order to estimate $\partial_{(\varepsilon,\lambda)} u_0$, we use that the inverse of the operator $\mathcal{L}_0(\varepsilon) = \mathcal{L}_0 - \varepsilon P_0 Df(\widetilde{u}_0)_{|H_0}$ defined in (7.31) ($\mathcal{L}_0$ is defined in (7.26)) satisfies, for $\lambda \in \mathcal{N}(\overline{\mathcal{G}}, 2N_0^{-\sigma})$,

$$|\mathcal{L}_0^{-1}(\varepsilon)|_s \leq N_0^{\tau' + \delta s} \,, \quad \forall s \in [s_1, S] \,. \tag{7.45}$$

Indeed, note that by (7.28), for $N = N_0$ and $\theta = 0$, the real numbers $|\delta^{\pm}_{l,j}(0)|$ defined after (7.40) are bounded from below by $\gamma N_0^{-\tau_1}/2 \geq N_0^{-\tau}$. Hence $\mathcal{L}_0 = Q_{N_0,0}$ satisfies (7.42), and Lemma 3.9 implies, $\forall s \in [s_1, S]$,

$$\begin{aligned}
|\mathcal{L}_0^{-1}(\varepsilon)|_s &\overset{(3.27),(7.42)}{\leq} \Big(1 + C(s)\varepsilon |Q^{-1}_{N_0,0}|_{s_0} |(Df)(\widetilde{u}_0)|_{s_0} \Big) \frac{N_0^{\tau' + \delta s}}{2} + C(s)\varepsilon (N_0^{\tau' + \delta s_0})^2 \, |(Df)(\widetilde{u}_0)|_s \\
&\overset{(7.42),(7.13),(S5)_0}{\leq} \Big(1 + C(s)\varepsilon N_0^{\tau' + \delta s_0} \Big) \frac{1}{2} N_0^{\tau' + \delta s} + C(s)\varepsilon N_0^{2(\tau' + \delta s_0) + 2(\tau' + \delta s_1 + 1)} \\
&\overset{(7.21),(7.16)}{\leq} N_0^{\tau' + \delta s}
\end{aligned}$$

since $4\tau' + 4\delta s_1 + 2 < S$. The bound $(S5)_0$-(ii) follows easily from (7.45). Let us give the details for $\partial_\varepsilon u_0$ (which is not small with ε). We have

$$\begin{aligned}
\|\partial_\varepsilon \widetilde{u}_0\|_S &\overset{(7.33)}{=} \|\mathcal{L}_0^{-1}(\varepsilon) P_0(f(\widetilde{u}_0) + g)\|_S \\
&\overset{(3.20)}{\leq} |\mathcal{L}_0^{-1}(\varepsilon)|_{s_1} \|f(\widetilde{u}_0) + g\|_S + C(S) |\mathcal{L}_0^{-1}(\varepsilon)|_S \|f(\widetilde{u}_0) + g\|_{s_1} \\
&\overset{(7.45),(F2),(7.14)}{\leq} C(S) N_0^{\tau' + \delta s_1} (\|\widetilde{u}_0\|_S + 1) + C'(S) N_0^{\tau' + \delta S} \\
&\overset{(S5)_0 - (i)}{\leq} C'(S) N_0^{3(\tau' + \delta s_1) + 2} + C'(S) N_0^{\tau' + \delta S} \leq N_0^{4\tau' + 2 s_1 + 4}
\end{aligned}$$

by (7.16) and $\delta = 1/4$. Then $(S5)_0$-(ii) is proved.

7.2 Iteration of the Nash-Moser scheme

Suppose, by induction, that we have already defined $u_n \in C^1([0, \varepsilon_0] \times \Lambda; H_n \cap \mathcal{U})$ and that properties $(S1)_k$-$(S5)_k$ hold for all $k \leq n$. We are going to define u_{n+1} and prove the statements $(S1)_{n+1}$-$(S5)_{n+1}$. Consider the operators $\mathcal{L}(u)$ (introduced in (2.1)),

$$\mathcal{L}(u) := \mathcal{L}(\omega, \varepsilon, u) := L_\omega - \varepsilon (Df)(u) \,. \tag{7.46}$$

In order to carry out a modified Nash-Moser scheme, we shall study the invertibility of

$$\mathcal{L}_{n+1}(u_n) := P_{n+1} \mathcal{L}(u_n)_{|H_{n+1}} \tag{7.47}$$

and the tame estimates of its inverse, applying Proposition 4.1. We distinguish two cases.
If $2^{n+1} > C_2$ (the constant C_2 is fixed in (7.17)), then there exists a unique $p \in [0, n]$ such that

$$N_{n+1} = N_p^\chi \,, \quad \chi = 2^{n+1-p} \in [C_2, 2C_2) \,. \tag{7.48}$$

If $2^{n+1} \leq C_2$ then there exists $\chi \in [C_2, 2C_2]$ such that

$$N_{n+1} = \bar{N}^\chi \,, \quad \bar{N} := [N_{n+1}^{1/C_2}] \in (N_0^{1/\chi}, N_0) \,. \tag{7.49}$$

If (7.48) holds we consider in Proposition 4.1 the two scales $N' = N_{n+1}$, $N = N_p$, see (4.2). If (7.49) holds, we set $N' = N_{n+1}$, $N = \bar{N}$.

A key point of the whole induction process is that the separation properties of the bad sites of $\mathcal{L}(u_n) + \theta Y$ hold uniformly for *all* $\theta \in \mathbb{R}$ and $j_0 \in \mathbb{Z}^d$.

Lemma 7.5. *For all*

$$(\varepsilon, \lambda) \in \bigcap_{k=1}^{n+1} \mathcal{G}^0_{N_k}(u_{k-1}) \,, \ \theta \in \mathbb{R} \,, \ j_0 \in \mathbb{Z}^d \,,$$

the hypothesis (H3) of Proposition 4.1 apply to $A_{N_{n+1}, j_0}(\varepsilon, \lambda, \theta)$ *where* $A(\varepsilon, \lambda, \theta) := \mathcal{L}(u_n) + \theta Y$.

Proof. We give the proof when (7.48) holds. By remark 5.1, a site

$$k \in E := \Big((0, j_0) + [-N_{n+1}, N_{n+1}]^b\Big) \times \{0, 1\} \,, \tag{7.50}$$

which is N_p-good for $A(\varepsilon, \lambda, \theta) := \mathcal{L}(u_n) + \theta Y$ (see Definition 5.1 with $A = A(\varepsilon, \lambda, \theta)$) is also

$$(A_{N_{n+1}, j_0}(\varepsilon, \lambda, \theta), N_p) - \text{good}$$

(see Definition 4.3 with $A = A_{N_{n+1}, j_0}(\varepsilon, \lambda, \theta)$). As a consequence the

$$\Big\{ \ (A_{N_{n+1}, j_0}(\varepsilon, \lambda, \theta), N_p) - \text{bad sites} \ \Big\} \ \subset \ \Big\{ N_p - \text{bad sites of } A(\varepsilon, \lambda, \theta) \text{ with } |l| \leq N_{n+1} \Big\}. \tag{7.51}$$

and (H3) is proved if the latter N_p-bad sites (in the right hand side of (7.51)) are contained in a disjoint union $\cup_\alpha \Omega_\alpha$ of clusters satisfying (4.6) (with $N = N_p$). This is a consequence of Proposition 5.1 applied to the infinite dimensional matrix $A(\varepsilon, \lambda, \theta)$. We claim that

$$\bigcap_{k=1}^{n+1} \mathcal{G}^0_{N_k}(u_{k-1}) \subset \mathcal{G}_{N_p}(u_n)\,, \text{ i.e. any } (\varepsilon,\lambda) \in \bigcap_{k=1}^{n+1} \mathcal{G}^0_{N_k}(u_{k-1}) \text{ is } N_p - \text{good for } A(\varepsilon,\lambda,\theta)\,, \tag{7.52}$$

and then assumption (i) of Proposition 5.1 holds. Indeed, if $p = 0$ then (7.52) is trivially true because $\mathcal{G}_{N_0}(u_n) = [0, \varepsilon_0] \times \Lambda$, by Lemma 7.3 and $(S1)_n$. If $p \geq 1$, we have

$$\|u_n - u_{p-1}\|_{s_1} \leq \sum_{k=p}^{n} \|u_k - u_{k-1}\|_{s_1} \stackrel{(S2)_k}{\leq} \sum_{k=p}^{n} N_k^{-\sigma-1} \leq N_p^{-\sigma} \sum_{k\geq p} N_k^{-1} \leq N_p^{-\sigma} \tag{7.53}$$

and so $(S3)_p$ implies

$$\bigcap_{k=1}^{p} \mathcal{G}^0_{N_k}(u_{k-1}) \subset \mathcal{G}_{N_p}(u_n)\,. \tag{7.54}$$

Assumption (ii) of Proposition 5.1 holds by (7.17), since $\chi \in [C_2, 2C_2)$.

When (7.49) holds the proof is analogous using Lemma 7.3 with $N = \bar N$ and $(S1)_n$. ■

Lemma 7.6. *Property $(S3)_{n+1}$ holds.*

Proof. We want to prove that

$$\|u - u_n\|_{s_1} \leq N_{n+1}^{-\sigma} \text{ and } (\varepsilon,\lambda) \in \bigcap_{k=1}^{n+1} \mathcal{G}^0_{N_k}(u_{k-1}) \quad \Longrightarrow \quad (\varepsilon,\lambda) \in \mathcal{G}_{N_{n+1}}(u)\,.$$

Since $(\varepsilon,\lambda) \in \mathcal{G}^0_{N_{n+1}}(u_n)$, by (6.3) and Definition 5.2 it is sufficient to prove that $\forall j_0 \in \mathbb{Z}^d$,

$$B_{N_{n+1}}(j_0;\varepsilon,\lambda)(u) \subset B^0_{N_{n+1}}(j_0;\varepsilon,\lambda)(u_n)\,,$$

(we highlight the dependence of these sets on u, u_n) or, equivalently, by (6.1), (5.2), that

$$\|A^{-1}_{N_{n+1},j_0}(\varepsilon,\lambda,\theta)(u_n)\|_0 \leq N_{n+1}^{\tau} \quad \Longrightarrow \quad A_{N_{n+1},j_0}(\varepsilon,\lambda,\theta)(u) \text{ is } N_{n+1} - \text{good}\,, \tag{7.55}$$

where $A(\varepsilon,\lambda,\theta)(u) = \mathcal{L}(u) + \theta Y = L_\omega + \theta Y - \varepsilon(Df)(u)$.

We prove (7.55) applying Proposition 4.1 to $A := A_{N_{n+1},j_0}(\varepsilon,\lambda,\theta)(u)$ with E defined in (7.50), $N' = N_{n+1}$, $N = N_p$ (resp. $N = \bar N$) if (7.48) (resp. (7.49)) is satisfied. Assumption (H1) holds with

$$\Upsilon \stackrel{(2.8),(7.13)}{=} C(1 + \|u_n\|_{s_1} + |V|_{s_1}) \stackrel{(S1)_n,(7.14)}{\leq} C'(V)\,. \tag{7.56}$$

By Lemma 7.5, for all $\theta \in \mathbb{R}$, $j_0 \in \mathbb{Z}^d$, the hypothesis (H3) of Proposition 4.1 holds for $A_{N_{n+1},j_0}(\varepsilon,\lambda,\theta)(u_n)$. Hence, by Proposition 4.1, for $s \in [s_0, s_1]$, if

$$\|A^{-1}_{N_{n+1},j_0}(\varepsilon,\lambda,\theta)(u_n)\|_0 \leq N_{n+1}^{\tau}$$

(which is assumption (H2)) then

$$|A^{-1}_{N_{n+1},j_0}(\varepsilon,\lambda,\theta)(u_n)|_s \leq \frac{1}{4} N_{n+1}^{\tau'}\Big(N_{n+1}^{\delta s} + |V|_s + \varepsilon|(Df)(u_n)|_s\Big)\,. \tag{7.57}$$

Finally, since $\|u - u_n\|_{s_1} \leq N_{n+1}^{-\sigma}$ we have

$$|A_{N_{n+1},j_0}(\varepsilon,\lambda,\theta)(u_n) - A_{N_{n+1},j_0}(\varepsilon,\lambda,\theta)(u)|_{s_1} \leq C\varepsilon\|u - u_n\|_{s_1} \leq N_{n+1}^{-\sigma}$$

and (7.55) follows by (7.57) and a standard perturbative argument (see for instance (3.26) in Lemma 3.9 with any $s \in [s_0, s_1]$ instead of s_0). ■

In order to define u_{n+1}, we write, for $h \in H_{n+1}$,

$$
\begin{aligned}
P_{n+1}\Big(L_\omega(u_n + h) - \varepsilon(f(u_n + h) + g)\Big) &= P_{n+1}\Big(L_\omega u_n - \varepsilon(f(u_n) + g)\Big) \\
&+ P_{n+1}\Big(L_\omega h - \varepsilon(Df)(u_n)h\Big) + R_n(h) \\
&= r_n + \mathcal{L}_{n+1}(u_n)h + R_n(h)
\end{aligned}
\tag{7.58}
$$

where $\mathcal{L}_{n+1}(u_n)$ is defined in (7.47) and

$$
r_n := P_{n+1}\Big(L_\omega u_n - \varepsilon(f(u_n) + g)\Big), \quad R_n(h) := -\varepsilon P_{n+1}\Big(f(u_n + h) - f(u_n) - (Df)(u_n)h\Big). \tag{7.59}
$$

By $(S4)_n$, if $(\varepsilon, \lambda) \in \mathcal{N}(\mathcal{C}_n, N_n^{-\sigma})$ then u_n solves the equation (P_n) and so

$$
r_n = P_{n+1}P_n^\perp\Big(L_\omega u_n - \varepsilon(f(u_n) + g)\Big) = P_{n+1}P_n^\perp\Big(V_0\, u_n - \varepsilon(f(u_n) + g)\Big), \tag{7.60}
$$

using also that $P_{n+1}P_n^\perp(D_\omega u_n) = 0$, see (2.7). Note that, by (7.2) and $\sigma \geq 2$ (see (7.20)), for $N_0 \geq 2$, we have the inclusion

$$
\mathcal{N}(\mathcal{C}_{n+1}, 2N_{n+1}^{-\sigma}) \subset \mathcal{N}(\mathcal{C}_n, N_n^{-\sigma}). \tag{7.61}
$$

Lemma 7.7. (Invertibility of $\mathcal{L}_{n+1}$) *For all $(\varepsilon, \lambda) \in \mathcal{N}(\mathcal{C}_{n+1}, 2N_{n+1}^{-\sigma})$ the operator $\mathcal{L}_{n+1}(u_n)$ is invertible and, for $s = s_1, S$,*

$$
|\mathcal{L}_{n+1}^{-1}(u_n)|_s \leq N_{n+1}^{\tau' + \delta s}. \tag{7.62}
$$

As a consequence, by (3.20), $\forall h \in H_{n+1}$,

$$
\|\mathcal{L}_{n+1}^{-1}(u_n)h\|_{s_1} \leq C(s_1)N_{n+1}^{\tau' + \delta s_1}\|h\|_{s_1}, \tag{7.63}
$$

$$
\|\mathcal{L}_{n+1}^{-1}(u_n)h\|_S \leq N_{n+1}^{\tau' + \delta s_1}\|h\|_S + C(S)N_{n+1}^{\tau' + \delta S}\|h\|_{s_1}. \tag{7.64}
$$

Proof. We give the proof when (7.48) holds. The other case is analogous. First assume $(\varepsilon, \lambda) \in \mathcal{C}_{n+1}$, see (7.23). Then since $(\varepsilon, \lambda) \in \mathtt{G}_{N_{n+1}}(u_n)$ (see (6.21) with $A_N(\varepsilon, \lambda) = \mathcal{L}_{n+1}(u_n)$), the operator $\mathcal{L}_{n+1}(u_n)$ is invertible and

$$
\|\mathcal{L}_{n+1}^{-1}(u_n)\|_0 \leq N_{n+1}^{\tau}. \tag{7.65}
$$

We now apply the multiscale Proposition 4.1 to $A := \mathcal{L}_{n+1}(u_n)$ with

$$
E := [-N_{n+1}, N_{n+1}]^b \times \{0, 1\}, \quad N' = N_{n+1}, \quad N = N_p, \text{ see (7.48)}.
$$

By remark 7.2 and since $\chi \in [C_2, 2C_2)$ (see (7.48)) the assumptions (4.3)-(4.5) hold. Assumption (H1) holds with (7.56). Assumption (H2) holds by (7.65). Moreover, by the definition of $\mathcal{C}_{n+1}$, as a particular case of Lemma 7.5 -for $\theta = 0$, $j_0 = 0$-, the hypothesis (H3) of Proposition 4.1 holds for $\mathcal{L}_{n+1}(u_n)$. Then Proposition 4.1 applies and we get that, $\forall(\varepsilon, \lambda) \in \mathcal{C}_{n+1}$, $\forall s \in \{s_1, S\}$,

$$
|\mathcal{L}_{n+1}^{-1}(u_n)|_s \overset{(4.7)}{\leq} \frac{1}{4}N_{n+1}^{\tau'}\Big(N_{n+1}^{\delta s} + |V|_s + \varepsilon|(Df)(u_n)|_s\Big),
$$

whence, for $s = s_1$,

$$
|\mathcal{L}_{n+1}^{-1}(u_n)|_{s_1} \overset{(7.13),(S1)_n,(7.14)}{\leq} \frac{1}{4}N_{n+1}^{\tau'}\Big(N_{n+1}^{\delta s_1} + |V|_{s_1} + \varepsilon C(s_1)\Big) \leq \frac{1}{2}N_{n+1}^{\tau' + \delta s_1} \tag{7.66}
$$

and, for $s = S$, recalling that $U_n := \|u_n\|_S$,

$$
\begin{aligned}
|\mathcal{L}_{n+1}^{-1}(u_n)|_S &\overset{(7.13),(7.14)}{\leq} \frac{1}{4}N_{n+1}^{\tau'}\Big(N_{n+1}^{\delta S} + |V|_S + \varepsilon C(S)(1 + U_n)\Big) \\
&\overset{(S5)_n}{\leq} \frac{1}{4}N_{n+1}^{\tau'}\Big(N_{n+1}^{\delta S} + C'(S)N_n^{2(\tau' + \delta s_1 + 1)}\Big) \leq \frac{1}{2}N_{n+1}^{\tau' + \delta S}
\end{aligned}
\tag{7.67}
$$

by (7.16) and $\delta = 1/4$. Assume next $(\varepsilon', \lambda') \in \mathcal{N}(\mathcal{C}_{n+1}, 2N_{n+1}^{-\sigma})$ and let $(\varepsilon, \lambda) \in \mathcal{C}_{n+1}$ be such that $|(\varepsilon', \lambda') - (\varepsilon, \lambda)| < 2N_{n+1}^{-\sigma}$. We write

$$\mathcal{L}_{n+1}(u_n(\varepsilon', \lambda')) = \mathcal{L}_{n+1}(u_n(\varepsilon, \lambda)) + \mathtt{R}_{n+1}$$

where $\mathcal{L}_{n+1}(u_n(\varepsilon, \lambda))$ satisfies (7.66)-(7.67) and

$$\mathtt{R}_{n+1} := \mathcal{L}_{n+1}(u_n(\varepsilon', \lambda')) - \mathcal{L}_{n+1}(u_n(\varepsilon, \lambda)) .$$

By (7.47), (7.13), (F2), (1.9), (7.21), $(S1)_n$, $(S5)_n$,

$$|\mathtt{R}_{n+1}|_{s_1} \leq C(s_1) N_{n+1}^{-\sigma+1} , \quad |\mathtt{R}_{n+1}|_S \leq C(S) N_n^{4\tau'+2s_1+4} N_{n+1}^{-\sigma} . \tag{7.68}$$

We apply Lemma 3.9 with

$$M = \mathcal{L}_{n+1}(u_n(\varepsilon, \lambda)) , \quad N = \mathcal{L}_{n+1}^{-1}(u_n(\varepsilon, \lambda)) , \quad P = \mathtt{R}_{n+1} .$$

By (7.66), (7.68) and (7.20) the perturbative assumption (3.25) holds with index s_1 instead of s_0. Then (3.26), (3.27) (with indices s_1, S instead of s_0, s) imply (7.62) for all $(\varepsilon', \lambda') \in \mathcal{N}(\mathcal{C}_{n+1}, 2N_{n+1}^{-\sigma})$, by (7.66), (7.67), (7.68), (7.20). ■

By (7.58), setting

$$F_{n+1} : H_{n+1} \to H_{n+1} , \qquad F_{n+1}(h) := -\mathcal{L}_{n+1}^{-1}(u_n)(r_n + R_n(h)) , \tag{7.69}$$

the equation (P_{n+1}) is equivalent to the fixed point problem $h = F_{n+1}(h)$.

Lemma 7.8. (Contraction in $\| \ \|_{s_1}$-norm) *$\forall (\varepsilon, \lambda) \in \mathcal{N}(\mathcal{C}_{n+1}, 2N_{n+1}^{-\sigma})$, F_{n+1} is a contraction in*

$$\mathtt{B}_{n+1}(s_1) := \Big\{ h \in H_{n+1} \ : \ \|h\|_{s_1} \leq \rho_{n+1} := N_{n+1}^{-\sigma-1} \Big\} . \tag{7.70}$$

The unique fixed point $\widetilde{h}_{n+1}(\varepsilon, \lambda)$ of F_{n+1} in $\mathtt{B}_{n+1}(s_1)$ belongs to $\mathcal{U}$ (see (1.13)) and satisfies

$$\|\widetilde{h}_{n+1}\|_{s_1} \leq K(S) N_{n+1}^{\tau'+\delta s_1} N_n^{-(S-s_1)} U_n . \tag{7.71}$$

Proof. For all $(\varepsilon, \lambda) \in \mathcal{N}(\mathcal{C}_{n+1}, 2N_{n+1}^{-\sigma})$, by (7.69) and (7.63), we have

$$\|F_{n+1}(h)\|_{s_1} \leq C(s_1) N_{n+1}^{\tau'+\delta s_1} (\|r_n\|_{s_1} + \|R_n(h)\|_{s_1}) \tag{7.72}$$

and r_n has the form (7.60) because of (7.61). Moreover (recall that $U_n := \|u_n\|_S$)

$$\begin{aligned} \|r_n\|_{s_1} + \|R_n(h)\|_{s_1} &\overset{(7.60),(7.5),(7.59),(7.12)}{\leq} N_n^{-(S-s_1)} (\|V_0 \, u_n\|_S + \varepsilon \|f(u_n)\|_S + \varepsilon \|g\|_S) + \varepsilon C(s_1) \|h\|_{s_1}^2 \\ &\overset{(7.9),(7.14)}{\leq} C(S) N_n^{-(S-s_1)} (U_n + 1) + \varepsilon \, C(s_1) \|h\|_{s_1}^2 \qquad (7.73) \\ &\overset{(S5)_n}{\leq} C(S) N_n^{-(S-s_1)} N_n^{2(\tau'+\delta s_1+1)} + \varepsilon \, C(s_1) \|h\|_{s_1}^2 . \qquad (7.74) \end{aligned}$$

(7.72) and (7.74) imply (using also (7.2)), for some $K(S), K(s_1) > 0$,

$$\begin{aligned} \|h\|_{s_1} \leq \rho_{n+1} \quad \Longrightarrow \quad \|F_{n+1}(h)\|_{s_1} &\leq K(S) N_{n+1}^{2(\tau'+\delta s_1)+1} N_n^{-(S-s_1)} + \varepsilon K(s_1) N_{n+1}^{\tau'+\delta s_1} \rho_{n+1}^2 \\ &\leq \rho_{n+1} := N_{n+1}^{-\sigma-1} , \end{aligned}$$

because the choice of S in (7.16) and of σ in (7.20) imply (for $N \geq N_0(S)$)

$$K(S) N_{n+1}^{2(\tau'+\delta s_1)+1} N_n^{-(S-s_1)} \leq \frac{\rho_{n+1}}{2} , \quad \varepsilon K(s_1) N_{n+1}^{\tau'+\delta s_1} \rho_{n+1} \leq \frac{1}{2} . \tag{7.75}$$

Next, differentiating (7.69) with respect to h and using (7.59) we get

$$D_h F_{n+1}(h)[v] = \mathcal{L}_{n+1}^{-1}(u_n)\varepsilon P_{n+1}\Big((Df)(u_n+h)[v] - (Df)(u_n)[v]\Big)$$

and, for all $\|h\|_{s_1} \le \rho_{n+1}$, using (7.10) with $s = s_1$,

$$\|D_h F_{n+1}(h)[v]\|_{s_1} \overset{(7.63)}{\le} \varepsilon K(s_1) N_{n+1}^{\tau'+\delta s_1}\rho_{n+1}\|v\|_{s_1} \overset{(7.75)}{\le} \frac{1}{2}\|v\|_{s_1}\,.$$

Hence F_{n+1} is a contraction in $\mathtt{B}_{n+1}(s_1)$. Since $u_n \in \mathcal{U}$, it is easy to check that F_{n+1} leaves $B_{n+1}(s_1)\cap\mathcal{U}$ invariant, hence $\widetilde{h}_{n+1} \in \mathcal{U}$. Finally, (7.69), (7.72), (7.73) and (7.75) imply (7.71). ■

Since $\widetilde{h}_{n+1}(\varepsilon,\lambda)$ solves, for all $(\varepsilon,\lambda) \in \mathcal{N}(\mathcal{C}_{n+1}, 2N_{n+1}^{-\sigma})$, the equation

$$Q_{n+1}(\varepsilon,\lambda,h) := P_{n+1}\Big(L_\omega(u_n+h) - \varepsilon\big(f(u_n+h)+g\big)\Big) = 0\,, \quad h \in H_{n+1}\,, \tag{7.76}$$

and $u_n(0,\lambda) \overset{(S1)_n}{=} 0$, we deduce, by the uniqueness of the fixed point, that

$$\widetilde{h}_{n+1}(0,\lambda) = 0\,, \quad \forall (0,\lambda) \in \mathcal{N}(\mathcal{C}_{n+1}, 2N_{n+1}^{-\sigma})\,.$$

Lemma 7.9. (Estimate in high norm) *$\forall(\varepsilon,\lambda) \in \mathcal{N}(\mathcal{C}_{n+1}, 2N_{n+1}^{-\sigma})$ we have*

$$\|\widetilde{h}_{n+1}\|_S \le K(S) N_{n+1}^{\tau'+\delta s_1} U_n\,. \tag{7.77}$$

PROOF. We have

$$\begin{aligned}
\|\widetilde{h}_{n+1}\|_S &\overset{(7.69)}{=} \Big\|\mathcal{L}_{n+1}^{-1}(u_n)(r_n + R_n(\widetilde{h}_{n+1}))\Big\|_S \qquad (7.78)\\
&\overset{(7.64)}{\le} N_{n+1}^{\tau'+\delta s_1}\Big(\|r_n\|_S + \|R_n(\widetilde{h}_{n+1})\|_S\Big) + C(S) N_{n+1}^{\tau'+\delta S}\Big(\|r_n\|_{s_1} + \|R_n(\widetilde{h}_{n+1})\|_{s_1}\Big)\,.
\end{aligned}$$

Now, by (7.60), $(S1)_n$, (F2), (F3), (7.14), (7.8), (7.59), and setting $U_n := \|u_n\|_S$ (we can suppose $U_n \ge 1$) we get

$$\|r_n\|_S + \|R_n(\widetilde{h}_{n+1})\|_S \le C(S)(U_n + \varepsilon\rho_{n+1}\|\widetilde{h}_{n+1}\|_S) \tag{7.79}$$

and, using also (7.73), (7.71) and the second inequality in (7.75),

$$\|r_n\|_{s_1} + \|R_n(\widetilde{h}_{n+1})\|_{s_1} \le C(S) N_n^{-(S-s_1)} U_n\,. \tag{7.80}$$

Then (7.78), (7.79), (7.80) imply that

$$\begin{aligned}
\|\widetilde{h}_{n+1}\|_S &\le C(S)\Big(N_{n+1}^{\tau'+\delta s_1} + N_{n+1}^{\tau'+\delta S} N_n^{-(S-s_1)}\Big)U_n + C(S)\varepsilon N_{n+1}^{\tau'+\delta s_1}\rho_{n+1}\|\widetilde{h}_{n+1}\|_S \qquad (7.81)\\
&\overset{(7.16),(7.70)}{\le} C'(S) N_{n+1}^{\tau'+\delta s_1} U_n + \varepsilon C(S) N_{n+1}^{\tau'+\delta s_1 - \sigma - 1}\|\widetilde{h}_{n+1}\|_S\\
&\overset{(7.20)}{\le} C'(S) N_{n+1}^{\tau'+\delta s_1} U_n + \frac{1}{2}\|\widetilde{h}_{n+1}\|_S
\end{aligned}$$

for $\varepsilon_0 \le \varepsilon_0(S)$ small. As a consequence we get $\|\widetilde{h}_{n+1}\|_S \le 2C'(S) N_{n+1}^{\tau'+\delta s_1} U_n$ and (7.77) follows. ■

Lemma 7.10. (Estimate of the derivatives) *The map $\widetilde{h}_{n+1} \in C^1(\mathcal{N}(\mathcal{C}_{n+1}, 2N_{n+1}^{-\sigma}), H_{n+1})$ and*

$$\|\partial_{(\varepsilon,\lambda)}\widetilde{h}_{n+1}\|_{s_1} \le N_{n+1}^{-1}\,, \quad \|\partial_{(\varepsilon,\lambda)}\widetilde{h}_{n+1}\|_S \le N_{n+1}^{\tau'+\delta s_1+1}\Big(N_{n+1}^{\tau'+\delta s_1+1} U_n + U'_n\Big)\,. \tag{7.82}$$

PROOF. For all $(\varepsilon,\lambda)\in\mathcal{N}(\mathcal{C}_{n+1},2N_{n+1}^{-\sigma})$, $\widetilde{h}_{n+1}(\varepsilon,\lambda)$ is a solution of $Q_{n+1}(\varepsilon,\lambda,\widetilde{h}_{n+1}(\varepsilon,\lambda))=0$, see (7.76). We have, see (7.47),

$$D_hQ_{n+1}(\varepsilon,\lambda,\widetilde{h}_{n+1})=\mathcal{L}_{n+1}(u_n+\widetilde{h}_{n+1})=\mathcal{L}_{n+1}(u_n)-\varepsilon P_{n+1}\Big((Df)(u_n+\widetilde{h}_{n+1})-(Df)(u_n)\Big) \tag{7.83}$$

which is invertible by Lemma 3.9 applied with

$$M\to\mathcal{L}_{n+1}(u_n)\,,P\to-\varepsilon P_{n+1}((Df)(u_n+\widetilde{h}_{n+1})-(Df)(u_n))\,,\ s_0\to s_1\,.$$

Indeed the hypothesis (3.25) follows from (7.62) with $s=s_1$, (F1), $(S1)_n$, Lemma 3.1, $\|\widetilde{h}_{n+1}\|_{s_1}\le\rho_{n+1}$ and (7.75). Therefore Lemma 3.9 with $s=s_1$ implies

$$\Big|\mathcal{L}_{n+1}^{-1}(u_n+\widetilde{h}_{n+1})\Big|_{s_1}\stackrel{(3.26)}{\le}2|\mathcal{L}_{n+1}^{-1}(u_n)|_{s_1}\stackrel{(7.62)}{\le}2N_{n+1}^{\tau'+\delta s_1} \tag{7.84}$$

and, by (3.28), (7.62) with $s=S$, (7.77), $(S5)_n$, (7.10), $\delta=1/4$, (7.16),

$$\Big|\mathcal{L}_{n+1}^{-1}(u_n+\widetilde{h}_{n+1})\Big|_S\le C(S)N_{n+1}^{\tau'+\delta S}\,. \tag{7.85}$$

Hence, the Implicit function theorem implies $\widetilde{h}_{n+1}\in C^1(\mathcal{N}(\mathcal{C}_{n+1},2N_{n+1}^{-\sigma}),H_{n+1})$ and

$$\partial_{(\varepsilon,\lambda)}\widetilde{h}_{n+1}\stackrel{(7.83)}{=}-\mathcal{L}_{n+1}^{-1}(u_n+\widetilde{h}_{n+1})\Big(\partial_{(\varepsilon,\lambda)}Q_{n+1}\Big)(\varepsilon,\lambda,\widetilde{h}_{n+1})\,. \tag{7.86}$$

By $(S4)_n$, $u_n(\varepsilon,\lambda)$ solves (P_n) for $(\varepsilon,\lambda)\in\mathcal{N}(\mathcal{C}_{n+1},2N_{n+1}^{-\sigma})\stackrel{(7.61)}{\subset}\mathcal{N}(\mathcal{C}_n,N_n^{-\sigma})$. Then

$$\begin{aligned}(\partial_\varepsilon Q_{n+1})(\varepsilon,\lambda,\widetilde{h}_{n+1})&=P_{n+1}P_n^{\perp}(V_0\,\partial_\varepsilon u_n)+P_n(f(u_n)+g)-P_{n+1}(f(u_n+\widetilde{h}_{n+1})+g)\\&+\ \varepsilon P_n(Df)(u_n)\partial_\varepsilon u_n-\varepsilon P_{n+1}(Df)(u_n+\widetilde{h}_{n+1})\partial_\varepsilon u_n\end{aligned} \tag{7.87}$$

(we use also that $P_{n+1}P_n^{\perp}(D_\omega u_n)=0$ since $u_n\in H_n$, see (2.7)) and

$$\begin{aligned}(\partial_\lambda Q_{n+1})(\varepsilon,\lambda,\widetilde{h}_{n+1})&=P_{n+1}P_n^{\perp}(V_0\,\partial_\lambda u_n)+(\partial_\lambda L_\omega)\widetilde{h}_{n+1}\\&+\ \varepsilon P_n(Df)(u_n)\partial_\lambda u_n-\varepsilon P_{n+1}(Df)(u_n+\widetilde{h}_{n+1})\partial_\lambda u_n\,.\end{aligned} \tag{7.88}$$

We deduce from (7.84)-(7.88) the estimates (7.82) using also (3.20), (F1), (F2), (F3), $(S1)_n$, (7.5), $(S5)_n$, (7.14), (7.16), (7.71), (7.77). We omit the details. ■

We now define a C^1-extension of $(\widetilde{h}_{n+1})_{|\mathcal{C}_{n+1}}$ onto the whole $[0,\varepsilon_0]\times\Lambda$.

Lemma 7.11. (Extension) *There is $h_{n+1}\in C^1([0,\varepsilon_0)\times\Lambda,H_{n+1}\cap\mathcal{U})$ satisfying $h_{n+1}(0,\lambda)=0$,*

$$\|h_{n+1}\|_{s_1}\le N_{n+1}^{-\sigma-1}\,,\quad\|\partial_{(\varepsilon,\lambda)}h_{n+1}\|_{s_1}\le N_{n+1}^{-1/2} \tag{7.89}$$

and h_{n+1} is equal to $\widetilde{h}_{n+1}$ on $\mathcal{N}(\mathcal{C}_{n+1},N_{n+1}^{-\sigma})$.

PROOF. Let

$$h_{n+1}(\varepsilon,\lambda):=\begin{cases}\psi_{n+1}(\varepsilon,\lambda)\widetilde{h}_{n+1}(\varepsilon,\lambda) & \text{if}\quad(\varepsilon,\lambda)\in\mathcal{N}(\mathcal{C}_{n+1},2N_{n+1}^{-\sigma})\\0 & \text{if}\quad(\varepsilon,\lambda)\notin\mathcal{N}(\mathcal{C}_{n+1},2N_{n+1}^{-\sigma})\end{cases} \tag{7.90}$$

where ψ_{n+1} is a C^∞ cut-off function satisfying

$$0\le\psi_{n+1}\le1\,,\quad\psi_{n+1}\equiv\begin{cases}1 & \text{if }(\varepsilon,\lambda)\in\mathcal{N}(\mathcal{C}_{n+1},N_{n+1}^{-\sigma})\\0 & \text{if }(\varepsilon,\lambda)\notin\mathcal{N}(\mathcal{C}_{n+1},2N_{n+1}^{-\sigma})\end{cases}\quad\text{and}\quad|\partial_{(\varepsilon,\lambda)}\psi_{n+1}|\le N_{n+1}^{\sigma}C\,.$$

Then $\|h_{n+1}\|_{s_1} \leq \|\widetilde{h}_{n+1}\|_{s_1} \leq N_{n+1}^{-\sigma-1}$ by Lemma 7.8, and,

$$\|\partial_{(\varepsilon,\lambda)} h_{n+1}\|_{s_1} \leq |\partial_{(\varepsilon,\lambda)} \psi_{n+1}| \, \|\widetilde{h}_{n+1}\|_{s_1} + \|\partial_{(\varepsilon,\lambda)} \widetilde{h}_{n+1}\|_{s_1} \leq N_{n+1}^{-1/2}$$

thanks to the first estimate in (7.82), and for N_0 large. ∎

Finally we define $u_{n+1} \in C^1([0,\varepsilon_0) \times \Lambda, H_{n+1} \cap \mathcal{U})$ as

$$u_{n+1} := u_n + h_{n+1} \,. \tag{7.91}$$

By Lemma 7.11, on $\mathcal{N}(\mathcal{C}_{n+1}, N_{n+1}^{-\sigma})$ we have $h_{n+1} = \widetilde{h}_{n+1}$ that solves equation (7.76) and so u_{n+1} solves equation (P_{n+1}). Hence $(S4)_{n+1}$ holds. By Lemma 7.11, property $(S2)_{n+1}$ holds. Property $(S1)_{n+1}$ follows as well because

$$\|u_{n+1}\|_{s_1} \leq \|u_0\|_{s_1} + \sum_{k=1}^{n+1} \|h_k\|_{s_1} \stackrel{(7.36),(S2)_{n+1}}{\leq} \frac{1}{2} + \sum_{k=1}^{n+1} N_k^{-\sigma-1} \leq \frac{1}{2} + N_1^{-1} \leq 1$$

and the estimate $\|\partial_{(\varepsilon,\lambda)} u_{n+1}\|_{s_1} \leq C(s_1) N_0^{\tau_1+s_1+1}\gamma^{-1}$ follows in the same way.

Lemma 7.12. *Property $(S5)_{n+1}$ holds.*

Proof. By the definition of U_n, and since $\|h_{n+1}\|_S \leq \|\widetilde{h}_{n+1}\|_S$, we get

$$U_{n+1} \leq U_n + \|\widetilde{h}_{n+1}\|_S \stackrel{(7.77)}{\leq} K'(S) N_{n+1}^{\tau'+\delta s_1} U_n \stackrel{(S5)_n}{\leq} K'(S) N_{n+1}^{\tau'+\delta s_1} N_n^{2(\tau'+\delta s_1+1)} \stackrel{(7.2)}{\leq} N_{n+1}^{2(\tau'+\delta s_1+1)} \,.$$

The estimate for U'_{n+1} follows similarly by (7.77), (7.82), $(S5)_n$. ∎

7.3 Proof of Theorem 1.1

By Theorem 7.1 it remains to prove that the measure estimate (1.10) holds.

Lemma 7.13. *The set $\mathcal{G}$ defined in* (7.19) *satisfies*

$$|\bar{\mathcal{G}}| = 1 - O(\gamma) \,. \tag{7.92}$$

Proof. The λ such that (7.19) is violated are

$$\bar{\mathcal{G}}^c \cap [1/2, 3/2] \subseteq \bigcup_{|l| \leq N_0, |j| \leq N_0} \mathcal{R}_{l,j} \quad \text{where} \quad \mathcal{R}_{l,j}^{\pm} := \Big\{ \lambda \in [1/2,3/2] \, : \, | \pm \lambda \bar{\omega} \cdot l + \mu_j | < \frac{\gamma}{N_0^{\tau_1}} \Big\} \,. \tag{7.93}$$

Dividing by λ, we have to estimate the $\xi := 1/\lambda \in [2/3, 2]$ such that

$$| \pm \bar{\omega} \cdot l + \xi \mu_j | < C \frac{\gamma}{N_0^{\tau_1}} \,.$$

The derivative of the functions $g_{lj}^{\pm}(\xi) := \pm \bar{\omega} \cdot l + \xi \mu_j$ satisfies $|\partial_\xi g_{lj}^{\pm}(\xi)| = |\mu_j| \geq \beta_0 > 0$, because $\Pi_0(-\Delta + V(x))_{|E_0} \geq \beta_0 I$ by (1.3). As a consequence, we estimate

$$|\mathcal{R}_{l,j}^{\pm}| \leq \frac{C}{\beta_0} \frac{\gamma}{N_0^{\tau_1}} \,. \tag{7.94}$$

Then (7.93), (7.94), imply

$$|\bar{\mathcal{G}}^c \cap [1/2, 3/2]| \leq \sum_{|l| \leq N_0, |j| \leq N_0, \pm} |\mathcal{R}_{l,j}^{\pm}| \leq C \frac{\gamma}{\beta_0} \frac{N_0^{d+\nu}}{N_0^{\tau_1}} = O(\gamma)$$

since $\tau_1 \geq d + \nu$. ∎

Finally we choose

$$\gamma := \varepsilon_0^\alpha \quad \text{with} \quad \alpha := 1/(S+1)\,, \quad N_0 := 4\gamma^{-1}\,, \tag{7.95}$$

so that (7.21) is fulfilled for ε_0 small enough. The complementary set of $\mathcal{C}_\infty$ in $[0,\varepsilon_0]\times\Lambda$ has measure

$$\begin{aligned}
|\mathcal{C}_\infty^c| &\overset{(7.25),(7.23)}{=} \Big|\bigcup_{k\geq1} \mathtt{G}^c_{N_k}(u_{k-1}) \bigcup_{k\geq1} (\mathcal{G}^0_{N_k}(u_{k-1}))^c \bigcup \Big([0,\varepsilon_0]\times\bar{\mathcal{G}}^c\Big)\Big| \\
&\leq \sum_{k\geq1} |\mathtt{G}^c_{N_k}(u_{k-1})| + \sum_{k\geq1} |(\mathcal{G}^0_{N_k}(u_{k-1}))^c| + \varepsilon_0|\bar{\mathcal{G}}^c| \\
&\overset{(6.22),(6.5),(7.17),(7.92)}{\leq} C\varepsilon_0 \sum_{k\geq1} N_k^{-1} + C\varepsilon_0\gamma \leq C\varepsilon_0(N_0^{-1}+\gamma) \overset{(7.95)}{\leq} C\varepsilon_0^{1+\alpha}
\end{aligned}$$

implying (1.10).

Theorem (1.1) is proved with $s(d,\nu) := s_1$ defined in (7.16) and $q(d,\nu) := S+3$, see (7.8).

Regularity

Finally, we prove that, if $V, f, g,$ are C^∞ then the solution $u(\varepsilon,\lambda)$ is in $C^\infty(\mathbb{T}^d\times\mathbb{T}^\nu)$. The argument is the one of Theorem 3 in [4]. The main point is the proof of the following lemma which gives an a-priori bound for the divergence of the Sobolev high norms of the approximate solutions u_n, extending property $(S5)_n$. Its proof requires only small modifications in Lemmata 7.7, 7.9, 7.12.

Lemma 7.14. *$\forall S' \geq S$,*

$$\|u_n\|_{S'} \leq C(S') N_n^{2(\tau'+\delta s_1+1)}\,. \tag{7.96}$$

Proof. First of all, by the arguments of Lemma 7.7, we get, the estimate

$$|\mathcal{L}^{-1}_{n+1}(u_n)|_{S'} \leq C(S')\Big(N_{n+1}^{\tau'+\delta S'} + N_{n+1}^{\tau'}\|u_n\|_{S'}\Big)\,. \tag{7.97}$$

Note that the multiscale Proposition 4.1 is valid for any $S' > s_1$, see (4.5). It requires also the condition $N \geq N_0(\Upsilon, S')$ which is verified for $N = N_n$ with $n \geq n_0(S')$ large enough.

Then, following the proof of Lemma 7.9 we obtain

$$\begin{aligned}
\|\widetilde{h}_{n+1}\|_{S'} &\leq N_{n+1}^{\tau'+\delta s_1}\Big(\|r_n\|_{S'} + \|R_n(\widetilde{h}_{n+1})\|_{S'}\Big) \\
&+ C(S')\Big(N_{n+1}^{\tau'+\delta S'} + N_{n+1}^{\tau'}\|u_n\|_{S'}\Big)\Big(\|r_n\|_{s_1} + \|R_n(\widetilde{h}_{n+1})\|_{s_1}\Big)\,.
\end{aligned} \tag{7.98}$$

We also have the analogue of (7.79)-(7.80), namely

$$\|r_n\|_{S'} + \|R_n(\widetilde{h}_{n+1})\|_{S'} \leq C(S')(\|u_n\|_{S'} + \varepsilon\rho_{n+1}\|\widetilde{h}_{n+1}\|_{S'})\,,$$

$$\|r_n\|_{s_1} + \|R_n(\widetilde{h}_{n+1})\|_{s_1} \leq C(S')N_n^{-(S'-s_1)}\|u_n\|_{S'}\,,$$

and, by (7.98), we deduce the analogue of (7.81), namely

$$\|\widetilde{h}_{n+1}\|_{S'} \leq C(S')N_{n+1}^{\tau'+\delta s_1}\|u_n\|_{S'} + C(S')N_{n+1}^{\tau'}N_n^{-(S'-s_1)}\|u_n\|^2_{S'} + \varepsilon C(S')N_{n+1}^{\tau'+\delta s_1}\rho_{n+1}\|\widetilde{h}_{n+1}\|_{S'}\,. \tag{7.99}$$

For $n \geq n_0(S')$ large enough,

$$\varepsilon C(S')N_{n+1}^{\tau'+\delta s_1}\rho_{n+1} \overset{(7.70)}{=} \varepsilon C(S')N_{n+1}^{\tau'+\delta s_1-\sigma-1} \overset{(7.20)}{\leq} \frac{1}{2}$$

and (7.99), (7.16) imply the analogue of (7.77), namely

$$\|\widetilde{h}_{n+1}\|_{S'} \leq K(S')N_{n+1}^{\tau'+\delta s_1}\|u_n\|_{S'} + K(S')N_{n+1}^{\tau'}N_n^{-(S'-s_1)}\|u_n\|^2_{S'}\,. \tag{7.100}$$

Of course, h_{n+1} defined in (7.90) satisfies (7.100) as well. Therefore, as in Lemma 7.12,

$$\|u_{n+1}\|_{S'} \le \|u_n\|_{S'} + \|h_{n+1}\|_{S'} \le 2K(S')N_{n+1}^{\tau'+\delta s_1}\|u_n\|_{S'} + K(S')N_{n+1}^{\tau'}N_n^{-(S'-s_1)}\|u_n\|_{S'}^2$$

and we deduce that the sequence $\|u_{n+1}\|_{S'}N_{n+1}^{-2(\tau'+\delta s_1+1)}$ is bounded, i.e. (7.96). ■

By (7.96) we deduce

$$\|h_n\|_{S'} \le K(S')N_n^{2(\tau_1+\delta s_1+1)}\,. \tag{7.101}$$

Now, consider *any* $s > s_1$ and write $s := (1-t)s_1 + tS'$ where $S' > s$, $t \in (0,1)$. By interpolation

$$\|h_n\|_s \le K(s_1,S')\|h_n\|_{s_1}^{1-t}\|h_n\|_{S'}^{t} \overset{(7.70),(7.101)}{\le} K(S')N_n^{-(\sigma+1)(1-t)}N_n^{\alpha t} = K(S')N_n^{-1} \tag{7.102}$$

having set $\alpha := 2(\tau_1 + \delta s_1 + 1)$, and choosing S' (large) such that

$$t = \frac{s-s_1}{S'-s_1} = \frac{\sigma+2}{\sigma+1+\alpha}\,.$$

In conclusion, (7.102) implies that $\sum_n \|h_n\|_s < +\infty$ and so $u(\varepsilon,\lambda) \in \mathbf{H}^s$, for *any* s.

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