Sobolev quasi periodic solutions of multidimensional wave equations with a multiplicative potential

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Abstract: We prove the existence of quasi-periodic solutions for wave equations with a multiplicative potential on $\mathbb{T}^d$, $d \geq 1$, and finitely differentiable nonlinearities, quasi-periodically forced in time. The only external parameter is the length of the frequency vector. The solutions have Sobolev regularity both in time and space. The proof is based on a Nash-Moser iterative scheme as in [5]. The key tame estimates for the inverse linearized operators are obtained by a multiscale inductive argument, which is more difficult than for NLS due to the dispersion relation of the wave equation. We prove the "separation properties" of the small divisors assuming weaker non-resonance conditions than in [11].

Keywords: Nonlinear wave equation, Nash-Moser Theory, KAM for PDE, Quasi-Periodic Solutions, Small Divisors, Infinite Dimensional Hamiltonian Systems.

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1 Introduction

The first existence results of quasi-periodic solutions for Hamiltonian PDE were proved by Kuksin [18] and Wayne [26] for one dimensional (1-d) nonlinear wave (NLW) and nonlinear Schrödinger (NLS) equations, extending KAM theory. This approach consists in generating iteratively a sequence of canonical changes of variables which bring the Hamiltonian into a normal form with an invariant torus at the origin. This procedure requires, at each step, to invert linear "homological equations", which have constant coefficients and can be solved by imposing the "second order Melnikov" non-resonance conditions. The final KAM torus is linearly stable. These pioneering results were limited to Dirichlet boundary conditions because the eigenvalues of ∂_{xx} had to be simple: the second order Melnikov non resonance conditions are violated already for periodic boundary conditions.

In such a case, the first existence results of quasi-periodic solutions were proved by Bourgain [8] extending the approach of Craig-Wayne [14] for periodic solutions. The search of the embedded torus is reduced to solving a functional equation in scales of Banach spaces, by some Newton implicit function procedure. The main advantage of this scheme is to require only the "first order Melnikov" non-resonance conditions to solve the homological equations. These conditions are essentially the minimal non-resonance assumptions. Translated in the KAM language this corresponds to allow a normal form with non-constant coefficients around the torus. The main difficulty is that the homological equations are PDEs with *non-constant* coefficients and are small perturbations of a diagonal operator having arbitrarily small eigenvalues.

At present, the theory for 1-d NLS and NLW equations has been sufficiently understood (see e.g. [19], [21], [20], [22], [13], [1]) but much work remains in higher space dimensions. The main difficulties are:

1. the eigenvalues of $-\Delta + V(x)$ appear in clusters of unbounded sizes,

2. the eigenfunctions are "not localized with respect to the exponentials".

Roughly speaking, an eigenfunction ψ_j of $-\Delta + V(x)$ is localized with respect to the exponentials, if its Fourier coefficients $(\hat{\psi}_j)_i$ rapidly converge to zero (when $|i-j| \to \infty$). This property always holds in 1 space dimension (see [14]) but may fail for $d \geq 2$, see [10]. It implies that the matrix which represents (in the eigenfunctions basis) the multiplication operator for an analytic function has an exponentially fast decay off the diagonal. It reflects into a "weak interaction" between different "clusters of small divisors". Problem 2 has been often bypassed replacing the multiplicative potential $V(x)$ by a "convolution potential" $V * (e^{\mathrm{i} j \cdot x}) := m_j e^{\mathrm{i} j \cdot x}$, $m_j \in \mathbb{R}$, $j \in \mathbb{Z}^d$. The "Fourier multipliers" m_j play the role of "external parameters".

The first existence results of quasi-periodic solutions for analytic NLS and NLW like

$$\frac{1}{\mathrm{i}} u_t = Bu + \varepsilon \partial_{\bar{u}} H(u, \bar{u}) \,, \quad u_{tt} + B^2 u + \varepsilon F'(u) = 0 \,, \quad x \in \mathbb{T}^d \,, \ \ d \geq 2 \,, \tag{1.1}$$

where B is a Fourier multiplier, have been proved by Bourgain [10], [11], by extending the Newton approach in [8] (see also [9] for periodic solutions). Actually this scheme is very convenient to overcome problem 1, because it requires only the first order Melnikov non-resonance conditions and therefore does not exclude multiplicity of normal frequencies (eigenvalues). The main difficulty concerns the multiscale inductive argument to estimate the off diagonal exponential decay of the inverse linearized operators in presence of huge clusters of small divisors. The proof is based on a repeated use of the resolvent identity and fine techniques of subharmonicity and semi-algebraic set theory, essentially to obtain refined measure and "complexity" estimates for sublevels of functions.

Also the KAM approach has been recently extended by Eliasson-Kuksin [15] for NLS on $\mathbb{T}^d$ with Fourier multipliers and analytic nonlinearities. The key issue is to control more accurately the perturbed frequencies after the KAM iteration and, in this way, verify the second order Melnikov non-resonance conditions, we refer also to [17], [23], [2] for related techniques. We also mention [16] which proves the reducibility of a linear Schrödinger equation forced by a small multiplicative potential, quasi-periodic in time.

On the other hand, a similar reducibility KAM result for NLW on $\mathbb{T}^d$ is still an open problem: the possibility of imposing the second order Melnikov conditions for wave equations in higher space dimensions is still uncertain.

In the recent paper [5] we proved the existence of quasi-periodic solutions for quasi-periodically forced NLS on $\mathbb{T}^d$ with finitely differentiable nonlinearities (all the previous results were valid for analytic nonlinearities, actually polynomials in [10], [11]) and a multiplicative potential $V(x)$ (not small). Clearly a difficulty is that the matrix which represents the multiplication operator has only a polynomial decay off the diagonal, and not exponential. The proof is based on a Nash-Moser iterative scheme in Sobolev scales (developed for periodic solutions also in [4], [3], [6], [7]) and novel techniques for estimating the high Sobolev norms of the solutions of the (non-constant coefficients) homological equations. In particular we assumed that $-\Delta + V(x) > 0$ in order to prove the "measure and complexity" estimates by means of elementary eigenvalue variations arguments, avoiding subharmonicity and semi-algebraic techniques as in [11].

The goal of this paper is to prove an analogous result -see Theorem 1.1- for d-dimensional nonlinear wave equations with a quasiperiodic-in-time nonlinearity like

$$u_{tt} - \Delta u + V(x) u = \varepsilon f(\omega t, x, u) \,, \quad x \in \mathbb{T}^d \,, \ \ \varepsilon > 0 \,, \tag{1.2}$$

where the multiplicative potential V is in $C^q(\mathbb{T}^d; \mathbb{R})$, $\omega \in \mathbb{R}^\nu$ is a non-resonant frequency vector (see (1.7), (1.8)), and

$$f \in C^q(\mathbb{T}^\nu \times \mathbb{T}^d \times \mathbb{R}; \mathbb{R}) \tag{1.3}$$

for some $q \in \mathbb{N}$ large enough (fixed in Theorem 1.1). The NLW equation is more difficult than NLS because the singular sites, namely the integers $(l, j) \in \mathbb{Z}^\nu \times \mathbb{Z}^d$ such that ($\Theta > 0$ being fixed)

$$| -(\omega \cdot l)^2 + j_1^2 + \ldots + j_d^2 + m | \leq \Theta \,, \quad m := (2\pi)^{-d} \int_{\mathbb{T}^d} V(x) dx \,, \tag{1.4}$$

stay "near a cone" and not a paraboloid as for NLS. Therefore it is harder to prove their "separation properties", see section 4. In this paper we use a non-resonance condition on ω which is weaker than in Bourgain [11], see remark 4.1. After the statement of Theorem 1.1 we explain in detail the main differences with respect to [11] and [5] (and other previous literature).

Concerning the potential we suppose that

$$\mathrm{Ker}(-\Delta + V(x)) = \{0\} \, . \tag{1.5}$$

Remark 1.1. *In [5] we assumed the stronger condition* $-\Delta+V(x) > 0$*. See comments after Theorem* 1.1*. Note that also in* (1.1) *the Fourier operator* $B^2 > 0$ *is positive.*

In (1.2) we use only one external parameter, namely the length of the frequency vector (time scaling). More precisely we assume that the frequency vector ω is co-linear with a fixed vector $\bar\omega \in \mathbb{R}^\nu$,

$$\omega = \lambda \bar\omega \, , \quad \lambda \in \Lambda := [1/2, 3/2] \subset \mathbb{R} \, , \quad |\bar\omega| \leq 1 \, , \tag{1.6}$$

where $\bar\omega$ is Diophantine, namely for some $\gamma_0 \in (0,1)$,

$$|\bar\omega \cdot l| \geq \frac{2\gamma_0}{|l|^\nu} \, , \quad \forall l \in \mathbb{Z}^\nu \setminus \{0\} \, , \tag{1.7}$$

and

$$\Big| \sum_{1 \leq i \leq j \leq \nu} \bar\omega_i \bar\omega_j p_{ij} \Big| \geq \frac{\gamma_0}{|p|^{\tau_0}} \, , \quad \forall p \in \mathbb{Z}^{\frac{\nu(\nu+1)}{2}} \setminus \{0\} \, . \tag{1.8}$$

There exists $\bar\omega$ satisfying (1.7) and (1.8) at least for $\tau_0 > \nu(\nu+1) - 1$ and γ_0 small, see Lemma 6.1. For definiteness we fix $\tau_0 := \nu(\nu+1)$.

Remark 1.2. *For NLS equations [5] only condition* (1.7) *is required, see comments after Theorem 1.1.*

The dynamics of the linear wave equation

$$u_{tt} - \Delta u + V(x) u = 0 \tag{1.9}$$

is well understood. The eigenfunctions of

$$(-\Delta + V(x))\psi_j(x) = \mu_j \psi_j(x)$$

form a Hilbert basis in $L^2(\mathbb{T}^d)$ and the eigenvalues $\mu_j \to +\infty$ as $j \to +\infty$. By assumption (1.5) all the eigenvalues μ_j are different from 0. We list them in non-decreasing order

$$\mu_1 \leq \ldots \leq \mu_{n^-} < 0 < \mu_{n^-+1} \leq \ldots \tag{1.10}$$

where n^- denotes the number of negative eigenvalues (counted with multiplicity).

All the solutions of (1.9) are the linear superpositions of normal mode oscillations, namely

$$u(t,x) = \sum_{j=1}^{n^-} (\beta_j^- e^{-\sqrt{|\mu_j|}t} + \beta_j^+ e^{\sqrt{|\mu_j|t}}) \psi_j(x) + \sum_{j \geq n^-+1} \mathrm{Re}(a_j e^{\mathrm{i}\sqrt{\mu_j}t}) \, \psi_j(x) \, , \ \beta_j^\pm \in \mathbb{R} \, , a_j \in \mathbb{C} \, .$$

The first n^- eigenfunctions correspond to hyperbolic directions where the dynamics is attractive/repulsive. The other infinitely many eigenfunctions correspond to elliptic directions.

- Question: for ε small enough, do there exist quasi-periodic solutions of the nonlinear wave equation (1.2) for positive measure sets of $\lambda \in [1/2, 3/2]$?

Note that, if $f(\varphi, x, 0) \not\equiv 0$ then $u = 0$ is not a solution of (1.2) for $\varepsilon \neq 0$.

The above question amounts to look for $(2\pi)^{d+\nu}$-periodic solutions $u(\varphi, x)$ of

$$(\omega \cdot \partial_\varphi)^2 u - \Delta u + V(x) u = \varepsilon f(\varphi, x, u) \tag{1.11}$$

in the Sobolev space

$$H^s := H^s(\mathbb{T}^\nu \times \mathbb{T}^d;\mathbb{R}) \ := \ \Big\{ u(\varphi,x) := \sum_{(l,j)\in\mathbb{Z}^\nu\times\mathbb{Z}^d} u_{l,j} e^{\mathrm{i}(l\cdot\varphi + j\cdot x)} \,:\, \|u\|_s^2 := K_0 \sum_{i\in\mathbb{Z}^{\nu+d}} |u_i|^2 \langle i\rangle^{2s} < +\infty\,,$$
$$u_{-i} = \overline{u_i}\ , \ \text{where} \ \ i := (l,j)\,,\ \langle i\rangle := \max(|l|,|j|,1) \Big\} \tag{1.12}$$

for some $(\nu+d)/2 < s \leq q$. Above $|j| := \max\{|j_1|,\dots,|j_d|\}$. For the sequel we fix $s_0 > (d+\nu)/2$ so that $H^s(\mathbb{T}^{\nu+d}) \hookrightarrow L^\infty(\mathbb{T}^{\nu+d})$, $\forall s \geq s_0$. The constant $K_0 > 0$ in (1.12) is fixed (large enough) so that $|u|_{L^\infty} \leq \|u\|_{s_0}$ and the interpolation inequality

$$\|u_1 u_2\|_s \leq \frac{1}{2}\|u_1\|_{s_0}\|u_2\|_s + \frac{C(s)}{2}\|u_1\|_s\|u_2\|_{s_0}\,, \quad \forall s \geq s_0\,,\ u_1,u_2 \in H^s\,, \tag{1.13}$$

holds with $C(s) \geq 1$, $\forall s \geq s_0$, and $C(s) = 1$, $\forall s \in [s_0, s_1]$; the constant $s_1 := s_1(d,\nu)$ is defined in (6.4).

The main result of the paper is:

Theorem 1.1. *Assume* (1.7)-(1.8). *There are* $s := s(d,\nu)$, $q := q(d,\nu) \in \mathbb{N}$, *such that:* $\forall f \in C^q$, $\forall V \in C^q$ *satisfying* (1.5), $\forall \varepsilon \in [0,\varepsilon_0)$ *small enough, there is a map*

$$u(\varepsilon,\cdot) \in C^1(\Lambda; H^s) \quad \text{with} \quad \sup_{\lambda\in\Lambda}\|u(\varepsilon,\lambda)\|_s \to 0 \text{ as } \varepsilon \to 0\,, \tag{1.14}$$

and a Cantor like set $\mathcal{C}_\varepsilon \subset \Lambda := [1/2, 3/2]$ *of asymptotically full Lebesgue measure, i.e.*

$$|\mathcal{C}_\varepsilon| \to 1 \quad \text{as} \quad \varepsilon \to 0, \tag{1.15}$$

such that, $\forall \lambda \in \mathcal{C}_\varepsilon$, $u(\varepsilon,\lambda)$ *is a solution of (1.11) with* $\omega = \lambda\bar\omega$. *Moreover, if* V, f *are of class* C^∞ *then* $\forall\lambda\,,\ u(\varepsilon,\lambda) \in C^\infty(\mathbb{T}^d \times \mathbb{T}^\nu;\mathbb{R})$.

Let us make some comments on the result.

1. Assumption (1.5) on the potential $V(x)$ is necessary in order to prove the existence result of Theorem 1.1 for *any* f. Actually, if there is an eigenfunction $\psi_0(x) \not\equiv 0$ such that $-\Delta\psi_0 + V\psi_0 = 0$, then equation (1.11) with nonlinearity $f(\varphi,x,u) := \psi_0(x)$ does not possess solutions. Indeed, multiplying (1.11) by ψ_0 and integrating in (φ,x) we get

$$0 = \varepsilon \int_{\mathbb{T}^\nu\times\mathbb{T}^d} f(\varphi,x,u(\varphi,x))\psi_0(x)\,d\varphi\,dx = \varepsilon\int_{\mathbb{T}^\nu\times\mathbb{T}^d}\psi_0^2(x)\,d\varphi\,dx \tag{1.16}$$

   which is impossible for $\varepsilon \neq 0$.

2. The main novelties of Theorem 1.1 with respect to previous results (which reduce essentially to [11], Chapter 20) are that we prove the existence of quasi-periodic solutions for quasi-periodically forced NLW on $\mathbb{T}^d$, $d \geq 2$, with a

   (i) *multiplicative* finitely differentiable potential $V(x)$,

   (ii) finitely *differentiable* nonlinearity, see (1.3),

   (iii) *pre-assigned* direction of the tangential frequencies, see (1.6).

   Moreover we weaken the non-resonance assumptions on ω which ensure the separation properties of the "bad" sites, see item 1 below.

   Theorem 1.1 generalizes [4] to the case of quasi-periodic solutions. We remark that the approaches developed in the previous papers [3], [4], [6], [7] for proving the tame estimates for the inverse linearized operator in the case of periodic solutions do NOT apply here. The main reason is that, for quasi-periodic solutions, the singular sites are NOT "separated at infinity", namely the distances between the integers $(l,j) \in \mathbb{Z}^\nu \times \mathbb{Z}^d$ such that (1.4) holds, do NOT increase when the Fourier indices tend to infinity. Hence the tame estimates for the inverses are obtained by an inductive multiscale approach (described shortly below).

3. The present Nash-Moser approach requires essentially no information about the localization of the eigenfunctions of $-\Delta + V(x)$ which, on the contrary, seem to be unavoidable to prove also reducibility with a KAM scheme, e.g. [15], [16]. Along the multiscale analysis we use (as in [5]) the exponential basis which diagonalizes $-\Delta + m$ where m is the average of $V(x)$. The key is to define "very regular" sites, namely take the constant Θ in (1.4) large enough (see also Definition 3.2), depending on the potential $V(x)$, see comments after Proposition 3.1. In this way the number of sites to be considered as "singular" increases. However, the separation properties of the singular sites obtained in Lemma 4.2 hold for any $\Theta > 0$, and this is sufficient for the applicability of the present multiscale approach.

4. Throughout this paper $\varepsilon \in [0, \varepsilon_0]$ is fixed (small) and $\lambda \in [1/2, 3/2]$ is the only external parameter in equation (1.2). Then the bound (1.15) is an improvement with respect to the analogous Theorem 1.1 in [5] (for NLS) where we only proved the existence of quasi-periodic solutions for a Cantor set, with asymptotically full measure, in the parameters $(\varepsilon, \lambda) \in [0, \varepsilon_0) \times [1/2, 3/2]$.

5. We have not tried to optimize the estimates for $q := q(d, \nu)$ and $s := s(d, \nu)$. In [3] we proved the existence of periodic solutions in $H^s_t H^1_x$ with $s > 1/2$, for one dimensional NLW equations with nonlinearities of class C^6, see the bounds (1.9), (4.28) in [3].

Let us make some comments about the proof, which is based on a general and systematic technique for estimating the inverses in high Sobolev norm of big matrices with polynomially off-diagonal decay (also called Green functions in Anderson localization theory).

Theorem 1.1 follows by an iterative procedure of Nash-Moser type (see section 6, Theorem 6.1) similar to the one used in [5]. Some parts are detailed in section 7 for the convenience of the reader (a minor difference with respect to [5] is that we argue for small fixed ε). One of the key points of this procedure is the inclusion (6.12) which, roughly, means that bounds in L^2-norm on the inverses of the linearized operators imply bounds in high norms for most values of the parameter λ. The multiscale Proposition 3.1, proved in [5], is the main tool for that. It uses assumption $(H3)$ about the separation of the "bad" sites, whose proof is the object of section 4. Section 5 is devoted to showing that, for most parameters λ, the required L^2-bounds for the inverse operators hold. Our measure estimates rely on Lemma 5.1 concerning the dependence of the eigenvalues of self-adjoint matrices with respect to a 1-dimensional parameter. Note that we can not provide directly good measure estimates to control the higher norms of the inverses, and this is why we repeatedly use the multiscale Proposition 3.1. Finally we remark that the subharmonicity and semi-algebraic techniques developed in [11], [12] (and references therein) for the measure and complexity estimates do not seem available in the present differentiable setting (the nonlinearities in [11], [12] are polynomials).

The main differences with respect to [5] which deals with the NLS equation are:

1. The proof of the separation properties of the bad sites (i.e. assumption $(H3)$ of Proposition 3.1) for the wave equation differs strongly from the one provided for the Schrödinger equation, due to the different form of the singular sites, see (1.4). The proof is inspired by the arguments in [11], but we use the non-resonance assumption (**NR**) (see (4.5), (1.8)), which is a Diophantine condition for polynomials in ω of degree 2, instead of the condition in [11] for polynomials of higher degree, see remark 4.1. A Diophantine condition like (**NR**) is necessary because the singular sites are integer points near a cone, see (4.10), and not a paraboloid as for NLS. Then it is necessary to assume an irrationality condition on the "slopes" of this cone. Assumption (**NR**) seems to be the weakest possible. The improvement is in the proof of Lemma 4.2 (different with respect to Lemma 20.14 of Bourgain [11]) which extends, to the quasi-periodic case, the arguments of [4]. We prove in Lemma 6.3 that, thanks to (1.8), condition (**NR**) holds for most $\lambda \in \Lambda$.

2. Since we do not assume that $-\Delta + V(x)$ is positive definite (as in [5]), but only the weaker assumption (1.5), the measure and complexity arguments in section 5 are more difficult than in [5], section 6. The main difference concerns Lemma 5.6 that we tackle with a Lyapunov-Schmidt type argument. This is possible because there is no small divisor associated to the negative eigenvalues μ_j of

$-\Delta + V(x)$: in fact (see (1.10))

$$-(\omega \cdot l)^2 + \mu_j \leq \mu_j \leq \mu_{n^-} < 0 \,, \quad \forall l \in \mathbb{Z}^\nu, \, j = 1, \ldots, n^- \,.$$

Note that Lemma 5.6 only holds for $j_0 \notin \mathcal{Q}_N$ defined in (3.7): in such a case the spectrum of the restricted operator $\Pi_{N,j_0}(-\Delta + V(x))_{E_{N,j_0}}$ in (5.22) is far away from zero by Lemma 2.3. This fact requires to modify also the definition of N-good sites, see Definition 3.4, with respect to the analogous Definition 5.1 of [5], see remark 3.2.

Finally we note that a technical simplification of the present approach with respect to [11], Chapter 20, is to study NLW in configuration space without regarding (1.2) as a first order Hamiltonian complex system. In this way we deal only with matrices of scalars and not of 2×2-matrices as in [5], [11]. The main difficulty working directly with the second order wave equation concerns the measure estimates: the derivative with respect to θ of the matrix in (2.7) is not positive definite (this affects Lemmata 5.3 and, especially, 5.6). The main technical trick that we use is the change of variables (5.20). We mention that also Bourgain-Wang [12], section 6, deals with NLW in configuration space, where the measure and complexity estimates are verified using subharmonicity and semi-algebraic techniques.

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2 The linearized equation

We look for solutions of the NLW equation (1.11) in H^s by means of a Nash-Moser iterative scheme. The main step concerns the invertibility of (any finite dimensional restriction of) the linearized operator

$$\mathcal{L}(u) := \mathcal{L}(\omega, \varepsilon, u) := L_\omega - \varepsilon g(\varphi, x) \tag{2.1}$$

where

$$L_\omega := (\omega \cdot \partial_\varphi)^2 - \Delta + V(x) \qquad \text{and} \qquad g(\varphi, x) := (\partial_u f)(\varphi, x, u) \,. \tag{2.2}$$

For the convergence of the Nash-Moser scheme (see sections 6-7) we need tame estimates for the inverse of (any finite dimensional restriction of) $\mathcal{L}(u)$ in high Sobolev norms (in particular (7.15)). For that, it is useful to work with the matrix representation of $\mathcal{L}(u)$. We decompose the multiplicative potential as

$$V(x) = m + V_0(x)$$

where m is the average of $V(x)$ and $V_0(x)$ has zero mean value. Then we write

$$L_\omega = D_\omega + V_0(x) \qquad \text{where} \qquad D_\omega := (\omega \cdot \partial_\varphi)^2 - \Delta + m \tag{2.3}$$

has constant coefficients. In the Fourier basis $(e^{\mathrm{i}(l\cdot\varphi + j\cdot x)})$, the operator $\mathcal{L}(u)$ is represented by the infinite dimensional self-adjoint matrix

$$A(\lambda) := A(\lambda\bar{\omega}, \varepsilon, u) := D + T \tag{2.4}$$

where

$$D := \mathrm{diag}_{(l,j)\in\mathbb{Z}^\nu\times\mathbb{Z}^d} \; -(\omega \cdot l)^2 + \|j\|^2 + m := \mathrm{diag}_{i\in\mathbb{Z}^b}\delta_i \,,$$

$$\|j\|^2 := j_1^2 + \ldots + j_d^2 \,, \quad i := (l, j) \in \mathbb{Z}^b := \mathbb{Z}^\nu \times \mathbb{Z}^d \,, \quad \delta_i := -(\omega \cdot l)^2 + \|j\|^2 + m \tag{2.5}$$

and

$$T := T_2 - \varepsilon T_1 \,, \quad T := (T_i^{i'})_{i,i'\in\mathbb{Z}^b} \,, \quad T_i^{i'} := (V_0)_{j-j'} - \varepsilon g_{i-i'} \tag{2.6}$$

represents the multiplication operator by $V_0(x) - \varepsilon g(\varphi, x)$. The matrix T is *Töplitz*, namely $T_i^{i'}$ depends only on the difference of the indices $i - i'$, and, since the functions g, $V \in H^s$, then $T_i^{i'} \to 0$ as $|i - i'| \to \infty$ at a polynomial rate. We introduce in the next section the s-norms $|\;|_s$ of a matrix (see Definition 2.1) which quantify the polynomial off-diagonal decay of its entries.

Along the iterative scheme of section 6, the function u (hence g) will depend on (ε, λ), so that $T := T(\varepsilon, \lambda)$ will be considered as a family of operators (or of infinite dimensional matrices representing them in the Fourier basis) parametrized by (ε, λ).

Introducing an additional parameter θ, we consider the family of infinite dimensional matrices

$$A(\varepsilon, \lambda, \theta) = D(\theta) + T(\varepsilon, \lambda) \tag{2.7}$$

where

$$D(\theta) := D(\lambda, \theta) := \mathrm{diag}_{i \in \mathbb{Z}^b} \Big(-(\lambda \bar{\omega} \cdot l + \theta)^2 + \|j\|^2 + m \Big) \tag{2.8}$$

and $|T|_{s_1} + |\partial_\lambda T|_{s_1} \leq C$, depending on V (the norm $|\ |_{s_1}$ is introduced in Definition 2.1). The role of the parameter θ is to exploit the co-variance property (2.11).

The main goal of the following sections is to prove polynomial off-diagonal decay for the inverse of the $(2N+1)^b$-dimensional sub-matrices of $A(\varepsilon, \lambda, \theta)$ centered at (l_0, j_0) denoted by

$$A_{N,l_0,j_0}(\varepsilon, \lambda, \theta) := A_{|l-l_0| \leq N, |j-j_0| \leq N}(\varepsilon, \lambda, \theta) \tag{2.9}$$

where $|l| := \max\{|l_1|, \ldots, |l_\nu|\}$, $|j| := \max\{|j_1|, \ldots, |j_d|\}$. In particular, when $l_0 = 0, j_0 = 0$ and $\theta = 0$, we get a bound for the s-norm of $A_{N,0,0}^{-1}$ which yields (via Lemma 2.2) the tame estimates in higher Sobolev norms needed in the Nash-Moser scheme, see Lemma 7.4.

If $l_0 = 0$ we use the simpler notation

$$A_{N,j_0}(\varepsilon, \lambda, \theta) := A_{N,0,j_0}(\varepsilon, \lambda, \theta)\,.$$

If also $j_0 = 0$, we simply write

$$A_N(\varepsilon, \lambda, \theta) := A_{N,0}(\varepsilon, \lambda, \theta)\,,$$

and, for $\theta = 0$, we denote

$$A_{N,j_0}(\varepsilon, \lambda) := A_{N,j_0}(\varepsilon, \lambda, 0)\,.$$

The relation between $|j| := \max\{|j_1|, \ldots, |j_d|\}$ and $\|j\|$ defined in (2.5) is

$$|j| \leq \|j\| \leq \sqrt{d}|j|\,. \tag{2.10}$$

By (2.9), (2.7), (2.8) and since T is Töplitz, the following *crucial* covariance property holds:

$$A_{N,l_1,j_1}(\varepsilon, \lambda, \theta) = A_{N,j_1}(\varepsilon, \lambda, \theta + \lambda \bar{\omega} \cdot l_1)\,. \tag{2.11}$$

Such property is exploited in Lemma 4.1 to bound the number of M-singular sites (with $M \in \{N, N - 2L_0\}$). It justifies the introduction of the parameter θ : complexity bounds as in (4.2) for the set $B_M(j_0; \lambda)$ of the "bad " θ will enable us, thanks to (1.7), to bound, for a given $\bar{\jmath}$, the number of time Fourier indices l such that the sub-matrix $A_{N,l,\bar{\jmath}}(\varepsilon, \lambda)$ has bad properties.

2.1 Matrices with off-diagonal decay

For $B \subset \mathbb{Z}^b$ we introduce the subspace

$$H^s_B := \Big\{ u = \sum_{i \in \mathbb{Z}^b} u_i e_i \in H^s \ : \ u_i = 0 \text{ if } i \notin B \Big\}$$

where $e_i := e^{\mathrm{i}(l \cdot \varphi + j \cdot x)}$. When B is finite, the space H^s_B does not depend on s and will be denoted H_B. For $B, C \subset \mathbb{Z}^b$ finite, we identify the space $\mathcal{L}^B_C$ of the linear maps $L : H_B \to H_C$ with the space of matrices

$$\mathcal{M}^B_C := \Big\{ M = (M^{i'}_i)_{i' \in B, i \in C}\,,\ M^{i'}_i \in \mathbb{C} \Big\}$$

identifying L with the matrix M with entries $M_i^{i'} := (Le_{i'}, e_i)_0$ where $(\ ,\ )_0 := (2\pi)^{-b}(\ ,\ )_{L^2}$ denotes the normalized L^2-scalar product. We consider also the L^2-operatorial norm

$$\|M_C^B\|_0 := \sup_{h \in H_B, h \neq 0} \frac{\|M_C^B h\|_0}{\|h\|_0}. \tag{2.12}$$

We now introduce stronger norms which quantify the polynomial decay off the diagonal of the matrix entries. These norms satisfy algebra and interpolation inequalities (see [5]) and control the higher Sobolev norms as stated in Lemma 2.2 below.

Definition 2.1. **(s-norm)** *The s-norm of a matrix $M \in \mathcal{M}_C^B$ is defined by*

$$|M|_s^2 := K_0 \sum_{n \in \mathbb{Z}^b} [M(n)]^2 \langle n \rangle^{2s}$$

where $\langle n \rangle := \max(|n|, 1)$ (see (1.12)),

$$[M(n)] := \begin{cases} \max_{i - i' = n} |M_i^{i'}| & \text{if } \ n \in C - B \\ 0 & \text{if } \ n \notin C - B \end{cases}$$

and the constant $K_0 > 0$ is the one of (1.12).

The s-norm is modeled on matrices which represent the multiplication operator.

Lemma 2.1. *The (Töplitz) matrix T which represents the multiplication operator by $g \in H^s$ satisfies $|T|_s \leq C\|g\|_s$.*

In analogy with the operators of multiplication by a function, the matrices with finite s-norm satisfy interpolation inequalities (see [5]). In particular we have (see also (1.13))

Lemma 2.2. **(Sobolev norm)** *$\forall s \geq s_0$ there is $C(s) \geq 1$ such that, for any finite subset $B, C \subset \mathbb{Z}^b$,*

$$\|Mw\|_s \leq (1/2)|M|_{s_0}\|w\|_s + (C(s)/2)|M|_s\|w\|_{s_0}, \quad \forall M \in \mathcal{M}_C^B, \ w \in H_B. \tag{2.13}$$

For further properties of the s-norms (and complete proofs) we refer to [5], section 3.

2.2 A spectral lemma

We denote

$$E_{N,j_0} := \Big\{ u(x) := \sum_{|j - j_0| \leq N} u_j e^{\mathrm{i} j \cdot x}, \ u_j \in \mathbb{C} \Big\} \tag{2.14}$$

(functions of the x-variable only) and the corresponding orthogonal projector

$$\Pi_{N,j_0} : H^{s_0}(\mathbb{T}^d) \to E_{N,j_0}. \tag{2.15}$$

More generally, for a finite non empty subset $B \subset \mathbb{Z}^d$ we denote by Π_B the L^2-orthogonal projector onto the space $E_B \subset L^2(\mathbb{T}^d)$ spanned by $\{e^{\mathrm{i} j \cdot x} : j \in B\}$.

As in this paper we consider restrictions of linear operators to finite dimensional subspaces, it is natural that we need informations on the spectral properties of the restricted self-adjoint operator

$$(-\Delta + V)_B := \Pi_B(-\Delta + V)_{|E_B}, \tag{2.16}$$

which are induced by spectral properties of the infinite dimensional operator $-\Delta + V$. This is the aim of Lemma 2.3 below, that will be used for the measure estimates of section 5, in particular in Lemma 5.6.

We shall denote (with a slight abuse of notation)

$$\partial B := \Big\{ j \in B \ : \ \mathrm{d}(j, \mathbb{Z}^d \backslash B) = 1 \Big\}$$

where $\mathrm{d}(j,j') := |j-j'|$ denotes the distance associated to the sup-norm. Note that, if $\mathrm{d}(0,\partial B) \geq L_0$, $L_0 \in \mathbb{N}$, then: either

$$\mathtt{B}(0,L_0-1) := \{j \in \mathbb{Z}^d \,:\, |j| \leq L_0-1\} \subset \mathbb{Z}^d \backslash B \qquad \text{or} \qquad \mathtt{B}(0,L_0) \subset B\,.$$

Recall (1.10) where n^- is the number of negative eigenvalues of $-\Delta + V(x)$ (counted with multiplicity).

Lemma 2.3. *Let $\beta_0 := \min\{|\mu_{n^-}|/2, \mu_{n^-+1}\}$. There is $L_0 \in \mathbb{N}$, such that, if $\mathrm{d}(0,\partial B) \geq L_0$, then*

1. *if $\mathtt{B}(0,L_0-1) \subset \mathbb{Z}^d \setminus B$, then $(-\Delta+V)_B \geq \beta_0 I$,*
2. *if $\mathtt{B}(0,L_0) \subset B$, then $(-\Delta+V)_B$ has n^- negative eigenvalues, all of them $\leq -\beta_0$. All the other eigenvalues of $(-\Delta+V)_B$ are $\geq \beta_0$.*

Proof The eigenvalues (1.10) of $-\Delta+V$ satisfy the min-max characterization

$$\mu_p = \inf_{\substack{G\subset H^1(\mathbb{T}^d),\\ \dim G=p}} \ \sup_{u\in G,\|u\|_{L^2}=1} Q(u)\,, \quad p=1,2,\ldots \tag{2.17}$$

where $Q : H^1(\mathbb{T}^d;\mathbb{R}) \to \mathbb{R}$ is the quadratic form

$$Q(u) := \|\nabla u\|_{L^2}^2 + \int_{\mathbb{T}^d} V(x)u^2(x)dx \tag{2.18}$$

and the infimum in (2.17) is taken over the subspaces G of $H^1(\mathbb{T}^d)$ of dimension p.

Let $\mathcal{H}^- \subset H^1(\mathbb{T}^d)$ be the n^--dimensional orthogonal sum of the eigenspaces associated to the negative eigenvalues $\mu_1,\ldots,\mu_{n^-}$. Then

$$Q(u) \leq \mu_{n^-}\|u\|_{L^2}^2 \leq -2\beta_0\|u\|_{L^2}^2\,, \quad \forall u \in \mathcal{H}^-\,,$$

by the definition of β_0. Moreover there is L_1 (large) such that $G^- := \Pi_{L_1,0}\mathcal{H}^-$ (recall (2.15)) has dimension n^- and

$$Q(u) \leq -\beta_0\|u\|_{L^2}^2\,, \quad \forall u \in G^-\,. \tag{2.19}$$

Let

$$L_0 := \max\{L_1, (\beta_0 + |V|_{L^\infty})^{1/2}\}\,. \tag{2.20}$$

1) Assume $\mathtt{B}(0,L_0-1) \subset \mathbb{Z}^d \setminus B$. Then (using that $\mathrm{d}(0,B) \geq L_0$)

$$\|\nabla u\|_{L^2}^2 \geq L_0^2\|u\|_{L^2}^2\,, \quad \forall u \in E_B\,,$$

and, by (2.18),

$$Q(u) \geq (L_0^2 - |V|_{L^\infty})\|u\|_{L^2}^2 \overset{(2.20)}{\geq} \beta_0\|u\|_{L^2}^2\,, \quad \forall u \in E_B\,.$$

Hence $(-\Delta+V)_B \geq \beta_0 I$.

2) Assume $\mathtt{B}(0,L_0) \subset B$. Let $(\mu_{B,p})$ be the non-decreasing sequence of the eigenvalues of the self-adjoint operator $(-\Delta+V)_B$, counted with multiplicity. They satisfy a variational characterization analogous to (2.17) with the only difference that the infimum is taken over the subspaces $G \subset E_B$. Since $\mathtt{B}(0,L_1) \subset \mathtt{B}(0,L_0) \subset B$, the subspace $G^- \subset E_B$ and, recalling that $\dim G^- = n^-$,

$$\mu_{B,n^-} = \inf_{\substack{G\subset E_B,\\ \dim G=n^-}} \ \sup_{u\in G,\|u\|_{L^2}=1} Q(u) \leq \sup_{u\in G^-,\|u\|_{L^2}=1} Q(u) \overset{(2.19)}{\leq} -\beta_0\,.$$

Moreover

$$\begin{aligned} \mu_{B,n^-+1} &= \inf_{\substack{G\subset E_B,\\ \dim G=n^-+1}} \ \sup_{u\in G,\|u\|_{L^2}=1} Q(u) \\ &\geq \inf_{\substack{G\subset H^1(\mathbb{T}^d),\\ \dim G=n^-+1}} \ \sup_{u\in G,\|u\|_{L^2}=1} Q(u) \overset{(2.17)}{=} \mu_{n^-+1} \geq \beta_0 \end{aligned}$$

by the definition of β_0. The proof of the lemma is complete. ■

3 The multiscale analysis

Using arguments on the variations of the eigenvalues (section 5) we will be able to prove that the matrices in (2.9) are invertible and their inverses satisfy appropriate bounds in L^2-matrix norm for most values of the parameters. However we need additional properties for their submatrices centered along the diagonal in order to obtain "good" bounds for the higher norms $| \ |_s$ of the inverses. These are properties of separation of the "singular" sites or of the "bad" sites (see Proposition 3.1 below). A few definitions are first in order.

Given $\Omega, \Omega' \subset E \subset \mathbb{Z}^b$ we define

$$\mathrm{diam}(E) := \sup_{i,i' \in E} |i - i'| , \qquad \mathrm{d}(\Omega, \Omega') := \inf_{i \in \Omega, i' \in \Omega'} |i - i'| .$$

Let $\delta \in (0,1)$ be fixed.

Definition 3.1. (N-good/bad matrix) *The matrix $A \in \mathcal{M}_E^E$, with $E \subset \mathbb{Z}^b$, $\mathrm{diam}(E) \leq 4N$, is N-good if A is invertible and*

$$\forall s \in [s_0, s_1] \ , \quad |A^{-1}|_s \leq N^{\tau' + \delta s}. \tag{3.1}$$

Otherwise A is N-bad.

Note that in (3.1) the "tame" exponent $\tau' + \delta s$ increases with s but δ is strictly < 1. This is a quite weak condition for the off-diagonal decay of A (for $\delta = 1$ there is no decay and the Nash-Moser scheme will not converge).

Definition 3.2. (Regular/Singular site) *Fix $\Theta \geq 1$. The index $i \in \mathbb{Z}^b$ is* REGULAR *for $A = A(\varepsilon, \lambda, \theta)$ if $|A_i^i| \geq \Theta$. Otherwise i is* SINGULAR.

Since for quasi-periodic solutions there is not an appropriate separation property of the singular sites in (1.4) (as in the periodic cases [14], [9], [5], see item 2 after Theorem 1.1), we need a stronger definition of "badness" for a site. This notion is adapted to the Nash-Moser inductive process and it does not only depend on the diagonal terms, but also on the off-diagonal entries.

Definition 3.3. ((A,N)-good/bad site) *For $A \in \mathcal{M}_E^E$, we say that $i \in E \subset \mathbb{Z}^b$ is*

- *(A, N)-*REGULAR *if there is $F \subset E$ such that $\mathrm{diam}(F) \leq 4N$, $\mathrm{d}(i, E \backslash F) \geq N/2$ and A_F^F is N-good.*
- *(A, N)-*GOOD *if it is regular for A or (A, N)-regular. Otherwise we say that i is (A, N)-*BAD.

Let us consider the new larger scale

$$N' = N^\chi \tag{3.2}$$

with $\chi > 1$. For a matrix $A \in \mathcal{M}_E^E$ we define $\mathrm{Diag}(A) := (\delta_{ii'} A_i^{i'})_{i,i' \in E}$.

The goal of the next multiscale proposition is to deduce that a matrix A at the larger scale N' is N'-good under the assumptions (H1)-(H3) below and the relations (3.3)-(3.5) between the constants $\delta, \tau, \tau', d, \nu, \chi, etc....$ Proposition 3.1 is proved in [5] by "resolvent identity"-type arguments.

Proposition 3.1. (Multiscale step, [5]) *Assume*

$$\delta \in (0, 1/2) \, , \ \tau' > 2\tau + b + 1 \, , \ C_1 \geq 2 \, , \tag{3.3}$$

and, setting $\kappa := \tau' + b + s_0$,

$$\chi(\tau' - 2\tau - b) > 3(\kappa + (s_0 + b)C_1) \, , \ \chi\delta > C_1 \, , \tag{3.4}$$

$$S \geq s_1 > 3\kappa + \chi(\tau + b) + C_1 s_0 \, . \tag{3.5}$$

$\Upsilon > 0$ being fixed , there exists $N_0(\Upsilon, S) \in \mathbb{N}$, $\Theta(\Upsilon, s_1) > 0$ large enough (see Definition 3.2), such that: $\forall N \geq N_0(\Upsilon, S)$, $\forall E \subset \mathbb{Z}^b$ with $\mathrm{diam}(E) \leq 4N' = 4N^\chi$, if $A \in \mathcal{M}_E^E$ satisfies

- **(H1) (Off-diagonal decay)** $|A - \mathrm{Diag}(A)|_{s_1} \leq \Upsilon$
- **(H2) (L^2-bound)** $\|A^{-1}\|_0 \leq (N')^\tau$
- **(H3) (Separation properties)** *There is a partition of the (A, N)-bad sites $B = \cup_\alpha \Omega_\alpha$ with*

$$\mathrm{diam}(\Omega_\alpha) \leq N^{C_1}\,, \quad \mathrm{d}(\Omega_\alpha, \Omega_\beta) \geq N^2\ ,\ \forall \alpha \neq \beta\,, \tag{3.6}$$

then A is N'-good. More precisely

$$\forall s \in [s_0, S]\ ,\ \ |A^{-1}|_s \leq \frac{1}{4}(N')^{\tau'}\Big((N')^{\delta s} + |A - \mathrm{Diag}(A)|_s\Big)\,.$$

Condition (H1) means that A is "polynomially localized" with respect to the diagonal. For the matrix A in (2.4), $\Upsilon = O(\|V\|_{s_1} + \varepsilon\|g\|_{s_1})$ and Θ introduced in Definition 3.2 has to verify $\Theta >> \Upsilon$. Condition (H2) is then verified (for most parameters λ) with an exponent $\tau \geq \tau(\nu, d)$ large enough (see e.g. Lemma 5.9) imposing lower bounds for the moduli of the eigenvalues of A.

Remark 3.1. i) *Since $\delta\chi > C_1$ (see* (3.4)*), the size N^{C_1} of a "bad" cluster Ω_α (see* (3.6)*) is small with respect to the new scale $N' = N^\chi$, see* (3.2)*. Condition* (3.5) *quantify a sufficiently fast off diagonal decay for the matrix A, see (H1).*

ii) *We could fix $\tau' = 3\tau + b$. Then, for $\tau > b$, the first inequality in* (3.4) *is satisfied if $\chi > 9 + (1 + s_0/b)(3 + C_1)$. As a consequence the constant χ is large independently of τ (it depends only on d, ν).*

We shall apply Proposition 3.1 to finite dimensional matrices A_{N,i_0} (recall the notation in (2.9)) which are obtained as restrictions of the infinite dimensional matrix $A(\varepsilon, \lambda, \theta)$ in (2.7). It is convenient to introduce a notion of N-good site for an infinite dimensional matrix (slightly different from the one in [5], see remark 3.2). Let

$$\mathcal{Q}_N := \Big\{j \in \mathbb{Z}^d\ :\ \mathrm{d}(0, \partial(j + [-N, N]^d)) < L_0\Big\}\,,\ \ \check{\mathcal{Q}}_N := \Big\{i = (l, j) \in \mathbb{Z}^{\nu+d}\ :\ j \in \mathcal{Q}_N\Big\} \tag{3.7}$$

where L_0 is defined in Lemma 2.3. We shall always assume that $N - 2L_0 \geq N/2$.

Definition 3.4. (N-good/bad site) *A site $i \in \mathbb{Z}^b$ is:*

- N-REGULAR *if $A_{N,i}$ is N-good (Definition 3.1). Otherwise we say that i is N-*SINGULAR.
- N-GOOD *if i is regular (Definition 3.2)* or *for all $M \in \{N - 2L_0, N\}$, all the sites i' with $|i' - i| \leq M$ and $i' \notin \check{\mathcal{Q}}_M$ are M-regular. Otherwise, we say that i is N-*BAD.

We now explain the main difference between Definition 3.4 and Definition 5.1 in [5].

Remark 3.2. *In [5], the definition of a good site i was "i is regular or all the sites i' with $|i' - i| \leq N$ are N-regular". Definition 3.4 is more involved because we do not assume the positivity condition $-\Delta + V > 0$. We restrict to the sites $i' \notin \check{\mathcal{Q}}_M$ in order to be able to apply the spectral Lemma 2.3 (in Lemma 5.6) and then prove the measure estimates of section 5. The cost is that we have to consider both the scales $M = N$ and $M = N - 2L_0$ in order to prove the following lemma, stated in view of the application of Proposition 3.1 (see Lemma 7.2).*

Lemma 3.1. *Let $A = A_{N',i_0}$ with $i_0 \notin \check{\mathcal{Q}}_{N'}$. Then any N-good site $i \in i_0 + [-N', N']^{d+\nu}$ is (A, N)-good.*

PROOF. We decompose

$$E := i_0 + [-N', N']^{\nu+d} = G \times H \quad \text{where} \quad G := \Pi_{p=1}^{\nu}[a_p, b_p]\,,\ H := \Pi_{q=1}^{d}[c_q, d_q] \tag{3.8}$$

and, writing $i_0 = (l_0, j_0)$,

$$a_p := (l_0)_p - N'\ ,\ b_p := (l_0)_p + N'\ ,\ c_q := (j_0)_q - N'\ ,\ d_q := (j_0)_q + N'\,.$$

Consider any N-good site $i := (l, j) \in E$ (see Definition 3.4). If i is a regular site, there is nothing to prove. If i is singular, we introduce its neighborhood

$$F_N := F_N(i) := G_N \times H_N \subset E \quad \text{where} \quad G_N := \Pi_{p=1}^{\nu} I_p \subset G\,, \quad H_N := \Pi_{q=1}^{d} J_q \subset H\,, \tag{3.9}$$

and the intervals $I_p \subset [a_p, b_p]$, $J_q \subset [c_q, d_q]$ are defined as follows:

- if $l_p - a_p > N$ and $b_p - l_p > N$ (resp. $j_q - c_q > N$ and $d_q - j_q > N$), then $I_p := [l_p - N, l_p + N]$ (resp. $J_q := [j_q - N, j_q + N]$);
- if $l_p - a_p \leq N$ (resp. $j_q - c_q \leq N$), then $I_p := [a_p, a_p + 2N]$ (resp. $J_q := [c_q, c_q + 2N]$);
- if $b_p - l_p \leq N$ (resp. $d_q - j_q \leq N$), then $I_p := [b_p - 2N, b_p]$ (resp. $J_q := [d_q - 2N, d_q]$).

By construction we have

$$\mathrm{d}(i, E \setminus F_N) \geq N \tag{3.10}$$

and we can write

$$F_N = \bar{\imath} + [-N, N]^{\nu+d} \quad \text{for some } \ \bar{\imath} = (\bar{l}, \bar{\jmath}) \in E \ \text{ with } \ |i - \bar{\imath}| \leq N\,. \tag{3.11}$$

For $M = N - 2L_0$, we define as in (3.9) the sets $F_M := G_M \times H_M$, $G_M := \Pi_{p=1}^{\nu} I_{M,p}$, $H_M := \Pi_{q=1}^{d} J_{M,q}$, and we write

$$F_M = \tilde{\imath} + [-M, M]^{\nu+d} \quad \text{for some } \ \tilde{\imath} = (\tilde{l}, \tilde{\jmath}) \ \text{ with } \ |i - \tilde{\imath}| \leq M\,. \tag{3.12}$$

We claim that

$$\mathrm{d}(\partial H_N \backslash \partial H, H_M) \geq 2L_0\,. \tag{3.13}$$

In fact, assume $j' \in \partial H_N \backslash \partial H$. Then there is some $q \in \{1, \ldots, d\}$ such that $j'_q \in \partial J_q \backslash \{c_q, d_q\}$. By construction, it is easy to see that $\mathrm{d}(J_{M,q}, [c_q, d_q] \backslash J_q) \geq 2L_0 + 1$. Hence $\mathrm{d}(j'_q, J_{M,q}) \geq 2L_0$ and $\mathrm{d}(j', H_M) \geq 2L_0$, proving (3.13).

We are now in position to prove that i is (A, N)-good. We distinguish two cases:

(i) $\mathrm{d}(0, \partial H_N) \geq L_0$. Since $H_N = \bar{\jmath} + [-N, N]^d$ (see (3.9)-(3.11)) we get $\bar{\jmath} \notin \mathcal{Q}_N$ (see (3.7)), namely $\bar{\imath} \notin \check{\mathcal{Q}}_N$. Since i is a singular N-good site (see Definition 3.4), $|i - \bar{\imath}| \leq N$ (see (3.11)), $\bar{\imath} \notin \check{\mathcal{Q}}_N$, we deduce that the matrix $A_{N,\bar{\imath}} = A_{F_N}^{F_N}$ is N-good . As a consequence, since $F_N \subset E$ (see (3.9)), $\mathrm{diam}(F_N) = 2N$ (see (3.11)) $\mathrm{d}(i, E \backslash F_N) \geq N$ (see (3.10)), the site i is (A, N)-good (see Definition 3.3).

(ii) $\mathrm{d}(0, \partial H_N) < L_0$. It is an assumption of the Lemma that $i_0 = (l_0, j_0) \notin \check{\mathcal{Q}}_{N'}$ which means $\mathrm{d}(0, \partial H) \geq L_0$ (by (3.8) we have $H = j_0 + [-N', N']^d$). Hence $\mathrm{d}(0, \partial H_N \backslash \partial H) = \mathrm{d}(0, \partial H_N) < L_0$. Hence, by (3.13), we deduce $\mathrm{d}(0, H_M) \geq L_0$ and therefore $\mathrm{d}(0, \partial H_M) \geq L_0$. Then $\tilde{\imath} \notin \check{\mathcal{Q}}_M$ (the site $\tilde{\imath}$ is defined in (3.12) and we have $H_M = \tilde{\jmath} + [-M, M]^d$). Since i is singular and N-good, $|i - \tilde{\imath}| \leq M$ (see (3.12)), $\tilde{\imath} \notin \check{\mathcal{Q}}_M$, then the matrix $A_{M,\tilde{\imath}} = A_{F_M}^{F_M}$ is N-good. As a consequence, since $\mathrm{d}(i, E \backslash F_M) \geq M \geq N/2$, the site i is (A, N)-good.

This concludes the proof of the Lemma. ■

4 Separation properties of the bad sites

We now verify the "separation properties" of the bad sites required in the multiscale Proposition 3.1.

Let $A := A(\varepsilon, \lambda, \theta)$ be the infinite dimensional matrix of (2.7). We define

$$B_M(j_0; \lambda) := B_M(j_0; \varepsilon, \lambda) := \Big\{ \theta \in \mathbb{R} \, : \, A_{M,j_0}(\varepsilon, \lambda, \theta) \text{ is } M - \text{bad} \Big\}\,. \tag{4.1}$$

Definition 4.1. **(N-good/bad parameters)** *A parameter $\lambda \in \Lambda$ is N-good for A if*

$$\forall M \in \{N, N - 2L_0\}\,, \quad \forall\, j_0 \in \mathbb{Z}^d \setminus \mathcal{Q}_M\,, \quad B_M(j_0; \lambda) \subset \bigcup_{q=1,\ldots,N^{2d+\nu+3}} I_q \tag{4.2}$$

where I_q are intervals with measure $|I_q| \leq N^{-\tau}$. Otherwise, we say λ is N-bad. We define

$$\mathcal{G}_N := \mathcal{G}_N(u) := \Big\{ \lambda \in \Lambda \, : \, \lambda \ \text{ is } \ N - \text{good for } A \Big\}\,. \tag{4.3}$$

In order to prove the separation properties of the N-bad sites we have to require that $\omega = \lambda\bar{\omega}$ satisfies a Diophantine type non-resonance condition. We assume:

- (**NR**) There exist $\gamma > 0$ such that, for any *non zero* polynomial $P(X) \in \mathbb{Z}[X_1, \ldots, X_\nu]$ of the form

$$P(X) = n + \sum_{1 \le i \le j \le \nu} p_{ij} X_i X_j \,, \quad n, p_{ij} \in \mathbb{Z} \,, \tag{4.4}$$

  we have

$$|P(\omega)| \ge \frac{\gamma}{1 + |p|^{\tau_0}} \,. \tag{4.5}$$

The non-resonance condition (**NR**) is satisfied by $\omega = \lambda\bar{\omega}$ for most $\lambda \in \Lambda$, see Lemma 6.3.

Remark 4.1. *In [11], Bourgain requires the non-resonance condition* (4.5) *for all non zero polynomials* $P(X) \in \mathbb{Z}[X_1, \ldots, X_\nu]$ *of degree* $\deg P \le 10d$.

The main result of this section is the following proposition. It will enable to verify the assumption (H3) of Proposition 3.1 for the submatrices $A_{N', j_0}(\varepsilon, \lambda, \theta)$ (see Lemma 7.2).

Proposition 4.1. (Separation properties of N-bad sites) *There exists* $C_1(d, \nu) \ge 2$, $N_0(\nu, d, \gamma_0, \Theta) \in \mathbb{N}$ *such that* $\forall N \ge N_0(\nu, d, \gamma_0, \Theta)$, *if*

- **(i)** λ *is* N*-good for* A,
- **(ii)** $\tau > \chi\nu$,
- **(iii)** $\omega = \lambda\bar{\omega}$ *satisfies* (**NR**),

then, $\forall \theta \in \mathbb{R}$, *the* N*-bad sites* $i := (l, j) \in \mathbb{Z}^\nu \times \mathbb{Z}^d$ *of* $A(\varepsilon, \lambda, \theta)$ *with* $|l| \le N' := N^\chi$ *admit a partition* $\cup_\alpha \Omega_\alpha$ *in disjoint clusters satisfying*

$$\mathrm{diam}(\Omega_\alpha) \le N^{C_1(d,\nu)} \,, \quad \mathrm{d}(\Omega_\alpha, \Omega_\beta) > N^2 \,, \ \forall \alpha \ne \beta \,. \tag{4.6}$$

The rest of this section is devoted to the proof of Proposition 4.1. Note that, by (1.7), the frequency vectors $\omega = \lambda\bar{\omega}$, $\forall \lambda \in [1/2, 3/2]$, are Diophantine, namely

$$|\omega \cdot l| \ge \frac{\gamma_0}{|l|^\nu} \,, \quad \forall l \in \mathbb{Z}^\nu \setminus \{0\} \,. \tag{4.7}$$

The outline of the proof of Proposition 4.1 is the following. As explained at the end of this section, it is sufficient to bound the length L of any N^2-chain of bad sites, i.e. a sequence $(i_q)_{1 \le q \le L}$ such that $|i_{q+1} - i_q| \le N^2$ (Definition 4.2), and whose time components have norm $\le N' = N^\chi$. In particular we aim to prove that the length L is bounded by some power of N (with an exponent depending only on d and ν), see (4.39). This is a consequence of the key Lemma 4.2 whose assumption (4.11) is verified thanks to Corollary 4.1. Actually, the goal of Corollary 4.1 is to bound the number of bad sites with a *fixed* spatial component and time components with norm $\le N'$. In turn Corollary 4.1 follows from Lemma 4.1 which uses assumptions (**i**) and (**ii**) of Proposition 4.1 and the diophantine property (4.7).

Note that, for a given χ, we may choose τ as large as we wish: this will affect only the smoothness required for the nonlinearity f and the potential V, see (6.2), (6.4). Then assumption **(ii)** can always be fulfilled, see also remark 3.1-**ii)**.

Lemma 4.1. *Assume that* λ *is* N*-good for* A *and let* $\tau > \chi\nu$. *Then, for all* $M \in \{N - 2L_0, N\}$, $\forall \bar{\jmath} \in \mathbb{Z}^d \backslash \mathcal{Q}_M$, *the number of* M*-singular sites* $(l_1, \bar{\jmath}) \in \mathbb{Z}^\nu \times \mathbb{Z}^d$ *with* $|l_1| \le 2N'$ *does not exceed* $N^{2d+\nu+3}$.

PROOF. If $(l_1, \bar{\jmath})$ is M-singular then $A_{M, l_1, \bar{\jmath}}(\varepsilon, \lambda, \theta)$ is M-bad (see Definitions 3.4 and 3.1 with $N = M$). By the co-variance property (2.11), we get that $A_{M, \bar{\jmath}}(\varepsilon, \lambda, \theta + \lambda\bar{\omega} \cdot l_1)$ is M-bad, namely $\theta + \lambda\bar{\omega} \cdot l_1 \in B_M(\bar{\jmath}; \lambda)$, see (4.1). By assumption, λ is N-good, and, therefore, (4.2) holds for $M = N$ and $M = N - 2L_0$.

We claim that in each interval I_q there is at most one element $\theta + \omega \cdot l_1$ with $\omega = \lambda\bar{\omega}$, $|l_1| \leq 2N'$. Then, since there are at most $N^{2d+\nu+3}$ intervals I_q (see (4.2)), the lemma follows.

We prove the previous claim by contradiction. Suppose that there exist $l_1 \neq l_1'$ with $|l_1|, |l_1'| \leq N'$, such that $\omega \cdot l_1 + \theta$, $\omega \cdot l_1' + \theta \in I_q$. Then

$$|\omega \cdot (l_1 - l_1')| = |(\omega \cdot l_1 + \theta) - (\omega \cdot l_1' + \theta)| \leq |I_q| \leq N^{-\tau} \,. \tag{4.8}$$

By (4.7) we also have

$$|\omega \cdot (l_1 - l_1')| \geq \frac{\gamma_0}{|l_1 - l_1'|^\nu} \geq \frac{\gamma_0}{(4N')^\nu} = 4^{-\nu}\gamma_0 N^{-\chi\nu} \,. \tag{4.9}$$

By assumption (ii) of Proposition 4.1 the inequalities (4.8) and (4.9) are in contradiction, for $N \geq N_0(\gamma_0)$ large enough. ■

Corollary 4.1. *Assume (i)-(ii) of Proposition 4.1. Then, $\forall \tilde{\jmath} \in \mathbb{Z}^d$, the number of N-bad sites $(l_1, \tilde{\jmath}) \in \mathbb{Z}^\nu \times \mathbb{Z}^d$ with $|l_1| \leq N'$ does not exceed $N^{3d+2\nu+4}$.*

Proof. By Lemma 4.1, for $M \in \{N - 2L_0, N\}$, the set S_M of M-singular sites $(l, j) \notin \check{\mathcal{Q}}_M$ (see (3.7) with $N = M$) with $|l| \leq N' + N$, $|j - \tilde{\jmath}| \leq M$ has cardinality at most $CN^{2d+\nu+3} \times N^d$. Each N-bad site $(l_1, \tilde{\jmath})$ with $|l_1| \leq N'$ is included, for some $M \in \{N - 2L_0, N\}$, in some M-ball centered at an element (l, j) of S_M which is not in $\check{\mathcal{Q}}_M$ (see Definition 3.4). Each of these balls contains at most CN^ν sites of the form $(l, \tilde{\jmath})$. Hence there are at most $C2N^{2d+\nu+3} \times N^d \times N^\nu$ such N-bad sites. ■

We underline that the bound on the N-bad sites given in Corollary 4.1 holds *for all* $\tilde{\jmath} \in \mathbb{Z}^d$, even if the complexity bound (4.2) holds for all $j_0 \notin \mathcal{Q}_M$. We now estimate also the spatial components of the singular sites. Here we use the form (4.10) of the small divisors.

Definition 4.2. (Γ-chain) *A sequence $i_0, \ldots, i_L \in \mathbb{Z}^{d+\nu}$ of distinct integer vectors satisfying*

$$|i_{q+1} - i_q| \leq \Gamma \,, \quad \forall q = 0, \ldots, L-1 \,,$$

for some $\Gamma \geq 2$, is called a Γ-chain of length L.

The next lemma provides the bound (4.12) on the length of a chain of singular sites by assumption (**iii**) of Proposition 4.1 and condition (4.11). It improves Lemma 20.14 of Bourgain [11] requiring the weaker non-resonance assumption (**NR**) (and giving a simpler proof).

Lemma 4.2. *Assume that $\omega = \lambda\bar{\omega}$ satisfies (**NR**). For all $\theta \in \mathbb{R}$, consider a Γ-chain $(l_q, j_q)_{q=0,\ldots,L}$ of θ-singular sites with $\Gamma \geq 2$, namely, $\forall q = 0, \ldots, L$,*

$$\Big| (\lambda\bar{\omega} \cdot l_q + \theta)^2 - \|j_q\|^2 - m \Big| < \Theta + 1 \,, \tag{4.10}$$

such that, $\forall \tilde{\jmath} \in \mathbb{Z}^d$, the cardinality

$$|\{(l_q, j_q)_{q=0,\ldots,L} \, : \, j_q = \tilde{\jmath}\}| \leq K \,. \tag{4.11}$$

Then its length is bounded by

$$L \leq (\Gamma K)^{C_2(d,\nu)} \,. \tag{4.12}$$

Proof. First note that it is sufficient to bound the length of a Γ-chain of singular sites when $\theta = 0$. Indeed, suppose first that $\theta = \omega \cdot \bar{l}$ for some $\bar{l} \in \mathbb{Z}^\nu$. For a Γ-chain of θ-singular sites $(l_q, j_q)_{q=0,\ldots,L}$, see (4.10), the translated Γ-chain $(l_q + \bar{l}, j_q)_{q=0,\ldots,L}$, is formed by 0-singular sites, namely

$$|(\omega \cdot (l_q + \bar{l}))^2 - \|j_q\|^2 - m| < \Theta \,.$$

For any $\theta \in \mathbb{R}$, we consider an approximating sequence $\omega \cdot \bar{l}_n \to \theta$, $\bar{l}_n \in \mathbb{Z}^\nu$. A Γ-chain of θ-singular sites (see (4.10)), is, for n large enough, also a Γ-chain of $\omega \cdot \bar{l}_n$-sites. Then we bound its length arguing as in the above case.

We now introduce the quadratic form $Q : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$Q(x,y) := -x^2 + \|y\|^2 \tag{4.13}$$

and the associated bilinear symmetric form $\Phi : (\mathbb{R} \times \mathbb{R}^d)^2 \to \mathbb{R}$ defined by

$$\Phi\Big((x,y),(x',y')\Big) := -xx' + y \cdot y' \,. \tag{4.14}$$

Note that Φ is the sum of the bilinear forms

$$\Phi = -\Phi_1 + \Phi_2 \tag{4.15}$$

$$\Phi_1\Big((x,y),(x',y')\Big) := xx' \,, \quad \Phi_2\Big((x,y),(x',y')\Big) := y \cdot y' \,. \tag{4.16}$$

Let $(l_q, j_q)_{q=0,\ldots,L}$ be a Γ-chain, namely

$$|l_{q+1} - l_q|, |j_{q+1} - j_q| \leq \Gamma \,, \quad \forall q = 0, \ldots, L-1 \,, \tag{4.17}$$

of 0-singular sites, see (4.10) with $\theta = 0$. Setting

$$x_q := \omega \cdot l_q \, \in \, \omega \cdot \mathbb{Z}^\nu \,, \tag{4.18}$$

we get that (see (4.13))

$$|Q(x_q, j_q)| < \Theta + 1 + |m| \,, \quad \forall q = 0, \ldots, L \,. \tag{4.19}$$

Lemma 4.3. $\forall q, q_0 \in [0, L]$ *we have*

$$\Big|\Phi\Big((x_{q_0}, j_{q_0}), (x_q - x_{q_0}, j_q - j_{q_0})\Big)\Big| \leq C|q - q_0|^2\Gamma^2 \,. \tag{4.20}$$

Proof. By bilinearity

$$Q(x_q, j_q) = Q(x_{q_0}, j_{q_0}) + 2\Phi\Big((x_{q_0}, j_{q_0}), (x_q - x_{q_0}, j_q - j_{q_0})\Big) + Q(x_q - x_{q_0}, j_q - j_{q_0}) \,. \tag{4.21}$$

We have

$$\begin{aligned} |Q(x_q - x_{q_0}, j_q - j_{q_0})| &\overset{(4.13)}{\leq} |x_q - x_{q_0}|^2 + \|j_q - j_{q_0}\|^2 \\ &\overset{(4.18),(2.10)}{\leq} |\omega|^2 |l_q - l_{q_0}|^2 + d|j_q - j_{q_0}|^2 \overset{(4.17)}{\leq} C|q - q_0|^2\Gamma^2 \,. \end{aligned} \tag{4.22}$$

Then (4.20) follows by (4.21), (4.22) and (4.19). ■

Proof of Lemma 4.2 continued. In the case when the vectors $(x_q - x_{q_0}, j_q - j_{q_0})$, $|q - q_0| \leq r$ (for some $r > 0$), form a basis of $\mathbb{R}^{d+1}$, we can deduce from (4.20) and the nondegeneracy of Φ a bound (depending on r) on (x_{q_0}, j_{q_0}). In the general case we must introduce the subspace of $\mathbb{R}^{d+1}$

$$G := \mathrm{Span}_{\mathbb{R}}\Big\{(x_q - x_{q'}, j_q - j_{q'}) \, : \, 0 \leq q, q' \leq L\Big\} = \mathrm{Span}_{\mathbb{R}}\Big\{(x_q - x_{q_0}, j_q - j_{q_0}) \, : \, 0 \leq q \leq L\Big\} \tag{4.23}$$

and we call $g \leq d + 1$ the dimension of G. Introducing a small parameter $\delta > 0$, to be specified later (see (4.38)), we distinguish two cases.
Case I. $\forall q_0 \in [0, L]$,

$$\mathrm{Span}_{\mathbb{R}}\{(x_q - x_{q_0}, j_q - j_{q_0}) \, : \, |q - q_0| \leq L^\delta \,, \; q \in [0, L]\} = G \,. \tag{4.24}$$

We select a basis of $G \subset \mathbb{R}^{d+1}$ from $(x_q - x_{q_0}, j_q - j_{q_0})$ with $|q - q_0| \leq L^\delta$, say

$$f_s := (x_{q_s} - x_{q_0}, j_{q_s} - j_{q_0}) = (\omega \cdot \Delta_s l, \Delta_s j) \,, \quad s = 1, \ldots, g \,, \tag{4.25}$$

where

$$(\Delta_s l, \Delta_s j) := (l_{q_s} - l_{q_0}, j_{q_s} - j_{q_0}) \quad \text{satisfies} \quad |(\Delta_s l, \Delta_s j)| \overset{(4.17)}{\leq} C\Gamma |q_s - q_0| \leq C\Gamma L^\delta . \tag{4.26}$$

Hence

$$|f_s| \leq C\,\Gamma L^\delta , \qquad \forall s = 1, \ldots, g . \tag{4.27}$$

Then, in order to derive from (4.20) a bound on (x_{q_0}, j_{q_0}) or its projection onto G, we need a nondegeneracy property for $Q_{|G}$. The following lemma states it.

Lemma 4.4. *Assume* (**NR**). *Then the matrix*

$$\Omega := (\Omega_s^{s'})_{s,s'=1}^g , \quad \Omega_s^{s'} := \Phi(f_{s'}, f_s) , \tag{4.28}$$

is invertible and

$$|(\Omega^{-1})_s^{s'}| \leq C(\Gamma L^\delta)^{C_3(d,\nu)} , \quad \forall s, s' = 1, \ldots, g . \tag{4.29}$$

Proof. According to the splitting (4.15) we write Ω like

$$\Omega := \Big(- \Phi_1(f_{s'}, f_s) + \Phi_2(f_{s'}, f_s) \Big)_{s,s'=1,\ldots,g} = -S + R \tag{4.30}$$

where, by (4.25),

$$S_s^{s'} := \Phi_1(f_{s'}, f_s) = (\omega \cdot \Delta_{s'} l)(\omega \cdot \Delta_s l) , \quad R_s^{s'} := \Phi_2(f_{s'}, f_s) = \Delta_{s'} j \cdot \Delta_s j . \tag{4.31}$$

The matrix $R = (R_1, \ldots, R_g)$ has integer entries (the $R_i \in \mathbb{Z}^g$ denote the columns). The matrix $S := (S_1, \ldots, S_g)$ has rank 1 since all its columns $S_s \in \mathbb{R}^g$ are colinear:

$$S_s = (\omega \cdot \Delta_s l)(\omega \cdot \Delta_1 l, \ldots, \omega \cdot \Delta_g)^t , \quad s = 1, \ldots g .$$

We develop the determinant

$$\begin{aligned} P(\omega) \; &:= \; \det \Omega \overset{(4.30)}{=} \det(-S + R) \\ &= \; \det(R) - \det(S_1, R_2, \ldots, R_g) - \ldots - \det(R_1, \ldots, R_{g-1}, S_g) \end{aligned} \tag{4.32}$$

using that the determinant of matrices with 2 columns S_i, S_j, $i \neq j$, is zero. The expression in (4.32) is a polynomial in ω of degree 2 of the form (4.4) with coefficients

$$|(n,p)| \overset{(4.31),(4.26)}{\leq} C(\Gamma L^\delta)^{C(d)} . \tag{4.33}$$

If $P \neq 0$ then the non-resonance condition (**NR**) implies

$$|\det \Omega| = |P(\omega)| \overset{(4.5)}{\geq} \frac{\gamma}{1 + |p|^{\tau_0}} \overset{(4.33)}{\geq} \frac{\gamma}{(\Gamma L^\delta)^{C'(d,\nu)}} \tag{4.34}$$

(recall that $\tau_0 := \nu(\nu+1)$). In order to conclude the proof of the lemma, we have to show that $P \neq 0$. By contradiction, if $P = 0$ then (compare with (4.30))

$$0 = P(\mathrm{i}\omega) = \det\Big(\Phi_1(f_{s'}, f_s) + \Phi_2(f_{s'}, f_s)\Big)_{s,s'=1,\ldots g} = \det(f_{s'} \cdot f_s)_{s,s'=1,\ldots g} > 0$$

because f_s is a basis of $\mathbb{R}^g$. This contradiction proves that P is not the zero polynomial.

By (4.34), the Cramer rule, and (4.27) we deduce (4.29). ■

Proof of Lemma 4.2 continued. We introduce

$$G^{\perp \Phi} := \Big\{ z \in \mathbb{R}^{d+1} \; : \; \Phi(z, f) = 0 , \; \forall f \in G \Big\} .$$

Since Ω is invertible (Lemma 4.4), $\Phi_{|G}$ is nondegenarate, hence

$$\mathbb{R}^{d+1} = G \oplus G^{\perp \Phi}$$

and we denote by $P_G : \mathbb{R}^{d+1} \to G$ the corresponding projector onto G.

We are going to estimate

$$P_G(x_{q_0}, j_{q_0}) = \sum_{s'=1}^{g} a_{s'} f_{s'} \,. \tag{4.35}$$

For all $s = 1, \ldots, g$, and since $f_s \in G$, we have

$$\Phi\Big((x_{q_0}, j_{q_0}), f_s\Big) = \Phi\Big(P_G(x_{q_0}, j_{q_0}), f_s\Big) \overset{(4.35)}{=} \Phi\Big(\sum_{s'=1}^{g} a_{s'} f_{s'}, f_s\Big) = \sum_{s'=1}^{g} a_{s'} \Phi(f_{s'}, f_s)$$

that we write as the linear system

$$\Omega a = b \,, \qquad a := \begin{pmatrix} a_1 \\ \ldots \\ a_g \end{pmatrix} , \quad b := \begin{pmatrix} \Phi\Big((x_{q_0}, j_{q_0}), f_1\Big) \\ \ldots \\ \Phi\Big((x_{q_0}, j_{q_0}), f_g\Big) \end{pmatrix} \tag{4.36}$$

and Ω is defined in (4.28).

Lemma 4.5. *For all $q_0 \in [0, L]$ we have*

$$|P_G(x_{q_0}, j_{q_0})| \leq (\Gamma L^\delta)^{C_4(d,\nu)} \,. \tag{4.37}$$

PROOF. By (4.36), (4.25), (4.20) and (4.24), we get $|b| \leq C(\Gamma L^\delta)^2$. Hence, using also (4.36) and (4.29), we get $|a| = |\Omega^{-1} b| \leq C(\Gamma L^\delta)^C$. This, with (4.35) and (4.27), implies (4.37). ∎

We now complete the proof of Lemma 4.2 when case I holds. As a consequence of Lemma 4.5, for all $q_1, q_2 \in [0, L]$,

$$|(x_{q_1}, j_{q_1}) - (x_{q_2}, j_{q_2})| = |P_G\Big((x_{q_1}, j_{q_1}) - (x_{q_2}, j_{q_2})\Big)| \leq (\Gamma L^\delta)^{C_5(d,\nu)} \,.$$

Therefore, for all $q_1, q_2 \in [0, L]$, $|j_{q_1} - j_{q_2}| \leq (\Gamma L^\delta)^{C_5(d,\nu)}$, and so

$$\text{diam}\{j_q \ ; \ 0 \leq q \leq L\} \leq (\Gamma L^\delta)^{C_5(d,\nu)} \,.$$

Since all the j_q are in $\mathbb{Z}^d$, their number (counted without multiplicity) does not exceed $C(\Gamma L^\delta)^{C_5(d,\nu) d}$. Thus we have obtained the bound

$$\sharp\{j_q \ : \ 0 \leq q \leq L\} \leq C(\Gamma L^\delta)^{C_5(d,\nu) d} \,.$$

By assumption (4.11), for each $q_0 \in [0, L]$, the number of $q \in [0, L]$ such that $j_q = j_{q_0}$ is at most K, and so

$$L \leq (\Gamma L^\delta)^{C_6(d,\nu)} K \,.$$

Choosing $\delta > 0$ such that

$$\delta C_6(d, \nu) < 1/2 \,, \tag{4.38}$$

we get $L \leq (\Gamma^{C_6(d,\nu)} K)^2$, proving (4.12).

Case II. There is $q_0 \in [0, L]$ such that

$$\mu := \dim \text{Span}_{\mathbb{R}}\{(x_q - x_{q_0}, j_q - j_{q_0}) \ : \ |q - q_0| \leq L^\delta \,, \ q \in [0, L] \,\} \leq g - 1 \,,$$

namely all the vectors (x_q, j_q) stay in a affine subspace of dimension $\mu \leq g - 1$. Then we repeat on the sub-chain (l_q, j_q), $|q - q_0| \leq L^\delta$, the argument of case I, to obtain a bound for L^δ (and hence for L).

Applying at most $(d+1)$-times the above procedure, we obtain a bound for L of the form $L \leq (\Gamma K)^{C(d,\nu)}$. This concludes the proof of Lemma 4.2. ∎

PROOF OF PROPOSITION 4.1 COMPLETED. Set $\Gamma := N^2$ in Definition 4.2 and introduce the following equivalence relation on the set of the N-bad sites :

Definition 4.3. *We say that $x \equiv y$ if there is a N^2-chain $\{i_q\}_{q=0,\ldots,L}$ of N-bad sites connecting x to y, namely $i_0 = x$, $i_L = y$.*

A N^2-chain $(l_q, j_q)_{q=0,\ldots,L}$ of N-bad sites of $A(\varepsilon,\lambda,\theta)$ is formed by θ-singular sites, namely (4.10) holds if ε is small enough, see Definition 3.4. Moreover, by Corollary 4.1 (remark it holds for all $\tilde\jmath \in \mathbb{Z}^\nu$), the condition (4.11) of Lemma 4.2 is satisfied with $K := N^{3d+2\nu+4}$. Hence Lemma 4.2 implies

$$L \overset{(4.12)}{\le} (N^2 N^{3d+2\nu+4})^{C_2(d,\nu)} \le N^{C'(d,\nu)} . \tag{4.39}$$

The equivalence relation in Definition 4.3 induces a partition of the N-bad sites of $A(\varepsilon,\lambda,\theta)$ with $|l| \le N'$, in disjoint equivalent classes (Ω_α), satisfying

$$\mathrm{d}(\Omega_\alpha,\Omega_\beta) > N^2 , \quad \mathrm{diam}(\Omega_\alpha) \le N^2 L \overset{(4.39)}{\le} N^2 N^{C'(d,\nu)} \le N^{C_1(d,\nu)} .$$

5 Measure and complexity estimates

We define

$$B_N^0(j_0;\lambda) \;:=\; B_N^0(j_0;\varepsilon,\lambda) := \Big\{\theta \in \mathbb{R} \,:\, \|A_{N,j_0}^{-1}(\varepsilon,\lambda,\theta)\|_0 > N^\tau\Big\} \tag{5.1}$$

$$= \Big\{\theta \in \mathbb{R} \,:\, \exists \text{ an eigenvalue of } A_{N,j_0}(\varepsilon,\lambda,\theta) \text{ with modulus less than } N^{-\tau}\Big\} \tag{5.2}$$

where $\| \ \|_0$ is the operatorial L^2-norm defined in (2.12). The equivalence between (5.1) and (5.2) is a consequence of the self-adjointness of $A_{N,j_0}(\varepsilon,\lambda,\theta)$. We also define

$$\mathcal{G}_N^0 := \mathcal{G}_N^0(u) \;:=\; \Big\{\lambda \in \Lambda \,:\, \forall M \in \{N, N-2L_0\}\,,\ \forall j_0 \in \mathbb{Z}^d \backslash \mathcal{Q}_M\,,\ B_M^0(j_0;\lambda) \subset \bigcup_{q=1,\ldots,N^{2d+\nu+3}} I_q \text{ where } I_q \text{ are intervals with measure } |I_q| \le N^{-\tau}\Big\} \tag{5.3}$$

(the set $\mathcal{Q}_N$ is defined in (3.7)). The aim of this section is to provide, for any large N, the bound (5.5) for the Lebesgue measure of the complementary set of $\mathcal{G}_N^0$. This will be used, along the Nash-Moser iteration, to estimate the measures of the complementary sets $\mathcal{G}_N^c$ (see (4.3)) by (6.12). On the other hand (6.12) itself will be a consequence of the multiscale Proposition 3.1, see Lemma 7.3.

Proposition 5.1. *There are constants $c, C > 0$, $N_0 \in \mathbb{N}$, depending on V, d, ν, such that, for all $N \ge N_0$ and*

$$\varepsilon_0(\|T_1\|_0 + \|\partial_\lambda T_1\|_0) \le c \tag{5.4}$$

(T_1 is defined in (2.6)), the set $\mathcal{B}_N^0 := \Lambda \setminus \mathcal{G}_N^0$ has measure

$$|\mathcal{B}_N^0| \le C\, N^{-1} . \tag{5.5}$$

The sequel of this section is devoted to the proof of Proposition 5.1. It is derived from several lemmas based on basic properties of eigenvalues of self-adjoint matrices, which are a consequence of their variational characterization. In the definitions below, when A is not invertible, we set $\|A^{-1}\|_0 := \infty$.

Lemma 5.1. *Let J be an interval of $\mathbb{R}$ and $A(\xi)$ be a family of self-adjoint square matrices in $\mathcal{M}_E^E$, C^1 in the real parameter $\xi \in J$, and such that $\partial_\xi A(\xi) \ge \beta I$ for some $\beta > 0$. Then, for any $\alpha > 0$, the Lebesgue measure*

$$\Big|\Big\{\xi \in J \,:\, \|A^{-1}(\xi)\|_0 \ge \alpha^{-1}\Big\}\Big| \le 2|E|\alpha\beta^{-1}$$

where $|E|$ denotes the cardinality of the set E.

More precisely there is a family $(I_q)_{1\le q\le |E|}$ of intervals such that

$$|I_q| \le 2\alpha\beta^{-1} \quad \text{and} \quad \Big\{\xi \in J \,:\, \|A^{-1}(\xi)\|_0 \ge \alpha^{-1}\Big\} \subset \bigcup_{1\le q\le |E|} I_q \tag{5.6}$$

PROOF. List the eigenvalues of the self-adjoint matrices $A(\xi)$ as C^1 functions $(\xi \mapsto \mu_q(\xi))$, $1 \le q \le |E|$. We have
$$\Big\{\xi \in J \, : \, \|A^{-1}(\xi)\|_0 \ge \alpha^{-1}\Big\} = \bigcup_{1\le q\le |E|} \Big\{\xi \in J \, : \, \mu_q(\xi) \in [-\alpha,\alpha]\Big\}.$$
Now, since $\partial_\xi A(\xi) \ge \beta I$, we have $\partial_\xi \mu_q(\xi) \ge \beta > 0$, which implies that $I_q := \{\xi \in J \, : \, \mu_q(\xi) \in [-\alpha,\alpha]\}$ is an interval, of length less than $2\alpha\beta^{-1}$. ■

Lemma 5.2. *Let A, A_1 be self adjoint matrices. Then their eigenvalues (ranked in nondecreasing order) satisfy the Lipschitz property*
$$|\mu_k(A) - \mu_k(A_1)| \le \|A - A_1\|_0 \, . \tag{5.7}$$

PROOF. The proof is standard. ■

We shall obtain complexity estimates for the sets $B^0_M(j_0;\lambda)$ when $M = N$, the case $M = N - 2L_0$ being similar. We shall argue differently for $|j_0| \ge 8N$ (Lemma 5.3) and $|j_0| < 8N$ (Corollary 5.1).

In the next lemmas we assume
$$N \ge N_0(V,\nu,d) > 0 \text{ large enough} \quad \text{and} \quad \varepsilon\|T_1\|_0 \le 1 \, . \tag{5.8}$$

Lemma 5.3. $\forall |j_0| \ge 8N$, $\forall \lambda \in \Lambda$, *we have*
$$B^0_N(j_0;\lambda) \subset \bigcup_{q=1,\dots,2(2N+1)^{d+\nu}} I_q \tag{5.9}$$
where I_q are intervals satisfying $|I_q| \le N^{-\tau}$.

PROOF. We first claim that, if $|j_0| \ge 8N$ and $N \ge N_0(V,d,\nu)$ (see (5.8)), then
$$B^0_N(j_0;\lambda) \subset \mathbb{R} \setminus [-4N, 4N] \, . \tag{5.10}$$
Indeed, by Lemma 5.2 the eigenvalues $\lambda_{l,j}(\theta)$ of $A_{N,j_0}(\varepsilon,\lambda,\theta)$ satisfy
$$\lambda_{l,j}(\theta) = \delta_{l,j}(\theta) + O(\varepsilon\|T_1\|_0 + \|V\|_0) \quad \text{where} \quad \delta_{l,j}(\theta) := -(\omega\cdot l + \theta)^2 + \|j\|^2 \, . \tag{5.11}$$
Since $|\omega| = |\lambda||\bar\omega| \le 3/2$ (see (1.6)), $\|j\| \ge |j|$ (see (2.10)), $|j - j_0| \le N$, $|l| \le N$, we get
$$\delta_{l,j}(\theta) \ge (|j_0| - |j - j_0|)^2 - (|\omega||l| + |\theta|)^2 \ge (|j_0| - N)^2 - (2N + |\theta|)^2 \, . \tag{5.12}$$
As a consequence, all the eigenvalues $\lambda_{l,j}(\theta)$ of $A_{N,j_0}(\varepsilon,\lambda,\theta)$ satisfy, for $|j_0| \ge 8N$ and $|\theta| \le 4N$,
$$\lambda_{l,j}(\theta) \overset{(5.11),(5.12)}{\ge} 10N^2 - O(\varepsilon\|T_1\|_0 + \|V\|_0) \overset{(5.8)}{\ge} N^2 \, ,$$
implying (5.10). We now estimate the complexity of
$$B^{0,-}_N := B^0_N(j_0;\lambda) \cap (-\infty, -4N) \quad \text{and} \quad B^{0,+}_N := B^0_N(j_0;\lambda) \cap (4N, \infty) \, .$$
Let us consider $B^{0,-}_N$. For $\theta < -4N$, the derivative
$$\partial_\theta A_{N,j_0}(\varepsilon,\lambda,\theta) = \operatorname{diag}_{|l|\le N, |j-j_0|\le N} - 2(\omega\cdot l + \theta) > 8N - 2|\omega||l| \ge 5N$$
and therefore Lemma 5.1 (applied with $\beta = 5N$, $\alpha = N^{-\tau}$) implies
$$B^{0,-}_N \cap (-\infty, -4N) \subset \bigcup_{1\le q\le (2N+1)^{d+\nu}} I^-_q \, ,$$
where I^-_q are intervals satisfying $|I^-_q| \le N^{-\tau}$. We get the same estimate for $B^{0,+}_N$ and (5.9) follows. ■

We now consider the case $|j_0| < 8N$. We can no longer argue directly as in Lemma 5.3. In this case the aim is to bound the measure of

$$B^0_{2,N}(j_0;\lambda) := B^0_{2,N}(j_0;\varepsilon,\lambda) := \Big\{\theta \in \mathbb{R} \,:\, \|A^{-1}_{N,j_0}(\varepsilon,\lambda,\theta)\|_0 > N^\tau/2\Big\} \tag{5.13}$$

for "most" λ. The continuity property (5.7) of the eigenvalues allows then to derive a "complexity estimate" for $B^0_N(j_0;\lambda)$ in terms of the measure $|B^0_{2,N}(j_0;\lambda)|$ (Lemma 5.5). Lemma 5.6 is devoted to the estimate of the bi-dimensional Lebesgue measure

$$\Big|\Big\{(\lambda,\theta) \in \Lambda \times \mathbb{R} \,:\, \theta \in B^0_{2,N}(j_0,\lambda)\Big\}\Big|$$

when $j_0 \notin \mathcal{Q}_N$. Such an estimate is then used in Lemma 5.10 to justify that the measure of the section $|B^0_{2,N}(j_0,\lambda)|$ has an appropriate bound for "most" λ (by a Fubini type argument).

We first show that, for $|j_0| < 8N$, the set $B^0_{2,N}(j_0;\lambda)$ is contained in an interval of size $O(N)$ centered at the origin.

Lemma 5.4. *$\forall |j_0| < 8N$, $\forall \lambda \in \Lambda$, we have*

$$B^0_{2,N}(j_0;\lambda) \subset I_N := [-12dN, 12dN]\,. \tag{5.14}$$

PROOF. The eigenvalues $\lambda_{l,j}(\theta)$ of $A_{N,j_0}(\varepsilon,\lambda,\theta)$ satisfy (5.11) where, for $|\theta| \geq 12dN$,

$$|\omega \cdot l + \theta| \geq |\theta| - |\omega \cdot l| \geq 12dN - 2N \geq 10dN\,, \tag{5.15}$$

and, by (2.10), we have $\|j\|^2 \leq d(|j_0| + |j - j_0|)^2 \leq d(9N)^2$. Hence

$$\begin{aligned}\lambda_{l,j}(\theta) = -(\omega \cdot l + \theta)^2 + \|j\|^2 + O(\varepsilon\|T_1\|_0 + \|V\|_0) \quad &\overset{(5.15),(5.4)}{\leq} \quad -(10dN)^2 + d(9N)^2 + C(1 + \|V\|_0) \\ &\leq \quad -16d^2N^2\end{aligned}$$

for $N \geq N(V,d,\nu)$ large enough (see (5.8)), implying (5.14). ■

Lemma 5.5. *There is $\hat{C} := \hat{C}(d) > 0$ such that $\forall |j_0| < 8N$, $\forall \lambda \in \Lambda$, we have*

$$B^0_N(j_0;\lambda) \subset \bigcup_{q=1,\ldots,[\hat{C}\,\mathtt{M}N^{\tau+1}]} I_q$$

where I_q are intervals of length $|I_q| \leq N^{-\tau}$ and $\mathtt{M} := |B^0_{2,N}(j_0;\lambda)|$.

PROOF. Assume $\theta \in B^0_N(j_0,\lambda)$, see (5.1). Then there is an eigenvalue of $A_{N,j_0}(\varepsilon,\lambda,\theta)$ with modulus less than $N^{-\tau}$. Now, for $|\Delta\theta| \leq 1$, (recall (2.7))

$$\begin{aligned}\|A_{N,j_0}(\varepsilon,\lambda,\theta+\Delta\theta) - A_{N,j_0}(\varepsilon,\lambda,\theta)\|_0 \quad &= \quad \|\mathrm{Diag}_{|l|\leq N, |j-j_0|\leq N}\,(\lambda\overline{\omega}\cdot l + \theta)^2 - (\lambda\overline{\omega}\cdot l + \theta + \Delta\theta)^2\|_0 \\ &\leq \quad (4N + 2|\theta| + 1)|\Delta\theta|.\end{aligned}$$

Hence, by Lemma 5.2,

$$(4N + 2|\theta| + 1)|\Delta\theta| \leq N^{-\tau} \quad \Longrightarrow \quad \theta + \Delta\theta \in B^0_{2,N}(j_0,\lambda) \tag{5.16}$$

because $A_{N,j_0}(\varepsilon,\lambda,\theta+\Delta\theta)$ has an eigenvalue with modulus less than $2N^{-\tau}$. Now by Lemma 5.4, $|\theta| \leq 12dN$. Hence, by (5.16), there is a positive constant $c := c(d)$ such that, for $\theta \in B^0_N(j_0;\lambda)$,

$$[\theta - cN^{-(\tau+1)}, \theta + cN^{-(\tau+1)}] \subset B^0_{2,N}(j_0,\lambda)\,.$$

Therefore $B^0_N(j_0,\lambda)$ is included in an union of intervals J_m with disjoint interiors,

$$B^0_N(j_0,\lambda) \subset \bigcup_m J_m \subset B^0_{2,N}(j_0,\lambda), \quad \text{with length} \quad |J_m| \geq 2cN^{-(\tau+1)} \tag{5.17}$$

(if some of the intervals $[\theta - cN^{-(\tau+1)}, \theta + cN^{-(\tau+1)}]$ overlap, then we glue them together). We decompose each J_m as an union of (non overlapping) intervals I_q of length between $cN^{-(\tau+1)}/2$ and $cN^{-(\tau+1)}$. Then, by (5.17), we get a new covering

$$B^0_N(j_0,\lambda) \subset \bigcup_{q=1,\ldots,Q} I_q \subset B^0_{2,N}(j_0,\lambda) \quad \text{with} \quad cN^{-(\tau+1)}/2 \le |I_q| \le cN^{-(\tau+1)} \le N^{-\tau}$$

and, since the intervals I_q do not overlap,

$$QcN^{-(\tau+1)}/2 \le \sum_{q=1}^{Q} |I_q| \le |B^0_{2,N}(j_0,\lambda)| =: \mathtt{M}\,.$$

As a consecuence $Q \le \hat{C}\,\mathtt{M}\,N^{\tau+1}$, proving the lemma. ∎

The next lemma has major importance. The main difference with respect to the analogous lemma in [5] is that we do not assume the positivity of $-\Delta + V(x)$, but only (1.5). Hence we have to require $j_0 \notin \mathcal{Q}_N$, in order to be able to apply the spectral Lemma 2.3. We use that the spectrum of the operator P_{N,j_0} in (5.22) is bounded away from zero (see (5.23)) in order to prove Lemma 5.8 by eigenvalue variation arguments.

Lemma 5.6. *$\forall |j_0| < 8N$, $j_0 \notin \mathcal{Q}_N$, the set*

$$\mathbf{B}^0_{2,N}(j_0) := \mathbf{B}^0_{2,N}(j_0;\varepsilon) := \Big\{(\lambda,\theta) \in \Lambda \times \mathbb{R} \,:\, \Big\| A^{-1}_{N,j_0}(\varepsilon,\lambda,\theta) \Big\|_0 > N^{\tau}/2 \Big\} \tag{5.18}$$

has measure

$$|\mathbf{B}^0_{2,N}(j_0)| \le CN^{-\tau+d+\nu+1}\,. \tag{5.19}$$

Proof. By Lemma 5.4, $\mathbf{B}^0_{2,N}(j_0) \subset \Lambda \times I_N$. In order to estimate the "bad" (λ,θ) where at least one eigenvalue of $A_{N,j_0}(\varepsilon,\lambda,\theta)$ has modulus less than $2N^{-\tau}$, we introduce the variables

$$\xi := \frac{1}{\lambda^2}\,, \quad \eta := \frac{\theta}{\lambda} \quad \text{where} \quad (\xi,\eta) \in [4/9, 4] \times 2I_N\,. \tag{5.20}$$

Hence $\theta = \lambda\eta$, $\lambda := 1/\sqrt{\xi}$, and we consider the self adjoint matrix

$$A(\xi,\eta) := \frac{1}{\lambda^2} A_{N,j_0}(\varepsilon,\lambda,\theta) = \mathrm{diag}_{|l|\le N, |j-j_0|\le N}\Big(-(\bar{\omega}\cdot l + \eta)^2 \Big) + \xi P_{N,j_0} - \varepsilon\xi T_1(\varepsilon, 1/\sqrt{\xi}) \tag{5.21}$$

where, according to the notations (2.14)-(2.16),

$$P_{N,j_0} := \Pi_{N,j_0}(-\Delta + V(x))_{|E_{N,j_0}}\,. \tag{5.22}$$

The self-adjoint operator P_{N,j_0} possesses a L^2-orthonormal basis of eigenvectors

$$P_{N,j_0}\Psi_j = \hat{\mu}_j \Psi_j$$

with real eigenvalues $(\hat{\mu}_j)_{j=1,\ldots(2N+1)^d}$ (depending on N) indexed in non-decreasing order. We define

$$\mathcal{I}_- := \Big\{ j \,:\, \hat{\mu}_j < 0 \Big\}\,, \qquad \mathcal{I}_+ := \Big\{ j \,:\, \hat{\mu}_j > 0 \Big\}\,.$$

Recalling the assumption $j_0 \notin \mathcal{Q}_N$ (see (3.7)) Lemma 2.3 implies that:

1. if $\mathtt{B}(0, L_0 - 1) \subset \mathbb{Z}^d \setminus \{|j - j_0| \le N\}$ then $P_{N,j_0} \ge \beta_0 I$. In this case $\mathcal{I}_- = \emptyset$, $\mathcal{I}_+ = \{1,\ldots,(2N+1)^d\}$ and $\min_{j\in\mathcal{I}_+} \hat{\mu}_j \ge \beta_0$.

2. if $\mathtt{B}(0,L_0)\subset\{|j-j_0|\le N\}$ then P_{N,j_0} has n^- negative eigenvalues $\hat\mu_j\le-\beta_0$ and the others $\hat\mu_j\ge\beta_0$ (we recall that n^- is the number of negative eigenvalues of $-\Delta+V(x)$). We shall use that

$$\max_{j\in\mathcal{I}_-}\hat\mu_j\le-\beta_0\quad\text{and}\quad\min_{j\in\mathcal{I}_+}\hat\mu_j\ge\beta_0\,. \tag{5.23}$$

We shall consider only the most difficult case 2 when $\mathcal{I}_-\neq\emptyset$. We denote

$$H_-:=H_{\mathcal{I}_-}:=\Big\{u:=\sum_{|l|\le N,j\in\mathcal{I}_-}u_{l,j}e^{\mathrm{i}l\cdot\varphi}\Psi_j\Big\},\quad H_+:=H_{\mathcal{I}_+}:=\Big\{u:=\sum_{|l|\le N,j\in\mathcal{I}_+}u_{l,j}e^{\mathrm{i}l\cdot\varphi}\Psi_j\Big\},$$

and Π_-, Π_+ the corresponding L^2-projectors. Correspondingly we represent $A:=A(\xi,\eta)$ in (5.21) as

$$A=\begin{pmatrix}A_- & A_-^+\\ A_+^- & A_+\end{pmatrix}:=\begin{pmatrix}\Pi_-A_{|H_-} & \Pi_-A_{|H_+}\\ \Pi_+A_{|H_-} & \Pi_+A_{|H_+}\end{pmatrix} \tag{5.24}$$

where $A_-^+=(A_+^-)^\dagger$, $A_-^\dagger:=A_-$, $A_+^\dagger=A_+$.

Lemma 5.7. *For all $\xi\in[4/9,4]$, $\eta\in\mathbb{R}$, the matrix $A_-:=\Pi_-A_{|H_-}$ is invertible and*

$$\|A_-^{-1}\|_0\le3\beta_0^{-1}\,. \tag{5.25}$$

Proof. By (5.21) and Lemma 5.2, the eigenvalues of the matrix A_- satisfy, for $|l|\le N$, $j\in\mathcal{I}_-$,

$$\begin{aligned}-(\bar\omega\cdot l+\eta)^2+\xi\hat\mu_j+O(\varepsilon\|T_1\|_0)\quad&\le\quad\xi\hat\mu_j+O(\varepsilon\|T_1\|_0)\le\xi\max_{j\in\mathcal{I}_-}\hat\mu_j+O(\varepsilon\|T_1\|_0)\\ &\overset{(5.23),(5.4)}{<}\quad-\beta_0/3\,,\end{aligned}$$

i.e. are negative and uniformly bounded away from zero. Then (5.25) follows. ■

Proof of Lemma 5.6 continued. The invertibility of the matrix in (5.24) is reduced to that of the self-adjoint matrix

$$L:=L(\xi,\eta):=A_+-A_+^-A_-^{-1}A_-^+ \tag{5.26}$$

via the "resolvent type" identity

$$A^{-1}=\begin{pmatrix}I & -A_-^{-1}A_-^+\\ 0 & I\end{pmatrix}\begin{pmatrix}A_-^{-1} & 0\\ 0 & L^{-1}\end{pmatrix}\begin{pmatrix}I & 0\\ -A_+^-A_-^{-1} & I\end{pmatrix}. \tag{5.27}$$

We now deduce the invertibility of the matrix $L(\xi,\eta)$ for "most" parameters (ξ,η) (with an appropriate L^2-bound for the inverse) showing that $\partial_\xi L(\xi,\eta)$ is positive definite and by eigenvalue variation arguments.

Lemma 5.8. $\|L(\xi,\eta)^{-1}\|_0\le N^\tau/20$ *except for $(\xi,\eta)\in[4/9,4]\times2I_N$ in a set of measure $O(N^{-\tau+d+\nu+1})$.*

Proof. The derivative with respect to ξ of the matrix $L(\xi,\eta)$ in (5.26) is

$$\begin{aligned}\partial_\xi L&=\partial_\xi A_+-(\partial_\xi A_+^-)A_-^{-1}A_-^+-A_+^-(\partial_\xi A_-^{-1})A_-^+-A_+^-A_-^{-1}(\partial_\xi A_-^+)\\ &=\partial_\xi A_+-(\partial_\xi A_+^-)A_-^{-1}A_-^++A_+^-A_-^{-1}(\partial_\xi A_-)A_-^{-1}A_-^+-A_+^-A_-^{-1}(\partial_\xi A_-^+)\,.\end{aligned} \tag{5.28}$$

Moreover, since $\Pi_+((\omega\cdot\partial_\varphi)^2-\Delta+V(x))_{|H_-}=0$ (and similarly exchanging $\pm$), we have

$$A_-^+=-\varepsilon\xi\Pi_+(T_1(\varepsilon,\xi^{-1/2}))_{|H_-}\,,\quad A_+^-=-\varepsilon\xi\Pi_-(T_1(\varepsilon,\xi^{-1/2}))_{|H_+}\,. \tag{5.29}$$

Hence, since $4\ge\xi\ge4/9$,

$$\|A_-^+\|_0+\|A_+^-\|_0+\|\partial_\xi A_-^+\|_0+\|\partial_\xi A_+^-\|_0=0(\varepsilon(\|T_1\|_0+\|\partial_\lambda T_1\|_0)). \tag{5.30}$$

In addition, by (5.21)-(5.22),

$$\|\partial_\xi A_-\|_0 = \|\Pi_- P_{N,j_0}|_{H_-}\|_0 + O(\varepsilon(\|T_1\|_0 + \|\partial_\lambda T_1\|_0)) \le C\,, \tag{5.31}$$

$$\partial_\xi A_+ = \Pi_+ P_{N,j_0}|_{H_+} + O(\varepsilon(\|T_1\|_0 + \|\partial_\lambda T_1\|_0)\,. \tag{5.32}$$

Hence by (5.28), (5.32), (5.30), (5.25), (5.31), for $\varepsilon(\|T_1\|_0 + \|\partial_\lambda T_1\|_0)$ small,

$$\partial_\xi L = \Pi_+ P_{N,j_0}|_{H_+} + O(\varepsilon(\|T_1\|_0 + \|\partial_\lambda T_1\|_0) \overset{(5.23),(5.4)}{\ge} \frac{\beta_0}{2} I\,. \tag{5.33}$$

By (5.33) and Lemma 5.1, for each fixed η, the set of $\xi \in [4/9, 4]$ such that at least one eigenvalue of the matrix $L(\xi,\eta)$ in (5.26) has modulus $\le 20N^{-\tau}$ has measure at most $O(N^{-\tau+d+\nu}\beta_0^{-1})$. Then, integrating on $\eta \in 2I_N$, whose length is $|I_N| = O(N)$, we prove the lemma. ■

End of the proof of Lemma 5.6. From (5.27), (5.25), (5.29), Lemma 5.8 and (5.4), we derive the bound

$$\|A^{-1}\|_0 \le 2(\|L^{-1}(\xi,\eta)\|_0 + \|A_-^{-1}\|_0) \le 2\Big(\frac{N^\tau}{20} + 3\beta_0^{-1}\Big) \overset{(5.8)}{\le} \frac{N^\tau}{9} \tag{5.34}$$

except in a set of (ξ,η) of measure $O(N^{-\tau+d+\nu+1})$. We finally turn to the original parameters (λ,θ). Since the change of variables (5.20) has Jacobian of modulus greater than 1/8, we have

$$\|A_{N,j_0}^{-1}(\varepsilon,\lambda,\theta)\|_0 \overset{(5.21)}{=} \lambda^{-2}\|A^{-1}\|_0 \overset{(1.6),(5.34)}{\le} 4\,\frac{N^\tau}{9} \le \frac{N^\tau}{2}\,,$$

except for $(\lambda,\theta) \in \Lambda \times \mathbb{R}$ in a set of measure $\le CN^{-\tau+d+\nu+1}$. The proof of Lemma 5.6 is complete. ■

By the same arguments we also get the following measure estimate that will be used in the Nash-Moser iteration, see (6.27).

Lemma 5.9. *The complementary of the set*

$$\mathtt{G}_N := \mathtt{G}_N(u) := \Big\{\lambda \in \Lambda \,:\, \|A_N^{-1}(\varepsilon,\lambda)\|_0 \le N^\tau\Big\} \tag{5.35}$$

has measure

$$|\Lambda \setminus \mathtt{G}_N| \le N^{-\tau+d+\nu+2}\,. \tag{5.36}$$

As a consequence of Lemma 5.6, for “most” λ the measure of $B^0_{2,N}(j_0;\lambda)$ is “small”.

Lemma 5.10. $\forall |j_0| < 8N$, $j_0 \notin \mathcal{Q}_N$, *the set*

$$\mathcal{F}_N(j_0) := \Big\{\lambda \in \Lambda \,:\, |B^0_{2,N}(j_0;\lambda)| \ge \hat{C}^{-1}N^{-\tau+2d+\nu+2}\Big\}\,,$$

where $\hat{C}$ is the positive constant of Lemma 5.5, has measure

$$|\mathcal{F}_N(j_0)| \le CN^{-d-1}\,. \tag{5.37}$$

Proof. By Fubini theorem (see (5.18) and (5.13))

$$|\mathbf{B}^0_{2,N}(j_0)| = \int_\Lambda |B^0_{2,N}(j_0;\lambda)|\,d\lambda\,. \tag{5.38}$$

Let $\mu := \tau - 2d - \nu - 2$. By (5.38) and (5.19),

$$\begin{aligned} CN^{-\tau+d+\nu+1} &\ge \int_\Lambda |B^0_{2,N}(j_0;\lambda)|\,d\lambda \\ &\ge \hat{C}^{-1}N^{-\mu}\Big|\Big\{\lambda \in \Lambda \,:\, |B^0_{2,N}(j_0;\lambda)| \ge \hat{C}^{-1}N^{-\mu}\Big\}\Big| := \hat{C}^{-1}N^{-\mu}|\mathcal{F}_N(j_0)| \end{aligned}$$

whence (5.37). ■

For all $\lambda \notin \mathcal{F}_N(j_0)$, $|B^0_{2,N}(j_0;\lambda)| < N^{-\tau+2d+\nu+2}\hat{C}^{-1}$. Then Lemma 5.5 implies

Corollary 5.1. *$\forall |j_0| < 8N$, $j_0 \notin \mathcal{Q}_N$, $\forall \lambda \notin \mathcal{F}_N(j_0)$, we have*

$$B_N^0(j_0;\lambda) \subset \bigcup_{q=1,\ldots,N^{2d+\nu+3}} I_q$$

with I_q intervals satisfying $|I_q| \leq N^{-\tau}$.

Proposition 5.1 is now a direct consequence of the following lemma.

Lemma 5.11. $\mathcal{B}_N^0 \subseteq \bigcup_{|j_0|<8N, j_0\notin\mathcal{Q}_N} \mathcal{F}_N(j_0)$.

Proof. Lemma 5.3 and Corollary 5.1 imply that

$$\lambda \notin \bigcup_{|j_0|<8N, j_0\notin\mathcal{Q}_N} \mathcal{F}_N(j_0) \quad \Longrightarrow \quad \lambda \in \mathcal{G}_N^0$$

(see the definition in (5.3)). The lemma follows. ■

Proof of Proposition 5.1 completed. By Lemma 5.11 and (5.37) we get

$$|\mathcal{B}_N^0| \leq \sum_{|j_0|<8N, j_0\notin\mathcal{Q}_N} |\mathcal{F}_N(j_0)| \leq C(8N)^d N^{-d-1} \leq CN^{-1}\,.$$

6 Nash Moser iterative scheme and proof of Theorem 1.1

Consider the orthogonal splitting

$$H^s = H_n \oplus H_n^\perp$$

where H^s is defined in (1.12) and

$$H_n := \Big\{ u = \sum_{|(l,j)|\leq N_n} u_{l,j}\, e^{\mathrm{i}(l\cdot\varphi + j\cdot x)} \Big\}, \quad H_n^\perp := \Big\{ u = \sum_{|(l,j)|> N_n} u_{l,j}\, e^{\mathrm{i}(l\cdot\varphi + j\cdot x)} \in H^s \Big\}$$

with

$$N_n := N_0^{2^n}, \qquad \text{namely} \quad N_{n+1} = N_n^2\,, \ \forall n \geq 0\,. \tag{6.1}$$

We shall take $N_0 \in \mathbb{N}$ large enough depending on ε_0 and V, d, ν. Moreover we always assume $N_0 > L_0$ defined in Lemma 2.3. We denote by

$$P_n : H^s \to H_n \qquad \text{and} \qquad P_n^\perp : H^s \to H_n^\perp$$

the orthogonal projectors onto H_n and $H_n^\perp$. The following "smoothing" properties hold, $\forall n \in \mathbb{N}$, $s \geq 0$, $r \geq 0$,

$$\|P_n u\|_{s+r} \leq N_n^r \|u\|_s\,, \ \forall u \in H^s\,, \qquad \|P_n^\perp u\|_s \leq N_n^{-r}\|u\|_{s+r}\,, \ \forall u \in H^{s+r}\,.$$

For $f \in C^q(\mathbb{T}^\nu \times \mathbb{T}^d \times \mathbb{R};\mathbb{R})$ with

$$q \geq S + 2\,, \tag{6.2}$$

the composition operator on Sobolev spaces

$$f : H^s \to H^s\,, \qquad f(u)(\varphi, x) := f(\varphi, x, u(\varphi, x))$$

satisfies the following standard properties: $\forall s \in [s_1, S]$, $s_1 > (d+\nu)/2$,

- **(F1)** (**Regularity**) $f \in C^2(H^s; H^s)$.

- **(F2)** (**Tame estimates**) $\forall u, h \in H^s$ with $\|u\|_{s_1} \leq 1$,

$$\|f(u)\|_s \leq C(s)(1 + \|u\|_s), \quad \|(Df)(u)h\|_s \leq C(s)(\|h\|_s + \|u\|_s\|h\|_{s_1}), \tag{6.3}$$

$$\|D^2 f(u)[h,v]\|_s \leq C(s)\Big(\|u\|_s\|h\|_{s_1}\|v\|_{s_1} + \|v\|_s\|h\|_{s_1} + \|v\|_{s_1}\|h\|_s\Big).$$

- **(F3)** (**Taylor Tame estimate**) $\forall u \in H^s$ with $\|u\|_{s_1} \leq 1$, $\forall h \in H^s$ with $\|h\|_{s_1} \leq 1$,

$$\|f(u+h) - f(u) - (Df)(u)\,h\|_s \leq C(s)(\|u\|_s\|h\|_{s_1}^2 + \|h\|_{s_1}\|h\|_s).$$

In particular, for $s = s_1$, $\|f(u+h) - f(u) - (Df)(u)\,h\|_{s_1} \leq C(s_1)\|h\|_{s_1}^2$.

We fix the Sobolev indices $s_0 < s_1 < S$ as

$$s_0 := b = d + \nu, \qquad s_1 := 10(\tau + b)C_2, \qquad S := 12\tau' + 8(s_1 + 1), \tag{6.4}$$

where

$$C_2 := 6(C_1 + 2), \ \tau := \max\{d + \nu + 3, 2C_2\nu + 1\}, \ \tau' := 3\tau + 2b, \tag{6.5}$$

and $C_1 := C_1(d,\nu) \geq 2$ is defined in Proposition 4.1. Note that s_0, s_1, S defined in (6.4) depend only on d and ν. We also fix the constant δ in Definition 3.1 as

$$\delta := 1/4. \tag{6.6}$$

Remark 6.1. *By (6.4)-(6.6) the hypotheses (3.3)-(3.5) of Proposition 3.1 are satisfied for any $\chi \in [C_2, 2C_2)$, as well as assumption (ii) of Proposition 4.1. We assume $\tau \geq d + \nu + 3$ in view of (5.36).*

Setting

$$\tau_1 := 3\nu + d + 1 \tag{6.7}$$

and $\gamma > 0$, we implement the first steps of the Nash-Moser iteration restricting λ to the set

$$\begin{aligned} \bar{\mathcal{G}} &:= \Big\{\lambda \in \Lambda \, : \, \Big\|\Big(-\lambda^2(\bar{\omega}\cdot l)^2 + \Pi_0(-\Delta + V(x))_{|E_0}\Big)^{-1}\Big\|_{L^2_x} \leq \frac{N_0^{\tau_1}}{\gamma}, \ \forall |l| \leq N_0\Big\} \\ &= \Big\{\lambda \in \Lambda \, : \, |-\lambda^2(\bar{\omega}\cdot l)^2 + \hat{\mu}_j| \geq \gamma N_0^{-\tau_1}, \ \forall |j| \leq N_0, \ |l| \leq N_0\Big\} \end{aligned} \tag{6.8}$$

where $\hat{\mu}_j$ are the eigenvalues of $\Pi_0(-\Delta + V(x))_{|E_0}$ and $\Pi_0 := \Pi_{N_0,0}$, $E_0 := E_{N_0,0}$ are defined in (2.14). We shall prove in Lemma 6.2 that $|\bar{\mathcal{G}}| = 1 - O(\gamma)$ (since $\tau_1 > 3\nu + d$).

We prove the separation properties of the small divisors for λ satisfying assumption (NR), namely in

$$\tilde{\mathcal{G}} \ := \ \Big\{\lambda \in \Lambda \, : \, \Big|n + \lambda^2 \sum_{1 \leq i \leq j \leq \nu} p_{ij}\bar{\omega}_i\bar{\omega}_j\Big| \geq \frac{\gamma}{1 + |p|^{\tau_0}}, \ \ \forall (n,p) \neq 0\Big\}. \tag{6.9}$$

The constant γ will be fixed in (6.26). We also set

$$\sigma := \tau' + \delta s_1 + 2. \tag{6.10}$$

Given a set A we denote $\mathcal{N}(A,\eta)$ the open neighborhood of A of width η (which is empty if A is empty).

Theorem 6.1. (Nash-Moser) *There exist $\varepsilon_0, \bar{c}, \bar{\gamma} > 0$ (depending on d, ν, V, γ_0) such that, if*

$$\gamma \in (0, \bar{\gamma}), \ N_0 \geq 2\gamma^{-1}, \qquad \textit{and} \qquad \varepsilon \in [0, \varepsilon_0), \ \varepsilon N_0^S \leq \bar{c}, \tag{6.11}$$

then there is a sequence $(u_n)_{n\geq 0}$ of C^1 maps $u_n(\varepsilon,\cdot) : \Lambda \to H^{s_1}$ satisfying

$(\mathbf{S1})_n$ *$u_n(\varepsilon,\lambda) \in H_n$, $u_n(0,\lambda) = 0$, $\|u_n\|_{s_1} \leq 1$, $\|u_0\|_{s_1} \leq N_0^{-\sigma}$ and $\|\partial_\lambda u_n\|_{s_1} \leq C(s_1)N_0^{\tau_1+s_1+1}\gamma^{-1}$.*

$(\mathbf{S2})_n$ *$(n \geq 1)$ For all $1 \leq k \leq n$, $\|u_k - u_{k-1}\|_{s_1} \leq N_k^{-\sigma-1}$, $\|\partial_\lambda(u_k - u_{k-1})\|_{s_1} \leq N_k^{-1/2}$.*

$(\mathbf{S3})_n \quad (n \geq 1)$

$$\|u - u_{n-1}\|_{s_1} \leq N_n^{-\sigma} \quad \Longrightarrow \quad \bigcap_{k=1}^{n} \mathcal{G}^0_{N_k}(u_{k-1}) \cap \tilde{\mathcal{G}} \subseteq \mathcal{G}_{N_n}(u) \tag{6.12}$$

where $\mathcal{G}^0_N(u)$ *(resp.* $\mathcal{G}_N(u)$*) is defined in (5.3) (resp. in (4.3)) and* $\tilde{\mathcal{G}}$ *in* (6.9).

$(\mathbf{S4})_n$ *Define the set*

$$\mathcal{C}_n := \bigcap_{k=1}^{n} \mathtt{G}_{N_k}(u_{k-1}) \bigcap_{k=1}^{n} \mathcal{G}^0_{N_k}(u_{k-1}) \bigcap \tilde{\mathcal{G}} \cap \bar{\mathcal{G}}\,, \tag{6.13}$$

where $\mathtt{G}_{N_k}(u_{k-1})$ *is defined in (5.35),* $\bar{\mathcal{G}}$ *in (6.8),* $\tilde{\mathcal{G}}$ *in* (6.9), $\mathcal{G}^0_{N_k}(u_{k-1})$ *in (5.3).*
If $\lambda \in \mathcal{N}(\mathcal{C}_n, N_n^{-\sigma})$ *then* $u_n(\varepsilon, \lambda)$ *solves the equation*

$$(P_n) \qquad\qquad P_n\Big(L_\omega u - \varepsilon f(u)\Big) = 0\,.$$

$(\mathbf{S5})_n \quad U_n := \|u_n\|_S$, $U'_n := \|\partial_\lambda u_n\|_S$ *(where* S *is defined in (6.4)) satisfy*

$$(i) \;\; U_n \leq N_n^{2(\tau' + \delta s_1 + 1)}\,, \qquad (ii) \;\; U'_n \leq N_n^{4\tau' + 2s_1 + 4}\,.$$

The sequence $(u_n)_{n \geq 0}$ *converges in* C^1 *norm to a map*

$$u(\varepsilon, \cdot) \in C^1(\Lambda, H^{s_1}) \quad \text{with} \quad u(0, \lambda) = 0 \tag{6.14}$$

and, if λ *belongs to the Cantor like set*

$$\mathcal{C}_\varepsilon := \bigcap_{n \geq 0} \mathcal{C}_n \tag{6.15}$$

then $u(\varepsilon, \lambda)$ *is a solution of* (1.11), *with* $\omega = \lambda \bar{\omega}$.

The proof of Theorem 6.1 follows exactly the steps in [5], section 7. A difference is that we do not need to estimate $\partial_\varepsilon u_n$. Another difference is that the frequencies in $\mathcal{C}_n$ (see (6.13)) belong also to $\tilde{\mathcal{G}}$ (in order to prove the separation properties). For the reader convenience, in the Appendix, we spell out the main steps indicating the other minor adaptations in the proof. The main one concerns the proof of Lemma 7.3 where we estimate $A^{-1}_{M,j_0}(\varepsilon, \lambda, \theta)$ for both $M = N_{n+1}$ and $N_{n+1} - 2L_0$ (and not only N_{n+1}).

The sets of parameters $\mathcal{C}_n$ in $(S4)_n$ are decreasing, i.e.

$$\ldots \subseteq \mathcal{C}_n \subseteq \mathcal{C}_{n-1} \subseteq \ldots \subseteq \mathcal{C}_0 \subset \tilde{\mathcal{G}} \cap \bar{\mathcal{G}} \subset \Lambda\,,$$

and it could happen that $\mathcal{C}_{n_0} = \emptyset$ for some $n_0 \geq 1$. In such a case $u_n = u_{n_0}$, $\forall n \geq n_0$ (however the map $u(\varepsilon, \cdot)$ in (6.14) is always defined), and $\mathcal{C}_\varepsilon = \emptyset$. We shall prove in (6.27) that (with the choices in (6.26)) the set $\mathcal{C}_\varepsilon$ has asymptotically full measure. For that we use in particular Proposition 5.1.

In order to prove Theorem 1.1, we first verify the existence of frequencies satisfying (1.8).

Lemma 6.1. *For* $\tau_0 > \nu(\nu+1) - 1$, *the complementary of the set of* $\omega \in \mathbb{R}^\nu$, $|\omega| \leq 1$, *verifying* (1.8) *has measure* $O(\gamma_0^{1/2})$.

Proof. We have to estimate the measure of

$$\bigcup_{p \in \mathbb{Z}^{\nu(\nu+1)/2} \setminus \{0\}} \mathcal{R}_p \quad \text{where} \quad \mathcal{R}_p := \Big\{ \omega \in \mathbb{R}^\nu\,,\, |\omega| \leq 1 \;:\; \Big| \sum_{1 \leq i \leq j \leq \nu} \omega_i \omega_j p_{ij} \Big| < \frac{\gamma_0}{|p|^{\tau_0}} \Big\}\,.$$

Let $M := M_p$ be the $(\nu \times \nu)-$*symmetric* matrix such that

$$\sum_{1 \leq i \leq j \leq \nu} \omega_i \omega_j p_{ij} = M\omega \cdot \omega\,, \quad \forall \omega \in \mathbb{R}^\nu\,.$$

The symmetric matrix M has coefficients

$$M_{ij} := \frac{p_{ij}}{2}(1+\delta_{ij}), \ \forall 1 \le i \le j \le \nu, \quad \text{and} \quad M_{ij} = M_{ji}. \tag{6.16}$$

There is an orthonormal basis of eigenvectors $V := (v_1, \ldots, v_k)$ of $Mv_k = \lambda_k v_k$ with real eigenvalues $\lambda_k := \lambda_k(p)$. Under the isometric change of variables $\omega = Vy$ we have to estimate

$$|\mathcal{R}_p| = \Big| \Big\{ y \in \mathbb{R}^\nu, \ |y| \le 1 \ : \ \Big| \sum_{1 \le k \le \nu} \lambda_k y_k^2 \Big| < \frac{\gamma_0}{|p|^{\tau_0}} \Big\} \Big|. \tag{6.17}$$

Since $M^2 v_k = \lambda_k^2 v_k$, $\forall k = 1, \ldots, \nu$, we get

$$\sum_{k=1}^{\nu} \lambda_k^2 = \mathrm{Tr}(M^2) = \sum_{i,j=1}^{\nu} M_{ij}^2 \overset{(6.16)}{\ge} |p|^2/2.$$

Hence there is an index $k_0 \in \{1, \ldots, \nu\}$ such that $|\lambda_{k_0}| \ge |p|/\sqrt{2\nu}$ and the derivative

$$\Big| \partial^2_{y_{k_0}} \Big(\sum_{1 \le i \le \nu} \lambda_k y_k^2 \Big) \Big| = |2\lambda_{k_0}| \ge \sqrt{2}\, |p|/\sqrt{\nu}. \tag{6.18}$$

As a consequence of (6.17) and (6.18) we deduce the measure estimate $|\mathcal{R}_p| \le C\sqrt{\frac{\gamma_0}{|p|^{\tau_0+1}}}$ (see e.g. Lemma 9.1 in [15]) and

$$\Big| \bigcup_{p \in \mathbb{Z}^{\nu(\nu+1)/2} \setminus \{0\}} \mathcal{R}_p \Big| \le \sum_{p \in \mathbb{Z}^{\nu(\nu+1)/2} \setminus \{0\}} |\mathcal{R}_p| \le \sum_{p \in \mathbb{Z}^{\nu(\nu+1)/2} \setminus \{0\}} C\sqrt{\frac{\gamma_0}{|p|^{\tau_0+1}}} \le C'\sqrt{\gamma_0}$$

for $\tau_0 > \nu(\nu+1) - 1$. ■

In Lemmata 6.2 and 6.3 we estimate the measures of the complementary sets of $\bar{\mathcal{G}}$ and $\widetilde{\mathcal{G}}$ (see (6.8), (6.9)) used in (6.27) to prove that the measure of the complementary set of $\mathcal{C}_\varepsilon$ tends to 0 as $\varepsilon \to 0$.

Lemma 6.2. *The complementary of the set $\bar{\mathcal{G}}$ defined in* (6.8) *satisfies*

$$|\Lambda \setminus \bar{\mathcal{G}}| = O(\gamma). \tag{6.19}$$

Proof. The λ such that (6.8) is violated are

$$\Lambda \setminus \bar{\mathcal{G}} = \bigcup_{|l|, |j| \le N_0} \mathcal{R}_{l,j} \quad \text{where} \quad \mathcal{R}_{l,j} := \Big\{ \lambda \in [1/2, 3/2] \ : \ |\lambda^2 (\bar{\omega} \cdot l)^2 - \hat{\mu}_j| < \frac{\gamma}{N_0^{\tau_1}} \Big\}. \tag{6.20}$$

By Lemma 2.3 the eigenvalues $|\hat{\mu}_j| > \beta_0$ (for $N_0 > L_0$). Therefore, $\mathcal{R}_{0,j} = \emptyset$ if $\gamma N_0^{-\tau_1} < \beta_0$. We have to estimate the $\xi := \lambda^2 \in [4/9, 4]$ such that $|\xi(\bar{\omega} \cdot l)^2 - \hat{\mu}_j| < \gamma N_0^{-\tau_1}$. The derivative of the function $g_{lj}(\xi) := \xi(\bar{\omega} \cdot l)^2 - \hat{\mu}_j$ satisfies $\partial_\xi g_{lj}(\xi) = (\bar{\omega} \cdot l)^2 \ge 4\gamma_0^2 N_0^{-2\nu}$ by (1.7). As a consequence

$$|\mathcal{R}_{l,j}| \le C\gamma_0^{-2} \gamma N_0^{-\tau_1 + 2\nu}. \tag{6.21}$$

Then (6.20), (6.21), imply

$$|\Lambda \setminus \bar{\mathcal{G}}| \le \sum_{|l| \le N_0, |j| \le N_0} |\mathcal{R}_{l,j}| \le C\gamma\gamma_0^{-2} N_0^{d+\nu} N_0^{-\tau_1 + 2\nu} = O(\gamma)$$

since $\tau_1 > 3\nu + d$ (see (6.7)). ■

Lemma 6.3. *Let $\gamma \in (0, 1/4)$. Then the complementary of the set $\tilde{\mathcal{G}}$ in* (6.9) *has a measure*

$$|\Lambda \setminus \tilde{\mathcal{G}}| = O(\gamma) \,. \tag{6.22}$$

Proof. For $p := (p_{ij})_{1 \le i \le j \le \nu} \in \mathbb{Z}^{\nu(\nu+1)/2}$, let

$$a_p := \sum_{1 \le i \le j \le \nu} p_{ij} \bar{\omega}_i \bar{\omega}_j \,, \quad g_{n,p}(\xi) := n + \xi a_p \,.$$

We have

$$|\Lambda \setminus \tilde{\mathcal{G}}| \le C \sum_{(n,p) \neq (0,0)} |\mathcal{R}_{n,p}| \quad \text{where} \quad \mathcal{R}_{n,p} := \Big\{ \xi := \lambda^2 \in [1/4, 9/4] \, : \, |g_{n,p}(\xi)| < \frac{\gamma}{1 + |p|^{\tau_0}} \Big\} \tag{6.23}$$

Case I: $n \neq 0$. If $\mathcal{R}_{n,p} \neq \emptyset$ then, since $\gamma \in (0, 1/4)$ and $|\xi| \le 3$, we deduce $|a_p| \ge 1/4$, $|n| \le 4|a_p|$ and

$$|\mathcal{R}_{n,p}| \le \frac{2\gamma}{(1 + |p|^{\tau_0})|a_p|} \,.$$

Hence

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} |\mathcal{R}_{n,p}| = \sum_{n \in \mathbb{Z} \setminus \{0\}, |n| \le 4|a_p|} |\mathcal{R}_{n,p}| \le \frac{C\gamma}{(1 + |p|)^{\tau_0}} \,. \tag{6.24}$$

Case II: $n = 0$. In this case, using (1.8) we obtain

$$\mathcal{R}_{0,p} \subset \Big(0, \frac{\gamma}{1 + |p|^{\tau_0}} \frac{|p|^{\tau_0}}{\gamma_0}\Big] \subset \Big(0, \frac{\gamma}{\gamma_0}\Big] \,. \tag{6.25}$$

From (6.23), (6.24), (6.25), $\tau_0 := \nu(\nu+1)$, we deduce (6.22). ■

Proof of Theorem 1.1. We now verify that $\mathcal{C}_\varepsilon$ has asymptotically full measure, i.e. (1.15) holds, choosing

$$\gamma := \varepsilon^\alpha \quad \text{with} \quad \alpha := 1/(S+1) \,, \quad N_0 := 4\gamma^{-1} \,, \tag{6.26}$$

so that (6.11) is fulfilled for ε small enough.

The complementary set of $\mathcal{C}_\varepsilon$ in Λ has measure

$$\begin{aligned} |\mathcal{C}_\varepsilon^c| \quad & \overset{(6.15),(6.13)}{=} \quad \Big| \bigcup_{k \ge 1} \mathtt{G}^c_{N_k}(u_{k-1}) \bigcup_{k \ge 1} (\mathcal{G}^0_{N_k}(u_{k-1}))^c \bigcup \tilde{\mathcal{G}}^c \bigcup \bar{\mathcal{G}}^c \Big| \\ & \le \quad \sum_{k \ge 1} |\mathtt{G}^c_{N_k}(u_{k-1})| + \sum_{k \ge 1} |(\mathcal{G}^0_{N_k}(u_{k-1}))^c| + |\tilde{\mathcal{G}}^c| + |\bar{\mathcal{G}}^c| \\ & \overset{(5.36),(5.5),(6.22),(6.19)}{\le} \quad C \sum_{k \ge 1} N_k^{-1} + C\gamma \le C'(N_0^{-1} + \gamma) \overset{(6.26)}{\le} C'' \varepsilon^\alpha \end{aligned} \tag{6.27}$$

implying (1.15). Finally (1.14) follows by (6.14) and

$$\begin{aligned} \|u(\varepsilon, \lambda)\|_{s_1} \quad & \le \quad \|u_0\|_{s_1} + \sum_{k=1}^{\infty} \|u_k - u_{k-1}\|_{s_1} \\ & \overset{(S1)_0,(S2)_n}{\le} \quad N_0^{-\sigma} + \sum_{k=1}^{\infty} N_k^{-\sigma-1} \le C N_0^{-\sigma} \overset{(6.26)}{\le} C\varepsilon^{\alpha\sigma} \,, \end{aligned}$$

hence $\|u(\varepsilon, \lambda)\|_{s_1} \to 0$, uniformly for $\lambda \in \Lambda$, as $\varepsilon \to 0$. Theorem 1.1 is proved with $s(d, \nu) := s_1$ defined in (6.4) and $q(d, \nu) := S + 3$, see (6.2). The C^∞-regularity result follows as in [5]-section 7.3.

7 Appendix: proof of the Nash-Moser Theorem 6.1

Step 1: Initialization. We take $\lambda \in \mathcal{N}(\bar{\mathcal{G}}, 2N_0^{-\sigma})$ (see (6.8)), so that

$$\mathcal{L}_0 := P_0(L_{\lambda\bar{\omega}})_{|H_0} \qquad \text{satisfies} \qquad \|\mathcal{L}_0^{-1}\|_{s_1} \leq 2N_0^{\tau_1+s_1}\gamma^{-1}$$

(see Lemma 7.1 in [5]), and we look for a solution of equation (P_0) as a fixed point of

$$F_0 : H_0 \to H_0 \,, \quad F_0(u) := \varepsilon \mathcal{L}_0^{-1} P_0 f(u) \,.$$

A contraction mapping argument (as in Lemma 7.2 of [5]) proves that, for $\varepsilon\gamma^{-1}N_0^{\tau_1+s_1+\sigma} \leq c(s_1)$ small, $\forall \lambda \in \mathcal{N}(\bar{\mathcal{G}}, 2N_0^{-\sigma})$, there exists a unique solution $\widetilde{u}_0(\varepsilon,\lambda)$ of (P_0) in

$$\mathtt{B}_0(s_1) := \{u \in H_0 \,:\, \|u\|_{s_1} \leq \rho_0 := N_0^{-\sigma}\} \,.$$

By uniqueness $\widetilde{u}_0(0,\lambda) = 0$. The implicit function theorem implies that $\widetilde{u}_0(\varepsilon,\cdot) \in C^1(\mathcal{N}(\bar{\mathcal{G}}, 2N_0^{-\sigma}); H_0)$ and $\partial_\lambda \widetilde{u}_0 = -\mathcal{L}_0^{-1}(\varepsilon)(\partial_\lambda \mathcal{L}_0)\widetilde{u}_0$ satisfies

$$\|\partial_\lambda \widetilde{u}_0\|_{s_1} \leq C N_0^{\tau_1+s_1+2-\sigma}\gamma^{-1} \,.$$

Then we define the C^1 map $u_0 := \psi_0 \widetilde{u}_0 : \Lambda \to H_0$ with cut-off function $\psi_0 : \Lambda \to [0,1]$,

$$\psi_0 := \begin{cases} 1 & \text{if } \lambda \in \mathcal{N}(\bar{\mathcal{G}}, N_0^{-\sigma}) \\ 0 & \text{if } \lambda \notin \mathcal{N}(\bar{\mathcal{G}}, 2N_0^{-\sigma}) \end{cases} \qquad \text{and} \qquad |D_\lambda \psi_0| \leq N_0^{\sigma} C \,.$$

We get $\|u_0\|_{s_1} \leq N_0^{-\sigma}$, $\|\partial_\lambda u_0\|_{s_1} \leq C(s_1)N_0^{\tau_1+s_1+1}\gamma^{-1}$. The statements $(S1)_0$, $(S4)_0$ are proved (note that $\mathcal{C}_0 = \bar{\mathcal{G}} \cap \bar{\mathcal{G}}$). Statement $(S5)_0$ follows in the same way using (6.11). Note that $(S2)_0$, $(S3)_0$ are empty.

For the next steps of the induction, the following lemma establishes a property which replaces $(S3)_n$ for the first steps. It is proved exactly as in Lemma 7.3 of [5], where we use the fact that, for $\varepsilon = 0$, the matrices $A_{N,j_0}(0,\lambda,\theta)$ are diagonal in time-Fourier basis.

Lemma 7.1. *There exists $N_0(S,V) \in \mathbb{N}$ and $c(s_1) > 0$ such that, if $N_0 \geq N_0(S,V)$ and $\varepsilon N_0^{\tau'+\delta s_1} \leq c(s_1)$, then $\forall N_0^{1/C_2} \leq N \leq N_0$, $\forall \|u\|_{s_1} \leq 1$, we have $\mathcal{G}_N(u) = \Lambda$.*

Step 2: Iteration of the Nash-Moser scheme. Suppose, by induction, that we have already defined $u_n \in C^1(\Lambda; H_n)$ and that properties $(S1)_k$-$(S5)_k$ hold for all $k \leq n$. This assumption will be implicit in all the subsequent lemmas. We are going to define u_{n+1} and prove the statements $(S1)_{n+1}$-$(S5)_{n+1}$.

In order to carry out a modified Nash-Moser scheme, we shall study the invertibility of

$$\mathcal{L}_{n+1}(u_n) := P_{n+1}\mathcal{L}(u_n)_{|H_{n+1}} \quad \text{where} \quad \mathcal{L}(u) := L_\omega - \varepsilon(Df)(u) \,, \tag{7.1}$$

(see (2.1)) and the tame estimates of its inverse, applying Proposition 3.1. We distinguish two cases. If $2^{n+1} > C_2$ (the constant C_2 is fixed in (6.5)), then there exists a unique $p \in [0,n]$ such that

$$N_{n+1} = N_p^{\chi} \,, \quad \chi = 2^{n+1-p} \in [C_2, 2C_2) \,, \text{ and } \quad N_{n+1} - 2L_0 = N_p^{\tilde{\chi}} \,, \quad \tilde{\chi} \in [C_2, 2C_2) \,. \tag{7.2}$$

If $2^{n+1} \leq C_2$ then there exists $\chi, \tilde{\chi} \in [C_2, 2C_2]$ such that

$$N_{n+1} = \bar{N}^{\chi} \,, \quad \bar{N} := [N_{n+1}^{1/C_2}] \in (N_0^{1/C_2}, N_0) \text{ and } N_{n+1} - 2L_0 = \bar{N}^{\tilde{\chi}} \,. \tag{7.3}$$

If (7.2) holds we consider in Proposition 3.1 the two scales $N' = N_{n+1}$ (resp. $N' = N_{n+1} - 2L_0$), $N = N_p$, see (3.2) . If (7.3) holds, we set $N' = N_{n+1}$ (resp. $N' = N_{n+1} - 2L_0$), $N = \bar{N}$.

Lemma 7.2. *Let $A(\varepsilon,\lambda,\theta)$ be defined in* (2.7)*, with $u = u_n$. For all*

$$\lambda \in \bigcap_{k=1}^{n+1} \mathcal{G}^0_{N_k}(u_{k-1}) \cap \tilde{\mathcal{G}}\,, \ \ \theta \in \mathbb{R}\,,$$

the hypothesis (H3) of Proposition 3.1 apply to $A_{M,j_0}(\varepsilon,\lambda,\theta)$, $\forall M \in \{N_{n+1}, N_{n+1}-2L_0\}$, $\forall j_0 \in \mathbb{Z}^d \setminus \mathcal{Q}_M$.

Proof. We give the proof when $M = N_{n+1}$ and (7.2) holds. Since $j_0 \notin \mathcal{Q}_{N_{n+1}}$ (i.e. $(0,j_0) \notin \check{\mathcal{Q}}_{N_{n+1}}$) Lemma 3.1 implies that, a site

$$i \in E := (0,j_0) + [-N_{n+1}, N_{n+1}]^b \tag{7.4}$$

which is N_p-good for $A(\varepsilon,\lambda,\theta)$ (see Definition 3.4) is also $(A_{N_{n+1},j_0}(\varepsilon,\lambda,\theta), N_p)$-good (see Definition 3.3). As a consequence,

$$\Big\{ \ (A_{N_{n+1},j_0}(\varepsilon,\lambda,\theta), N_p)\text{−bad sites } \Big\} \ \subset \ \Big\{ N_p\text{−bad sites of } A(\varepsilon,\lambda,\theta) \text{ with } |l| \le N_{n+1} \Big\}. \tag{7.5}$$

and (H3) is proved if the latter N_p-bad sites (in the right hand side of (7.5)) are contained in a disjoint union $\cup_\alpha \Omega_\alpha$ of clusters satisfying (3.6) (with $N = N_p$). This is a consequence of Proposition 4.1 applied to the infinite dimensional matrix $A(\varepsilon,\lambda,\theta)$. We claim that

$$\bigcap_{k=1}^{n+1} \mathcal{G}^0_{N_k}(u_{k-1}) \subset \mathcal{G}_{N_p}(u_n)\,, \ \text{i.e. any } \ \lambda \in \bigcap_{k=1}^{n+1} \mathcal{G}^0_{N_k}(u_{k-1}) \ \text{ is } \ N_p - \text{good for } \ A(\varepsilon,\lambda,\theta)\,, \tag{7.6}$$

and then assumption (i) of Proposition 4.1 holds. Indeed, if $p = 0$ then (7.6) is trivially true because $\mathcal{G}_{N_0}(u_n) = \Lambda$, by Lemma 7.1 and $(S1)_n$. If $p \ge 1$, we have

$$\|u_n - u_{p-1}\|_{s_1} \le \sum_{k=p}^{n} \|u_k - u_{k-1}\|_{s_1} \overset{(S2)_k}{\le} \sum_{k=p}^{n} N_k^{-\sigma-1} \le N_p^{-\sigma} \sum_{k \ge p} N_k^{-1} \le N_p^{-\sigma}$$

and so $(S3)_p$ implies

$$\bigcap_{k=1}^{p} \mathcal{G}^0_{N_k}(u_{k-1}) \subset \mathcal{G}_{N_p}(u_n)\,.$$

Assumption (ii) of Proposition 4.1 holds by (6.5), since $\chi \in [C_2, 2C_2)$. Assumption (iii) of Proposition 4.1 holds for all $\lambda \in \tilde{\mathcal{G}}$, see (6.9).

When (7.3) holds the proof is analogous using Lemma 7.1 with $N = \bar{N}$ and $(S1)_n$. ■

Lemma 7.3. *Property $(S3)_{n+1}$ holds.*

Proof. We want to prove that

$$\|u - u_n\|_{s_1} \le N_{n+1}^{-\sigma} \ \text{ and } \ \lambda \in \bigcap_{k=1}^{n+1} \mathcal{G}^0_{N_k}(u_{k-1}) \cap \tilde{\mathcal{G}} \quad \Longrightarrow \quad \lambda \in \mathcal{G}_{N_{n+1}}(u)\,.$$

Since $\lambda \in \mathcal{G}^0_{N_{n+1}}(u_n)$, by (5.3) and Definition 4.1 it is sufficient to prove that

$$B_M(j_0;\lambda)(u) \subset B^0_M(j_0;\lambda)(u_n)\,, \quad \forall M \in \{N_{n+1}, N_{n+1}-2L_0\}\,, \ j_0 \in \mathbb{Z}^d \setminus \mathcal{Q}_M$$

(we highlight the dependence of these sets on u, u_n) or, equivalently, by (5.1), (4.1), that

$$(\|A^{-1}_{M,j_0}(\varepsilon,\lambda,\theta)(u_n)\|_0 \le M^\tau \implies A_{M,j_0}(\varepsilon,\lambda,\theta)(u) \text{ is } M - \text{good})\,, \ \forall M \in \{N_{n+1}, N_{n+1}-2L_0\}\,, \tag{7.7}$$

where $A(\varepsilon,\lambda,\theta)(u)$ is in (2.7).

Let us make the case $M = N_{n+1}$, the other is similar. We prove (7.7) applying Proposition 3.1 to $A := A_{N_{n+1},j_0}(\varepsilon,\lambda,\theta)(u)$ with E defined in (7.4), $N' = N_{n+1}$, $N = N_p$ (resp. $N = \bar{N}$) if (7.2) (resp. (7.3)) is satisfied.

Using Lemma 2.1, $\|V\|_{C^q} \le C$, assumption (H1) holds with

$$\Upsilon \le C(1 + \|u_n\|_{s_1} + |V|_{s_1}) \overset{(S1)_n}{\le} C'(V)\,. \tag{7.8}$$

By Lemma 7.2, for all $\theta \in \mathbb{R}$, $j_0 \in \mathbb{Z}^d \setminus \mathcal{Q}_{N_{n+1}}$, the hypothesis (H3) of Proposition 3.1 holds for $A_{N_{n+1},j_0}(\varepsilon,\lambda,\theta)(u_n)$. Hence, by Proposition 3.1, for $s \in [s_0, s_1]$, if

$$\|A^{-1}_{N_{n+1},j_0}(\varepsilon,\lambda,\theta)(u_n)\|_0 \le N^{\tau}_{n+1}$$

(which is assumption (H2)) then

$$|A^{-1}_{N_{n+1},j_0}(\varepsilon,\lambda,\theta)(u_n)|_s \le \frac{1}{4} N^{\tau'}_{n+1}\Big(N^{\delta s}_{n+1} + |V|_s + \varepsilon|(Df)(u_n)|_s\Big)\,. \tag{7.9}$$

Finally, since $\|u - u_n\|_{s_1} \le N^{-\sigma}_{n+1}$ we have

$$|A_{N_{n+1},j_0}(\varepsilon,\lambda,\theta)(u_n) - A_{N_{n+1},j_0}(\varepsilon,\lambda,\theta)(u)|_{s_1} \le C\varepsilon\|u - u_n\|_{s_1} \le N^{-\sigma}_{n+1}$$

and (7.7) follows by (7.9) and a standard perturbative argument (see e.g. [5]). ■

From now on the convergence proof of the Nash-Moser iteration follows [5] with no changes.

In order to define u_{n+1}, we write, for $h \in H_{n+1}$,

$$P_{n+1}\Big(L_\omega(u_n + h) - \varepsilon f(u_n + h)\Big) = r_n + \mathcal{L}_{n+1}(u_n)h + R_n(h) \tag{7.10}$$

where $\mathcal{L}_{n+1}(u_n)$ is defined in (7.1) and

$$r_n := P_{n+1}\Big(L_\omega u_n - \varepsilon f(u_n)\Big)\,, \quad R_n(h) := -\varepsilon P_{n+1}\Big(f(u_n + h) - f(u_n) - (Df)(u_n)h\Big)\,. \tag{7.11}$$

By $(S4)_n$, if $\lambda \in \mathcal{N}(\mathcal{C}_n, N_n^{-\sigma})$ then u_n solves Equation (P_n) and so

$$r_n = P_{n+1}P_n^{\perp}\Big(L_\omega u_n - \varepsilon f(u_n)\Big) = P_{n+1}P_n^{\perp}\Big(V_0\, u_n - \varepsilon f(u_n)\Big)\,, \tag{7.12}$$

using also that $P_{n+1}P_n^{\perp}(D_\omega u_n) = 0$, see (2.3). Note that, by (6.1) and $\sigma \ge 2$ (see (6.10)), for $N_0 \ge 2$, we have the inclusion $\mathcal{N}(\mathcal{C}_{n+1}, 2N^{-\sigma}_{n+1}) \subset \mathcal{N}(\mathcal{C}_n, N_n^{-\sigma})$.

Lemma 7.4. (Invertibility of $\mathcal{L}_{n+1}$) *For all $\lambda \in \mathcal{N}(\mathcal{C}_{n+1}, 2N^{-\sigma}_{n+1})$ the operator $\mathcal{L}_{n+1}(u_n)$ is invertible and, for $s = s_1, S$,*

$$|\mathcal{L}^{-1}_{n+1}(u_n)|_s \le N^{\tau'+\delta s}_{n+1}\,. \tag{7.13}$$

As a consequence, by (2.13), $\forall h \in H_{n+1}$,

$$\|\mathcal{L}^{-1}_{n+1}(u_n)h\|_{s_1} \le C(s_1) N^{\tau'+\delta s_1}_{n+1}\|h\|_{s_1}\,, \tag{7.14}$$

$$\|\mathcal{L}^{-1}_{n+1}(u_n)h\|_{S} \le N^{\tau'+\delta s_1}_{n+1}\|h\|_S + C(S)N^{\tau'+\delta S}_{n+1}\|h\|_{s_1}\,. \tag{7.15}$$

Proof. We apply the multiscale Proposition 3.1 to $A_{N_{n+1}} = \mathcal{L}_{n+1}(u_n)$ as in Lemma 7.3. Assumption (H1) holds by (7.8). For all $\lambda \in \mathtt{G}_{N_{n+1}}(u_n)$ (see (5.35)) $\|\mathcal{L}^{-1}_{n+1}(u_n)\|_0 \le N^{\tau}_{n+1}$ and (H2) holds. The hypothesis (H3) holds, for $\lambda \in \mathcal{C}_{n+1}$ (see (6.13)), as a particular case of Lemma 7.2, for $\theta = 0$, $j_0 = 0$, $M = N_{n+1}$, and since $0 \notin \mathcal{Q}_{N_{n+1}}$. Then Proposition 3.1 applies $\forall\lambda \in \mathcal{C}_{n+1}$, implying (7.13). For all $\lambda \in \mathcal{N}(\mathcal{C}_{n+1}, 2N^{-\sigma}_{n+1})$ the proof of (7.13) follows by a perturbative argument as in Lemma 7.7 in [5]. ■

By (7.10), the equation (P_{n+1}) is equivalent to the fixed point problem $h = F_{n+1}(h)$ where

$$F_{n+1} : H_{n+1} \to H_{n+1}\,, \qquad F_{n+1}(h) := -\mathcal{L}_{n+1}^{-1}(u_n)(r_n + R_n(h))\,.$$

By a contraction mapping argument as in Lemma 7.8 in [5] (using (7.14), (7.12), (7.11)) we prove the existence, $\forall \lambda \in \mathcal{N}(\mathcal{C}_{n+1}, 2N_{n+1}^{-\sigma})$, of a unique fixed point $\widetilde{h}_{n+1}(\varepsilon, \lambda)$ of F_{n+1} in

$$\mathtt{B}_{n+1}(s_1) := \Big\{ h \in H_{n+1} \ : \ \|h\|_{s_1} \le \rho_{n+1} := N_{n+1}^{-\sigma-1} \Big\}\,.$$

Since $u_n(0,\lambda) = 0$ (by $(S1)_n$), we deduce, by the uniqueness of the fixed point, that $\widetilde{h}_{n+1}(0,\lambda) = 0$. Moreover, as in Lemma 7.9 of [5] (using the tame estimate (7.15)), one deduces the following bound on the high norm

$$\|\widetilde{h}_{n+1}\|_S \le K(S) N_{n+1}^{\tau' + \delta s_1} U_n\,.$$

By the implicit function theorem as in Lemma 7.10 in [5] (using (7.14)-(7.15)) the map $\widetilde{h}_{n+1}$ is in $C^1(\mathcal{N}(\mathcal{C}_{n+1}, 2N_{n+1}^{-\sigma}), H_{n+1})$ and

$$\|\partial_\lambda \widetilde{h}_{n+1}\|_{s_1} \le N_{n+1}^{-1}\,, \quad \|\partial_\lambda \widetilde{h}_{n+1}\|_S \le N_{n+1}^{\tau' + \delta s_1 + 1} \Big(N_{n+1}^{\tau' + \delta s_1 + 1} U_n + U_n' \Big)\,.$$

Finally we define the C^1-extension onto the whole Λ as

$$h_{n+1}(\lambda) := \begin{cases} \psi_{n+1}(\lambda) \widetilde{h}_{n+1}(\lambda) & \text{if} \quad \lambda \in \mathcal{N}(\mathcal{C}_{n+1}, 2N_{n+1}^{-\sigma}) \\ 0 & \text{if} \quad \lambda \notin \mathcal{N}(\mathcal{C}_{n+1}, 2N_{n+1}^{-\sigma}) \end{cases}$$

where ψ_{n+1} is a C^∞ cut-off function satisfying

$$0 \le \psi_{n+1} \le 1\,, \quad \psi_{n+1} \equiv \begin{cases} 1 & \text{if } \lambda \in \mathcal{N}(\mathcal{C}_{n+1}, N_{n+1}^{-\sigma}) \\ 0 & \text{if } \lambda \notin \mathcal{N}(\mathcal{C}_{n+1}, 2N_{n+1}^{-\sigma}) \end{cases} \quad \text{and} \quad |\partial_\lambda \psi_{n+1}| \le N_{n+1}^{\sigma} C\,.$$

Then (see Lemma 7.11 in [5])

$$\|h_{n+1}\|_{s_1} \le N_{n+1}^{-\sigma-1}\,, \quad \|\partial_\lambda h_{n+1}\|_{s_1} \le N_{n+1}^{-1/2}\,.$$

In conclusion, $u_{n+1} := u_n + h_{n+1}$ satisfies $(S1)_{n+1}$, $(S2)_{n+1}$, $(S4)_{n+1}$, $(S5)_{n+1}$ (see Lemma 7.12 in [5]).

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