KAM theory for the Hamiltonian derivative wave equation

Massimiliano Berti, Luca Biasco, Michela Procesi

Abstract: We prove an infinite dimensional KAM theorem which implies the existence of Cantor families of small-amplitude, reducible, elliptic, analytic, invariant tori of Hamiltonian derivative wave equations.

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1 Introduction

In the last years many progresses have been done concerning KAM theory for nonlinear Hamiltonian PDEs. The first existence results were given by Kuksin [18] and Wayne [29] for semilinear wave (NLW) and Schrödinger equations (NLS) in one space dimension ($1d$) under Dirichlet boundary conditions, see [24]-[25] and [21] for further developments. The approach of these papers consists in generating iteratively a sequence of symplectic changes of variables which bring the Hamiltonian into a constant coefficients (=reducible) normal form with an elliptic (=linearly stable) invariant torus at the origin. Such a torus is filled by quasi-periodic solutions with zero Lyapunov exponents. This procedure requires to solve, at each step, constant-coefficients linear "homological equations" by imposing the "second order Melnikov" non-resonance conditions. Unfortunately these (infinitely many) conditions are violated already for periodic boundary conditions.

In this case, existence of quasi-periodic solutions for semilinear $1d$-NLW and NLS equations, was first proved by Bourgain [3] by extending the Newton approach introduced by Craig-Wayne [9] for periodic solutions. Its main advantage is to require only the "first order Melnikov" non-resonance conditions (the minimal assumptions) for solving the homological equations. Actually, developing this perspective, Bourgain was able to prove in [4], [6] also the existence of quasi-periodic solutions for NLW and NLS (with Fourier multipliers) in higher space dimensions, see also the recent extensions in [1], [28]. The main drawback of this approach is that the homological equations are linear PDEs with non-constant coefficients. Translated in the KAM language this implies a non-reducible normal form around the torus and then a lack of informations about the stability of the quasi-periodic solutions.

Later on, existence of reducible elliptic tori was proved by Chierchia-You [7] for semilinear $1d$-NLW, and, more recently, by Eliasson-Kuksin [12] for NLS (with Fourier multipliers) in any space dimension, see also Procesi-Xu [27], Geng-Xu-You [14].

An important problem concerns the study of PDEs where the nonlinearity involves derivatives. A comprehension of this situation is of major importance since most of the models coming from Physics are of this kind.

In this direction KAM theory has been extended to deal with KdV equations by Kuksin [19]-[20], Kappeler-Pöschel [17], and, for the $1d$-derivative NLS (DNLS) and Benjiamin-Ono equations, by Liu-Yuan [22]. The key idea of these results is again to provide only a non-reducible normal form around the torus. However, in this cases, the homological equations with non-constant coefficients are only *scalar* (not an infinite system as in the Craig-Wayne-Bourgain approach). We remark that the KAM proof is more delicate for DNLS and Benjiamin-Ono, because these equations are less "dispersive" than KdV, i.e. the eigenvalues of the principal part of the differential operator grow only quadratically at infinity, and not cubically as for KdV. As a consequence of this difficulty, the quasi-periodic solutions in [19], [17] are analytic, in [22], only C^∞. Actually, for the applicability of these KAM schemes, the more dispersive the equation is, the more derivatives in the nonlinearity can be supported. The limit case of the derivative nonlinear wave equation (DNLW) -which is not dispersive at all- is excluded by these approaches.

In the paper [3] (which proves the existence of quasi-periodic solutions for semilinear $1d$-NLS and NLW), Bourgain claims, in the last remark, that his analysis works also for the Hamiltonian "derivation" wave equation

$$y_{tt} - y_{xx} + g(x)y = \Big(- \frac{d^2}{dx^2} \Big)^{1/2} F(x, y) \, ,$$

see also [5], page 81. Unfortunately no details are given. However, Bourgain [5] provided a detailed proof of the existence of periodic solutions for the non-Hamiltonian equation

$$y_{tt} - y_{xx} + \mathrm{m} y + y_t^2 = 0 \, , \quad \mathrm{m} \neq 0 \, .$$

These kind of problems have been then reconsidered by Craig in [8] for more general Hamiltonian derivative wave equations like

$$y_{tt} - y_{xx} + g(x)y + f(x, D^\beta y) = 0 \, , \quad x \in \mathbb{T} \, ,$$

where $g(x) \geq 0$ and D is the first order pseudo-differential operator $D := \sqrt{-\partial_{xx} + g(x)}$. The perturbative analysis of Craig-Wayne [9] for the search of periodic solutions works when $\beta < 1$. The main reason is that the wave equation vector field gains one derivative and then the nonlinear term $f(D^\beta u)$ has a strictly weaker effect on the dynamics for $\beta < 1$. The case $\beta = 1$ is left as an open problem. Actually, in this case, the small divisors problem for periodic solutions has the same level of difficulty of quasi-periodic solutions with 2 frequencies.

The goal of this paper is to extend KAM theory to deal with the Hamiltonian derivative wave equation

$$y_{tt} - y_{xx} + \mathrm{m} y + f(Dy) = 0 \, , \quad \mathrm{m} > 0 \, , \quad D := \sqrt{-\partial_{xx} + \mathrm{m}} \, , \quad x \in \mathbb{T} \, , \tag{1.1}$$

with real analytic nonlinearities (see Remark 7.1)

$$f(s) = a s^3 + \sum_{k \geq 5} f_k s^k \, , \quad a \neq 0 \, . \tag{1.2}$$

We write equation (1.1) as the infinite dimensional Hamiltonian system

$$u_t = -\mathrm{i} \partial_{\bar u} H \, , \qquad \bar u_t = \mathrm{i} \partial_u H \, ,$$

with Hamiltonian

$$H(u, \bar u) := \int_{\mathbb{T}} \bar u D u + F\Big(\frac{u + \bar u}{\sqrt{2}} \Big) \, dx \, , \quad F(s) := \int_0^s f \, , \tag{1.3}$$

in the complex unknown

$$u := \frac{1}{\sqrt{2}} (Dy + \mathrm{i} y_t) \, , \qquad \bar u := \frac{1}{\sqrt{2}} (Dy - \mathrm{i} y_t) \, , \qquad \mathrm{i} := \sqrt{-1} \, .$$

Setting $u = \sum_{j \in \mathbb{Z}} u_j e^{\mathrm{i} j x}$ (similarly for $\bar u$), we obtain the Hamiltonian in infinitely many coordinates

$$H = \sum_{j \in \mathbb{Z}} \lambda_j u_j \bar u_j + \int_{\mathbb{T}} F\Big(\frac{1}{\sqrt{2}} \sum_{j \in \mathbb{Z}} (u_j e^{\mathrm{i} j x} + \bar u_j e^{-\mathrm{i} j x}) \Big) \, dx \tag{1.4}$$

where

$$\lambda_j := \sqrt{j^2 + \mathrm{m}} \tag{1.5}$$

are the eigenvalues of the diagonal operator D. Note that the nonlinearity in (1.1) is x-independent implying, for (1.3), the conservation of the momentum $-\mathrm{i}\int_{\mathbb{T}} \bar{u}\partial_x u\, dx$. This symmetry allows to simplify somehow the KAM proof (a similar idea was used by Geng-You [13]).

For every choice of the *tangential sites* $\mathcal{I} := \{\mathtt{j}_1, \ldots, \mathtt{j}_n\} \subset \mathbb{Z}$, $n \geq 2$, the integrable Hamiltonian $\sum_{j\in\mathbb{Z}} \lambda_j u_j \bar{u}_j$ has the invariant tori $\{u_j\bar{u}_j = \xi_j, \text{ for } j \in \mathcal{I}\,,\ u_j = \bar{u}_j = 0 \text{ for } j \notin \mathcal{I}\}$ parametrized by the actions $\xi = (\xi_j)_{j\in\mathcal{I}} \in \mathbb{R}^n$. The next KAM result states the existence of nearby invariant tori for the complete Hamiltonian H in (1.4).

Theorem 1.1. *The equation* (1.1)-(1.2) *admits Cantor families of small-amplitude, analytic, quasi-periodic solutions with zero Lyapunov exponents and whose linearized equation is reducible to constant coefficients. Such Cantor families have asymptotically full measure at the origin in the set of parameters.*

The proof of Theorem 1.1 is based on the abstract KAM Theorem 4.1, which provides a reducible normal form (see (4.12)) around the elliptic invariant torus, and on the measure estimates Theorem 4.2. The key point in proving Theorem 4.2 is the asymptotic bound (4.15) on the perturbed normal frequencies $\Omega^\infty(\xi)$ after the KAM iteration. This allows to prove that the second order Melnikov non-resonance conditions (4.11) are fulfilled for an asymptotically full measure set of parameters (see (4.19)). The estimate (4.15), in turn, is achieved by exploiting the *quasi-Töplitz* property of the perturbation. This notion has been introduced by Procesi-Xu [27] in the context of NLS in higher space dimensions and it is similar, in spirit, to the Töplitz-Lipschitz property in Eliasson-Kuksin [12]. The precise formulation of quasi-Töplitz functions, adapted to the DNLW setting, is given in Definition 3.4 below.

Let us roughly explain the main ideas and techniques for proving Theorems 4.1, 4.2. These theorems concern, as usual, a parameter dependent family of analytic Hamiltonians of the form

$$H = \omega(\xi)\cdot y + \Omega(\xi)\cdot z\bar{z} + P(x, y, z, \bar{z}; \xi) \tag{1.6}$$

where $(x, y) \in \mathbb{T}^n \times \mathbb{R}^n$, $z, \bar{z}$ are infinitely many variables, $\omega(\xi) \in \mathbb{R}^n$, $\Omega(\xi) \in \mathbb{R}^\infty$ and $\xi \in \mathbb{R}^n$. The frequencies $\Omega_j(\xi)$ are close to the unperturbed frequencies λ_j in (1.5).

As well known, the main difficulty of the KAM iteration which provides a reducible KAM normal form like (4.12) is to fulfill, at each iterative step, the second order Melnikov non-resonance conditions. Actually, following the formulation of the KAM theorem given in [2], it is sufficient to verify

$$|\omega^\infty(\xi)\cdot k + \Omega_i^\infty(\xi) - \Omega_j^\infty(\xi)| \geq \frac{\gamma}{1 + |k|^\tau}\,, \quad \gamma > 0\,, \tag{1.7}$$

only for the "final" frequencies $\omega^\infty(\xi)$ and $\Omega^\infty(\xi)$, see (4.11), and not along the inductive iteration.

The application of the usual KAM theory (see e.g. [18], [24]-[25]), to the DNLW equation provides only the asymptotic decay estimate

$$\Omega_j^\infty(\xi) = j + O(1) \quad \text{for} \quad j \to +\infty\,. \tag{1.8}$$

Such a bound is not enough: the set of parameters ξ satisfying (1.7) could be empty. Note that for the semilinear NLW equation (see e.g. [24]) the frequencies decay asymptotically faster, namely like $\Omega_j^\infty(\xi) = j + O(1/j)$.

The key idea for verifying the second order Melnikov non-resonance conditions (1.7) for DNLW is to prove the higher order asymptotic decay estimate (see (4.15), (4.2))

$$\Omega_j^\infty(\xi) = j + a_+(\xi) + \frac{\mathrm{m}}{2j} + O\Big(\frac{\gamma^{2/3}}{j}\Big) \quad \text{for} \quad j \geq O(\gamma^{-1/3}) \tag{1.9}$$

where $a_+(\xi)$ is a constant independent of j (an analogous expansion holds for $j \to -\infty$ with a possibly different limit constant $a_-(\xi)$). In this way infinitely many conditions in (1.7) are verified by imposing only first order Melnikov conditions like $|\omega^\infty(\xi) \cdot k + h| \geq 2\gamma^{2/3}/|k|^\tau$, $h \in \mathbb{Z}$. Indeed, for $i > j > O(|k|^\tau \gamma^{-1/3})$, we get

$$\begin{aligned} |\omega^\infty(\xi) \cdot k + \Omega_i^\infty(\xi) - \Omega_j^\infty(\xi)| &= |\omega^\infty(\xi) \cdot k + i - j + \frac{\mathrm{m}(i-j)}{2ij} + O(\gamma^{2/3}/j)| \\ &\geq 2\gamma^{2/3}|k|^{-\tau} - O(|k|/j^2) - O(\gamma^{2/3}/j) \geq \gamma^{2/3}|k|^{-\tau} \end{aligned}$$

noting that $i-j$ is integer and $|i-j| = O(|k|)$ (otherwise no small divisors occur). We refer to section 6 for the precise arguments, see in particular Lemma 6.2.

The asymptotic decay (4.15) for the perturbed frequencies $\Omega^\infty(\xi)$ is achieved thanks to the "quasi-Töplitz" property of the perturbation (Definition 3.4). Let us roughly explain this notion. The new normal frequencies after each KAM step are $\Omega_j^+ = \Omega_j + P_j^0$ where the corrections P_j^0 are the coefficients of the quadratic form

$$P^0 z\bar{z} := \sum_j P_j^0 z_j \bar{z}_j \, , \quad P_j^0 := \int_{\mathbb{T}^n} (\partial^2_{z_j \bar{z}_j} P)(x, 0, 0, 0; \xi)\, dx \, .$$

We say that a quadratic form P^0 is quasi-Töplitz if it has the form

$$P^0 = T + R$$

where T is a Töplitz matrix (i.e. constant on the diagonals) and R is a "small" remainder satisfying $R_{jj} = O(1/j)$ (see Lemma 5.2). Then (1.9) follows with $a := T_{jj}$ which is independent of j.

Since the quadratic perturbation P^0 along the KAM iteration does not depend only on the quadratic perturbation at the previous steps, we need to extend the notion of quasi-Töplitz to general (non-quadratic) analytic functions.

The preservation of the quasi-Töplitz property of the perturbations P at each KAM step (with just slightly modified parameters) holds in view of the following key facts:

1. the Poisson bracket of two quasi-Töplitz functions is quasi-Töplitz (Proposition 3.1),
2. the hamiltonian flow generated by a quasi-Töplitz function preserves the quasi-Töplitz property (Proposition 3.2),
3. the solution of the homological equation with a quasi-Töplitz perturbation is quasi-Töplitz (Proposition 5.1).

We note that, in [12], the analogous properties 1 (and therefore 2) for Töpliz-Lipschitz functions is proved only when one of them is quadratic.

The definition of quasi-Töplitz functions heavily relies on properties of projections. However, for an analytic function in infinitely many variables, such projections may not be well defined unless the Taylor-Fourier series (see (2.28)) is *absolutely* convergent. For such reason, instead of the sup-norm, we use the majorant norm (see (2.12), (2.54)), for which the bounds (2.14) and (2.55) on projections hold (see also Remark 2.4).

We underline that the majorant norm of a vector field introduced in (2.54) is very different from the weighted norm introduced by Pöschel in [23]-Appendix C, which works *only* in finite dimension, see comments in [23] after Lemma C.2 and Remark 2.3. As far as we know this majorant norm of vector fields is new. In Section 2 we show its properties, in particular the key estimate of the majorant norm of the commutator of two vector fields (see Lemma 2.15).

Before concluding this introduction we also mention the recent KAM theorem of Grebért-Thomann [16] for the quantum harmonic oscillator with semilinear nonlinearity. Also here the eigenvalues grow to infinity only linearly. We quote the normal form results of Delort-Szeftel [10], Delort [11], for quasi-linear wave equations, where only finitely many steps of normal form can be performed. Finally we mention also the recent work by Gérard-Grellier [15] on Birkhoff normal form for a degenerate "half-wave" equation.

The paper is organized as follows:

- In SECTION 2 we define the *majorant* norm of formal power series of scalar functions (Definition 2.2) and vector fields (Definition 2.6) and we investigate the relations with the notion of analiticity, see Lemmata 2.1, 2.2, 2.3, 2.11 and Corollary 2.1. Then we prove Lemma 2.15 on commutators.

- In SECTION 3 we define the Töplitz (Definition 3.3) and Quasi-Töplitz functions (Definition 3.4). Then we prove that this class of functions is closed under Poisson brackets (Proposition 3.1) and composition with the Hamiltonian flow (Proposition 3.2).

- In SECTION 4 we state the abstract KAM Theorem 4.1. The first part of Theorem 4.1 follows by the KAM Theorem 5.1 in [2]. The main novelty is part II, in particular the asymptotic estimate (4.15) of the normal frequencies.

- In SECTION 5 we prove the abstract KAM Theorem 4.1.

  We first perform (as in Theorem 5.1 in [2]) a first normal form step, which makes Theorem 4.1 suitable for the direct application to the wave equation.

  In Proposition 5.1 we prove that the solution of the homological equation with a quasi-Töplitz perturbation is quasi-Töplitz. Then the main results of the KAM step concerns the asymptotic estimates of the perturbed frequencies (section 5.2.3) and the Töplitz estimates of the new perturbation (section 5.2.4).

- In SECTION 6 we prove Theorem 4.2: the second order Melnikov non-resonance conditions are fulfilled for a set of parameters with large measure, see (4.19). We use the conservation of momentum to avoid the presence of double eigenvalues.

- In SECTION 7 we finally apply the abstract KAM Theorem 4.1 to the DNLW equation (1.1)-(1.2), proving Theorem 1.1. We first verify that the Hamiltonian (1.4) is quasi-Töplitz (Lemma 7.1), as well as the Birkhoff normal form Hamiltonian (7.8) of Proposition 7.1. The main technical difficulties concern the proof in Lemma 7.4 that the generating function (7.17) of the Birkhoff symplectic transformation is also quasi-Töplitz (and the small divisors Lemma 7.2). In section 7.2 we prove that the perturbation, obtained after the introduction of the action-angle variables, is still quasi-Töplitz (Proposition 7.2). Finally in section 7.3 we prove Theorem 1.1 applying Theorems 4.1 and 4.2.

2 Functional setting

Given a finite subset $\mathcal{I} \subset \mathbb{Z}$ (possibly empty), $a \geq 0, p > 1/2$, we define the Hilbert space

$$\ell^{a,p}_{\mathcal{I}} := \Big\{ z = \{z_j\}_{j \in \mathbb{Z} \setminus \mathcal{I}} \,,\ z_j \in \mathbb{C} \quad : \ \|z\|^2_{a,p} := \sum_{j \in \mathbb{Z} \setminus \mathcal{I}} |z_j|^2 e^{2a|j|} \langle j \rangle^{2p} < \infty \Big\} \,.$$

When $\mathcal{I} = \emptyset$ we denote $\ell^{a,p} := \ell^{a,p}_{\mathcal{I}}$. We consider the direct product

$$E := \mathbb{C}^n \times \mathbb{C}^n \times \ell^{a,p}_{\mathcal{I}} \times \ell^{a,p}_{\mathcal{I}} \tag{2.1}$$

where n is the cardinality of $\mathcal{I}$. We endow the space E with the (s,r)-weighted norm

$$v = (x, y, z, \bar z) \in E \,, \quad \|v\|_E := \|v\|_{E,s,r} = \frac{|x|_\infty}{s} + \frac{|y|_1}{r^2} + \frac{\|z\|_{a,p}}{r} + \frac{\|\bar z\|_{a,p}}{r} \tag{2.2}$$

where, $0 < s, r < 1$, and $|x|_\infty := \max_{h=1,\dots,n} |x_h|$, $|y|_1 := \sum_{h=1}^{n} |y_h|$. Note that, for all $s' \le s$, $r' \le r$,

$$\|v\|_{E,s',r'} \le \max\{s/s', (r/r')^2\} \|v\|_{E,s,r} \,. \tag{2.3}$$

We shall also use the notations

$$z_j^+ = z_j \,, \quad z_j^- = \bar{z}_j \,.$$

We identify a vector $v \in E$ with the sequence $\{v^{(j)}\}_{j \in \mathcal{J}}$ with indices in

$$\mathcal{J} := \left\{ j = (j_1, j_2), \ j_1 \in \{1,2,3,4\}, \ j_2 \in \begin{cases} \{1, \dots, n\} & \text{if } j_1 = 1,2 \\ \mathbb{Z} \setminus \mathcal{I} & \text{if } j_1 = 3,4 \end{cases} \right\} \tag{2.4}$$

and components

$$v^{(1,j_2)} := x_{j_2} \,, \ v^{(2,j_2)} := y_{j_2} \ \ (1 \le j_2 \le n), \ v^{(3,j_2)} := z_{j_2} \,, \ v^{(4,j_2)} := \bar{z}_{j_2} \ \ (j_2 \in \mathbb{Z} \setminus \mathcal{I}) \,,$$

more compactly

$$v^{(1,\cdot)} := x \,, \ v^{(2,\cdot)} := y, \,, \ v^{(3,\cdot)} := z, \,, \ v^{(4,\cdot)} := \bar{z} \,.$$

We denote by $\{e_j\}_{j \in \mathcal{J}}$ the orthogonal basis of the Hilbert space E, where e_j is the sequence with all zeros, except the j_2-th entry of its j_1-th components, which is 1. Then every $v \in E$ writes $v = \sum_{j \in \mathcal{J}} v^{(j)} e_j$, $v^{(j)} \in \mathbb{C}$. We also define the toroidal domain

$$D(s,r) := \mathbb{T}^n_s \times D(r) := \mathbb{T}^n_s \times B_{r^2} \times B_r \times B_r \subset E \tag{2.5}$$

where $D(r) := B_{r^2} \times B_r \times B_r$,

$$\mathbb{T}^n_s := \Big\{ x \in \mathbb{C}^n \ : \ \max_{h=1,\dots,n} |\mathrm{Im}\, x_h| < s \Big\}, \quad B_{r^2} := \Big\{ y \in \mathbb{C}^n \ : \ |y|_1 < r^2 \Big\} \tag{2.6}$$

and $B_r \subset \ell^{a,p}_{\mathcal{I}}$ is the open ball of radius r centered at zero. We think $\mathbb{T}^n$ as the n-dimensional torus $\mathbb{T}^n := 2\pi\mathbb{R}^n/\mathbb{Z}^n$, namely $f : D(s,r) \to \mathbb{C}$ means that f is 2π-periodic in each x_h-variable, $h = 1, \dots, n$.

Remark 2.1. *If $n = 0$ then $D(s,r) \equiv B_r \times B_r \subset \ell^{a,p} \times \ell^{a,p}$.*

2.1 Majorant norm

2.1.1 Scalar functions

We consider *formal* power series with infinitely many variables

$$f(v) = f(x,y,z,\bar{z}) = \sum_{(k,i,\alpha,\beta) \in \mathbb{I}} f_{k,i,\alpha,\beta} \, e^{\mathrm{i} k \cdot x} y^i z^\alpha \bar{z}^\beta \tag{2.7}$$

with coefficients $f_{k,i,\alpha,\beta} \in \mathbb{C}$ and multi-indices in

$$\mathbb{I} := \mathbb{Z}^n \times \mathbb{N}^n \times \mathbb{N}^{(\mathbb{Z} \setminus \mathcal{I})} \times \mathbb{N}^{(\mathbb{Z} \setminus \mathcal{I})} \tag{2.8}$$

where

$$\mathbb{N}^{(\mathbb{Z} \setminus \mathcal{I})} := \Big\{ \alpha := (\alpha_j)_{j \in \mathbb{Z} \setminus \mathcal{I}} \in \mathbb{N}^{\mathbb{Z}} \ \text{ with } \ |\alpha| := \sum_{j \in \mathbb{Z} \setminus \mathcal{I}} \alpha_j < +\infty \Big\} \,. \tag{2.9}$$

In (2.7) we use the standard multi-indices notation $z^\alpha \bar{z}^\beta := \Pi_{j \in \mathbb{Z} \setminus \mathcal{I}} \, z_j^{\alpha_j} \bar{z}_j^{\beta_j}$. We denote the monomials

$$\mathfrak{m}_{k,i,\alpha,\beta}(v) = \mathfrak{m}_{k,i,\alpha,\beta}(x,y,z,\bar{z}) := e^{\mathrm{i} k \cdot x} y^i z^\alpha \bar{z}^\beta \,. \tag{2.10}$$

Remark 2.2. *If $n = 0$ the set $\mathbb{I}$ reduces to $\mathbb{N}^{\mathbb{Z}} \times \mathbb{N}^{\mathbb{Z}}$ and the formal series to* $f(z,\bar z) = \sum_{(\alpha,\beta)\in\mathbb{I}} f_{\alpha,\beta}\, z^\alpha \bar z^\beta$.

We define the "majorant" of f as

$$\big(Mf\big)(v) = \big(Mf\big)(x,y,z,\bar z) := \sum_{(k,i,\alpha,\beta)\in\mathbb{I}} |f_{k,i,\alpha,\beta}| e^{\mathrm{i}k\cdot x} y^i z^\alpha \bar z^\beta \,. \tag{2.11}$$

We now discuss the convergence of formal series.

Definition 2.1. *A series*

$$\sum_{(k,i,\alpha,\beta)\in\mathbb{I}} c_{k,i,\alpha,\beta}\,, \quad c_{k,i,\alpha,\beta} \in \mathbb{C}\,,$$

is absolutely convergent if the function $\mathbb{I} \ni (k,i,\alpha,\beta) \mapsto c_{k,i,\alpha,\beta} \in \mathbb{C}$ is in $L^1(\mathbb{I},\mu)$ where μ is the counting measure of $\mathbb{I}$. Then we set

$$\sum_{(k,i,\alpha,\beta)\in\mathbb{I}} c_{k,i,\alpha,\beta} := \int_{\mathbb{I}} c_{k,i,\alpha,\beta}\, d\mu\,.$$

By the properties of the Lebesgue integral, given any sequence $\{I_l\}_{l\ge0}$ of finite subsets $I_l \subset \mathbb{I}$ with $I_l \subset I_{l+1}$ and $\cup_{l\ge0} I_l = \mathbb{I}$, the absolutely convergent series

$$\sum_{k,i,\alpha,\beta} c_{k,i,\alpha,\beta} := \sum_{(k,i,\alpha,\beta)\in\mathbb{I}} c_{k,i,\alpha,\beta} = \lim_{l\to\infty} \sum_{(k,i,\alpha,\beta)\in I_l} c_{k,i,\alpha,\beta}\,.$$

Definition 2.2. (Majorant-norm: scalar functions) *The majorant-norm of a formal power series* (2.7) *is*

$$\|f\|_{s,r} := \sup_{(y,z,\bar z)\in D(r)} \sum_{k,i,\alpha,\beta} |f_{k,i,\alpha,\beta}| e^{|k|s} |y^i||z^\alpha||\bar z^\beta| \tag{2.12}$$

where $|k| := |k|_1 := |k_1| + \ldots + |k_n|$.

By (2.7) and (2.12) we clearly have $\|f\|_{s,r} = \|Mf\|_{s,r}$.

For every subset of indices $I \subset \mathbb{I}$, we define the projection

$$(\Pi_I f)(x,y,z,\bar z) := \sum_{(k,i,\alpha,\beta)\in I} f_{k,i,\alpha,\beta} e^{\mathrm{i}k\cdot x} y^i z^\alpha \bar z^\beta \tag{2.13}$$

of the formal power series f in (2.7). Clearly

$$\|\Pi_I f\|_{s,r} \le \|f\|_{s,r} \tag{2.14}$$

and, for any $I, I' \subset \mathbb{I}$, it results

$$\Pi_I \Pi_{I'} = \Pi_{I\cap I'} = \Pi_{I'}\Pi_I\,. \tag{2.15}$$

Property (2.14) is one of the main advantages of the majorant-norm with respect to the usual sup-norm

$$|f|_{s,r} := \sup_{v\in D(s,r)} |f(v)|\,. \tag{2.16}$$

We now define useful projectors on the time Fourier indices.

Definition 2.3. *Given $\varsigma = (\varsigma_1,\ldots,\varsigma_n) \in \{+,-\}^n$ we define*

$$f_\varsigma := \Pi_\varsigma f := \Pi_{\mathbb{Z}^n_\varsigma \times \mathbb{N}^n \times \mathbb{N}^{(\mathbb{Z}\setminus\mathcal{I})} \times \mathbb{N}^{(\mathbb{Z}\setminus\mathcal{I})}} f = \sum_{k\in\mathbb{Z}^n_\varsigma, i,\alpha,\beta} f_{k,i,\alpha,\beta}\, e^{\mathrm{i}k\cdot x} y^i z^\alpha \bar z^\beta \tag{2.17}$$

where

$$\mathbb{Z}^n_\varsigma := \left\{ k \in \mathbb{Z}^n \quad \text{with} \quad \begin{cases} k_h \ge 0 & \text{if} \quad \varsigma_h = + \\ k_h < 0 & \text{if} \quad \varsigma_h = - \end{cases} \quad \forall\, 1 \le h \le n \right\}. \tag{2.18}$$

Then any formal series f can be decomposed as

$$f = \sum_{\varsigma \in \{+,-\}^n} \Pi_\varsigma f \tag{2.19}$$

and (2.14) implies $\|\Pi_\varsigma f\|_{s,r} \le \|f\|_{s,r}$.

We now investigate the relations between formal power series with finite majorant norm and analytic functions. We recall that a function $f : D(s,r) \to \mathbb{C}$ is

- ANALYTIC, if $f \in C^1(D(s,r), \mathbb{C})$, namely the Fréchet differential $D(s,r) \ni v \mapsto df(v) \in \mathcal{L}(E, \mathbb{C})$ is continuous,
- WEAKLY ANALYTIC, if $\forall v \in D(s,r)$, $v' \in E \setminus \{0\}$, there exists $\varepsilon > 0$ such that the function

  $$\{\xi \in \mathbb{C}\,,\ |\xi| < \varepsilon\ \} \ \mapsto\ f(v + \xi v') \in \mathbb{C}$$

  is analytic in the usual sense of one complex variable.

A well known result (see e.g. Theorem 1, page 133 of [26]) states that a function f is

$$\text{analytic} \quad \Longleftrightarrow \quad \text{weakly analytic and locally bounded}\,. \tag{2.20}$$

Lemma 2.1. *Suppose that the formal power series* (2.7) *is absolutely convergent for all* $v \in D(s,r)$. *Then* $f(v)$ *and* $Mf(v)$, *defined in* (2.7) *and* (2.11), *are well defined and weakly analytic in* $D(s,r)$. *If, moreover, the sup-norm* $|f|_{s,r} < \infty$, *resp.* $|Mf|_{s,r} < \infty$, *then* f, *resp.* Mf, *is analytic in* $D(s,r)$.

PROOF. Since the series (2.7) is absolutely convergent the functions f, Mf, and, for all $\varsigma \in \{+,-\}^n$, $f_\varsigma := \Pi_\varsigma f$, Mf_ς (see (2.17)) are well defined (also the series in (2.17) is absolutely convergent).

We now prove that each Mf_ς is weakly analytic, namely $\forall v \in D(s,r)$, $v' \in E \setminus \{0\}$,

$$Mf_\varsigma(v + \xi v') = \sum_{k \in \mathbb{Z}^n_\varsigma, i, \alpha, \beta} |f_{k,i,\alpha,\beta}| \mathfrak{m}_{k,i,\alpha,\beta}(v + \xi v') \tag{2.21}$$

is analytic in $\{|\xi| < \varepsilon\}$, for ε small enough (recall the notation (2.10)). Since each $\xi \mapsto \mathfrak{m}_{k,i,\alpha,\beta}(v + \xi v')$ is entire, the analyticity of $Mf_\varsigma(v + \xi v')$ follows once we prove that the series (2.21) is totally convergent, namely

$$\sum_{k \in \mathbb{Z}^n_\varsigma, i, \alpha, \beta} |f_{k,i,\alpha,\beta}| \sup_{|\xi| < \varepsilon} |\mathfrak{m}_{k,i,\alpha,\beta}(v + \xi v')| < +\infty\,. \tag{2.22}$$

Let us prove (2.22). We claim that, for ε small enough, there is $v^\varsigma \in D(s,r)$ such that

$$\sup_{|\xi| < \varepsilon} \left| \mathfrak{m}_{k,i,\alpha,\beta}(v + \xi v') \right| \le \mathfrak{m}_{k,i,\alpha,\beta}(v^\varsigma)\,, \quad \forall k \in \mathbb{Z}^n_\varsigma, i, \alpha, \beta\,. \tag{2.23}$$

Therefore (2.22) follows by

$$\begin{aligned} \sum_{k \in \mathbb{Z}^n_\varsigma, i, \alpha, \beta} |f_{k,i,\alpha,\beta}| \sup_{|\xi| < \varepsilon} |\mathfrak{m}_{k,i,\alpha,\beta}(v + \xi v')| \quad &\le \quad \sum_{k \in \mathbb{Z}^n_\varsigma, i, \alpha, \beta} |f_{k,i,\alpha,\beta}| \mathfrak{m}_{k,i,\alpha,\beta}(v^\varsigma) \\ &= \quad Mf_\varsigma(v^\varsigma) < +\infty\,. \end{aligned}$$

Let us construct $v^\varsigma \in D(s,r)$ satisfying (2.23). Since $v = (x, y, z, \bar z) \in D(s,r)$ we have $x \in \mathbb{T}^n_s$ and, since $\mathbb{T}^n_s$ is open, there is $0 < s' < s$ such that $|\mathrm{Im}(x_h)| < s'$, $\forall 1 \le h \le n$. Hence, for ε small enough,

$$\sup_{|\xi| < \varepsilon} \left| \mathrm{Im}(x + \xi x')_h \right| \le s' < s\,, \quad \forall 1 \le h \le n\,. \tag{2.24}$$

The vector $v^\varsigma := (x^\varsigma, y^\varsigma, z^\varsigma, \bar{z}^\varsigma)$ with components

$$
\begin{aligned}
x^\varsigma_h &:= -\mathrm{i}\varsigma_h s' , \qquad & y^\varsigma_h &:= |y_h| + \varepsilon |y'_h| , \qquad & 1 \le h \le n , \\
z^\varsigma_h &:= |z_h| + \varepsilon |z'_h| , \qquad & \bar{z}^\varsigma_h &:= |\bar{z}_h| + \varepsilon |\bar{z}'_h| , \qquad & h \in \mathbb{Z} ,
\end{aligned}
\tag{2.25}
$$

belongs to $D(s,r)$ because $|\mathrm{Im}\, x^\varsigma_h| = s' < s$, $\forall\, 1 \le h \le n$, and also $(y^\varsigma, z^\varsigma, \bar{z}^\varsigma) \in D(r)$ for ε small enough, because $(y,z,\bar{z}) \in D(r)$ and $D(r)$ is open. Moreover, $\forall k \in \mathbb{Z}^n_\varsigma$, by (2.24), (2.18) and (2.25),

$$
\sup_{|\xi|<\varepsilon} \left| e^{\mathrm{i}k\cdot(x+\xi x')} \right| \le e^{|k|s'} = e^{\mathrm{i}k\cdot x^\varsigma} . \tag{2.26}
$$

By (2.10), (2.25), (2.26), we get (2.23). Hence each Mf_ς is weakly analytic and, by the decomposition (2.19), also f and Mf are weakly analytic. The final statement follows by (2.20). ■

Corollary 2.1. *If $\|f\|_{s,r} < +\infty$ then f and Mf are analytic and*

$$
|f|_{s,r}, |Mf|_{s,r} \ \le \ \|f\|_{s,r} . \tag{2.27}
$$

Proof. For all $v = (x,y,z,\bar{z}) \in \mathbb{T}^n_s \times D(r)$, we have $|e^{\mathrm{i}k\cdot x}| \le e^{|k|s}$ and

$$
|f(v)| , \ |Mf(v)| \ \le \ \sum_{k,i,\alpha,\beta} |f_{k,i,\alpha,\beta}| e^{|k|s} |y^i| |z^\alpha| |\bar{z}^\beta| \stackrel{(2.12)}{\le} \|f\|_{s,r} < +\infty
$$

by assumption. Lemma 2.1 implies that f, Mf are analytic. ■

Now, we associate to any analytic function $f : D(s,r) \to \mathbb{C}$ the formal Taylor-Fourier power series

$$
\mathtt{f}(v) := \sum_{(k,i,\alpha,\beta)\in\mathbb{I}} f_{k,i,\alpha,\beta}\, e^{\mathrm{i}k\cdot x} y^i z^\alpha \bar{z}^\beta \tag{2.28}
$$

(as (2.7)) with Taylor-Fourier coefficients

$$
f_{k,i,\alpha,\beta} := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} e^{-\mathrm{i}k\cdot x} \frac{1}{i!\alpha!\beta!} (\partial^i_y \partial^\alpha_z \partial^\beta_{\bar{z}} f)(x,0,0,0)\, dx \tag{2.29}
$$

where $\partial^i_y \partial^\alpha_z \partial^\beta_{\bar{z}} f$ are the partial derivatives[1].

What is the relation between f and its formal Taylor-Fourier series $\mathtt{f}$?

Lemma 2.2. *Let $f : D(s,r) \to \mathbb{C}$ be analytic. If its associated Taylor-Fourier power series* (2.28)-(2.29) *is absolutely convergent in $D(s,r)$, and the sup-norm*

$$
\Big| \sum_{k,i,\alpha,\beta} f_{k,i,\alpha,\beta}\, e^{\mathrm{i}k\cdot x} y^i z^\alpha \bar{z}^\beta \Big|_{s,r} < \infty , \tag{2.31}
$$

then $f = \mathtt{f}$, $\forall\, v \in D(s,r)$.

Proof. Since the Taylor-Fourier series (2.28)-(2.29) is absolutely convergent and (2.31) holds, by Lemma 2.1 the function $\mathtt{f} : D(s,r) \to \mathbb{C}$ is analytic. The functions $f = \mathtt{f}$ are equal if the Taylor-Fourier coefficients

$$
f_{k,i,\alpha,\beta} = \mathtt{f}_{k,i,\alpha,\beta} , \quad \forall\, k, i, \alpha, \beta , \tag{2.32}
$$

[1] For a multi-index $\alpha = \sum_{1\le j\le k} e_{i_j}$, $|\alpha| = k$, the partial derivative is

$$
\partial^\alpha_z f(x,y,z,\bar{z}) := \frac{\partial^k}{\partial\tau_1 \ldots \partial\tau_k}_{|\tau=0} f(x,y,z+\tau_1 e_{i_1} + \ldots + \tau_k e_{i_k}, \bar{z}) . \tag{2.30}
$$

where the coefficients $\mathtt{f}_{k,i,\alpha,\beta}$ are defined from $\mathtt{f}$ as in (2.29). Let us prove (2.32). Indeed, for example,

$$\begin{aligned} \mathtt{f}_{0,0,e_h,0} &= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{d}{d\xi}_{|\xi=0}\sum_{k\in\mathbb{Z}^n,\, m\in\mathbb{N}} f_{k,0,me_h,0}e^{\mathrm{i}k\cdot x}\xi^m \\ &= \sum_{k\in\mathbb{Z}^n,\, m\in\mathbb{N}}\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{d}{d\xi}_{|\xi=0} f_{k,0,me_h,0}e^{\mathrm{i}k\cdot x}\xi^m = f_{0,0,e_h,0}\,, \end{aligned} \tag{2.33}$$

using that the above series totally converge for $r'<r$, namely

$$\begin{aligned} \sum_{k\in\mathbb{Z}^n,\, m\in\mathbb{N}}\sup_{x\in\mathbb{R},\,|\xi|\le r'}|f_{k,0,me_h,0}e^{\mathrm{i}k\cdot x}\xi^m| &\le \sum_{k\in\mathbb{Z}^n,\, m\in\mathbb{N}}|f_{k,0,me_h,0}|(r')^m \\ &\le \sum_{k,i,\alpha,\beta}|f_{k,i,\alpha,\beta}\mathfrak{m}_{k,i,\alpha,\beta}(0,0,r'e_h,0)|<\infty \end{aligned}$$

recall (2.10). For the others k,i,α,β in (2.32) is analogous. ■

The above arguments also show the unicity of the Taylor-Fourier expansion.

Lemma 2.3. *If an analytic function $f: D(s,r)\to\mathbb{C}$ equals an absolutely convergent formal series, i.e. $f(v)=\sum_{k,i,\alpha,\beta}\tilde f_{k,i,\alpha,\beta}e^{\mathrm{i}k\cdot x}y^iz^\alpha\bar z^\beta$, then its Taylor-Fourier coefficients* (2.29) *are $f_{k,i,\alpha,\beta}=\tilde f_{k,i,\alpha,\beta}$.*

The majorant norm of f is equivalent to the sup-norm of its majorant Mf.

Lemma 2.4.

$$|Mf|_{s,r}\le\|f\|_{s,r}\le 2^n|Mf|_{s,r}\,. \tag{2.34}$$

Proof. The first inequality in (2.34) is (2.27). The second one follows by

$$\|\Pi_\varsigma f\|_{s,r}\le|Mf|_{s,r}\,,\quad\forall\varsigma\in\{+,-\}^n\,, \tag{2.35}$$

where $\Pi_\varsigma f$ is defined in (2.17). Let us prove (2.35). Let

$$D^+(r):=\Big\{(y,z,\bar z)\in D(r)\ :\ y_h\ge0\,,\forall 1\le h\le n\,,\ z_l,\bar z_l\ge0\,,\forall l\in\mathbb{Z}\setminus\mathcal{I}\Big\}\,.$$

For any $0\le\sigma<s$, we have

$$\begin{aligned} |Mf|_{s,r} &= \sup_{(x,y,z,\bar z)\in D(s,r)}\Big|\sum_{k,i,\alpha,\beta}|f_{k,i,\alpha,\beta}|e^{\mathrm{i}k\cdot x}y^iz^\alpha\bar z^\beta\Big| \\ &\ge \sup_{x_1=-\mathrm{i}\varsigma_1\sigma,\ldots,x_n=-\mathrm{i}\varsigma_n\sigma,(y,z,\bar z)\in D^+(r)}\Big|\sum_{k,i,\alpha,\beta}|f_{k,i,\alpha,\beta}|e^{\mathrm{i}k\cdot x}y^iz^\alpha\bar z^\beta\Big| \\ &\overset{(2.18)}{\ge} \sup_{(y,z,\bar z)\in D^+(r)}\sum_{k\in\mathbb{Z}^n_\varsigma,i,\alpha,\beta}|f_{k,i,\alpha,\beta}|e^{|k|\sigma}|y^i||z^\alpha||\bar z^\beta| \\ &= \sup_{(y,z,\bar z)\in D(r)}\sum_{k\in\mathbb{Z}^n_\varsigma,i,\alpha,\beta}|f_{k,i,\alpha,\beta}|e^{|k|\sigma}|y^i||z^\alpha||\bar z^\beta|=\|\Pi_\varsigma f\|_{\sigma,r}\,. \end{aligned}$$

Then (2.35) follows since for every function g we have $\sup_{0\le\sigma<s}\|g\|_{\sigma,r}=\|g\|_{s,r}$. ■

Definition 2.4. (Order relation: scalar functions) *Given formal power series*

$$f=\sum_{k,i,\alpha,\beta}f_{k,i,\alpha,\beta}\,e^{\mathrm{i}k\cdot x}y^iz^\alpha\bar z^\beta\,,\quad g=\sum_{k,i,\alpha,\beta}g_{k,i,\alpha,\beta}\,e^{\mathrm{i}k\cdot x}y^iz^\alpha\bar z^\beta\,,$$

with $g_{k,i,\alpha,\beta}\in\mathbb{R}^+$, we say that

$$f\prec g\quad\text{if}\quad|f_{k,i,\alpha,\beta}|\le g_{k,i,\alpha,\beta}\,,\ \forall k,i,\alpha,\beta\,. \tag{2.36}$$

Note that, by the definition (2.11) of majorant series,

$$f \prec g \quad \Longleftrightarrow \quad f \prec Mf \prec g\,. \tag{2.37}$$

Moreover, if $\|g\|_{s,r} < +\infty$, then $f \prec g \Longrightarrow \|f\|_{s,r} \le \|g\|_{s,r}$.

For any $\varsigma \in \{+,-\}^n$ define $q_\varsigma := (q_\varsigma^{(j)})_{j\in\mathcal{J}}$ as

$$q_\varsigma^{(j)} := \begin{cases} -\varsigma_h\,\mathrm{i} & \text{if} \quad j = (1,h)\,, \quad 1 \le h \le n\,, \\ 1 & \text{otherwise}\,. \end{cases} \tag{2.38}$$

Lemma 2.5. *Assume $\|f\|_{s,r}, \|g\|_{s,r} < +\infty$. Then*

$$f + g \prec Mf + Mg\,, \qquad f \cdot g \prec Mf \cdot Mg \tag{2.39}$$

and

$$M\big(\partial_j(\Pi_\varsigma f)\big) = q_\varsigma^{(j)}\partial_j\big(M(\Pi_\varsigma f)\big)\,, \quad j \in \mathcal{J}\,, \tag{2.40}$$

where ∂_j is short for $\partial_{v^{(j)}}$ and $q_\varsigma^{(j)}$ are defined in (2.38).

Proof. Since the series which define f and g are absolutely convergent, the bounds (2.39) follow by summing and multiplying the series term by term. Next (2.40) follows by differentiating the series term by term. ■

An immediate consequence of (2.39) is

$$\|f+g\|_{s,r} \le \|f\|_{s,r} + \|g\|_{s,r}\,, \quad \|f\,g\|_{s,r} \le \|f\|_{s,r}\|g\|_{s,r}\,. \tag{2.41}$$

The next lemma extends property (2.39) for infinite series.

Lemma 2.6. *Assume that $f^{(j)}$, $g^{(j)}$ are formal power series satisfying*

1. *$f^{(j)} \prec g^{(j)}$, $\forall j \in \mathcal{J}$,*
2. *$\|g^{(j)}\|_{s,r} < \infty$, $\forall j \in \mathcal{J}$,*
3. *$\sum_{j\in\mathcal{J}} |g^{(j)}(v)| < \infty$, $\forall\, v \in D(s,r)$,*
4. *$g(v) := \sum_{j\in\mathcal{J}} g^{(j)}(v)$ is bounded in $D(s,r)$, namely $|g|_{s,r} < \infty$.*

Then the function $g : D(s,r) \to \mathbb{C}$ is analytic, its Taylor-Fourier coefficients (defined as in (2.29)*) are*

$$g_{k,i,\alpha,\beta} = \sum_{j\in\mathcal{J}} g^{(j)}_{k,i,\alpha,\beta} \ge 0\,, \qquad \forall\,(k,i,\alpha,\beta) \in \mathbb{I}\,, \tag{2.42}$$

and $\|g\|_{s,r} < \infty$. Moreover

1. *$\sum_{j\in\mathcal{J}} |f^{(j)}(v)| < \infty$, $\forall\, v \in D(s,r)$,*
2. *$f(v) := \sum_{j\in\mathcal{J}} f^{(j)}(v)$ is analytic in $D(s,r)$,*
3. *$f \prec g$ and $\|f\|_{s,r} \le \|g\|_{s,r} < \infty$.*

PROOF. For each monomial $\mathfrak{m}_{k,i,\alpha,\beta}(v)$ (see (2.10)) and $v = (x, y, z, \bar{z}) \in D(s, r)$, we have

$$|\mathfrak{m}_{k,i,\alpha,\beta}(v)| = \mathfrak{m}_{k,i,\alpha,\beta}(v_+)\,, \tag{2.43}$$

where $v_+ := (\mathrm{i}\,\mathrm{Im}\,x, |y|, |z|, |\bar{z}|) \in D(s,r)$ with $|y| := (|y_1|, \ldots, |y_n|)$ and $|z|$, $|\bar{z}|$ are similarly defined.

Since $\|g^{(j)}\|_{s,r} < \infty$ (and $f^{(j)} \prec g^{(j)}$) the series

$$g^{(j)}(v) := \sum_{k,i,\alpha,\beta} g^{(j)}_{k,i,\alpha,\beta}\mathfrak{m}_{k,i,\alpha,\beta}(v)\,, \qquad g^{(j)}_{k,i,\alpha,\beta} \geq 0 \tag{2.44}$$

is absolutely convergent. For all $v \in D(s,r)$ we prove that

$$\begin{aligned}\sum_{j\in\mathcal{J}}\sum_{k,i,\alpha,\beta} |g^{(j)}_{k,i,\alpha,\beta}\mathfrak{m}_{k,i,\alpha,\beta}(v)| &\overset{(2.44),(2.43)}{=} \sum_{j\in\mathcal{J}}\sum_{k,i,\alpha,\beta} g^{(j)}_{k,i,\alpha,\beta}\mathfrak{m}_{k,i,\alpha,\beta}(v_+) \\ &\overset{(2.44)}{=} \sum_{j\in\mathcal{J}} g^{(j)}(v_+) = g(v_+) < \infty \end{aligned}\tag{2.45}$$

by assumption 3. Therefore, by Fubini's theorem, we exchange the order of the series

$$g(v) = \sum_{j\in\mathcal{J}}\sum_{k,i,\alpha,\beta} g^{(j)}_{k,i,\alpha,\beta}\mathfrak{m}_{k,i,\alpha,\beta}(v) = \sum_{k,i,\alpha,\beta}\Big(\sum_{j\in\mathcal{J}} g^{(j)}_{k,i,\alpha,\beta}\Big)\mathfrak{m}_{k,i,\alpha,\beta}(v) \tag{2.46}$$

proving that g is equal to an absolutely convergent series. Lemma 2.1 and the assumption $|g|_{s,r} < \infty$ imply that g is analytic in $D(s,r)$. Moreover (2.46) and Lemma 2.3 imply (2.42). The $g_{k,i,\alpha,\beta} \geq 0$ because $g^{(j)}_{k,i,\alpha,\beta} \geq 0$, see (2.44). Therefore $Mg = g$, and, by (2.34) and the assumption $|g|_{s,r} < \infty$, we deduce $\|g\|_{s,r} < \infty$.

Concerning f we have

$$\sum_{j\in\mathcal{J}} |f^{(j)}(v)| \leq \sum_{j\in\mathcal{J}}\sum_{k,i,\alpha,\beta}\Big|f^{(j)}_{k,i,\alpha,\beta}\mathfrak{m}_{k,i,\alpha,\beta}(v)\Big| \leq \sum_{j\in\mathcal{J}}\sum_{k,i,\alpha,\beta} g^{(j)}_{k,i,\alpha,\beta}|\mathfrak{m}_{k,i,\alpha,\beta}(v)| \overset{(2.45)}{<} \infty$$

and, arguing as for g, its Taylor-Fourier coefficients are $f_{k,i,\alpha,\beta} = \sum_{j\in\mathcal{J}} f^{(j)}_{k,i,\alpha,\beta}$, $\forall (k,i,\alpha,\beta) \in \mathbb{I}$. Then

$$|f_{k,i,\alpha,\beta}| \leq \sum_{j\in\mathcal{J}} |f^{(j)}_{k,i,\alpha,\beta}| \leq \sum_{j\in\mathcal{J}} g^{(j)}_{k,i,\alpha,\beta} \overset{(2.42)}{=} g_{k,i,\alpha,\beta}\,.$$

Hence $f \prec g$ and $\|f\|_{s,r} \leq \|g\|_{s,r} < \infty$. Finally f is analytic by Lemma 2.1. ■

Lemma 2.7. *Let $\|f\|_{s,r} < \infty$. Then, $\forall 0 < s' < s$, $0 < r' < r$, we have $\|\partial_j f\|_{s',r'} < \infty$.*

PROOF. It is enough to prove the lemma for each $f_\varsigma = \Pi_\varsigma f$ defined in (2.17). By $\|f\|_{s,r} < \infty$ and Corollary 2.1 the functions f_ς, Mf_ς are analytic and

$$\|\partial_j f_\varsigma\|_{s',r'} \overset{(2.34)}{\leq} 2^n |M(\partial_j f_\varsigma)|_{s',r'} \overset{(2.40)}{=} 2^n|\partial_j(Mf_\varsigma)|_{s',r'} \leq c|Mf_\varsigma|_{s,r} \overset{(2.34)}{\leq} c\|f_\varsigma\|_{s,r}$$

for a suitable $c := c(n,s,s',r,r')$, having used the Cauchy estimate (in one variable). ■

We conclude this subsection with a simple result on representation of differentials.

Lemma 2.8. *Let $f : D(s,r) \to \mathbb{C}$ be Fréchet differentiable at v_0. Then*

$$df(v_0)[v] = \sum_{j\in\mathcal{J}} \partial_j f(v_0) v^{(j)}\,, \quad \forall v = \sum_{j\in\mathcal{J}} v^{(j)} e_j \in E\,, \tag{2.47}$$

and

$$\sum_{j\in\mathcal{J}} |\partial_j f(v_0) v^{(j)}| \leq \|df(v_0)\|_{\mathcal{L}(E,\mathbb{C})}\|v\|_E\,. \tag{2.48}$$

PROOF. (2.47) follows by the continuity of the differential $df(v_0) \in \mathcal{L}(E, \mathbb{C})$. Next, consider a vector $\tilde v = (\tilde v^{(j)})_{j \in \mathcal{J}} \in E$ such that $|\tilde v_j| = |v_j|$ and

$$\tilde v^{(j)} (\partial_j f)(v_0) = |(\partial_j f)(v_0) v^{(j)}| \, , \quad \forall \, j \in \mathcal{J} .$$

Hence $df(v_0)[\tilde v] = \sum_{j \in \mathcal{J}} \tilde v^{(j)} (\partial_j f)(v_0) = \sum_{j \in \mathcal{J}} |(\partial_j f)(v_0) v^{(j)}|$ which gives (2.48) because $\|\tilde v\|_E = \|v\|_E$. ■

2.1.2 Vector fields

We now consider a *formal* vector field

$$X(v) := \Big(X^{(j)}(v) \Big)_{j \in \mathcal{J}} \tag{2.49}$$

where each component $X^{(j)}$ is a formal power series

$$X^{(j)}(v) = X^{(j)}(x, y, z, \bar z) = \sum_{k,i,\alpha,\beta} X^{(j)}_{k,i,\alpha,\beta} \, e^{\mathrm{i} k \cdot x} y^i z^\alpha \bar z^\beta \tag{2.50}$$

as in (2.7). We define its "majorant" vector field componentwise, namely

$$MX(v) := \Big((MX)^{(j)}(v) \Big)_{j \in \mathcal{J}} := \Big(MX^{(j)}(v) \Big)_{j \in \mathcal{J}} . \tag{2.51}$$

We consider vector fields $X : D(s, r) \subset E \to E$, see (2.1).

Definition 2.5. *The vector field X is absolutely convergent at v if every component $X^{(j)}(v)$, $j \in \mathcal{J}$, is absolutely convergent (see Definition 2.1) and*

$$\Big\| \big(X^{(j)}(v) \big)_{j \in \mathcal{J}} \Big\|_E < +\infty .$$

The properties of the space E in (2.1) (as target space), that we will use are:

1. E is a separable Hilbert space times a finite dimensional space,
2. the "monotonicity property" of the norm

$$v_0, v_1 \in E \quad \text{with} \quad |v_0^{(j)}| \le |v_1^{(j)}| \, , \ \forall \, j \in \mathcal{J} \quad \Longrightarrow \quad \|v_0\|_E \le \|v_1\|_E . \tag{2.52}$$

For $X : D(s, r) \to E$ we define the sup-norm

$$|X|_{s,r} := \sup_{v \in D(s,r)} \|X(v)\|_{E,s,r} . \tag{2.53}$$

Definition 2.6. (Majorant-norm: vector field) *The majorant norm of a formal vector field X as in* (2.49) *is*

$$\begin{aligned} \|X\|_{s,r} \quad &:= \quad \sup_{(y,z,\bar z) \in D(r)} \Big\| \Big(\sum_{k,i,\alpha,\beta} |X^{(j)}_{k,i,\alpha,\beta}| e^{|k|s} |y^i| |z^\alpha| |\bar z^\beta| \Big)_{j \in \mathcal{J}} \Big\|_{E,s,r} \\ &= \quad \sup_{(y,z,\bar z) \in D(r)} \Big\| \sum_{k,i,\alpha,\beta} |X_{k,i,\alpha,\beta}| e^{|k|s} |y^i| |z^\alpha| |\bar z^\beta| \Big\|_{E,s,r} \end{aligned} \tag{2.54}$$

where

$$X_{k,i,\alpha,\beta} := \big(X^{(j)}_{k,i,\alpha,\beta} \big)_{j \in \mathcal{J}} \qquad \text{and} \qquad |X_{k,i,\alpha,\beta}| := \big(|X^{(j)}_{k,i,\alpha,\beta}| \big)_{j \in \mathcal{J}} .$$

Remark 2.3. *The stronger norm (see [24])*

$$\lfloor X \rceil_{s,r} := \Big\| \Big(\sup_{(y,z,\bar z)\in D(r)} \sum_{k,i,\alpha,\beta} |X^{(j)}_{k,i,\alpha,\beta}| e^{|k|s} |y^i| |z^\alpha| |\bar z^\beta| \Big)_{j\in\mathcal{J}} \Big\|_{E,s,r}$$

is not suited for infinite dimensional systems: for $X = Id$ we have $\lfloor X \rceil_{s,r} = +\infty$.

By (2.54) and (2.51) we get $\|X\|_{s,r} = \|MX\|_{s,r}$. For a subset of indices $I \subset \mathbb{I}$ we define the projection

$$(\Pi_I X)(x,y,z,\bar z) := \sum_{(k,i,\alpha,\beta)\in I} X_{k,i,\alpha,\beta}\, e^{\mathrm{i}k\cdot x} y^i z^\alpha \bar z^\beta \,.$$

Lemma 2.9. (Projection) $\forall I \subset \mathbb{I}$,

$$\|\Pi_I X\|_{s,r} \le \|X\|_{s,r} \,. \tag{2.55}$$

PROOF. See (2.54). ■

Remark 2.4. *The estimate* (2.55) *may fail for the sup-norm* $|\ |_{s,r}$ *and suitable* I.

Let $\Pi_{|k|\ge K}$ the "ultraviolet" projection

$$(\Pi_{|k|\ge K} X)(x,y,z,\bar z) := \sum_{|k|\ge K, i,\alpha,\beta} X_{k,i,\alpha,\beta}\, e^{\mathrm{i}k\cdot x} y^i z^\alpha \bar z^\beta \,.$$

Lemma 2.10. (Smoothing) $\forall\, 0 < s' < s$,

$$\|\Pi_{|k|\ge K} X\|_{s',r} \le \frac{s}{s'}\, e^{-K(s-s')} \|X\|_{s,r} \,. \tag{2.56}$$

PROOF. Recall (2.54) and use $e^{|k|s'} \le e^{|k|s} e^{-K(s-s')}$, $\forall |k| \ge K$. ■

We decompose each formal vector field

$$X = \sum_{\varsigma\in\{+,-\}^n} \Pi_\varsigma X \tag{2.57}$$

applying (2.19) componentwise

$$X_\varsigma := \Pi_\varsigma X := \Big(\Pi_\varsigma X^{(j)} \Big)_{j\in\mathcal{J}} \tag{2.58}$$

recall (2.17). Clearly (2.55) implies

$$\|X_\varsigma\|_{s,r} \le \|X\|_{s,r} \,. \tag{2.59}$$

In the next lemma we prove that, if X has finite majorant norm, then it is analytic.

Lemma 2.11. *Assume*

$$\|X\|_{s,r} < +\infty \,. \tag{2.60}$$

Then the series in (2.49)-(2.50), *resp.* (2.51), *absolutely converge to the* analytic *vector field $X(v)$, resp. $MX(v)$, for every $v \in D(s,r)$. Moreover the sup-norm defined in* (2.53) *satisfies*

$$|X|_{s,r},\ |MX|_{s,r} \ \le\ \|X\|_{s,r} \,. \tag{2.61}$$

PROOF. By (2.60) and Definition 2.6, for each $j \in \mathcal{J}$, we have

$$\sup_{(y,z,\bar z)\in D(r)} \sum_{k,i,\alpha,\beta} |X^{(j)}_{k,i,\alpha,\beta}| e^{|k|s} |y^i| |z^\alpha| |\bar z^\beta| < +\infty$$

and Lemma 2.1 (and Corollary 2.1) implies that each coordinate function $X^{(j)}$, $(MX)^{(j)} : D(s,r) \to \mathbb{C}$ is analytic. Moreover (2.61) follows applying (2.27) componentwise. By (2.60) the maps

$$X\,,\; MX : D(s,r) \to E$$

are bounded. Since E is a separable Hilbert space (times a finite dimensional space), Theorem 3-Appendix A in [26], implies that X, $MX : D(s,r) \to E$ are analytic. ■

Viceversa, we associate to an analytic vector field $X : D(s,r) \to E$ a formal Taylor-Fourier vector field (2.49)-(2.50) developing each component $X^{(j)}$ as in (2.28)-(2.29).

Definition 2.7. (Order relation: vector fields) *Given formal vector fields X, Y, we say that*

$$X \prec Y$$

if each coordinate $X^{(j)} \prec Y^{(j)}$, $j \in \mathcal{J}$, according to Definition 2.4.

If $\|Y\|_{s,r} < +\infty$ and

$$X \prec Y \quad \Longrightarrow \quad \|X\|_{s,r} \le \|Y\|_{s,r}\,. \tag{2.62}$$

Applying Lemma 2.5 component-wise we get

Lemma 2.12. *If $\|X\|_{s,r}, \|Y\|_{s,r} < \infty$ then $X + Y \prec MX + MY$ and $\|X+Y\|_{s,r} \le \|X\|_{s,r} + \|Y\|_{s,r}$.*

Lemma 2.13.

$$|MX|_{s,r} \le \|X\|_{s,r} \le 2^n |MX|_{s,r}\,. \tag{2.63}$$

PROOF. As for Lemma 2.4 with $f \rightsquigarrow X$, $|\sum_{k,i,\alpha,\beta}| \rightsquigarrow \|\sum_{k,i,\alpha,\beta}\|_E$ and using (2.52). ■

We define the space of analytic vector fields

$$\mathcal{V}_{s,r} := \mathcal{V}_{s,r,E} := \Big\{ X : D(s,r) \to E \;\; \text{with norm } \|X\|_{s,r} < +\infty \Big\}\,.$$

By Lemma 2.11 if $X \in \mathcal{V}_{s,r}$ then X is analytic, namely the Fréchet differential $D(s,r) \ni v \mapsto dX(v) \in \mathcal{L}(E,E)$ is continuous. The next lemma bounds its operator norm from $(E,s,r) := (E, \|\ \|_{E,s,r})$ to (E,s',r'), see (2.2).

Lemma 2.14. (Cauchy estimate) *Let $X \in \mathcal{V}_{s,r}$. Then, for $s/2 \le s' < s$, $r/2 \le r' < r$,*

$$\sup_{v \in D(s',r')} \|dX(v)\|_{\mathcal{L}((E,s,r),(E,s',r'))} \le 4\delta^{-1} |X|_{s,r} \tag{2.64}$$

where the sup-norm $|X|_{s,r}$ is defined in (2.53) *and*

$$\delta := \min\Big\{ 1 - \frac{s'}{s}, 1 - \frac{r'}{r} \Big\}\,. \tag{2.65}$$

PROOF. In the Appendix. ■

The commutator of two vector fields $X, Y : D(s,r) \to E$ is

$$[X,Y](v) := dX(v)[Y(v)] - dY(v)[X(v)]\,, \quad \forall\, v \in D(s,r)\,. \tag{2.66}$$

The next lemma is the fundamental result of this section.

Lemma 2.15. (Commutator) *Let $X, Y \in \mathcal{V}_{s,r}$. Then, for $r/2 \le r' < r$, $s/2 \le s' < s$,*

$$\|[X,Y]\|_{s',r'} \le 2^{2n+3} \delta^{-1} \|X\|_{s,r} \|Y\|_{s,r} \tag{2.67}$$

where δ is defined in (2.65).

Proof. The lemma follows by

$$\|dX[Y]\|_{s',r'} \le 4^{n+2}\delta^{-1}\|X\|_{s,r}\|Y\|_{s,r}\,, \tag{2.68}$$

the analogous estimate for $dY[X]$ and (2.66).

We claim that, for each $\varsigma \in \{+,-\}^n$, the vector field X_ς defined in (2.58) satisfies

$$\|dX_\varsigma[Y]\|_{s',r'} \le 2^{n+2}\delta^{-1}\|X_\varsigma\|_{s,r}\|Y\|_{s,r} \tag{2.69}$$

which implies (2.68) because

$$\begin{aligned}\|dX[Y]\|_{s',r'} &\overset{(2.57)}{\le} \sum_{\varsigma\in\{+,-\}^n}\|dX_\varsigma[Y]\|_{s',r'} \overset{(2.69)}{\le} \sum_{\varsigma\in\{+,-\}^n} 2^{n+2}\delta^{-1}\|X_\varsigma\|_{s,r}\|Y\|_{s,r}\\ &\overset{(2.59)}{\le} \sum_{\varsigma\in\{+,-\}^n} 2^{n+2}\delta^{-1}\|X\|_{s,r}\|Y\|_{s,r} \le 4^{n+2}\delta^{-1}\|X\|_{s,r}\|Y\|_{s,r}\,.\end{aligned}$$

Let us prove (2.69). First note that, since $\|X_\varsigma\|_{s,r} \overset{(2.59)}{\le} \|X\|_{s,r} < +\infty$ and $\|Y\|_{s,r} < +\infty$ by assumption, Lemma 2.11 implies that the vector fields

$$X_\varsigma, MX_\varsigma, Y, MY : D(s,r) \to E\,, \quad \forall \varsigma \in \{+,-\}^n\,, \tag{2.70}$$

are analytic, as well as each component $X_\varsigma^{(i)}, MX_\varsigma^{(i)}, Y^{(i)}, MY^{(i)} : D(s,r) \to \mathbb{C}$, $i \in \mathcal{J}$.

The key for proving the lemma is the following chain of inequalities:

$$\begin{aligned} dX_\varsigma[Y]^{(i)} \prec M(dX_\varsigma[Y])^{(i)} &\overset{(2.47)}{=} M\Big(\sum_{j\in\mathcal{J}}(\partial_j X_\varsigma^{(i)})Y^{(j)}\Big)\\ &\overset{\text{Lemma 2.6}}{\prec} \sum_{j\in\mathcal{J}} M(\partial_j X_\varsigma^{(i)})MY^{(j)}\\ &\overset{(2.40)}{=} \sum_{j\in\mathcal{J}} q_\varsigma^{(j)}\partial_j\big(MX_\varsigma^{(i)}\big)MY^{(j)} \overset{(2.47)}{=} d\big(MX_\varsigma^{(i)}\big)[\tilde Y_q]\end{aligned} \tag{2.71}$$

where

$$\tilde Y_q := (\tilde Y_q^{(j)})_{j\in\mathcal{J}} := (q_\varsigma^{(j)} MY^{(j)})_{j\in\mathcal{J}} \in E\,. \tag{2.72}$$

Actually, since $|q_\varsigma^{(j)}| = 1$ (see (2.38)), then

$$\|\tilde Y_q(v)\|_E = \|MY(v)\|_E \overset{(2.70)}{<} +\infty\,, \quad \forall v \in D(s,r)\,. \tag{2.73}$$

In (2.71) above we applied Lemma 2.6 with

$$s \rightsquigarrow s'\,,\ r \rightsquigarrow r'\,,\ f^{(j)} \rightsquigarrow (\partial_j X_\varsigma^{(i)})Y^{(j)}\,,\ g^{(j)} \rightsquigarrow M(\partial_j X_\varsigma^{(i)})MY^{(j)}\,. \tag{2.74}$$

Let us verify that the hypotheses of Lemma 2.6 hold:

1. $f^{(j)} \prec g^{(j)}$ follows by (2.39) and since $\|f^{(j)}\|_{s',r'}, \|g^{(j)}\|_{s',r'} < +\infty$ because $\|X_\varsigma^{(i)}\|_{s,r} \le \|X\|_{s,r} < +\infty$, $\|Y^{(j)}\|_{s,r} \le \|Y\|_{s,r} < +\infty$, and Lemma 2.7.
2. $\|g^{(j)}\|_{s',r'} < \infty$ is proved above.

3. We have $\sum_{j\in\mathcal{J}} |g^{(j)}(v)| < \infty$, for all $v \in D(s', r')$, because

$$\sum_{j\in\mathcal{J}} |g^{(j)}(v)| \overset{(2.74)}{=} \sum_{j\in\mathcal{J}} |M(\partial_j X_\varsigma^{(i)})(v) MY^{(j)}(v)| \overset{(2.40)}{=} \sum_{j\in\mathcal{J}} |q_\varsigma^{(j)} \partial_j \big(MX_\varsigma^{(i)}\big)(v) MY^{(j)}(v)|$$
$$\overset{(2.38)}{=} \sum_{j\in\mathcal{J}} |\partial_j \big(MX_\varsigma^{(i)}\big)(v) MY^{(j)}(v)| \overset{(2.48)}{\leq} \|dMX_\varsigma^{(i)}(v)\|_{\mathcal{L}(E,\mathbb{C})} \|MY(v)\|_E < +\infty$$

by (2.70), (2.73). Actually we also proved that $g^{(j)} = q_\varsigma^{(j)} \partial_j \big(MX_\varsigma^{(i)}\big) MY^{(j)}$.

4. The function

$$g(v) := \sum_{j\in\mathcal{J}} g^{(j)}(v) = \sum_{j\in\mathcal{J}} q_\varsigma^{(j)} \partial_j \big(MX_\varsigma^{(i)}\big) MY^{(j)} \overset{(2.47)}{=} d\big(MX_\varsigma^{(i)}\big)\big[\tilde{Y}_q\big]$$

since $MX_\varsigma^{(i)}$ is differentiable (see (2.70)) and $\tilde{Y}_q \in E$ (see (2.73)).
Moreover the bound $|g|_{s',r'} < \infty$ follows by

$$|g|_{s',r'} = |d\big(MX_\varsigma^{(i)}\big)\big[\tilde{Y}_q\big]|_{s',r'} \leq |d\big(MX_\varsigma\big)\big[\tilde{Y}_q\big]|_{s',r'}$$

and

$$\begin{aligned}
|d\big(MX_\varsigma\big)\big[\tilde{Y}_q\big]|_{s',r'} &\overset{(2.53)}{=} \sup_{v\in D(s',r')} \Big\| d\big(MX_\varsigma\big)(v)\big[\tilde{Y}_q(v)\big] \Big\|_{E,s',r'} \\
&\leq \sup_{v\in D(s',r')} \Big\| d\big(MX_\varsigma\big)(v) \Big\|_{\mathcal{L}((E,s,r),(E,s',r'))} \|\tilde{Y}_q(v)\|_{E,s,r} \\
&\overset{(2.64)}{\leq} 4\delta^{-1} |MX_\varsigma|_{s,r} \sup_{v\in D(s',r')} \|\tilde{Y}_q(v)\|_{E,s,r} \\
&\overset{(2.61),(2.73)}{\leq} 4\delta^{-1} \|X_\varsigma\|_{s,r} \sup_{v\in D(s',r')} \|\big(MY\big)(v)\|_{E,s,r} \\
&\overset{(2.53)}{\leq} 4\delta^{-1} \|X_\varsigma\|_{s,r} |MY|_{s,r} \overset{(2.63)}{\leq} 4\delta^{-1} \|X_\varsigma\|_{s,r} \|Y\|_{s,r} < +\infty
\end{aligned} \tag{2.75}$$

because $\|Y\|_{s,r} < +\infty$ and $\|X_\varsigma\|_{s,r} \leq \|X\|_{s,r} < +\infty$ by assumption.

Hence Lemma 2.6 implies

$$dX_\varsigma^{(i)}[Y] \overset{(2.47)}{=} \sum_j (\partial_j X_\varsigma^{(i)}) Y^{(j)} =: f \overset{\text{Lemma 2.6}}{\prec} g := d\big(MX_\varsigma^{(i)}\big)\big[\tilde{Y}_q\big], \quad \forall i \in \mathcal{J},$$

namely, by (2.37) and Definition 2.7,

$$dX_\varsigma[Y] \prec M(dX_\varsigma[Y]) \prec d\big(MX_\varsigma\big)\big[\tilde{Y}_q\big]. \tag{2.76}$$

Hence (2.72) is fully justified. By (2.76) and (2.62) we get

$$\begin{aligned}
\|dX_\varsigma[Y]\|_{s',r'} \leq \|d\big(MX_\varsigma\big)\big[\tilde{Y}_q\big]\|_{s',r'} &\overset{(2.63)}{\leq} 2^n \Big| M\Big(d\big(MX_\varsigma\big)\big[\tilde{Y}_q\big]\Big)\Big|_{s',r'} \\
&= 2^n \Big| d\big(MX_\varsigma\big)\big[\tilde{Y}_q\big]\Big|_{s',r'}
\end{aligned} \tag{2.77}$$

because $d\big(MX_\varsigma\big)\big[\tilde{Y}_q\big]$ coincides with its majorant by (2.76). Finally (2.69) follows by (2.77), (2.75). ■

2.2 Hamiltonian formalism

Given a function $H : D(s,r) \subset E \to \mathbb{C}$ we define the associated Hamiltonian vector field

$$X_H := (\partial_y H, -\partial_x H, -\mathrm{i}\partial_{\bar z} H, \mathrm{i}\partial_z H) \tag{2.78}$$

where the partial derivatives are defined as in (2.30).

For a subset of indices $I \subset \mathbb{I}$, the bound (2.55) implies

$$\|X_{\Pi_I H}\|_{s,r} \le \|X_H\|_{s,r}\,. \tag{2.79}$$

The Poisson brackets are defined by

$$\begin{aligned} \{H,K\} &:= \{H,K\}^{x,y} + \{H,K\}^{z,\bar z} \\ &:= \Big(\partial_x H \cdot \partial_y K - \partial_x K \cdot \partial_y H\Big) + \mathrm{i}\Big(\partial_z H \cdot \partial_{\bar z} K - \partial_{\bar z} H \cdot \partial_z K\Big) \\ &= \partial_x H \cdot \partial_y K - \partial_x K \cdot \partial_y H + \mathrm{i}\partial_{z^+} H \cdot \partial_{z^-} K - \mathrm{i}\partial_{z^-} H \cdot \partial_{z^+} K \\ &= \partial_x H \cdot \partial_y K - \partial_x K \cdot \partial_y H + \mathrm{i} \sum_{\sigma=\pm,\, j\in\mathbb{Z}\setminus\mathcal{I}} \sigma \partial_{z_j^\sigma} H\, \partial_{z_j^{-\sigma}} K \end{aligned} \tag{2.80}$$

where "$\cdot$" denotes the standard pairing $a \cdot b := \sum_j a_j b_j$. We recall the Jacobi identity

$$\{\{K,G\},H\} + \{\{G,H\},K\} + \{\{H,K\},G\} = 0\,. \tag{2.81}$$

Along this paper we shall use the Lie algebra notations

$$\mathrm{ad}_F := \{\ ,F\}\,, \quad e^{\mathrm{ad}_F} := \sum_{k=0}^{\infty} \frac{\mathrm{ad}_F^k}{k!}\,. \tag{2.82}$$

Given a set of indices

$$\mathcal{I} := \{\mathtt{j}_1, \ldots, \mathtt{j}_n\} \subset \mathbb{Z}\,, \tag{2.83}$$

we define the *momentum*

$$\mathcal{M} := \mathcal{M}_{\mathcal{I}} := \sum_{l=1}^{n} \mathtt{j}_l\, y_l + \sum_{j\in\mathbb{Z}\setminus\mathcal{I}} j z_j \bar z_j = \sum_{l=1}^{n} \mathtt{j}_l\, y_l + \sum_{j\in\mathbb{Z}\setminus\mathcal{I}} j z_j^+ z_j^-\,.$$

We say that a function H satisfies momentum conservation if $\{H, \mathcal{M}\} = 0$.

By (2.80), any monomial $e^{\mathrm{i}k\cdot x} y^i z^\alpha \bar z^\beta$ is an eigenvector of the operator $\mathrm{ad}_{\mathcal{M}}$, namely

$$\{e^{\mathrm{i}k\cdot x} y^i z^\alpha \bar z^\beta, \mathcal{M}\} = \pi(k,\alpha,\beta) e^{\mathrm{i}k\cdot x} y^i z^\alpha \bar z^\beta \tag{2.84}$$

where

$$\pi(k,\alpha,\beta) := \sum_{l=1}^{n} \mathtt{j}_l k_l + \sum_{j\in\mathbb{Z}\setminus\mathcal{I}} j(\alpha_j - \beta_j)\,. \tag{2.85}$$

We refer to $\pi(k,\alpha,\beta)$ as the *momentum of the monomial* $e^{\mathrm{i}k\cdot x} y^i z^\alpha \bar z^\beta$. A monomial satisfies momentum conservation if and only if $\pi(k,\alpha,\beta) = 0$. Moreover, a power series (2.7) with $\|f\|_{s,r} < +\infty$ satisfies momentum conservation if and only if all its monomials have zero momentum.

Let $\mathcal{O} \subset \mathbb{R}^n$ be a subset of *parameters*, and

$$f : D(s,r) \times \mathcal{O} \to \mathbb{C} \qquad \text{with} \qquad X_f : D(s,r) \times \mathcal{O} \to E\,. \tag{2.86}$$

For $\lambda > 0$, we consider

$$|X_f|^{\lambda}_{s,r,\mathcal{O}} := |X_f|^{\lambda}_{s,r} := \sup_{\mathcal{O}} |X_f|_{s,r} + \lambda |X_f|^{\rm lip}_{s,r} := \sup_{\xi \in \mathcal{O}} |X_f(\xi)|_{s,r} + \lambda \sup_{\xi,\eta \in \mathcal{O},\ \xi \neq \eta} \frac{|X_f(\xi) - X_f(\eta)|_{s,r}}{|\xi - \eta|} . \tag{2.87}$$

Note that $| \cdot |^{\lambda}_{s,r}$ is only a semi-norm on spaces of functions f because the Hamiltonian vector field $X_f = 0$ when f is constant.

Definition 2.8. *A function f as in* (2.86) *is called*

- **regular***, if the sup-norm* $|X_f|_{s,r,\mathcal{O}} := \sup_{\mathcal{O}} |X_f|_{s,r} < \infty$*, see* (2.53).
- **M-regular***, if the majorant norm* $\|X_f\|_{s,r,\mathcal{O}} := \sup_{\mathcal{O}} \|X_f\|_{s,r} < \infty$*, see* (2.54).
- **λ-regular***, if the Lipschitz semi norm* $|X_f|^{\lambda}_{s,r,\mathcal{O}} < \infty$*, see* (2.87).

We denote by $\mathcal{H}_{s,r}$ the space of M-regular Hamiltonians and by $\mathcal{H}^{\rm null}_{s,r}$ its subspace of functions satisfying momentum conservation.

When $\mathcal{I} = \emptyset$ (namely there are no (x, y)-variables) we denote the space of M-regular functions simply by $\mathcal{H}_r$, similarly $\mathcal{H}^{\rm null}_r$, and we drop s form the norms, i.e. $| \cdot |_r$, $\| \cdot \|_r$, $| \cdot |_{r,\mathcal{O}}$, etc.

Note that, by (2.61) and (2.87), we have

$$\mathrm{M-regular} \quad \Longrightarrow \quad \mathrm{regular} \quad \Longleftarrow \quad \lambda - \mathrm{regular} . \tag{2.88}$$

If H, F satisfy momentum conservation, the same holds for $\{H, K\}$. Indeed by the Jacobi identity (2.81),

$$\{\mathcal{M}, H\} = 0 \ \text{ and } \ \{\mathcal{M}, K\} = 0 \quad \Longrightarrow \quad \{\mathcal{M}, \{H, K\}\} = 0 . \tag{2.89}$$

For $H, K \in \mathcal{H}_{s,r}$ we have

$$X_{\{H,K\}} = dX_H[X_K] - dX_K[X_H] = [X_H, X_K] \tag{2.90}$$

and the commutator Lemma 2.15 implies the fundamental lemma below.

Lemma 2.16. *Let $H, K \in \mathcal{H}_{s,r}$. Then, for all $r/2 \leq r' < r$, $s/2 \leq s' < s$*

$$\|X_{\{H,K\}}\|_{s',r'} = \|[X_H, X_K]\|_{s',r'} \leq 2^{2n+3} \delta^{-1} \|X_H\|_{s,r} \|X_K\|_{s,r} \tag{2.91}$$

where δ is defined in (2.65).

Unlike the sup-norm, the majorant norm of a function is very sensitive to coordinate transformations. For our purposes, we only need to consider close to identity canonical transformations that are generated by an M-regular Hamiltonian flow. We show below that the M-regular functions are closed under this group and we estimate the majorant norm of the transformed Hamiltonian vector field.

Lemma 2.17. (Hamiltonian flow) *Let $r/2 \leq r' < r$, $s/2 \leq s' < s$, and $F \in \mathcal{H}_{s,r}$ with*

$$\|X_F\|_{s,r} < \eta := \delta/(2^{2n+5} e) \tag{2.92}$$

with δ defined in (2.65)*. Then the time 1-hamiltonian flow*

$$\Phi^1_F := e^{\mathrm{ad}_F} : D(s', r') \to D(s, r)$$

is well defined, analytic, symplectic, and, $\forall H \in \mathcal{H}_{s,r}$, we have $H \circ \Phi^1_F \in \mathcal{H}_{s',r'}$ and

$$\|X_{H \circ \Phi^1_F}\|_{s',r'} \leq \frac{\|X_H\|_{s,r}}{1 - \eta^{-1} \|X_F\|_{s,r}} . \tag{2.93}$$

Finally if $F, H \in \mathcal{H}^{\rm null}_{s,r}$ then $H \circ \Phi^1_F \in \mathcal{H}^{\rm null}_{s',r'}$.

Proof. We estimate by Lie series the Hamiltonian vector field of

$$H' = H \circ \Phi_F^1 = e^{\mathrm{ad_F}} H = \sum_{k=0}^{\infty} \frac{\mathrm{ad}_F^k H}{k!} = \sum_{k=0}^{\infty} \frac{H^{(k)}}{k!} \, , \quad \text{i.e. } X_{H'} = \sum_{k=0}^{\infty} \frac{X_{H^{(k)}}}{k!} \, , \tag{2.94}$$

where $H^{(i)} := \mathrm{ad}_F^i(H) = \mathrm{ad}_F(H^{(i-1)})$, $H^{(0)} := H$.

For each $k \geq 0$, divide the intervals $[s', s]$ and $[r', r]$ into k equal segments and set

$$s_i := s - i\, \frac{s - s'}{k} \, , \qquad r_i := r - i\, \frac{r - r'}{k} \, , \qquad i = 0, \ldots, k \, .$$

By (2.91) we have

$$\|X_{H^{(i)}}\|_{s_i, r_i} = \|[X_F, X_{H^{(i-1)}}]\|_{s_i, r_i} \leq 2^{2n+3} \delta_i^{-1} \|X_{H^{(i-1)}}\|_{s_{i-1}, r_{i-1}} \|X_F\|_{s_{i-1}, r_{i-1}} \tag{2.95}$$

where

$$\delta_i \quad := \quad \min\left\{ 1 - \frac{s_i}{s_{i-1}} \, , \, 1 - \frac{r_i}{r_{i-1}} \right\} \geq \frac{\delta}{k} \, . \tag{2.96}$$

By (2.95)-(2.96) we deduce

$$\|X_{H^{(i)}}\|_{s_i, r_i} \leq 2^{2n+3} k \delta^{-1} \|X_{H^{(i-1)}}\|_{s_{i-1}, r_{i-1}} \|X_F\|_{s_{i-1}, r_{i-1}} \, , \quad i = 1, \ldots, k \, .$$

Iterating k-times, and using $\|X_F\|_{s_{i-1}, r_{i-1}} \leq 4 \|X_F\|_{s,r}$ (see (2.3))

$$\|X_{H^{(k)}}\|_{s', r'} \leq (2^{2n+5} k \delta^{-1})^k \|X_H\|_{s,r} \|X_F\|_{s,r}^k \, . \tag{2.97}$$

By (2.94), using $k^k \leq e^k k!$ and recalling the definition of η in (2.92), we estimate

$$\begin{aligned} \|X_{H'}\|_{s', r'} &\overset{(2.94)}{\leq} \sum_{k=0}^{\infty} \frac{\|X_{H^{(k)}}\|_{s', r'}}{k!} \overset{(2.97)}{\leq} \|X_H\|_{s,r} \sum_{k=0}^{\infty} \frac{(2^{2n+5} k \delta^{-1} \|X_F\|_{s,r})^k}{k!} \\ &\leq \|X_H\|_{s,r} \sum_{k=0}^{\infty} (\eta^{-1} \|X_F\|_{s,r})^k \overset{(2.92)}{=} \frac{\|X_H\|_{s,r}}{1 - \eta^{-1} \|X_F\|_{s,r}} \end{aligned}$$

proving (2.93).

Finally, if F and H satisfy momentum conservation then each $\mathrm{ad}_F^k H$, $k \geq 1$, satisfy momentum conservation. For $k = 1$ it is proved in (2.89) and, for $k > 1$, it follows by induction and the Jacobi identity (2.81). By (2.94) we conclude that also $H \circ \Phi_F^1$ satisfies momentum conservation. ■

We conclude this section with two simple lemmata.

Lemma 2.18. *Let* $P = \sum_{|k| \leq K, i, \alpha, \beta} P_{k,i,\alpha,\beta} e^{\mathrm{i} k \cdot x} y^i z^\alpha \bar{z}^\beta$ *and* $|\Delta_{k,i,\alpha,\beta}| \geq \gamma \langle k \rangle^{-\tau}$, $\forall |k| \leq K, i, \alpha, \beta$. *Then*

$$F := \sum_{|k| \leq K, i, \alpha, \beta} \frac{P_{k,i,\alpha,\beta}}{\Delta_{k,i,\alpha,\beta}} e^{\mathrm{i} k \cdot x} y^i z^\alpha \bar{z}^\beta \quad \text{satisfies} \quad \|X_F\|_{s,r} \leq \gamma^{-1} K^\tau \|X_P\|_{s,r} \, .$$

Proof. By Definition 2.6 and $|\Delta_{k,i,\alpha,\beta}| \geq \gamma K^{-\tau}$ for all $|k| \leq K$. ■

Lemma 2.19. *Let* $P = \sum_{j \in \mathbb{Z} \setminus \mathcal{I}} P_j z_j \bar{z}_j$ *with* $\|X_P\|_r < \infty$. *Then* $|P_j| \leq \|X_P\|_r$.

Proof. By (2.78) and Definition 2.6 we have

$$\|X_P\|_r^2 = 2 \sup_{\|z\|_{a,p} < r} \sum_{h \in \mathbb{Z} \setminus \mathcal{I}} |P_h|^2 \frac{|z_h|^2}{r^2} e^{2a|h|} \langle h \rangle^{2p} \geq |P_j|^2$$

by evaluating at $z_h^{(j)} := \delta_{jh} e^{-a|j|} \langle j \rangle^p r / \sqrt{2}$. ■

3 Quasi-Töplitz functions

Let $N_0 \in \mathbb{N}$, $\theta, \mu \in \mathbb{R}$ be parameters such that

$$1 < \theta, \mu < 6 \,, \quad 12 N_0^{L-1} + 2\kappa N_0^{b-1} < 1 \,, \quad \kappa := \max_{1 \le l \le n} |\mathtt{j}_l| \,, \tag{3.1}$$

(the $\mathtt{j}_l$ are defined in (2.83)) where

$$0 < b < L < 1 \,. \tag{3.2}$$

For $N \ge N_0$, we decompose

$$\ell_{\mathcal{I}}^{a,p} \times \ell_{\mathcal{I}}^{a,p} = \ell_L^{a,p} \oplus \ell_R^{a,p} \oplus \ell_H^{a,p} \tag{3.3}$$

where

$$\ell_L^{a,p} := \ell_L^{a,p}(N) := \Big\{ w = (z^+, z^-) \in \ell_{\mathcal{I}}^{a,p} \times \ell_{\mathcal{I}}^{a,p} \ : \ z_j^\sigma = 0 \,, \ \sigma = \pm \,, \ \forall |j| \ge 6 N^L \Big\}$$

$$\ell_R^{a,p} := \ell_R^{a,p}(N) := \Big\{ w = (z^+, z^-) \in \ell_{\mathcal{I}}^{a,p} \times \ell_{\mathcal{I}}^{a,p} \ : \ z_j^\sigma = 0 \,, \ \sigma = \pm \,, \ \text{unless } 6N^L < |j| < N \Big\}$$

$$\ell_H^{a,p} := \ell_H^{a,p}(N) := \Big\{ w = (z^+, z^-) \in \ell_{\mathcal{I}}^{a,p} \times \ell_{\mathcal{I}}^{a,p} \ : \ z_j^\sigma = 0 \,, \ \sigma = \pm \,, \ \forall |j| \le N \Big\} \,.$$

Note that by (3.1)-(3.2) the subspaces $\ell_L^{a,p} \cap \ell_H^{a,p} = 0$ and $\ell_R^{a,p} \neq 0$. Accordingly we decompose any

$$w \in \ell^{a,p} \times \ell^{a,p} \qquad \text{as} \qquad w = w_L + w_R + w_H$$

and we call $w_L \in \ell_L^{a,p}$ the "low momentum variables" and $w_H \in \ell_H^{a,p}$ the "high momentum variables".

We split the Poisson brackets in (2.80) as

$$\{\cdot, \cdot\} = \{\cdot, \cdot\}^{x,y} + \{\cdot, \cdot\}^L + \{\cdot, \cdot\}^R + \{\cdot, \cdot\}^H$$

where

$$\{H, K\}^H := \mathrm{i} \sum_{\sigma = \pm, \, |j| > cN} \sigma \partial_{z_j^\sigma} H \, \partial_{z_j^{-\sigma}} K \,. \tag{3.4}$$

The other Poisson brackets $\{\cdot, \cdot\}^L$, $\{\cdot, \cdot\}^R$ are defined analogously with respect to the splitting (3.3).

Lemma 3.1. *Consider two monomials* $\mathfrak{m} = c_{k,i,\alpha,\beta} e^{\mathrm{i} k \cdot x} y^i z^\alpha \bar{z}^\beta$ *and* $\mathfrak{m}' = c'_{k',i',\alpha',\beta'} e^{\mathrm{i} k' \cdot x} y^{i'} z^{\alpha'} \bar{z}^{\beta'}$. *The momentum of* $\mathfrak{m}\mathfrak{m}'$, $\{\mathfrak{m}, \mathfrak{m}'\}$, $\{\mathfrak{m}, \mathfrak{m}'\}^{x,y}$, $\{\mathfrak{m}, \mathfrak{m}'\}^L$, $\{\mathfrak{m}, \mathfrak{m}'\}^R$, $\{\mathfrak{m}, \mathfrak{m}'\}^H$, *equals the sum of the momenta of each monomial* $\mathfrak{m}$, $\mathfrak{m}'$.

Proof. By (2.85), (2.80), and

$$\pi(k + k', \alpha + \alpha', \beta + \beta') = \pi(k, \alpha, \beta) + \pi(k', \alpha', \beta') = \pi(k, \alpha - e_j, \beta) + \pi(k', \alpha', \beta' - e_j) \,,$$

for any $j \in \mathbb{Z}$. ■

We now define subspaces of $\mathcal{H}_{s,r}$ (recall Definition 2.8).

Definition 3.1. (Low-momentum) *A monomial* $e^{\mathrm{i} k \cdot x} y^i z^\alpha \bar{z}^\beta$ *is* (N, μ)*-low momentum if*

$$\sum_{j \in \mathbb{Z} \setminus \mathcal{I}} |j| (\alpha_j + \beta_j) < \mu N^L \,, \quad |k| < N^b \,. \tag{3.5}$$

We denote by

$$\mathcal{L}_{s,r}(N, \mu) \subset \mathcal{H}_{s,r}$$

the subspace of functions

$$g = \sum g_{k,i,\alpha,\beta} e^{\mathrm{i} k \cdot x} y^i z^\alpha \bar{z}^\beta \in \mathcal{H}_{s,r} \tag{3.6}$$

whose monomials are (N,μ)*-low momentum. The corresponding projection*

$$\Pi^L_{N,\mu} : \mathcal{H}_{s,r} \to \mathcal{L}_{s,r}(N,\mu) \tag{3.7}$$

is defined as $\Pi^L_{N,\mu} := \Pi_I$ *(see* (2.13)*) where* I *is the subset of* $\mathbb{I}$ *(see* (2.8)*) satisfying* (3.5)*. Finally, given* $h \in \mathbb{Z}$*, we denote by*

$$\mathcal{L}_{s,r}(N,\mu,h) \subset \mathcal{L}_{s,r}(N,\mu)$$

the subspace of functions whose monomials satisfy

$$\pi(k,\alpha,\beta) + h = 0\,. \tag{3.8}$$

By (3.5), (3.1)-(3.2), any function in $\mathcal{L}_{s,r}(N,\mu)$, $1 < \mu < 6$, only depends on x, y, w_L and therefore

$$g, g' \in \mathcal{L}_{s,r}(N,\mu) \quad \Longrightarrow \quad gg',\ \{g,g'\}^{x,y},\ \{g,g'\}^L \ \text{ do not depend on } w_H\,. \tag{3.9}$$

Moreover, by (2.85), (3.1), (3.5), if

$$|h| \geq \mu N^L + \kappa N^b \quad \Longrightarrow \quad \mathcal{L}_{s,r}(N,\mu,h) = \emptyset\,. \tag{3.10}$$

Definition 3.2. ((N,θ,μ)-bilinear) *We denote by*

$$\mathcal{B}_{s,r}(N,\theta,\mu) \subset \mathcal{H}^{\rm null}_{s,r}$$

the subspace of the (N,θ,μ)*-bilinear functions defined as*

$$f := \sum_{|m|,|n|>\theta N, \sigma,\sigma'=\pm} f^{\sigma,\sigma'}_{m,n}(x,y,w_L) z^\sigma_m z^{\sigma'}_n \quad \text{with} \quad f^{\sigma,\sigma'}_{m,n} \in \mathcal{L}_{s,r}(N,\mu,\sigma m + \sigma' n) \tag{3.11}$$

and we denote the projection

$$\Pi_{N,\theta,\mu} : \mathcal{H}_{s,r} \to \mathcal{B}_{s,r}(N,\theta,\mu)\,.$$

Explicitely, for $g \in \mathcal{H}_{s,r}$ *as in* (3.6)*, the coefficients in* (3.11) *of* $f := \Pi_{N,\theta,\mu} g$ *are*

$$f^{\sigma,\sigma'}_{m,n}(x,y,w^L) := \sum_{\substack{(k,i,\alpha,\beta) \text{ s.t. } (3.5) \text{ holds} \\ \text{and } \pi(k,\alpha,\beta) = -\sigma m - \sigma' n}} f^{\sigma,\sigma'}_{k,i,\alpha,\beta,m,n} e^{\mathrm{i} k\cdot x} y^i z^\alpha \bar{z}^\beta \tag{3.12}$$

where

$$\begin{aligned} f^{+,+}_{k,i,\alpha,\beta,m,n} &:= (2-\delta_{mn})^{-1} g_{k,i,\alpha+e_m+e_n,\beta}\,, \qquad & f^{+,-}_{k,i,\alpha,\beta,m,n} &:= g_{k,i,\alpha+e_m,\beta+e_n}\,, \\ f^{-,-}_{k,i,\alpha,\beta,m,n} &:= (2-\delta_{mn})^{-1} g_{k,i,\alpha,\beta+e_m+e_n}\,, \qquad & f^{-,+}_{k,i,\alpha,\beta,m,n} &:= g_{k,i,\alpha+e_n,\beta+e_m}\,. \end{aligned} \tag{3.13}$$

For parameters $1 < \theta < \theta'$, $6 > \mu > \mu'$, we have

$$\mathcal{B}_{s,r}(N,\theta',\mu') \subset \mathcal{B}_{s,r}(N,\theta,\mu)\,.$$

Remark 3.1. *The projection* $\Pi_{N,\theta,\mu}$ *can be written in the form* Π_I*, see* (2.13)*, for a suitable* $I \subset \mathbb{I}$*. The representation in* (3.11) *is not unique. It becomes unique if we impose the "symmetric" conditions*

$$f^{\sigma,\sigma'}_{m,n} = f^{\sigma',\sigma}_{n,m}\,. \tag{3.14}$$

Note that the coefficients in (3.12)-(3.13) *satisfy* (3.14).

3.1 Töplitz functions

Let $N \geq N_0$.

Definition 3.3. (Töplitz) *A function $f \in \mathcal{B}_{s,r}(N,\theta,\mu)$ is* (N,θ,μ)-Töplitz *if the coefficients in* (3.11) *have the form*

$$f^{\sigma,\sigma'}_{m,n} = f^{\sigma,\sigma'}\big(\mathtt{s}(m), \sigma m + \sigma' n\big) \quad \text{for some} \quad f^{\sigma,\sigma'}(\varsigma, h) \in \mathcal{L}_{s,r}(N,\mu,h)\,, \tag{3.15}$$

with $\mathtt{s}(m) := \mathrm{sign}(m)$, $\varsigma = +,-$ and $h \in \mathbb{Z}$. We denote by

$$\mathcal{T}_{s,r} := \mathcal{T}_{s,r}(N,\theta,\mu) \subset \mathcal{B}_{s,r}(N,\theta,\mu)$$

the space of the (N,θ,μ)-Töplitz functions.

For parameters $N' \geq N$, $\theta' \geq \theta$, $\mu' \leq \mu$, $r' \leq r$, $s' \leq s$ we have

$$\mathcal{T}_{s,r}(N,\theta,\mu) \subseteq \mathcal{T}_{s',r'}(N',\theta',\mu')\,. \tag{3.16}$$

Lemma 3.2. *Consider $f, g \in \mathcal{T}_{s,r}(N,\theta,\mu)$ and $p \in \mathcal{L}_{s,r}(N,\mu_1,0)$ with $1 < \mu, \mu_1 < 6$. For all $0 < s' < s$, $0 < r' < r$ and $\theta' \geq \theta, \mu' \leq \mu$ one has*

$$\Pi_{N,\theta',\mu'}\{f,p\}^L\,,\ \Pi_{N,\theta',\mu'}\{f,p\}^{x,y} \in \mathcal{T}_{s',r'}(N,\theta',\mu')\,. \tag{3.17}$$

If moreover

$$\mu N^L + \kappa N^b < (\theta' - \theta) N \tag{3.18}$$

then

$$\Pi_{N,\theta',\mu'}\{f,g\}^H \in \mathcal{T}_{s',r'}(N,\theta',\mu')\,. \tag{3.19}$$

Proof. Write $f \in \mathcal{T}_{s,r}(N,\theta,\mu)$ as in (3.11) where $f^{\sigma,\sigma'}_{m,n}$ satisfy (3.15) and (3.14), namely

$$f^{\sigma,\sigma'}_{m,n} = f^{\sigma',\sigma}_{n,m} = f^{\sigma,\sigma'}(\mathtt{s}(m), \sigma m + \sigma' n) \in \mathcal{L}_{s,r}(N,\mu,\sigma m + \sigma' n)\,, \tag{3.20}$$

similarly for g.

Proof of (3.17). Since the variables z^σ_m, $z^{\sigma'}_n$, $|m|, |n| > \theta N$, are high momentum,

$$\{f^{\sigma,\sigma'}_{m,n} z^\sigma_m z^{\sigma'}_n\,,\, p\}^L = \{f^{\sigma,\sigma'}_{m,n}\,,\, p\}^L\, z^\sigma_m z^{\sigma'}_n$$

and $\{f^{\sigma,\sigma'}_{m,n}\,,\ p\}^L$ does not depend on w_H by (3.9) (recall that $f^{\sigma,\sigma'}_{m,n}$, $p \in \mathcal{L}_{s,r}(N,\mu)$). The coefficient of $z^\sigma_m z^{\sigma'}_n$ in $\Pi_{N,\theta',\mu'}\{f,p\}^L$ is

$$\Pi^L_{N,\mu'}\{f^{\sigma,\sigma'}_{m,n}\,,\ p\}^L \stackrel{(3.20)}{=} \Pi^L_{N,\mu'}\{f^{\sigma,\sigma'}(\mathtt{s}(m), \sigma m + \sigma' n)\,,\, p\}^L \in \mathcal{L}_{s',r'}(N,\mu',\sigma m + \sigma' n)$$

using Lemma 3.1 (recall that p has zero momentum). The proof that $\Pi_{N,\theta',\mu'}\{f,p\}^{x,y} \in \mathcal{T}_{s',r'}(N,\theta',\mu')$ is analogous.

Proof of (3.19). A direct computation, using (3.4), gives

$$\{f,g\}^H = \sum_{|m|,|n|>\theta N,\, \sigma,\sigma'=\pm} p^{\sigma,\sigma'}_{m,n} z^\sigma_m z^{\sigma'}_n$$

with

$$p^{\sigma,\sigma'}_{m,n} = 2\mathrm{i} \sum_{|l|>\theta N\,,\ \sigma_1=\pm} \sigma_1 \Big(f^{\sigma,\sigma_1}_{m,l}\, g^{-\sigma_1,\sigma'}_{l,n} + f^{\sigma',\sigma_1}_{n,l}\, g^{-\sigma_1,\sigma}_{l,m} \Big)\,. \tag{3.21}$$

By (3.9) the coefficient $p^{\sigma,\sigma'}_{m,n}$ does not depend on w_H. Therefore

$$\Pi_{N,\theta',\mu'}\{f,g\}^H = \sum_{|m|,|n|>\theta' N,\, \sigma,\sigma'=\pm} q^{\sigma,\sigma'}_{m,n} z^\sigma_m z^{\sigma'}_n \quad \text{with} \quad q^{\sigma,\sigma'}_{m,n} := \Pi^L_{N,\mu'} p^{\sigma,\sigma'}_{m,n} \tag{3.22}$$

(recall (3.7)). It results $q^{\sigma,\sigma'}_{m,n} \in \mathcal{L}_{s',r'}(N,\mu',\sigma m+\sigma' n)$ by (3.22), (3.21), and Lemma 3.1 since, i.e.,

$$f^{\sigma,\sigma_1}_{m,l} \in \mathcal{L}_{s,r}(N,\mu,\sigma m+\sigma_1 l) \quad \text{and} \quad g^{-\sigma_1,\sigma'}_{l,n} \in \mathcal{L}_{s,r}(N,\mu,-\sigma_1 l+\sigma' n)\,.$$

Hence the (N,θ',μ')-bilinear function $\Pi_{N,\theta',\mu'}\{f,g\}^H$ in (3.22) is written in the form (3.11). It remains to prove that it is (N,θ',μ')-Töplitz, namely that for all $|m|,|n|>\theta' N$, $\sigma,\sigma'=\pm$,

$$q^{\sigma,\sigma'}_{m,n} = q^{\sigma,\sigma'}\big(\mathtt{s}(m),\sigma m+\sigma' n\big) \quad \text{for some} \quad q^{\sigma,\sigma'}(\varsigma,h) \in \mathcal{L}_{s,r}(N,\mu',h)\,. \tag{3.23}$$

Let us consider in (3.21)-(3.22) the term (with $m,n,\sigma,\sigma',\sigma_1$ fixed)

$$\Pi^L_{N,\mu'} \sum_{|l|>\theta N} f^{\sigma,\sigma_1}_{m,l} g^{-\sigma_1,\sigma'}_{l,n} \tag{3.24}$$

(the other is analogous). Since $f,g \in \mathcal{T}_{s,r}(N,\theta,\mu)$ we have

$$f^{\sigma,\sigma_1}_{m,l} = f^{\sigma,\sigma_1}\big(\mathtt{s}(m),\sigma m+\sigma_1 l\big) \in \mathcal{L}_{s,r}(N,\mu,\sigma m+\sigma_1 l) \tag{3.25}$$

$$g^{-\sigma_1,\sigma'}_{l,n} = g^{-\sigma_1,\sigma'}\big(\mathtt{s}(l),-\sigma_1 l+\sigma' n\big) \in \mathcal{L}_{s,r}(N,\mu,-\sigma_1 l+\sigma' n)\,. \tag{3.26}$$

By (3.10), (3.25), (3.26), if the coefficients $f^{\sigma,\sigma_1}_{m,l}$, $g^{-\sigma_1,\sigma'}_{l,n}$ are not zero then

$$|\sigma m+\sigma_1 l|\,,\ |-\sigma_1 l+\sigma' n| < \mu N^L + \kappa N^b\,. \tag{3.27}$$

By (3.27), (3.1), we get $cN > |\sigma m+\sigma_1 l| = |\sigma\sigma_1 \mathtt{s}(m)|m| + \mathtt{s}(l)|l||$, which implies, since $|m|>\theta' N > N$ (see (3.22)), that the sign

$$\mathtt{s}(l) = -\sigma\sigma_1 \mathtt{s}(m)\,. \tag{3.28}$$

Moreover

$$|l| \geq |m| - |\sigma m+\sigma_1 l| \overset{(3.27)}{>} \theta' N - \mu N^L - \kappa N^b \overset{(3.18)}{>} \theta N\,.$$

This shows that the restriction $|l|>\theta N$ in the sum (3.24) is automatically met. Then

$$\begin{aligned}
\Pi^L_{N,\mu'} \sum_{|l|>\theta N} f^{\sigma,\sigma_1}_{m,l} g^{-\sigma_1,\sigma'}_{l,n} &\overset{(3.26)}{=} \Pi^L_{N,\mu'} \sum_{l\in\mathbb{Z}} f^{\sigma,\sigma_1}\big(\mathtt{s}(m),\sigma m+\sigma_1 l\big) g^{-\sigma_1,\sigma'}\big(\mathtt{s}(l),-\sigma_1 l+\sigma' n\big) \\
&= \Pi^L_{N,\mu'} \sum_{j\in\mathbb{Z}} f^{\sigma,\sigma_1}\big(\mathtt{s}(m),j\big) g^{-\sigma_1,\sigma'}\big(\mathtt{s}(l),\sigma m+\sigma' n-j\big) \\
&\overset{(3.28)}{=} \Pi^L_{N,\mu'} \sum_{j\in\mathbb{Z}} f^{\sigma,\sigma_1}\big(\mathtt{s}(m),j\big) g^{-\sigma_1,\sigma'}\big(-\sigma\sigma_1 \mathtt{s}(m),\sigma m+\sigma' n-j\big)
\end{aligned}$$

depends only on $\mathtt{s}(m)$ and $\sigma m+\sigma' n$, i.e. (3.23). ■

3.2 Quasi-Töplitz functions

Given $f \in \mathcal{H}_{s,r}$ and $\tilde f \in \mathcal{T}_{s,r}(N,\theta,\mu)$ we set

$$\hat f := N(\Pi_{N,\theta,\mu} f - \tilde f)\,. \tag{3.29}$$

All the functions $f \in \mathcal{H}_{s,r}$ below possibly depend on parameters $\xi \in \mathcal{O}$, see (2.86). For simplicity we shall often omit this dependence and denote $\| \ \|_{s,r,\mathcal{O}} = \| \ \|_{s,r}$.

Definition 3.4. (Quasi-Töplitz) *A function $f \in \mathcal{H}^{\rm null}_{s,r}$ is called (N_0,θ,μ)-quasi-Töplitz if the quasi-Töplitz semi-norm*

$$\|f\|^T_{s,r} := \|f\|^T_{s,r,N_0,\theta,\mu} := \sup_{N\geq N_0}\Big[\inf_{\tilde f\in\mathcal{T}_{s,r}(N,\theta,\mu)}\Big(\max\{\|X_f\|_{s,r},\|X_{\tilde f}\|_{s,r},\|X_{\hat f}\|_{s,r}\}\Big)\Big] \tag{3.30}$$

is finite. We define

$$\mathcal{Q}^T_{s,r} := \mathcal{Q}^T_{s,r}(N_0,\theta,\mu) := \Big\{f\in\mathcal{H}^{\rm null}_{s,r}\ :\ \|f\|^T_{s,r,N_0,\theta,\mu}<\infty\Big\}.$$

In other words, a function f is (N_0,θ,μ)-quasi-Töplitz with semi-norm $\|f\|^T_{s,r}$ if, for all $N\geq N_0$, $\forall\varepsilon>0$, there is $\tilde f\in\mathcal{T}_{s,r}(N,\theta,\mu)$ such that

$$\Pi_{N,\theta,\mu}f = \tilde f + N^{-1}\hat f \quad \text{and} \quad \|X_f\|_{s,r}\,,\ \|X_{\tilde f}\|_{s,r}\,,\ \|X_{\hat f}\|_{s,r}\leq \|f\|^T_{s,r}+\varepsilon\,. \tag{3.31}$$

We call $\tilde f\in\mathcal{T}_{s,r}(N,\theta,\mu)$ a "*Töplitz approximation*" of f and $\hat f$ the "*Töplitz-defect*". Note that, by Definition 3.3 and (3.29)

$$\Pi_{N,\theta,\mu}\tilde f = \tilde f\,,\quad \Pi_{N,\theta,\mu}\hat f = \hat f\,.$$

By the definition (3.30) we get

$$\|X_f\|_{s,r}\leq\|f\|^T_{s,r} \tag{3.32}$$

and we complete (2.88) noting that

$$\text{quasi-Töplitz} \implies \text{M}-\text{regular} \implies \text{regular} \impliedby \lambda-\text{regular}\,. \tag{3.33}$$

Clearly, if f is (N_0,θ,μ)-Töplitz then f is (N_0,θ,μ)-quasi-Töplitz and

$$\|f\|^T_{s,r,N_0,\theta,\mu} = \|X_f\|_{s,r}\,. \tag{3.34}$$

Then we have the following inclusions

$$\mathcal{T}_{s,r}\subset\mathcal{Q}^T_{s,r}\,,\ \mathcal{B}_{s,r}\subset\mathcal{H}^{\rm null}_{s,r}\subset\mathcal{H}_{s,r}\,.$$

Note that neither $\mathcal{B}_{s,r}\subseteq\mathcal{Q}^T_{s,r}$ nor $\mathcal{B}_{s,r}\supseteq\mathcal{Q}^T_{s,r}$.

Lemma 3.3. *For parameters $N_1\geq N_0$, $\mu_1\leq\mu$, $\theta_1\geq\theta$, $r_1\leq r$, $s_1\leq s$, we have*

$$\mathcal{Q}^T_{s,r}(N_0,\theta,\mu)\subset\mathcal{Q}^T_{s_1,r_1}(N_1,\theta_1,\mu_1)$$

and

$$\|f\|^T_{s_1,r_1,N_1,\theta_1,\mu_1}\leq\max\{s/s_1,(r/r_1)^2\}\|f\|^T_{s,r,N_0,\theta,\mu}\,. \tag{3.35}$$

Proof. By (3.31), for all $N\geq N_1\geq N_0$ (since $\theta_1\geq\theta$, $\mu_1\leq\mu$)

$$\Pi_{N,\theta_1,\mu_1}f = \Pi_{N,\theta_1,\mu_1}\Pi_{N,\theta,\mu}f = \Pi_{N,\theta_1,\mu_1}\tilde f + N^{-1}\Pi_{N,\theta_1,\mu_1}\hat f\,.$$

The function $\Pi_{N,\theta_1,\mu_1}\tilde f\in\mathcal{T}_{s_1,r_1}(N,\theta_1,\mu_1)$ and

$$\|X_{\Pi_{N,\theta_1,\mu_1}\tilde f}\|_{s_1,r_1}\overset{(2.79)}{\leq}\|X_{\tilde f}\|_{s_1,r_1}\overset{(3.31)}{\leq}\|f\|^T_{s_1,r_1}+\varepsilon\,,$$

$$\|X_{\Pi_{N,\theta_1,\mu_1}\hat f}\|_{s_1,r_1}\overset{(2.79)}{\leq}\|X_{\hat f}\|_{s_1,r_1}\overset{(3.31)}{\leq}\|f\|^T_{s_1,r_1}+\varepsilon\,.$$

Hence, $\forall N\geq N_1$,

$$\inf_{\tilde f\in\mathcal{T}_{s_1,r_1}(N,\theta_1,\mu_1)}\Big(\max\{\|X_f\|_{s_1,r_1},\|X_{\tilde f}\|_{s_1,r_1},\|X_{\hat f}\|_{s_1,r_1}\}\Big)\leq\|f\|^T_{s_1,r_1}+\varepsilon\,,$$

applying (2.3) we have (3.35), because $\varepsilon > 0$ is arbitrary. ■

For $f \in \mathcal{H}_{s,r}$ we define its homogeneous component of degree $l \in \mathbb{N}$,

$$f^{(l)} := \Pi^{(l)} f := \sum_{k \in \mathbb{Z}^n \,,\, 2|i|+|\alpha|+|\beta|=l} f_{k,i,\alpha,\beta}\, e^{\mathrm{i} k \cdot x} y^i z^\alpha \bar{z}^\beta \,, \tag{3.36}$$

and the projections

$$f_K := \Pi_{|k| \leq K} f := \sum_{|k| \leq K, i, \alpha, \beta} f_{k,i,\alpha,\beta}\, e^{\mathrm{i} k \cdot x} y^i z^\alpha \bar{z}^\beta \,, \quad \Pi_{>K} f := f - \Pi_{|k| \leq K} f \,. \tag{3.37}$$

We also set

$$f_K^{\leq 2} := \Pi_{|k| \leq K} f^{\leq 2} \,, \quad f^{\leq 2} := f^{(0)} + f^{(1)} + f^{(2)} \,. \tag{3.38}$$

The above projectors $\Pi^{(l)}$, $\Pi_{|k| \leq K}$, $\Pi_{>K}$ have the form Π_I, see (2.13), for suitable subsets $I \subset \mathbb{I}$.

Lemma 3.4. (Projections) *Let $f \in \mathcal{Q}^T_{s,r}(N_0, \theta, \mu)$. Then, for all $l \in \mathbb{N}$, $K \in \mathbb{N}$,*

$$\|\Pi^{(l)} f\|^T_{s,r,N_0,\theta,\mu} \leq \|f\|^T_{s,r,N_0,\theta,\mu} \tag{3.39}$$

$$\|f^{\leq 2}\|^T_{s,r,N_0,\theta,\mu} \,, \ \|f - f_K^{\leq 2}\|^T_{s,r,N_0,\theta,\mu} \leq \|f\|^T_{s,r,N_0,\theta,\mu} \tag{3.40}$$

$$\|\Pi_{|k| \leq K} f\|^T_{s,r,N_0,\theta,\mu} \leq \|f\|^T_{s,r,N_0,\theta,\mu} \tag{3.41}$$

$$\|\Pi_{k=0} \Pi_{|\alpha|=|\beta|=1} \Pi^{(2)} f\|^T_{r,N_0,\theta,\mu} \leq \|\Pi^{(2)} f\|^T_{s,r,N_0,\theta,\mu} \tag{3.42}$$

and, $\forall\, 0 < s' < s$,

$$\|\Pi_{>K} f\|^T_{s',r,N_0,\theta,\mu} \leq e^{-K(s-s')} \frac{s}{s'} \|f\|^T_{s,r,N_0,\theta,\mu} \,. \tag{3.43}$$

Proof. We first note that by (2.15) (recall also Remark 3.1) we have

$$\Pi^{(l)}\, \Pi_{N,\theta,\mu} g = \Pi_{N,\theta,\mu}\, \Pi^{(l)} g \,, \quad \forall\, g \in \mathcal{H}_{s,r} \,. \tag{3.44}$$

Then, applying $\Pi^{(l)}$ in (3.31), we deduce that, $\forall N \geq N_0$, $\forall \varepsilon > 0$, there is $\tilde{f} \in \mathcal{T}_{s,r}(N, \theta, \mu)$ such that

$$\Pi^{(l)} \Pi_{N,\theta,\mu} f = \Pi_{N,\theta,\mu} \Pi^{(l)} f = \Pi^{(l)} \tilde{f} + N^{-1} \Pi^{(l)} \hat{f} \tag{3.45}$$

and, by (2.79), (3.31),

$$\|X_{\Pi^{(l)} f}\|_{s,r} \,, \ \|X_{\Pi^{(l)} \tilde{f}}\|_{s,r} \,, \ \|X_{\Pi^{(l)} \hat{f}}\|_{s,r} \leq \|f\|^T_{s,r} + \varepsilon \,. \tag{3.46}$$

We claim that $\Pi^{(l)} \tilde{f} \in \mathcal{T}_{s,r}(N, \theta, \mu)$, $\forall l \geq 0$. Hence (3.45)-(3.46) imply $\Pi^{(l)} f \in \mathcal{Q}^T_{s,r}(N_0, \theta, \mu)$ and

$$\|\Pi^{(l)} f\|^T_{s,r} \leq \|f\|^T_{s,r} + \varepsilon \,,$$

i.e. (3.39). Let us prove our claim. For $l = 0, 1$ the projection $\Pi^{(l)} \tilde{f} = 0$ because $\tilde{f} \in \mathcal{T}_{s,r}(N, \theta, \mu)$ is bilinear. For $l \geq 2$, write $\tilde{f}$ in the form (3.11) with coefficients $\tilde{f}^{\sigma,\sigma'}_{m,n}$ satisfying (3.15). Then also $g := \Pi^{(l)} \tilde{f}$ has the form (3.11) with coefficients

$$g^{\sigma,\sigma'}_{m,n} = \Pi^{(l-2)} \tilde{f}^{\sigma,\sigma'}_{m,n}$$

which satisfy (3.15) noting that $\Pi^{(l)} \mathcal{L}_{s,r}(N, \mu, h) \subset \mathcal{L}_{s,r}(N, \mu, h)$. Hence $g \in \mathcal{T}_{s,r}(N, \theta, \mu)$, $\forall l \geq 0$, proving the claim. The proof of (3.40), (3.41), (3.42), and (3.43) are similar (use also (2.56)). ■

Lemma 3.5. *Assume that,* $\forall N \geq N_*$*, we have the decomposition*

$$G = G'_N + G''_N \quad \text{with} \quad \|G'_N\|^T_{s,r,N,\theta,\mu} \leq K_1 \,, \quad N\|X_{\Pi_{N,\theta,\mu}G''_N}\|_{s,r} \leq K_2 \,. \tag{3.47}$$

Then $\|G\|^T_{s,r,N_*,\theta,\mu} \leq \max\{\|X_G\|_{s,r}, K_1 + K_2\}$.

Proof. By assumption, $\forall N \geq N_*$, we have $\|G'_N\|^T_{s,r,N,\theta,\mu} \leq K_1$. Then, $\forall \varepsilon > 0$, there exist $\tilde{G}'_N \in \mathcal{T}_{s,r}(N,\theta,\mu)$, $\hat{G}'_N$, such that

$$\Pi_{N,\theta,\mu}G'_N = \tilde{G}'_N + N^{-1}\hat{G}'_N \quad \text{and} \quad \|X_{\tilde{G}'_N}\|_{s,r}, \|X_{\hat{G}'_N}\|_{s,r} \leq K_1 + \varepsilon \,. \tag{3.48}$$

Therefore, $\forall N \geq N_*$,

$$\Pi_{N,\theta,\mu}G = \tilde{G}_N + N^{-1}\hat{G}_N \,, \quad \tilde{G}_N := \tilde{G}'_N \,, \quad \hat{G}_N := \hat{G}'_N + N\Pi_{N,\theta,\mu}G''_N$$

where $\tilde{G}_N \in \mathcal{T}_{s,r}(N,\theta,\mu)$ and

$$\|X_{\tilde{G}_N}\|_{s,r} = \|X_{\tilde{G}'_N}\|_{s,r} \overset{(3.48)}{\leq} K_1 + \varepsilon, \tag{3.49}$$

$$\|X_{\hat{G}_N}\|_{s,r} \leq \|X_{\hat{G}'_N}\|_{s,r} + N\|X_{\Pi_{N,\theta,\mu}G''_N}\|_{s,r} \overset{(3.48),(3.47)}{\leq} K_1 + \varepsilon + K_2 \,. \tag{3.50}$$

Then $G \in \mathcal{Q}^T_{s,r,N_*,\theta,\mu}$ and

$$\begin{aligned}\|G\|^T_{s,r,N_*,\theta,\mu} &\leq \sup_{N\geq N_*} \max\left\{\|X_G\|_{s,r}, \|X_{\tilde{G}_N}\|_{s,r}, \|X_{\hat{G}_N}\|_{s,r}\right\} \\ &\overset{(3.49),(3.50)}{\leq} \max\{\|X_G\|_{s,r}, K_1 + K_2 + \varepsilon\} \,.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary the lemma follows. ∎

The Poisson bracket of two quasi-Töplitz functions is quasi-Töplitz.

Proposition 3.1. (Poisson bracket) *Assume that* $f^{(1)}, f^{(2)} \in \mathcal{Q}^T_{s,r}(N_0,\theta,\mu)$ *and* $N_1 \geq N_0$, $\mu_1 \leq \mu$, $\theta_1 \geq \theta$, $s/2 \leq s_1 < s$, $r/2 \leq r_1 < r$ *satisfy*

$$\kappa N_1^{b-L} < \mu - \mu_1, \ \ \mu_1 N_1^{L-1} + \kappa N_1^{b-1} < \theta_1 - \theta, \ \ 2N_1 e^{-N_1^b \frac{s-s_1}{2}} < 1, \ \ b(s-s_1)N_1^b > 2 \,. \tag{3.51}$$

Then

$$\{f^{(1)}, f^{(2)}\} \in \mathcal{Q}^T_{s_1,r_1}(N_1,\theta_1,\mu_1)$$

and

$$\|\{f^{(1)}, f^{(2)}\}\|^T_{s_1,r_1,N_1,\theta_1,\mu_1} \leq C(n)\delta^{-1}\|f^{(1)}\|^T_{s,r,N_0,\theta,\mu}\|f^{(2)}\|^T_{s,r,N_0,\theta,\mu} \tag{3.52}$$

where $C(n) \geq 1$ *and*

$$\delta := \min\left\{1 - \frac{s_1}{s}, 1 - \frac{r_1}{r}\right\}. \tag{3.53}$$

The proof is based on the following splitting Lemma for the Poisson brackets.

Lemma 3.6. (Splitting lemma) *Let* $f^{(1)}, f^{(2)} \in \mathcal{Q}^T_{s,r}(N_0,\theta,\mu)$ *and* (3.51) *hold. Then, for all* $N \geq N_1$,

$$\begin{aligned}&\Pi_{N,\theta_1,\mu_1}\{f^{(1)}, f^{(2)}\} = \\ &\Pi_{N,\theta_1,\mu_1}\Big(\Big\{\Pi_{N,\theta,\mu}f^{(1)}, \Pi_{N,\theta,\mu}f^{(2)}\Big\}^H + \Big\{\Pi_{N,\theta,\mu}f^{(1)}, \Pi^L_{N,2\mu}f^{(2)}\Big\}^L + \Big\{\Pi^L_{N,2\mu}f^{(1)}, \Pi_{N,\theta,\mu}f^{(2)}\Big\}^L \\ &\quad + \Big\{\Pi_{N,\theta,\mu}f^{(1)}, \Pi^L_{N,\mu}f^{(2)}\Big\}^{x,y} + \Big\{\Pi^L_{N,\mu}f^{(1)}, \Pi_{N,\theta,\mu}f^{(2)}\Big\}^{x,y} \\ &\quad + \Big\{\Pi_{|k|\geq N^b}f^{(1)}, f^{(2)}\Big\} + \Big\{\Pi_{|k|<N^b}f^{(1)}, \Pi_{|k|\geq N^b}f^{(2)}\Big\}\Big) \,.\end{aligned} \tag{3.54}$$

PROOF. We have

$$\{f^{(1)}, f^{(2)}\} = \{\Pi_{|k|<N^b} f^{(1)}, \Pi_{|k|<N^b} f^{(2)}\} + \{\Pi_{|k|\ge N^b} f^{(1)}, f^{(2)}\} + \{\Pi_{|k|<N^b} f^{(1)}, \Pi_{|k|\ge N^b} f^{(2)}\} . \tag{3.55}$$

The last two terms correspond to the last line in (3.54). We now study the first term in the right hand side of (3.55). We replace each $f^{(i)}$, $i = 1, 2$, with single monomials (with zero momentum) and we analyze under which conditions the projection

$$\Pi_{N,\theta_1,\mu_1} \Big\{ e^{\mathrm{i}k^{(1)}\cdot x} y^{i^{(1)}} z^{\alpha^{(1)}} \bar{z}^{\beta^{(1)}}, e^{\mathrm{i}k^{(2)}\cdot x} y^{i^{(2)}} z^{\alpha^{(2)}} \bar{z}^{\beta^{(2)}} \Big\} , \quad |k^{(1)}|, |k^{(2)}| < N^b ,$$

is not zero. By direct inspection, recalling the Definition 3.2 of Π_{N,θ_1,μ_1} and the expression (2.80) of the Poisson brackets $\{\,,\,\} = \{\,,\,\}^{x,y} + \{\,,\,\}^{z,\bar{z}}$, one of the following situations (apart from a trivial permutation of the indexes $1, 2$) must hold:

1. one has $z^{\alpha^{(1)}} \bar{z}^{\beta^{(1)}} = z^{\tilde\alpha^{(1)}} \bar{z}^{\tilde\beta^{(1)}} z_m^{\sigma} z_j^{\sigma_1}$ and $z^{\alpha^{(2)}} \bar{z}^{\beta^{(2)}} = z^{\tilde\alpha^{(2)}} \bar{z}^{\tilde\beta^{(2)}} z_n^{\sigma'} z_j^{-\sigma_1}$ where $|m|, |n| \ge \theta_1 N$, $\sigma, \sigma_1, \sigma' = \pm$, and $z^{\tilde\alpha^{(1)}} \bar{z}^{\tilde\beta^{(1)}} z^{\tilde\alpha^{(2)}} \bar{z}^{\tilde\beta^{(2)}}$ is of (N, μ_1)-low momentum. We consider the Poisson bracket $\{\,,\,\}^{z,\bar{z}}$ (in the variables (z_j^+, z_j^-)) of the monomials.
2. one has $z^{\alpha^{(1)}} \bar{z}^{\beta^{(1)}} = z^{\tilde\alpha^{(1)}} \bar{z}^{\tilde\beta^{(1)}} z_m^{\sigma} z_n^{\sigma'} z_j^{\sigma_1}$ and $z^{\alpha^{(2)}} \bar{z}^{\beta^{(2)}} = z^{\tilde\alpha^{(2)}} \bar{z}^{\tilde\beta^{(2)}} z_j^{-\sigma_1}$ where $|m|, |n| \ge \theta_1 N$ and $z^{\tilde\alpha^{(1)}} \bar{z}^{\tilde\beta^{(1)}} z^{\tilde\alpha^{(2)}} \bar{z}^{\tilde\beta^{(2)}}$ is of (N, μ_1)–low momentum. We consider the Poisson bracket $\{\,,\}^{z,\bar{z}}$.
3. one has $z^{\alpha^{(1)}} \bar{z}^{\beta^{(1)}} = z^{\tilde\alpha^{(1)}} \bar{z}^{\tilde\beta^{(1)}} z_m^{\sigma} z_n^{\sigma'}$ and $z^{\alpha^{(2)}} \bar{z}^{\beta^{(2)}} = z^{\tilde\alpha^{(2)}} \bar{z}^{\tilde\beta^{(2)}}$, where $|m|, |n| \ge \theta_1 N$ and $z^{\tilde\alpha^{(1)}} \bar{z}^{\tilde\beta^{(1)}} z^{\tilde\alpha^{(2)}} \bar{z}^{\tilde\beta^{(2)}}$ is of (N, μ_1)-low momentum. We consider the Poisson bracket $\{\,,\,\}^{x,y}$, i.e. in the variables (x, y).

Note that when we consider the $\{\,,\,\}^{x,y}$ Poisson bracket, the case

$$z^{\alpha^{(1)}} \bar{z}^{\beta^{(1)}} = z^{\tilde\alpha^{(1)}} \bar{z}^{\tilde\beta^{(1)}} z_m^{\sigma} \quad \text{and} \quad z^{\alpha^{(2)}} \bar{z}^{\beta^{(2)}} = z^{\tilde\alpha^{(2)}} \bar{z}^{\tilde\beta^{(2)}} z_n^{\sigma'} , \quad |m|, |n| \ge \theta_1 N ,$$

and $z^{\tilde\alpha^{(1)}} \bar{z}^{\tilde\beta^{(1)}} z^{\tilde\alpha^{(2)}} \bar{z}^{\tilde\beta^{(2)}}$ is of (N, μ_1)-low momentum, does not appear. Indeed, the momentum conservation $-\sigma m = \pi(\tilde\alpha^{(1)}, \tilde\beta^{(1)}, k^{(1)})$, (2.85) and $|k^{(1)}| < N^b$, give

$$\theta_1 N < |m| \le \sum_{l \in \mathbb{Z} \setminus \mathcal{I}} |l| (|\tilde\alpha_l^{(1)}| + |\tilde\beta_l^{(1)}|) + \kappa N^b \le \mu_1 N^L + \kappa N^b ,$$

which contradicts (3.51).

CASE 1. The momentum conservation of each monomial gives

$$\sigma_1 j = -\sigma m - \pi(\tilde\alpha^{(1)}, \tilde\beta^{(1)}, k^{(1)}) = \sigma' n + \pi(\tilde\alpha^{(2)}, \tilde\beta^{(2)}, k^{(2)}) . \tag{3.56}$$

Since $z^{\tilde\alpha^{(1)}} \bar{z}^{\tilde\beta^{(1)}} z^{\tilde\alpha^{(2)}} \bar{z}^{\tilde\beta^{(2)}}$ is of (N, μ_1)-low momentum (Definition 3.1),

$$\sum_{l \in \mathbb{Z} \setminus \mathcal{I}} |l| (\tilde\alpha_l^{(1)} + \tilde\beta_l^{(1)} + \tilde\alpha_l^{(2)} + \tilde\beta_l^{(2)}) \le \mu_1 N^L \implies \sum_{l \in \mathbb{Z} \setminus \mathcal{I}} |l| (\tilde\alpha_l^{(i)} + \tilde\beta_l^{(i)}) \le \mu_1 N^L , \ i = 1, 2 ,$$

which implies, by (3.56), (2.85), $|k^{(1)}| < N^b$, $|j| \ge \theta_1 N - \mu_1 N^L - \kappa N^b > \theta N$ by (3.51). Hence $|m|, |n|, |j| > \theta N$. Then $e^{\mathrm{i}k^{(h)}\cdot x} y^{i^{(h)}} z^{\alpha^{(h)}} \bar{z}^{\beta^{(h)}}$, $h = 1, 2$, are (N, θ, μ)-bilinear. Moreover the $(z_j, \bar{z}_j)$ are high momentum variables, namely $\{\,,\,\}^{z,\bar{z}} = \{\,,\,\}^H$, see (3.4). As m, n run over all $\mathbb{Z} \setminus \mathcal{I}$ with $|m|, |n| \ge \theta_1 N$, we obtain the first term in formula (3.54).

CASE 2. The momentum conservation of the second monomial reads

$$-\sigma_1 j = -\pi(\tilde\alpha^{(2)}, \tilde\beta^{(2)}, k^{(2)}) . \tag{3.57}$$

Then, using also (2.85), $|k^{(2)}| < N^b$, that $z^{\tilde\alpha^{(1)}} \bar z^{\tilde\beta^{(1)}} z^{\tilde\alpha^{(2)}} \bar z^{\tilde\beta^{(2)}}$ is of (N,μ_1)-low momentum,

$$|j| + \sum_{l\in\mathbb{Z}\setminus\mathcal{I}} |l|(\tilde\alpha_l^{(1)} + \tilde\beta_l^{(1)}) \overset{(3.57)}{=} |\pi(\tilde\alpha^{(2)}, \tilde\beta^{(2)}, k^{(2)})| + \sum_{l\in\mathbb{Z}\setminus\mathcal{I}} |l|(\tilde\alpha_l^{(1)} + \tilde\beta_l^{(1)}) \leq$$

$$\sum_{l\in\mathbb{Z}\setminus\mathcal{I}} |l|(\tilde\alpha_l^{(1)} + \tilde\beta_l^{(1)} + \tilde\alpha_l^{(2)} + \tilde\beta_l^{(2)}) + \kappa N^b \leq \mu_1 N^L + \kappa N^b \overset{(3.51)}{<} \mu N^L .$$

Then $z^{\tilde\alpha^{(1)}} \bar z^{\tilde\beta^{(1)}} z_j^{\sigma_1}$ is of (N,μ_1)-low momentum and the first monomial

$$e^{\mathrm{i}k^{(1)}\cdot x} y^{i^{(1)}} z^{\alpha^{(1)}} \bar z^{\beta^{(1)}} = e^{\mathrm{i}k^{(1)}\cdot x} y^{i^{(1)}} z^{\tilde\alpha^{(1)}} \bar z^{\tilde\beta^{(1)}} z_j^{\sigma_1} z_m^{\sigma} z_n^{\sigma'}$$

is (N,θ,μ)-bilinear ($\mu_1 \leq \mu$). The second monomial

$$e^{\mathrm{i}k^{(2)}\cdot x} y^{i^{(2)}} z^{\alpha^{(2)}} \bar z^{\beta^{(2)}} = e^{\mathrm{i}k^{(2)}\cdot x} y^{i^{(2)}} z^{\tilde\alpha^{(2)}} \bar z^{\tilde\beta^{(2)}} z_j^{-\sigma_1}$$

is $(N,2\mu)$-low-momentum because, arguing as above,

$$\begin{aligned} |j| + \sum_l |l|(\tilde\alpha_l^{(2)} + \tilde\beta_l^{(2)}) &\overset{(3.57)}{=} |\pi(\tilde\alpha^{(2)}, \tilde\beta^{(2)}, k^{(2)})| + \sum_l |l|(\tilde\alpha_l^{(2)} + \tilde\beta_l^{(2)}) \\ &\leq 2\mu_1 N^L + \kappa N^b \overset{(3.51)}{<} 2\mu N^L . \end{aligned}$$

The $(z_j, \bar z_j)$ are low momentum variables, namely $\{\,,\,\}^{z,\bar z} = \{\,,\,\}^L$, and we obtain the second and third contribution in formula (3.54).

Case 3. We have, for $i = 1,2$, that

$$\sum_l |l|(\tilde\alpha_l^{(i)} + \tilde\beta_l^{(i)}) \leq \sum_l |l|(\tilde\alpha_l^{(1)} + \tilde\beta_l^{(1)} + \tilde\alpha_l^{(2)} + \tilde\beta_l^{(2)}) \leq \mu_1 N^L \leq \mu N^L .$$

Then $e^{\mathrm{i}k^{(1)}\cdot x} y^{i^{(1)}} z^{\alpha^{(1)}} \bar z^{\beta^{(1)}}$ is (N,θ,μ)-bilinear and $e^{\mathrm{i}k^{(2)}\cdot x} y^{i^{(2)}} z^{\alpha^{(2)}} \bar z^{\beta^{(2)}}$ is (N,μ)-low-momentum. We obtain the fourth and fifth contribution in formula (3.54). ■

Proof of Proposition 3.1. Since $f^{(i)} \in \mathcal{Q}^T_{s,r}(N_0,\theta,\mu)$, $i = 1,2$, for all $N \geq N_1 \geq N_0$ there exist $\tilde f^{(i)} \in \mathcal{T}_{s,r}(N,\theta,\mu)$ and $\hat f^{(i)}$ such that (see (3.31))

$$\Pi_{N,\theta,\mu} f^{(i)} = \tilde f^{(i)} + N^{-1} \hat f^{(i)} , \quad i = 1,2 , \tag{3.58}$$

and

$$\|X_{f^{(i)}}\|_{s,r},\ \|X_{\tilde f^{(i)}}\|_{s,r},\ \|X_{\hat f^{(i)}}\|_{s,r} \leq 2\|f^{(i)}\|^T_{s,r} . \tag{3.59}$$

In order to show that $\{f^{(1)}, f^{(2)}\} \in \mathcal{Q}^T_{s_1,r_1}(N_1,\theta_1,\mu_1)$ and prove (3.52) we have to provide a decomposition

$$\Pi_{N,\theta_1,\mu_1}\{f^{(1)}, f^{(2)}\} = \tilde f^{(1,2)} + N^{-1} \hat f^{(1,2)} , \quad \forall N \geq N_1 ,$$

so that $\tilde f^{(1,2)} \in \mathcal{T}_{s_1,r_1}(N,\theta_1,\mu_1)$ and

$$\|X_{\{f^{(1)},f^{(2)}\}}\|_{s_1,r_1},\ \|X_{\tilde f^{(1,2)}}\|_{s_1,r_1}, \|X_{\hat f^{(1,2)}}\|_{s_1,r_1} < C(n)\delta^{-1} \|f^{(1)}\|^T_{s,r} \|f^{(2)}\|^T_{s,r} \tag{3.60}$$

(for brevity we omit the indices $N_1, \theta_1, \mu_1, N_0, \theta, \mu$). By (2.91) we have ($\delta$ is defined in (3.53))

$$\|X_{\{f^{(1)},f^{(2)}\}}\|_{s_1,r_1} \leq 2^{2n+3} \delta^{-1} \|X_{f^{(1)}}\|_{s,r} \|X_{f^{(2)}}\|_{s,r} .$$

Considering (3.58) and (3.54), we define the candidate Töplitz approximation

$$\begin{aligned} \tilde f^{(1,2)} &:= \Pi_{N,\theta_1,\mu_1}\Big(\Big\{\tilde f^{(1)},\tilde f^{(2)}\Big\}^H+\Big\{\tilde f^{(1)},\Pi^L_{N,2\mu}f^{(2)}\Big\}^L+\Big\{\Pi^L_{N,2\mu}f^{(1)},\tilde f^{(2)}\Big\}^L \\ &\qquad +\Big\{\tilde f^{(1)},\Pi^L_{N,\mu}f^{(2)}\Big\}^{x,y}+\Big\{\Pi^L_{N,\mu}f^{(1)},\tilde f^{(2)}\Big\}^{x,y} \end{aligned} \tag{3.61}$$

and Töplitz-defect

$$\hat f^{(1,2)} := N\Big(\Pi_{N,\theta_1,\mu_1}\{f^{(1)},f^{(2)}\}-\tilde f^{(1,2)}\Big)\,. \tag{3.62}$$

Lemma 3.2 and (3.51) imply that $\tilde f^{(1,2)}\in\mathcal{T}_{s_1,r_1}(N,\theta_1,\mu_1)$. The estimate (3.60) for $\tilde f^{(1,2)}$ follows by (3.61), (2.91), (2.79), (3.59). Next

$$\begin{aligned} \hat f^{(1,2)} &= \Pi_{N,\theta_1,\mu_1}\Big(\Big\{\tilde f^{(1)},\hat f^{(2)}\Big\}^H+\Big\{\hat f^{(1)},\tilde f^{(2)}\Big\}^H+N^{-1}\Big\{\hat f^{(1)},\hat f^{(2)}\Big\}^H \\ &+ \Big\{\hat f^{(1)},\Pi^L_{N,2\mu}f^{(2)}\Big\}^L+\Big\{\Pi^L_{N,2\mu}f^{(1)},\hat f^{(2)}\Big\}^L \\ &+ \Big\{\hat f^{(1)},\Pi^L_{N,\mu}f^{(2)}\Big\}^{x,y}+\Big\{\Pi^L_{N,\mu}f^{(1)},\hat f^{(2)}\Big\}^{x,y} \\ &+ N\Big\{\Pi_{|k|\ge N^b}f^{(1)},f^{(2)}\Big\}+N\Big\{\Pi_{|k|<N^b}f^{(1)},\Pi_{|k|\ge N^b}f^{(2)}\Big\}\Big) \end{aligned}$$

and the bound (3.60) follows again by (2.91), (2.79), (3.59), (2.56), (3.51). Let consider only the term $N\Big\{\Pi_{|k|\ge N^b}f^{(1)},f^{(2)}\Big\}=:g$, the last one being analogous. We first use Lemma 2.16 with $r'\rightsquigarrow r_1$, $r\rightsquigarrow r$, $s'\rightsquigarrow s_1$ and $s\rightsquigarrow s_1+\sigma/2$, where $\sigma:=s-s_1$. Since $\Big(1-\dfrac{s_1}{s_1+\sigma/2}\Big)^{-1}\le 2\Big(1-\dfrac{s_1}{s}\Big)^{-1}\le 2\delta^{-1}$ with the δ in (3.53), by (2.91) we get

$$\begin{aligned} \|X_g\|_{s_1,r_1} &\le C(n)\delta^{-1}N\|X_{\Pi_{|k|\ge N^b}f^{(1)}}\|_{s_1+\sigma/2,r}\|X_{f^{(2)}}\|_{s,r} \\ &\overset{(2.56)}{\le} C(n)\delta^{-1}N\frac{s}{s_1}e^{-N^b(s-s_1)/2}\|X_{f^{(1)}}\|_{s,r}\|X_{f^{(2)}}\|_{s,r} \\ &\overset{(3.51)}{\le} C(n)\delta^{-1}\|X_{f^{(1)}}\|_{s,r}\|X_{f^{(2)}}\|_{s,r}\,, \end{aligned}$$

for every $N\ge N_1$. The proof of Proposition 3.1 is complete. ■

The quasi-Töplitz character of a function is preserved under the flow generated by a quasi-Töplitz Hamiltonian.

Proposition 3.2. (Lie transform) *Let $f,g\in\mathcal{Q}^T_{s,r}(N_0,\theta,\mu)$ and let $s/2\le s'<s$, $r/2\le r'<r$. There is $c(n)>0$ such that, if*

$$\|f\|^T_{s,r,N_0,\theta,\mu}\le c(n)\,\delta\,, \tag{3.63}$$

with δ defined in (2.65)*, then the hamiltonian flow of f at time $t=1$,*

$$e^{\mathrm{ad}_f}:D(s',r')\to D(s,r)\,,$$

is well defined, analytic and symplectic, and, for

$$N_0'\ge\max\{N_0,\bar N\}\,,\quad \bar N:=\exp\Big(\max\Big\{\frac{2}{b},\frac{1}{L-b},\frac{1}{1-L},8\Big\}\Big)\,, \tag{3.64}$$

(recall (3.2)*), $\mu'<\mu$, $\theta'>\theta$, satisfying*

$$\kappa(N_0')^{b-L}\ln N_0'\le\mu-\mu'\,,\ (6+\kappa)(N_0')^{L-1}\ln N_0'\le\theta'-\theta\,,\ 2(N_0')^{-b}\ln^2N_0'\le b(s-s')\,, \tag{3.65}$$

we have $e^{\mathrm{ad}_f} g \in \mathcal{Q}^T_{s',r'}(N_0', \theta', \mu')$ *and*

$$\|e^{\mathrm{ad}_f} g\|^T_{s',r',N_0',\theta',\mu'} \le 2\|g\|^T_{s,r,N_0,\theta,\mu} \,. \tag{3.66}$$

Moreover, for $h = 0, 1, 2$, *and coefficients* $0 \le b_j \le 1/j!$, $j \in \mathbb{N}$,

$$\Big\| \sum_{j \ge h} b_j \,\mathrm{ad}_f^j(g) \Big\|^T_{s',r',N_0',\theta',\mu'} \le 2(C\delta^{-1}\|f\|^T_{s,r,N_0,\theta,\mu})^h \|g\|^T_{s,r,N_0,\theta,\mu} \,. \tag{3.67}$$

Note that (3.66) is (3.67) with $h = 0$, $b_j := 1/j!$

Proof. Let us prove (3.67). We define

$$G^{(0)} := g \,, \quad G^{(j)} := \mathrm{ad}_f^j(g) := \mathrm{ad}_f(G^{(j-1)}) = \{f, G^{(j-1)}\} \,, \ j \ge 1 \,,$$

and we split, for $h = 0, 1, 2$,

$$G^{\ge h} := \sum_{j \ge h} b_j G^{(j)} = \sum_{j=h}^{J-1} b_j G^{(j)} + \sum_{j \ge J} b_j G^{(j)} =: G^{\ge h}_{<J} + G_{\ge J} \,. \tag{3.68}$$

As in (2.97) we deduce

$$\|X_{G^{(j)}}\|_{s',r'} \le (C(n) j \delta^{-1})^j \|X_f\|^j_{s,r} \|X_g\|_{s,r} \,, \quad \forall j \ge 0 \,, \tag{3.69}$$

where δ is defined in (2.65). Let

$$\eta := C(n) e \delta^{-1} \|X_f\|_{s,r} < 1/(2e) \tag{3.70}$$

(namely take $c(n)$ small in (3.63)). By 3.69, using $j^j b_j \le j^j/j! < e^j$, we get

$$\|X_{G_{\ge J}}\|_{s',r'} \le \sum_{j \ge J} b_j (C(n) j \delta^{-1} \|X_f\|_{s,r})^j \|X_g\|_{s,r} \le 2\eta^J \|X_g\|_{s,r} \,. \tag{3.71}$$

In particular, for $J = h = 0, 1, 2$, we get

$$\|X_{G^{\ge h}}\|_{s',r'} \le 2\eta^h \|X_g\|_{s,r} \,. \tag{3.72}$$

For any $N \ge N_0'$ we choose

$$J := J(N) := \ln N \,, \tag{3.73}$$

and we set

$$G_N' := G^{\ge h}_{<J} \,, \quad G_N'' := G_{\ge J} \,, \quad G^{\ge h} = G_N' + G_N'' \,.$$

Then (3.67) follows by Lemma 3.5 (with $N_* \rightsquigarrow N_0', s \rightsquigarrow s', r \rightsquigarrow r', \theta \rightsquigarrow \theta', \mu \rightsquigarrow \mu'$) and (3.72), once we show that

$$\|G_N'\|^T_{s',r',N,\theta',\mu'} \le \frac{3}{2}\eta^h \|g\|^T_{s,r} \,, \quad N \|X_{G_N''}\|_{s',r'} \le \frac{1}{2}\eta^h \|g\|^T_{s,r} \tag{3.74}$$

with $h = 0, 1, 2$ (for simplicity $\|g\|^T_{s,r} := \|g\|^T_{s,r,N_0,\theta,\mu}$).

For all $N \ge N_0' \ge e^8$ (recall (3.64)),

$$\begin{aligned} N \|X_{G_{\ge J}}\|_{s',r'} &\overset{(3.71)}{\le} N 2\eta^J \|X_g\|_{s,r} \le \eta^h (N 2\eta^{J-h}) \|g\|^T_{s,r} \\ &\overset{(3.70)}{\le} \eta^h 2^{-J+h+1} e^h N e^{-J} \|g\|^T_{s,r} \le \frac{\eta^h}{2} \|g\|^T_{s,r} \,, \end{aligned} \tag{3.75}$$

proving the second inequality in (3.74). Let us prove the first inequality in (3.74).

Claim: $\forall j = 1, \ldots, J-1$, *we have* $G^{(j)} \in \mathcal{Q}^T_{s',r'}(N, \theta', \mu')$ *and*

$$\|G^{(j)}\|^T_{r',s',N,\theta',\mu'} \le \|g\|^T_{s,r}(C' j \delta^{-1} \|f\|^T_{s,r})^j \tag{3.76}$$

(*for simplicity* $\|f\|^T_{s,r} := \|f\|^T_{s,r,N_0,\theta,\mu}$). This claim implies (using $j^j b_j < e^j$)

$$\begin{aligned}\Big\| \sum_{j=h}^{J-1} b_j \, G^{(j)} \Big\|^T_{s',r',N,\theta',\mu'} &\overset{(3.76)}{\le} \sum_{j=h}^{J-1} b_j \|g\|^T_{s,r}(C' j \delta^{-1} \|f\|^T_{s,r})^j \\ &\overset{(3.70)}{\le} \|g\|^T_{s,r} \sum_{j=h}^{+\infty} \eta^j \le \frac{3}{2} \eta^h \|g\|^T_{s,r}\end{aligned}$$

for c small enough in (3.63). This proves the first inequality in (3.74).

Let us prove the claim. Fix $0 \le j \le J-1$. We define, $\forall i = 0, \ldots, j$,

$$\mu_i := \mu - i\,\frac{\mu - \mu'}{j}, \ \theta_i := \theta + i \frac{\theta' - \theta}{j}, \ r_i := r - i\,\frac{r - r'}{j}, \ s_i := s - i\,\frac{s - s'}{j}\,, \tag{3.77}$$

and we prove inductively that, for all $i = 0, \ldots, j$,

$$\|\mathrm{ad}^i_f(g)\|^T_{s_i,r_i,N,\theta_i,\mu_i} \le (C' j \delta^{-1} \|f\|^T_{s,r})^i \|g\|^T_{s,r}\,, \tag{3.78}$$

which, for $i = j$, gives (3.76). For $i = 0$, formula (3.78) follows because $g \in \mathcal{Q}^T_{s,r}(N_0, \theta, \mu)$ and Lemma 3.3.

Now assume that (3.78) holds for i and prove it for $i+1$. We want to apply Proposition 3.1 to the functions f and $\mathrm{ad}^i_f(g)$ with $N_1 \rightsquigarrow N$, $s \rightsquigarrow s_i, s_1 \rightsquigarrow s_{i+1}$, $\theta \rightsquigarrow \theta_i, \theta_1 \rightsquigarrow \theta_{i+1}$, etc. We have to verify conditions (3.51) that reads

$$\kappa N^{b-L} < \mu_i - \mu_{i+1}\,, \quad \mu_{i+1} N^{L-1} + \kappa N^{b-1} < \theta_{i+1} - \theta_i\,, \tag{3.79}$$

$$2N e^{-N^b \frac{s_i - s_{i+1}}{2}} < 1\,, \quad b(s_i - s_{i+1}) N^b > 2\,. \tag{3.80}$$

Since, by (3.77),

$$\mu_i - \mu_{i+1} = \frac{\mu - \mu'}{j}\,, \quad \theta_{i+1} - \theta_i = \frac{\theta - \theta'}{j}\,, \quad s_i - s_{i+1} = \frac{s - s'}{j}$$

and $j < J = \ln N$ (see (3.73)), $0 < b < L < 1$ (recall (3.2)), $\mu' \le \mu \le 6$, the above conditions (3.79)-(3.80) are implied by

$$\begin{aligned}&\kappa N^{b-L} \ln N < \mu - \mu'\,, \quad (6+\kappa) N^{L-1} \ln N < \theta' - \theta\,, \\ &2N e^{-N^b (s-s')/2 \ln N} < 1\,, \quad b(s - s') N^b > 2 \ln N\,.\end{aligned} \tag{3.81}$$

The last two conditions (3.81) are implied by $b(s - s')N^b > 2 \ln^2 N$ and since $N \ge e^{1/1-b}$ (recall (3.64)). Recollecting we have to verify

$$\kappa N^{b-L} \ln N \le \mu - \mu'\,, \quad (6+\kappa) N^{L-1} \ln N \le \theta' - \theta\,, \quad 2N^{-b} \ln^2 N \le b(s - s')\,. \tag{3.82}$$

Since the function $N \mapsto N^{-\gamma} \ln N$ is decreasing for $N \ge e^{1/\gamma}$, we have that (3.82) follows by (3.64)-(3.65). Therefore Proposition 3.1 implies that $\mathrm{ad}^{i+1}_f(g) \in \mathcal{Q}^T_{s_{i+1},r_{i+1}}(N, \theta_{i+1}, \mu_{i+1})$ and, by (3.52), (3.35), we get

$$\|\mathrm{ad}^{i+1}_f(g)\|^T_{s_{i+1},r_{i+1},N,\theta_{i+1},\mu_{i+1}} \le C' \delta_i^{-1} \|f\|^T_{s,r} \|\mathrm{ad}^i_f(g)\|^T_{s_i,r_i,N,\theta_i,\mu_i} \tag{3.83}$$

where

$$\delta_i := \min\left\{1 - \frac{s_{i+1}}{s_i}\,, 1 - \frac{r_{i+1}}{r_i}\right\} \ge \frac{\delta}{j} \tag{3.84}$$

and δ is defined in (2.65). Then

$$\|\mathrm{ad}_f^{i+1}(g)\|^T_{s_{i+1},r_{i+1},N,\theta_{i+1},\mu_{i+1}} \overset{(3.83),(3.84)}{\leq} C'j\delta^{-1}\|f\|^T_{s,r,N_0,\theta,\mu}\|\mathrm{ad}^i_f(g)\|^T_{s_i,r_i,N,\theta_i,\mu_i}$$
$$\overset{(3.78)}{\leq} (C'j\delta^{-1}\|f\|^T_{s,r})^{i+1}\|g\|^T_{s,r}$$

proving (3.78) by induction. ■

4 An abstract KAM theorem

We consider a family of integrable Hamiltonians

$$\mathcal{N} := \mathcal{N}(x, y, z, \bar{z}; \xi) := e(\xi) + \omega(\xi) \cdot y + \Omega(\xi) \cdot z\bar{z} \tag{4.1}$$

defined on $\mathbb{T}^n_s \times \mathbb{C}^n \times \ell^{a,p}_{\mathcal{I}} \times \ell^{a,p}_{\mathcal{I}}$, where $\mathcal{I}$ is defined in (2.83), the tangential frequencies $\omega := (\omega_1, \ldots, \omega_n)$ and the normal frequencies $\Omega := (\Omega_j)_{j\in\mathbb{Z}\setminus\mathcal{I}}$ depend on n-parameters

$$\xi \in \mathcal{O} \subset \mathbb{R}^n\,, \qquad \mathcal{O} \text{ bounded with positive Lebesgue measure}\,.$$

For each ξ there is an invariant n-torus

$$\mathcal{T}_0 = \mathbb{T}^n \times \{0\} \times \{0\} \times \{0\}$$

with frequency $\omega(\xi)$. In its normal space, the origin $(z, \bar{z}) = 0$ is an elliptic fixed point with proper frequencies $\Omega(\xi)$. The aim is to prove the persistence of a large portion of this family of linearly stable tori under small analytic perturbations $H = \mathcal{N} + P$.

(**A1**) Parameter dependence. The map $\omega : \mathcal{O} \to \mathbb{R}^n$, $\xi \mapsto \omega(\xi)$, is Lipschiz continuous.

With in mind the application to NLW we assume

(**A2**) Frequency asymptotics. We have

$$\Omega_j(\xi) = \sqrt{j^2 + \mathrm{m}} + a(\xi) \in \mathbb{R}\,, \quad j \in \mathbb{Z}\setminus\mathcal{I}\,, \tag{4.2}$$

for some Lipschiz continuous functions $a(\xi) \in \mathbb{R}$.

By (A1) and (A2), the Lipschiz semi-norms of the frequency maps satisfy, for some $1 \leq M_1 < \infty$,

$$|\omega|^{\mathrm{lip}} + |\Omega|^{\mathrm{lip}}_\infty \leq M_1 \tag{4.3}$$

where the Lipschiz semi-norm is

$$|\Omega|^{\mathrm{lip}}_\infty := \sup_{\xi,\eta\in\mathcal{O},\xi\neq\eta} \frac{|\Omega(\xi) - \Omega(\eta)|_\infty}{|\xi - \eta|}\,. \tag{4.4}$$

(**A3**) Regularity. The perturbation $P : D(s, r) \times \mathcal{O} \to \mathbb{C}$ is λ-regular (see Definition 2.8).

In order to obtain the asymptotic expansion (4.15) for the perturbed frequencies we also assume

(**A4**) Quasi-Töplitz. The perturbation P (preserves momentum and) is quasi-Töplitz, see (4.13).

Thanks to the conservation of momentum we restrict to the set of indices

$$\mathbf{I} := \Big\{ \quad (k,l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty, (k,l) \neq (0,0)\,, |l| \leq 2, \text{ where} \tag{4.5}$$
$$\text{or } l = 0\,,\ k \cdot \mathtt{j} = 0\,,$$
$$\text{or } l = \sigma e_m\,, m \in \mathbb{Z} \setminus \mathcal{I}\,,\ k \cdot \mathtt{j} + \sigma m = 0\,,$$
$$\text{or } l = \sigma e_m + \sigma' e_n\,, m,n \in \mathbb{Z} \setminus \mathcal{I}\,,\ k \cdot \mathtt{j} + \sigma m + \sigma' n = 0 \Big\}\,.$$

For $\eta > 0$ we define the set of Diophantine vectors

$$\mathcal{D}_\eta := \Big\{ \omega \in \mathbb{R}^n \ :\ |\omega \cdot k| \geq \frac{\eta}{1+|k|^n}\,,\ \forall\, k \in \mathbb{Z}^n \setminus \{0\} \Big\}\,. \tag{4.6}$$

Let

$$P = P_{00}(x) + \bar{P}(x,y,z,\bar{z}) \quad \text{where} \quad \bar{P}(x,0,0,0) = 0\,. \tag{4.7}$$

Theorem 4.1. (KAM theorem)

I) *Suppose that $H = \mathcal{N} + P$ satisfies* (A1)-(A3). *Let $\gamma \in (0,1)$ be a parameter and $\lambda := \gamma/M_1$. If*

$$\epsilon := \max \Big\{ \gamma^{-2/3} |X_{P_{00}}|^{\lambda}_{s,r}\,, \gamma^{-1} |X_{\bar{P}}|^{\lambda}_{s,r} \Big\} \tag{4.8}$$

is small enough, then there exist:

- **(Frequencies)** *Lipschiz functions $\omega^\infty : \mathcal{O} \to \mathbb{R}^n$, $\Omega^\infty : \mathcal{O} \to \ell_\infty$ such that*

$$|\omega^\infty - \omega| + \lambda |\omega^\infty - \omega|^{\rm lip}\,, \quad |\Omega^\infty - \Omega|_\infty + \lambda |\Omega^\infty - \Omega|^{\rm lip}_\infty \ \leq\ C\gamma\epsilon\,, \tag{4.9}$$

and $|\omega^\infty|^{\rm lip}$, $|\Omega^\infty|^{\rm lip}_\infty \leq 2M_1$.

- **(KAM normal form)** *A Lipschiz family of analytic symplectic maps*

$$\Phi : D(s/4, r/4) \times \mathcal{O}_\infty \ni (x_\infty, y_\infty, w_\infty; \xi) \mapsto (x,y,w) \in D(s,r) \tag{4.10}$$

close to the identity where

$$\mathcal{O}_\infty \ := \ \Big\{ \xi \in \mathcal{O} \cap \omega^{-1}(\mathcal{D}_{\gamma^{2/3}}) \ :\ |\omega^\infty(\xi) \cdot k + \Omega^\infty(\xi) \cdot l| \geq \frac{2\gamma}{1+|k|^\tau}\,, (k,l) \in \mathbf{I}$$
$$\text{where } \mathbf{I} \text{ is defined in (4.5) and } \mathcal{D}_{\gamma^{2/3}} \text{ in (4.6) with } \eta = \gamma^{2/3} \Big\} \tag{4.11}$$

such that,

$$H^\infty(\cdot;\xi) := H \circ \Phi(\cdot;\xi) = \omega^\infty(\xi) \cdot y_\infty + \Omega^\infty(\xi) \cdot z_\infty \bar{z}_\infty + P^\infty \quad \text{has} \quad P^\infty_{\leq 2} = 0\,. \tag{4.12}$$

Then, $\forall \xi \in \mathcal{O}_\infty$, the map $x_\infty \mapsto \Phi(x_\infty, 0, 0; \xi)$ is a real analytic embedding of an elliptic, n-dimensional torus with frequency $\omega^\infty(\xi)$ for the system with Hamiltonian H.

II) *Assume (A4). If, for some $1 < \theta, \mu < 6$, $N > 0$,*

$$\varepsilon := \max \Big\{ \gamma^{-2/3} \|X_{P_{00}}\|_{s,r}\,, \gamma^{-1} \|\bar{P}\|^T_{s,r,N,\theta,\mu} \Big\} \tag{4.13}$$

is small enough, then

- **(Asymptotic of frequencies)** *There exist $a^\infty_\pm : \mathcal{O}^*_\infty \to \mathbb{R}$ where*

$$\mathcal{O}^*_\infty := \Big\{ \xi \in \mathcal{O}_\infty \ :\ |\omega^\infty(\xi) \cdot k + p| \geq \frac{2\gamma^{2/3}}{1+|k|^\tau}\,,\ \forall\, k \in \mathbb{Z}^n\,,\ p \in \mathbb{Z}\,,\ (k,p) \neq (0,0) \Big\} \tag{4.14}$$

with $\tau > 1/b$ (recall (3.2)*) and*

$$\sup_{\xi \in \mathcal{O}^*_\infty} |\Omega^\infty_j(\xi) - \Omega_j(\xi) - \hat{a}^\infty_{\mathtt{s}(j)}(\xi)| \leq \gamma^{2/3} \varepsilon \frac{C}{|j|}\,, \quad \forall |j| \geq C_\star \gamma^{-1/3}\,. \tag{4.15}$$

Part **I)** of Theorem 4.1 follows by Theorem 5.1 of [2] because the KAM condition (4.8) implies hypothesis (H3) with $d = 1$ and $\mu = 2/3$ of Theorem 5.1-[2]. Note that condition (4.8) is weaker than the KAM condition in [24], allowing a direct application to the nonlinear wave equation.

There is only a minor difference in the settings of Theorems 4.1 and Theorem 5.1-[2]. In assumption (A1) the "tail" $\Omega_j - \sqrt{j^2 + \mathrm{m}} = a(\xi)$ is independent of j, unlike in [2] it tends to zero as $j \to \infty$. This difference does not affect the iterative part of the KAM theorem. The decay of the tail was used in [2] (as in [24]) only to prove the measure estimates.

The main novelty of Theorem 4.1 is part **II)**. In the next Theorem 4.2 we verify the second order Melnikov non-resonance conditions thanks to

1. the asymptotic decay (4.15) of the perturbed frequencies,
2. the restriction to indices $(k, l) \in \mathbf{I}$ in (4.11) which is a consequence of the momentum conservation, see (A4).

As in [2], the Cantor set of "good" parameters $\mathcal{O}_\infty$ in (4.11) and $\mathcal{O}^*_\infty$ in (4.14), are expressed in terms of the final frequencies only (and not inductively as in [24]). This simplifies the measure estimates.

Theorem 4.2. (Measure estimate) *Suppose*

$$\omega(\xi) = \bar{\omega} + A\xi\,,\ \bar{\omega} \in \mathbb{R}^n\,,\ A \in \mathrm{Mat}(n \times n)\ ,\quad \Omega_j(\xi) = \sqrt{j^2 + \mathrm{m}} + \vec{a} \cdot \xi\,,\ a \in \mathbb{R}^n \tag{4.16}$$

and assume the non-degeneracy condition:

$$A \ \text{ invertible } \quad \text{and} \quad 2(A^{-1})^T \vec{a} \notin \mathbb{Z}^n \setminus \{0\}\,. \tag{4.17}$$

Then, the Cantor like set $\mathcal{O}^*_\infty$ *defined in* (4.14), *with exponent*

$$\tau > \max\{2n + 1, 1/b\} \tag{4.18}$$

(b is fixed in (3.2)*), satisfies*

$$|\mathcal{O} \setminus \mathcal{O}^*_\infty| \leq C(\tau)\rho^{n-1}\gamma^{2/3} \qquad \text{where} \qquad \rho := \mathrm{diam}(\mathcal{O})\,. \tag{4.19}$$

Theorem 4.2 is proved in section 6. The asymptotic estimate (4.15) is used for proving the key inclusion (6.11).

5 Proof of the KAM Theorem 4.1

In this section we revisit the KAM scheme of [2] for proving part **II** of Theorem 4.1.

5.1 First step

We perform a preliminary change of variables in order to improve the smallness conditions. For all

$$\xi \in \omega^{-1}(\mathcal{D}_{\gamma^{2/3}}) \cap \mathcal{O} =: \mathcal{O}_0 \tag{5.1}$$

(see (4.6)) we consider the solution

$$F_{00}(x) := \sum_{k \neq 0} \frac{P_{00,k}}{\mathrm{i}\omega(\xi) \cdot k} e^{\mathrm{i}k \cdot x} \tag{5.2}$$

of the homological equation

$$-\,\mathrm{ad}_{\mathcal{N}} F_{00} + P_{00}(x) = \langle P_{00} \rangle\,. \tag{5.3}$$

Note that for any function $F_{00}(x)$ we have $\|F_{00}\|^T_{s,r} = \|X_{F_{00}}\|_{s,r}$, see Definition 3.4.

We want to apply Proposition 3.2 with $s, r, s', r' \rightsquigarrow 3s/4, 3r/4, s/2, r/2$. The condition (3.63) is verified because

$$\|F_{00}\|^T_{3s/4,r} = \|X_{F_{00}}\|_{3s/4,r} \overset{(5.2),(4.6)}{\leq} C(s)\gamma^{-2/3}\|X_{P_{00}}\|_{s,r} \overset{(4.13)}{\leq} C(s)\varepsilon_0$$

and ε_0 is sufficiently small. Hence the time–one flow

$$\Phi_{00} := e^{\mathrm{ad}_{F_{00}}} : D(s_0, r_0) \times \mathcal{O}_0 \to D(s, r) \quad \text{with} \quad s_0 := s/2\,, \ r_0 := r/2\,, \tag{5.4}$$

is well defined, analytic, symplectic. Let $\mu_0 < \mu$, $\theta_0 > \theta$, $N_0 > N$ large enough, so that (3.65) is satisfied with $s, r, N_0, \theta, \mu, \rightsquigarrow s, r, N, \theta, \mu$ and $s', r', N_0', \theta', \mu' \rightsquigarrow s_0, r_0, N_0, \theta_0, \mu_0$. Hence (3.66) implies

$$\|e^{\mathrm{ad}_{F_{00}}}\bar{P}\|^T_{s_0,r_0,N_0,\theta_0,\mu_0} \leq 2\|\bar{P}\|^T_{s,r,N,\theta,\mu}\,. \tag{5.5}$$

Noting that $e^{\mathrm{ad}_{F_{00}}}P_{00} = P_{00}$ and $e^{\mathrm{ad}_{F_{00}}}\mathcal{N} = \mathcal{N} + \mathrm{ad}_{F_{00}}\mathcal{N}$ the new Hamiltonian is

$$\begin{aligned} H^0 := e^{\mathrm{ad}_{F_{00}}}H = e^{\mathrm{ad}_{F_{00}}}\mathcal{N} + e^{\mathrm{ad}_{F_{00}}}P_{00} + e^{\mathrm{ad}_{F_{00}}}\bar{P} \quad &= \quad \mathcal{N} + \mathrm{ad}_{F_{00}}\mathcal{N} + P_{00} + e^{\mathrm{ad}_{F_{00}}}\bar{P} \\ &\overset{(5.3)}{=} \quad \langle P_{00}\rangle + \mathcal{N} + e^{\mathrm{ad}_{F_{00}}}\bar{P} =: \mathcal{N}_0 + P_0\,. \end{aligned} \tag{5.6}$$

By (5.5) and (4.13) we have that

$$\|P_0\|^T_{s_0,r_0,N_0,\theta_0,\mu_0} < 2\gamma\varepsilon\,. \tag{5.7}$$

5.2 KAM step

We now consider the generic KAM step for an Hamiltonian

$$H = \mathcal{N} + P = \mathcal{N} + P_K^{\leq 2} + (P - P_K^{\leq 2}) \tag{5.8}$$

where $P_K^{\leq 2}$ are defined as in (3.38).

5.2.1 Homological equation

Lemma 5.1. *Assume that*

$$|\Omega_j - \sqrt{j^2 + \mathrm{m}} - a_{\mathtt{s}(j)}| \leq \frac{\gamma}{|j|}\,, \quad \forall\, |j| \geq j_*\,, \tag{5.9}$$

for some $a_+, a_- \in \mathbb{R}$. *Let*

$$\Delta_{k,m,n} := \omega \cdot k + \Omega_m - \Omega_n\,, \qquad \tilde{\Delta}_{k,m,n} := \omega \cdot k + |m| - |n|\,.$$

If $|m|, |n| \geq \max\{j_*, \sqrt{\mathrm{m}}\}$ *and* $\mathtt{s}(m) = \mathtt{s}(n)$, *then*

$$|\Delta_{k,m,n} - \tilde{\Delta}_{k,m,n}| \leq \frac{\mathrm{m}}{2}\frac{|m-n|}{|n||m|} + \gamma\Big(\frac{1}{|m|} + \frac{1}{|n|}\Big) + \frac{\mathrm{m}^2}{2}\left(\frac{1}{|m|^3} + \frac{1}{|n|^3}\right)\,. \tag{5.10}$$

PROOF. For $0 \leq x \leq 1$ we have $|\sqrt{1+x} - 1 - x/2| \leq x^2/2$. Setting $x := \mathrm{m}/n^2$ (which is ≤ 1) and using (5.9), we get

$$\left|\Omega_n - |n| - \frac{\mathrm{m}}{2|n|} - a_{\mathtt{s}(n)}\right| \leq \frac{\gamma}{|n|} + \frac{\mathrm{m}^2}{2|n|^3}\,.$$

An analogous estimates holds for Ω_m. Since $|\Delta_{k,m,n} - \tilde{\Delta}_{k,m,n}| = |\Omega_m - |m| - \Omega_n + |n||$ the estimate (5.10) follows noting that $a_{\mathtt{s}(m)} = a_{\mathtt{s}(n)}$. ■

For a monomial $\mathfrak{m}_{k,i,\alpha,\beta} := e^{\mathrm{i}k\cdot x}y^i z^\alpha \bar{z}^\beta$ we set

$$[\mathfrak{m}_{k,i,\alpha,\beta}] := \begin{cases} \mathfrak{m}_{k,i,\alpha,\beta} & \text{if} \quad k = 0\,, \ \alpha = \beta \\ 0 & \qquad \text{otherwise.} \end{cases} \tag{5.11}$$

The following key proposition proves that the solution of the homological equation is quasi-Töplitz.

Proposition 5.1. (Homological equation)
I) *Let $K \in \mathbb{N}$. For all $\xi \in \mathcal{O}$ such that*

$$|\omega(\xi) \cdot k + \Omega(\xi) \cdot l| \geq \frac{\gamma}{\langle k \rangle^\tau}\,, \quad \forall (k,l) \in \mathbf{I} \text{ (see (4.5))},\ |k| \leq K\,, \tag{5.12}$$

then $\forall P_K^{(h)} \in \mathcal{H}_{s,r}^{\rm null}$, $h = 0,1,2$ *(see (3.36), (3.37)), the homological equations*

$$-\operatorname{ad}_{\mathcal{N}} F_K^{(h)} + P_K^{(h)} = [P_K^{(h)}]\,, \qquad h = 0,1,2\,, \tag{5.13}$$

have a unique solution of the same form $F_K^{(h)} \in \mathcal{H}_{s,r}^{\rm null}$ *with* $[F_K^{(h)}] = 0$ *and*

$$\|X_{F_K^{(h)}}\|_{s,r} < \gamma^{-1} K^\tau \|X_{P_K^{(h)}}\|_{s,r}\,. \tag{5.14}$$

In particular $F_K^{\leq 2} := F_K^{(0)} + F_K^{(1)} + F_K^{(2)}$ *solves*

$$-\operatorname{ad}_{\mathcal{N}} F_K^{\leq 2} + P_K^{\leq 2} = [P_K^{\leq 2}]\,. \tag{5.15}$$

II) *Assume now that* $P_K^{(h)} \in \mathcal{Q}_{s,r}^T(N_0, \theta, \mu)$ *and* $\Omega(\xi)$ *satisfies* (5.9) *for all* $|j| \geq \theta N_0^*$ *where*

$$N_0^* := \max\Big\{N_0\,,\ \hat{c}\gamma^{-1/3} K^{\tau+1}\Big\} \tag{5.16}$$

for a constant $\hat{c} := \hat{c}(\mathrm{m}, \kappa) \geq 1$. *Then,* $\forall \xi \in \mathcal{O}$ *such that*

$$|\omega(\xi) \cdot k + p| \geq \frac{\gamma^{2/3}}{\langle k \rangle^\tau}\,, \quad \forall |k| \leq K,\ p \in \mathbb{Z}\,, \tag{5.17}$$

we have $F_K^{(h)} \in \mathcal{Q}_{s,r}^T(N_0^*, \theta, \mu)$, $h = 0,1,2$, *and*

$$\|F_K^{(h)}\|_{s,r,N_0^*,\theta,\mu}^T \leq 4\hat{c}\gamma^{-1} K^{2\tau} \|P_K^{(h)}\|_{s,r,N_0,\theta,\mu}^T\,. \tag{5.18}$$

Proof. The solution of the homological equation (5.13) is

$$F_K^{(h)} := -\mathrm{i} \sum_{\substack{|k| \leq K, (k,i,\alpha,\beta) \neq (0,i,\alpha,\alpha) \\ 2i + |\alpha| + |\beta| = h}} \frac{P_{k,i,\alpha,\beta}}{\Delta_{k,i,\alpha,\beta}} e^{\mathrm{i} k \cdot x} y^i z^\alpha \bar{z}^\beta\,, \qquad \Delta_{k,i,\alpha,\beta} := \omega(\xi) \cdot k + \Omega(\xi) \cdot (\alpha - \beta)\,.$$

The divisors $\Delta_{k,i,\alpha,\beta} \neq 0$, $\forall (k,i,\alpha,\beta) \neq (0,i,\alpha,\alpha)$, because $(k,i,\alpha,\beta) \neq (0,i,\alpha,\alpha)$ is equivalent to $(k, \alpha - \beta) \in \mathbf{I}$, and the bounds (5.12) hold. Item **I**) follows by Lemma 2.18.

In item **II**) we notice that the cases $h = 0,1$ are trivial since $\Pi_{N,\theta,\mu} F_K^{\leq 1} = 0$.

When $h = 2$ we first consider the subtlest case when $P_K^{(2)}$ contains only the monomials with $i = 0$, $|\alpha| = |\beta| = 1$ (see (3.36)), namely

$$\mathcal{P} := P_K^{(2)} = \sum_{|k| \leq K, m,n \in \mathbb{Z} \setminus \mathcal{I}} P_{k,m,n} e^{\mathrm{i} k \cdot x} z_m \bar{z}_n\,, \tag{5.19}$$

and, because of the conservation of momentum, the indices k, m, n in (5.19) are restricted to

$$\mathtt{j} \cdot k + m - n = 0\,. \tag{5.20}$$

The unique solution $F_K^{(2)}$ of (5.13) with $[F_K^{(2)}] = 0$ is

$$\mathcal{F} := F_K^{(2)} := -\mathrm{i} \sum_{|k| \leq K, (k,m,n) \neq (0,m,m)} \frac{P_{k,m,n}}{\Delta_{k,m,n}} e^{\mathrm{i} k \cdot x} z_m \bar{z}_n\,, \quad \Delta_{k,m,n} := \omega(\xi) \cdot k + \Omega_m(\xi) - \Omega_n(\xi) \tag{5.21}$$

Note that by (5.12) and (5.20) we have $\Delta_{k,m,n} \neq 0$ if and only if $(k,m,n) \neq (0,m,m)$.

Let us prove (5.18). For all $N \geq N_0^*$

$$\Pi_{N,\theta,\mu}\mathcal{F} = -\mathrm{i} \sum_{|k|\leq K, |m|,|n|>\theta N} \frac{P_{k,m,n}}{\Delta_{k,m,n}} e^{\mathrm{i}k\cdot x} z_m \bar{z}_n, \tag{5.22}$$

and note that $e^{\mathrm{i}k\cdot x}$ is (N,μ)-low momentum since $|k| \leq K < (N_0^*)^b \leq N^b$ by (5.16) and $\tau > 1/b$. By assumption $\mathcal{P} \in \mathcal{Q}^T_{s,r,N_0,\theta,\mu}$ and so, recalling formula (3.45), we may write, $\forall N \geq N_0^* \geq N_0$,

$$\Pi_{N,\theta,\mu}\mathcal{P} = \tilde{\mathcal{P}} + N^{-1}\hat{\mathcal{P}} \quad \text{with} \quad \tilde{\mathcal{P}} := \sum_{|k|\leq K, |m|,|n|>\theta N} \tilde{P}_{k,m-n} e^{\mathrm{i}k\cdot x} z_m \bar{z}_n \in \mathcal{T}_{s,r}(N,\theta,\mu) \tag{5.23}$$

and

$$\|X_{\mathcal{P}}\|_{s,r}, \|X_{\tilde{\mathcal{P}}}\|_{s,r}, \|X_{\hat{\mathcal{P}}}\|_{s,r} \leq 2\|\mathcal{P}\|^T_{s,r}\,. \tag{5.24}$$

We now prove that

$$\tilde{\mathcal{F}} := \sum_{|k|\leq K, |m|,|n|>\theta N} \frac{\tilde{P}_{k,m-n}}{\tilde{\Delta}_{k,m,n}} e^{\mathrm{i}k\cdot x} z_m \bar{z}_n\,, \quad \tilde{\Delta}_{k,m,n} := \omega(\xi)\cdot k + |m| - |n|\,, \tag{5.25}$$

is a Töplitz approximation of $\mathcal{F}$. Since $|m|, |n| > \theta N \geq \theta N_0^* > N_0^* \overset{(5.16)}{>} \kappa K \geq |\mathtt{j}\cdot k|$ by (3.1), we deduce by (5.20) that m,n have the same sign. Then

$$\tilde{\Delta}_{k,m,n} = \omega(\xi)\cdot k + |m| - |n| = \omega(\xi)\cdot k + \mathtt{s}(m)(m-n)\,, \quad \mathtt{s}(m) := \mathrm{sign}(m)\,,$$

and $\tilde{\mathcal{F}}$ in (5.25) is (N,θ,μ)-Töplitz (see (3.15)). Moreover, since $|m| - |n| \in \mathbb{Z}$, by (5.17), we get

$$|\tilde{\Delta}_{k,m,n}| \geq \gamma^{2/3}\langle k\rangle^{-\tau}\,, \quad \forall |k| \leq K,\ m,n, \tag{5.26}$$

and Lemma 2.18 and (5.25) imply

$$\|X_{\tilde{\mathcal{F}}}\|_{s,r} \leq \gamma^{-2/3} K^\tau \|X_{\tilde{\mathcal{P}}}\|_{s,r}\,. \tag{5.27}$$

The Töplitz defect is

$$\begin{aligned} N^{-1}\hat{\mathcal{F}} \quad &:= \quad \Pi_{N,\theta,\mu}\mathcal{F} - \tilde{\mathcal{F}} \qquad (5.28)\\ &\overset{(5.22),(5.25)}{=} \sum_{|k|\leq K, |m|,|n|>\theta N} \Big(\frac{P_{k,m,n}}{\Delta_{k,m,n}} - \frac{\tilde{P}_{k,m-n}}{\tilde{\Delta}_{k,m,n}}\Big) e^{\mathrm{i}k\cdot x} z_m \bar{z}_n \\ &= \sum_{|k|\leq K, |m|,|n|>\theta N} \Big[\Big(\frac{P_{k,m,n}}{\Delta_{k,m,n}} - \frac{P_{k,m,n}}{\tilde{\Delta}_{k,m,n}}\Big) + \Big(\frac{P_{k,m,n} - \tilde{P}_{k,m-n}}{\tilde{\Delta}_{k,m,n}}\Big)\Big] e^{\mathrm{i}k\cdot x} z_m \bar{z}_n \\ &\overset{(5.23)}{=} \sum_{|k|\leq K, |m|,|n|>\theta N} \Big[P_{k,m,n}\Big(\frac{\tilde{\Delta}_{k,m,n} - \Delta_{k,m,n}}{\Delta_{k,m,n}\tilde{\Delta}_{k,m,n}}\Big) + N^{-1}\frac{\hat{P}_{k,m,n}}{\tilde{\Delta}_{k,m,n}}\Big] e^{\mathrm{i}k\cdot x} z_m \bar{z}_n\,. \end{aligned}$$

By (5.10), $|m|, |n| \geq \theta N \geq N$, and $|m-n| \leq \kappa K$ (see (5.20)) we get, taking $\hat{c}$ large enough,

$$|\tilde{\Delta}_{k,m,n} - \Delta_{k,m,n}| \quad \leq \quad \frac{\mathrm{m}\kappa K}{2N^2} + \frac{2\gamma}{N} + \frac{\mathrm{m}^2}{N^3} \leq \frac{\hat{c}}{4N}\left(\frac{K}{N} + \gamma\right) \overset{(5.16)}{\leq} \min\left\{\frac{\hat{c}\gamma^{1/3}}{2N}\,,\ \frac{\gamma^{2/3}}{2K^\tau}\right\}\,. \tag{5.29}$$

Hence

$$|\Delta_{k,m,n}| \geq |\tilde{\Delta}_{k,m,n}| - |\tilde{\Delta}_{k,m,n} - \Delta_{k,m,n}| \overset{(5.26),(5.29)}{\geq} \frac{\gamma^{2/3}}{\langle k\rangle^\tau} - \frac{\gamma^{2/3}}{2K^\tau} \geq \frac{\gamma^{2/3}}{2\langle k\rangle^\tau}\,. \tag{5.30}$$

Therefore (5.29), (5.26), (5.30) imply

$$\frac{|\tilde{\Delta}_{k,m,n} - \Delta_{k,m,n}|}{|\Delta_{k,m,n}||\tilde{\Delta}_{k,m,n}|} \le \frac{\hat{c}\gamma^{1/3}}{2N} \frac{2\langle k\rangle^\tau}{\gamma^{2/3}} \frac{\langle k\rangle^\tau}{\gamma^{2/3}} \le \frac{\hat{c}}{N\gamma} K^{2\tau}$$

and (5.28), (5.26), and Lemma 2.18, imply

$$\|X_{\tilde{\mathcal{F}}}\|_{s,r} \le \hat{c}\gamma^{-1} K^{2\tau} \|X_{\mathcal{P}}\|_{s,r} + \gamma^{-2/3} K^\tau \|X_{\hat{\mathcal{P}}}\|_{s,r} \stackrel{(5.24)}{\le} 4\hat{c}\gamma^{-1} K^{2\tau} \|\mathcal{P}\|^T_{s,r} \,. \tag{5.31}$$

In conclusion (5.14), (5.27), (5.31) prove (5.18) for $\mathcal{F}$.

Let us briefly discuss the case when $h = 2$ and $P^{(2)}_K$ contains only the monomials with $i = 0$, $|\alpha| = 2$, $|\beta| = 0$ or viceversa (see (3.36)). Denoting

$$\mathcal{P} := P^{(2)}_K := \sum_{|k|\le K, m,n \in \mathbb{Z}\setminus\mathcal{I}} P_{k,m,n} e^{\mathrm{i} k\cdot x} z_m z_n \,, \tag{5.32}$$

we have

$$\Pi_{N,\theta,\mu} \mathcal{F} = -\mathrm{i} \sum_{|k|\le K, |m|,|n| > \theta N} \frac{P_{k,m,n}}{\omega\cdot k + \Omega_m + \Omega_n} e^{\mathrm{i} k\cdot x} z_m z_n$$

where $|\omega\cdot k + \Omega_m + \Omega_n| > (|m| + |n|)/2 > \theta N/2$ since $|m|, |n| > \theta N$ and $|k| \le K < N^b$. In this case we may take as Töplitz approximation $\tilde{\mathcal{F}} = 0$. ■

5.2.2 The new Hamiltonian H^+

Let $F = F^{\le 2}_K$ be the solution of the homological equation (5.15). If, for $s/2 \le s_+ < s$, $r/2 \le r_+ < r$, the condition

$$\|F\|^T_{s,r,N_0^*,\theta,\mu} \le c(n)\, \delta_+ \,, \quad \delta_+ := \min\Big\{1 - \frac{s_+}{s}, 1 - \frac{r_+}{r}\Big\} \tag{5.33}$$

holds (see (3.63)), then Proposition 3.2 (with $s' \rightsquigarrow s_+, r' \rightsquigarrow r_+, N_0 \rightsquigarrow N_0^*$ defined in (5.16)) implies that the Hamiltonian flow $e^{\mathrm{ad}_F} : D(s_+, r_+) \to D(s,r)$ is well defined, analytic and symplectic. We transform the Hamiltonian H in (5.8), obtaining

$$\begin{aligned} H^+ := e^{\mathrm{ad}_F} H &\stackrel{(2.82)}{=} H + \mathrm{ad}_F(H) + \sum_{j\ge 2} \frac{1}{j!} \mathrm{ad}^j_F(H) \\ &\stackrel{(5.8)}{=} \mathcal{N} + P^{\le 2}_K + (P - P^{\le 2}_K) + \mathrm{ad}_F \mathcal{N} + \mathrm{ad}_F P + \sum_{j\ge 2} \frac{1}{j!} \mathrm{ad}^j_F(H) \\ &\stackrel{(5.15)}{=} \mathcal{N} + [P^{\le 2}_K] + P - P^{\le 2}_K + \mathrm{ad}_F P + \sum_{j\ge 2} \frac{1}{j!} \mathrm{ad}^j_F(H) := \mathcal{N}^+ + P^+ \end{aligned}$$

with new normal form

$$\mathcal{N}^+ := \mathcal{N} + \hat{\mathcal{N}} \,, \quad \hat{\mathcal{N}} := [P^{\le 2}_K] = \hat{e} + \hat{\omega}\cdot y + \hat{\Omega} z\cdot \bar{z}$$

$$\hat{\omega}_i := \partial_{y_i |\, y=0, z=0} \langle P\rangle \,, \ i = 1, \ldots n \,, \quad \hat{\Omega} := (\hat{\Omega}_j)_{j\in\mathbb{Z}\setminus\mathcal{I}} \,, \quad \hat{\Omega}_j := [P]_j := \partial^2_{z_j \bar{z}_j \,|\, y=0, z=0} \langle P\rangle \tag{5.34}$$

(the $\langle\ \rangle$ denotes the average with respect to the angles x) and new perturbation

$$P^+ := P - P^{\le 2}_K + \mathrm{ad}_F P^{\le 2} + \mathrm{ad}_F P^{\ge 3} + \sum_{j\ge 2} \frac{1}{j!} \mathrm{ad}^j_F(H) \tag{5.35}$$

having decomposed $P = P^{\le 2} + P^{\ge 3}$ with $P^{\ge 3} := \sum_{h\ge 3} P^{(h)}$, see (3.36).

5.2.3 The new normal form $\mathcal{N}^+$

The next lemma holds uniformly in the parameters ξ.

Lemma 5.2. *Let $P \in \mathcal{Q}^T_{s,r}(N_0, \theta, \mu)$ with $1 < \theta, \mu < 6$, $N_0 \geq 9$. Then*

$$|\hat{\omega}|, |\hat{\Omega}|_\infty \leq 2\|P^{(2)}\|^T_{s,r,N_0,\theta,\mu} \tag{5.36}$$

and there exist $\hat{a}_\pm \in \mathbb{R}$ satisfying

$$|\hat{a}_\pm| \leq 2\|P^{(2)}\|^T_{s,r,N_0,\theta,\mu}$$

such that

$$|\hat{\Omega}_j - \hat{a}_{\mathtt{s}(j)}| \leq \frac{40}{|j|}\|P^{(2)}\|^T_{s,r,N_0,\theta,\mu}\,, \quad \forall\, |j| \geq 6(N_0+1)\,. \tag{5.37}$$

Lemma 5.2 is based on the following elementary Lemma, whose proof is postponed.

Lemma 5.3. *Suppose that, $\forall N \geq N_0 \geq 9$, $j \geq \theta N$,*

$$\Omega_j = a_N + b_{N,j}N^{-1} \quad \text{with} \quad a_N, b_{N,j} \in \mathbb{R}\,, \quad |a_N| \leq c_1\,, \ |b_{N,j}| \leq c_1\,, \tag{5.38}$$

for some $c_1 > 0$ (indipendent of j). Then there exists $a \in \mathbb{R}$, satisfying $|a| \leq c_1$, such that

$$|\Omega_j - a| \leq \frac{20c_1}{|j|}\,, \quad \forall\, |j| \geq 6(N_0+1)\,. \tag{5.39}$$

PROOF OF LEMMA 5.2. The estimate on $\hat{\omega}$ is trivial. Regarding $\hat{\Omega}$ we set (recall (3.36), (3.42))

$$P_0^{(2)} := \Pi_{k=0}\Pi_{|\alpha|=|\beta|=1}\Pi^{(2)}P = \sum_j [P]_j z_j \bar{z}_j$$

since, by the momentum conservation (2.85), all the monomials in $P_0^{(2)}$ have $\alpha = \beta = e_j$. Note that $[P]_j$ is defined in (5.34). By Lemma 2.19

$$|[P]_j| \leq \|X_{P_0^{(2)}}\|_r \overset{(3.30)}{\leq} \|P_0^{(2)}\|^T_r \overset{(3.42)}{\leq} \|P^{(2)}\|^T_{s,r}\,. \tag{5.40}$$

We now prove (5.37) for $j > 0$ (the case $j < 0$ is similar). Since $P_0^{(2)} \in \mathcal{Q}^T_r(N, \theta, \mu)$, for all $N \geq N_0$, we may write $\Pi_{N,\theta,\mu}P_0^{(2)} = \tilde{P}^{(2)}_{0,N} + N^{-1}\hat{P}^{(2)}_{0,N}$ with

$$\tilde{P}^{(2)}_{0,N} := \sum_{j>\theta N} \tilde{P}_j z_j \bar{z}_j \in \mathcal{T}_r(N, \theta, \mu)\,, \quad \hat{P}^{(2)}_{0,N} := \sum_{j>\theta N} \hat{P}_j z_j \bar{z}_j$$

and

$$\|X_{P_0^{(2)}}\|_r\,, \|X_{\tilde{P}^{(2)}_{0,N}}\|_r\,, \|X_{\hat{P}^{(2)}_{0,N}}\|_r \leq 2\|P_0^{(2)}\|^T_r \leq 2\|P^{(2)}\|^T_{s,r}\,. \tag{5.41}$$

For $|j| > \theta N$, since all the quadratic forms in (5.41) are diagonal, we have

$$\hat{\Omega}_j = [P]_j = \tilde{P}_j + N^{-1}\hat{P}_j := a_{N,+} + N^{-1}b_{N,j}$$

where $a_{N,+} := \tilde{P}_j$ is independent of $j > 0$ because $\tilde{P}^{(2)}_{0,N} \in \mathcal{T}_r(N, \theta, \mu)$ (see (3.15)). Applying Lemma 2.19 to $\tilde{P}^{(2)}_{0,N}$ and $\hat{P}^{(2)}_{0,N}$, we obtain

$$|a_{N,+}| \leq \|X_{\tilde{P}^{(2)}_{0,N}}\|_{s,r} \overset{(5.41)}{\leq} 2\|P^{(2)}\|^T_{s,r}\,, \quad |b_{N,j}| = |\hat{P}_j| \leq \|X_{\hat{P}^{(2)}_{0,N}}\|_r \overset{(5.41)}{\leq} 2\|P^{(2)}\|^T_r\,.$$

Hence the assumptions of Lemma 5.3 are satisfied with $c_1 = 2\|P^{(2)}\|^T_{s,r}$ and (5.37) follows. ■

Proof of Lemma 5.3. For all $N_1 > N \geq N_0$, $j \geq \theta N_1$ we get, by (5.38),

$$|a_N - a_{N_1}| = |b_{N_1,j} N_1^{-1} - b_{N,j} N^{-1}| \leq 2c_1 N^{-1} . \tag{5.42}$$

Therefore a_N is a Cauchy sequence. Let $a := \lim_{N\to+\infty} a_N$ be its limit. Since $|a_N| \leq c_1$ we have $|a| \leq c_1$. Moreover, letting $N_1 \to +\infty$ in (5.42), we derive $|a - a_N| \leq 2c_1 N^{-1}$, $\forall N \geq N_0$, and, using also (5.38),

$$|\Omega_j - a| \leq |\Omega_j - a_N| + |a_N - a| \leq 3c_1 N^{-1}, \quad \forall\, N \geq N_0\,,\ j \geq 6N\,. \tag{5.43}$$

For all $j \geq 6(N_0+1)$ let $N := [j/6]$ (where $[\cdot]$ denotes the integer part). Since $N \geq N_0$, $j \geq 6N$,

$$|\Omega_j - a| \stackrel{(5.43)}{\leq} \frac{3c_1}{[j/6]} \leq \frac{3c_1}{(j/6)-1} \leq \frac{18c_1}{j}\Big(1 + \frac{1}{N_0}\Big) \leq \frac{20c_1}{j}$$

for all $j \geq 6(N_0+1)$. ■

5.2.4 The new perturbation P^+

We introduce, for $h = 0,1,2$,

$$\varepsilon^{(h)} := \gamma^{-1}\|P^{(h)}\|^T_{s,r,N_0,\theta,\mu}\,, \quad \bar\varepsilon := \sum_{h=0}^{2}\varepsilon^{(h)}\,, \quad \Theta := \gamma^{-1}\|P\|^T_{s,r,N_0,\theta,\mu} \tag{5.44}$$

and the corresponding quantities for P^+ with indices r_+, $s_+, N_0^+, \theta_+, \mu_+$.

Proposition 5.2. (KAM step) *Suppose* (s,r,N_0,θ,μ), $(s_+,r_+,N_0^+,\theta_+,\mu_+)$ *satisfy* $s/2 \leq s_+ < s$, $r/2 \leq r_+ < r$,

$$N_0^+ > \max\{N_0^*, \bar N\} \text{ (recall (5.16), (3.64))}, \quad 2(N_0^+)^{-b}\ln^2 N_0^+ \leq b(s - s_+)\,, \tag{5.45}$$

$$\kappa (N_0^+)^{b-L}\ln N_0^+ \leq \mu - \mu_+, \quad (6+\kappa)(N_0^+)^{L-1}\ln N_0^+ \leq \theta_+ - \theta\,. \tag{5.46}$$

Assume that

$$\bar\varepsilon K^{\bar\tau}\delta_+^{-1} \leq c \text{ small enough}\,, \quad \Theta \leq 1\,, \tag{5.47}$$

where $\bar\tau := 2\tau + n + 1$ *and* δ_+ *is defined in* (5.33). *Suppose also that* (5.9) *holds for* $|j| \geq \theta N_0^*$.

Then, for all $\xi \in \mathcal{O}$ *satisfying* (5.12),(5.17), *denoting by* $F := F_K^{\leq 2}$ *the solution of the homological equation* (5.15), *the Hamiltonian flow* $e^{\mathrm{ad}_F} : D(s_+,r_+) \to D(s,r)$, *and the transformed Hamiltonian*

$$H^+ := e^{\mathrm{ad}_F} H = \mathcal{N}_+ + P_+$$

satisfies

$$\begin{aligned}
\varepsilon_+^{(0)} &\lessdot \delta_+^{-2}K^{2\bar\tau}\bar\varepsilon^2 + \varepsilon^{(0)}\, s\, s_+^{-1} e^{-(s-s_+)K} \\
\varepsilon_+^{(1)} &\lessdot \delta_+^{-2}K^{2\bar\tau}\big(\varepsilon^{(0)} + \bar\varepsilon^2\big) + \varepsilon^{(1)}\, s\, s_+^{-1} e^{-(s-s_+)K} \\
\varepsilon_+^{(2)} &\lessdot \delta_+^{-2}K^{2\bar\tau}\big(\varepsilon^{(0)} + \varepsilon^{(1)} + \bar\varepsilon^2\big) + \varepsilon^{(2)}\, s\, s_+^{-1} e^{-(s-s_+)K}
\end{aligned} \tag{5.48}$$

$$\Theta_+ \leq \Theta(1 + C\delta_+^{-2}K^{2\bar\tau}\bar\varepsilon)\,. \tag{5.49}$$

The proof of this proposition is split in several lemmas where we analyze each term of P^+ in (5.35). We note first that

$$\|P_K^{\leq 2}\|^T_{s,r,N_0,\theta,\mu} \stackrel{(3.41)}{\leq} \|P^{\leq 2}\|^T_{s,r,N_0,\theta,\mu} \stackrel{(3.38),(5.44)}{\leq} \gamma\bar\varepsilon\,. \tag{5.50}$$

Moreover, the solution $F = F^{(0)} + F^{(1)} + F^{(2)}$ of the homological equation (5.15) (for brevity $F^{(h)} \equiv F_K^{(h)}$ and $F \equiv F_K^{\leq 2}$) satisfies, by (5.18) (with N_0^* defined in (5.16)), (3.41), (5.44),

$$\|F^{(h)}\|^T_{s,r,N_0^*,\theta,\mu} \lessdot K^{\bar\tau}\varepsilon^{(h)}\,,\ h = 0,1,2, \qquad \|F\|^T_{s,r,N_0^*,\theta,\mu} \lessdot K^{\bar\tau}\bar\varepsilon\,. \tag{5.51}$$

Hence (5.47) and (5.51) imply condition (5.33) and therefore $e^{\mathrm{ad}_F} : D(s_+,r_+) \to D(s,r)$ is well defined. We now estimate the terms of the new perturbation P^+ in (5.35).

Lemma 5.4.

$$\Big\|\mathrm{ad}_F(P^{\le 2})\Big\|^T_{s_+,r_+,N_0^+,\theta_+,\mu_+} + \Big\|\sum_{j\ge 2}\frac{1}{j!}\mathrm{ad}^j_F(H)\Big\|^T_{s_+,r_+,N_0^+,\theta_+,\mu_+} \lessdot \delta_+^{-2}\gamma K^{2\bar\tau}\bar\varepsilon^2 .$$

Proof. We have

$$\begin{aligned}\sum_{j\ge 2}\frac{1}{j!}\mathrm{ad}^j_F(H) &= \sum_{j\ge 2}\frac{1}{j!}\mathrm{ad}^j_F(\mathcal{N}+P) = \sum_{j\ge 2}\frac{1}{j!}\mathrm{ad}^{j-1}_F(\mathrm{ad}_F\mathcal{N}) + \sum_{j\ge 2}\frac{1}{j!}\mathrm{ad}^j_F(P)\\ &\overset{(5.15)}{=} \sum_{j\ge 2}\frac{1}{j!}\mathrm{ad}^{j-1}_F([P^{\le 2}_K] - P^{\le 2}_K) + \sum_{j\ge 2}\frac{1}{j!}\mathrm{ad}^j_F(P) .\end{aligned}$$

By (5.45), (5.46) and (5.33) we can apply Proposition 3.2 with $N_0, N_0', s', r', \theta', \mu', \delta \rightsquigarrow N_0^*, N_0^+, s_+, r_+, \theta_+, \mu_+, \delta_+$. We get (recall $N_0^* \ge N_0$)

$$\begin{aligned}\Big\|\sum_{j\ge 2}\frac{1}{j!}\mathrm{ad}^j_F(P)\Big\|^T_{s_+,r_+,N_0^+,\theta_+,\mu_+} &\overset{(3.67),(3.35)}{\lessdot} \Big(\delta_+^{-1}\|F\|^T_{s,r,N_0^*,\theta,\mu}\Big)^2\|P\|^T_{s,r,N_0,\theta,\mu}\\ &\overset{(5.51),(5.44)}{\lessdot} \delta_+^{-2}K^{2\bar\tau}\bar\varepsilon^2\gamma\,\Theta \end{aligned}\tag{5.52}$$

and, similarly,

$$\begin{aligned}\Big\|\sum_{j\ge 2}\frac{1}{j!}\mathrm{ad}^{j-1}_F(P^{\le 2}_K)\Big\|^T_{s_+,r_+,N_0^+,\theta_+,\mu_+} &= \Big\|\sum_{j\ge 1}\frac{1}{(j+1)!}\mathrm{ad}^j_F(P^{\le 2}_K)\Big\|^T_{s_+,r_+,N_0^+,\theta_+,\mu_+}\\ &\overset{(3.67)}{\lessdot} \delta_+^{-1}\|F\|^T_{s,r,N_0^*,\theta,\mu}\|P^{\le 2}_K\|^T_{s,r,N_0,\theta,\mu}\\ &\overset{(5.51),(5.50)}{\lessdot} \delta_+^{-1}K^{\bar\tau}\gamma\bar\varepsilon^2 . \end{aligned}\tag{5.53}$$

Finally, by Proposition 3.1, applied with

$$N_0, N_1, s_1, r_1, \theta_1, \mu_1, \delta \rightsquigarrow N_0^*, N_0^+, s_+, r_+, \theta_+, \mu_+, \delta_+ ,\tag{5.54}$$

we get

$$\begin{aligned}\Big\|\mathrm{ad}_F(P^{\le 2})\Big\|^T_{s_+,r_+,N_0^+,\theta_+,\mu_+} &\overset{(3.52)}{\lessdot} \delta_+^{-1}\|F\|^T_{s,r,N_0^*,\theta,\mu}\|P^{\le 2}\|^T_{s,r,N_0,\theta,\mu}\\ &\overset{(5.51),(5.50)}{\lessdot} \delta_+^{-1}K^{\bar\tau}\gamma\,\bar\varepsilon^2 . \end{aligned}\tag{5.55}$$

The bounds (5.52), (5.53), (5.55), and $\Theta \le 1$ (see (5.47)), prove the lemma. ■

Lemma 5.5. (5.49) *holds.*

Proof. By Proposition 3.1 (applied with (5.54)) we have

$$\begin{aligned}\Big\|\mathrm{ad}_F(P^{\ge 3})\Big\|^T_{s_+,r_+,N_0^+,\theta_+,\mu_+} &\lessdot \delta_+^{-1}\|F\|^T_{s,r,N_0^*,\theta,\mu}\|P^{\ge 3}\|^T_{s,r,N_0,\theta,\mu}\\ &\overset{(5.51),(3.40),(5.44)}{\lessdot} \delta_+^{-1}K^{\bar\tau}\gamma\,\bar\varepsilon\,\Theta , \end{aligned}\tag{5.56}$$

and (5.49) follows by (5.35), (3.40), (3.35), (5.44) (5.56), Lemma 5.4 and $\bar\varepsilon \le 3\Theta$ (which follows by (5.44) and (3.39)). ■

We now consider $P_+^{(h)}$, $h = 0, 1, 2$. The term $\mathrm{ad}_F P^{\ge 3}$ in (5.35) does not contribute to $P_+^{(0)}$. On the contrary, its contribution to $P_+^{(1)}$ is

$$\{F^{(0)}, P^{(3)}\}\tag{5.57}$$

and to $P_+^{(2)}$ is

$$\{F^{(1)}, P^{(3)}\} + \{F^{(0)}, P^{(4)}\} .\tag{5.58}$$

Lemma 5.6. $\|\{F^{(0)}, P^{(3)}\}\|^T_{s_+,r_+,N_0^+,\theta_+,\mu_+} \lessdot \delta_+^{-1}\gamma K^{\bar\tau}\varepsilon^{(0)}\Theta$ *and*

$$\Big\|\{F^{(1)}, P^{(3)}\} + \{F^{(0)}, P^{(4)}\}\Big\|^T_{s_+,r_+,N_0^+,\theta_+,\mu_+} \lessdot \delta_+^{-1}K^{\bar\tau}\gamma(\varepsilon^{(0)}+\varepsilon^{(1)})\Theta\,.$$

Proof. By (3.52) (applied with (5.54)), (5.51), (5.44) and (3.39). ■

The contribution of $P - P_K^{\leq 2}$ in (5.35) to $P_+^{(h)}$, $h = 0, 1, 2$, is $P^{(h)}_{>K}$.

Lemma 5.7. $\|P^{(h)}_{>K}\|^T_{s_+,r_+,N_0^+,\theta_+,\mu_+} \leq s\, s_+^{-1} e^{-K(s-s_+)}\gamma\varepsilon^{(h)}$

Proof. By (3.43) and (5.44). ■

Finally, (5.48) follows by (5.35), Lemmata 5.4, 5.6 (and (5.57)-(5.58)), Lemma 5.7 and $\Theta \leq 1$.

5.3 KAM iteration

Lemma 5.8. *Suppose that* $\varepsilon_i^{(0)}, \varepsilon_i^{(1)}, \varepsilon_i^{(2)} \in (0,1)$, $i = 0, \dots, \nu$, *satisfy*

$$\begin{aligned} \varepsilon^{(0)}_{i+1} &\leq C_* \mathtt{K}^i\, \bar\varepsilon_i^2 + C_*\varepsilon_i^{(0)}\, e^{-K_* 2^i} \\ \varepsilon^{(1)}_{i+1} &\leq C_* \mathtt{K}^i \big(\, \varepsilon_i^{(0)} + \bar\varepsilon_i^2\big) + C_*\varepsilon_i^{(1)}\, e^{-K_* 2^i} \\ \varepsilon^{(2)}_{i+1} &\leq C_* \mathtt{K}^i \big(\varepsilon_i^{(0)} + \varepsilon_i^{(1)} + \bar\varepsilon_i^2\big) +\, C_*\varepsilon_i^{(2)}\, e^{-K_* 2^i}\,, \quad i = 0, \dots, \nu-1\,, \end{aligned} \tag{5.59}$$

where $\bar\varepsilon_i := \varepsilon_i^{(0)} + \varepsilon_i^{(1)} + \varepsilon_i^{(2)}$, *for some* $\mathtt{K}, C_*, K_* > 1$. *Then there exist* $\bar\varepsilon_\star, C_\star > 0$, $\chi \in (1,2)$ *(depending on* $\mathtt{K}, C_*, K_* > 0$*), such that, if*

$$\bar\varepsilon_0 \leq \bar\varepsilon_\star \quad \Longrightarrow \quad \bar\varepsilon_i \leq C_\star\, \bar\varepsilon_0\, e^{-K_*\chi^i}\,, \quad \forall i = 0, \dots, \nu\,. \tag{5.60}$$

Proof. We first note that $\bar\varepsilon_{j+1} \lessdot \mathtt{K}^j\bar\varepsilon_j + \bar\varepsilon_j e^{-K_* 2^j}$. Then, applying (5.59) three times, we deduce

$$\begin{aligned} \varepsilon^{(0)}_{j+3} &\lessdot \mathtt{K}^{4j+3}\bar\varepsilon_j^2 + \varepsilon_j^{(0)} e^{-K_* 2^j} \\ \varepsilon^{(1)}_{j+3} &\lessdot \mathtt{K}^{4j+3}\bar\varepsilon_j^2 + \varepsilon_j^{(1)} e^{-K_* 2^j} \\ \varepsilon^{(2)}_{j+3} &\lessdot \mathtt{K}^{4j+3}\bar\varepsilon_j^2 + \varepsilon_j^{(2)} e^{-K_* 2^j} \end{aligned}$$

and, therefore $a_j := \bar\varepsilon_{3j}$

$$a_{j+1} \leq C_1 \mathtt{K}^{4j+3} a_j^2 + a_j\, C_1 e^{-K_* 2^j} \tag{5.61}$$

for some $C_1 := C_1(C_*) > 1$.

Claim: There is $\varepsilon_0 > 0$, $C_2 > 1$, such that, if $a_0 \leq \varepsilon_0$, then, for all $j \in \mathbb{N}$,

$$(\mathbf{S})_j \qquad a_j \leq C_2\, a_0 (2C_1)^j\, e^{-K_*\chi^j}\,, \quad \chi := 3/2\,.$$

We proceed by induction. The statement $(\mathbf{S})_0$ follows by the assumption $a_0 \leq \varepsilon_0$, for $C_2 e^{-K_*} > 1$. Now suppose $(\mathbf{S})_j$ holds true. Then $(\mathbf{S})_{j+1}$ follows by (5.61) and

$$\begin{aligned} a_{j+1} &\leq C_1 \mathtt{K}^{4j+3} C_2^2 a_0^2 (2C_1)^{2j} e^{-2K_*\chi^j} + C_2 a_0 (2C_1)^j C_1 e^{-K_*(2^j + \chi^j)} \\ &< C_2\, a_0\, (2C_1)^{j+1} e^{-K_*\chi^{j+1}} \end{aligned}$$

because, choosing $\varepsilon_0 C_2$ small enough, $\forall j \in \mathbb{N}$,

$$C_1 \kappa^{4j+3} C_2^2 a_0^2 (2C_1)^{2j} e^{-2K_*\chi^j} \leq C_1 \kappa^{4j+3} C_2^2 \varepsilon_0 a_0 (2C_1)^{2j} e^{-2K_*\chi^j} < (2C_1)^{j+1}\frac{C_2}{2}\, a_0\, e^{-K_*\chi^{j+1}}$$

and

$$C_1(2C_1)^j C_2 a_0 e^{-K_*(2^j+\chi^j)} < (2C_1)^{j+1}\frac{C_2}{2}\, a_0\, e^{-K_*\chi^{j+1}} .$$

The claim and (5.59) imply that also $\bar{\varepsilon}_{j+1}$, $\bar{\varepsilon}_{j+2}$ satisfy a bound like (5.60). ■

For $\nu \in \mathbb{N}$, we define

- $K_\nu := K_0 4^\nu$,
- $s_{\nu+1} := s_\nu - s_0 2^{-\nu-2} \searrow \frac{s_0}{2}$, $\quad r_{\nu+1} := r_\nu - r_0 2^{-\nu-2} \searrow \frac{r_0}{2}$, $\quad D_\nu := D(s_\nu, r_\nu)$,
- $\mu_{\nu+1} := \mu_\nu - \mu_0 2^{-\nu-2} \searrow \frac{\mu_0}{2}$, $\quad \theta_{\nu+1} := \theta_\nu + \theta_0 2^{-\nu-2} \nearrow 3\frac{\theta_0}{2}$,
- $N_\nu := N_0 2^{\nu\rho}$ with $N_0 := \hat{c}\gamma^{-1/3} K_0^{\tau+1}$, $\quad \rho := \max\Big\{2(\tau+1), \frac{1}{L-b}, \frac{1}{1-L}\Big\}$. (5.62)

Lemma 5.9. (Iterative lemma) *Let $\mathcal{O}_0 \subset \mathbb{R}^n$ and consider $H^0 = \mathcal{N}_0 + P_0 : D_0 \times \mathcal{O}_0 \to \mathbb{C}$ with $\mathcal{N}_0 := e_0 + \omega^{(0)}(\xi)\cdot y + \Omega^{(0)}(\xi)\cdot z\bar{z}$ in normal form such that $\Omega^{(0)}$ satisfies (4.2). Then there is $K_0 > 0$ large enough, $\varepsilon_0 \in (0, 1/2)$ such that, if*

$$\Theta_0 := \gamma^{-1}\|P_0\|^T_{s_0,r_0,N_0,\theta_0,\mu_0} \le \varepsilon_0 \,, \tag{5.63}$$

then

$(\mathbf{S1})_\nu$ *$\forall 0 \le i \le \nu$, there exist $H^i := \mathcal{N}_i + P_i \,:\, D_i \times \mathcal{O}^*_i \to \mathbb{C}$ with $\mathcal{N}_i := e_i + \omega^{(i)}(\xi)\cdot y + \Omega^{(i)}(\xi)\cdot z\bar{z}$ in normal form, $\Omega^{(i)} = (\Omega^{(i)}_j)_{j\in\mathbb{Z}\setminus\mathcal{I}}$ fulfills (5.9) for some $a^{(i)}_\pm$, for all $|j| \ge \theta_i N_i$. Above $\mathcal{O}^*_0 := \mathcal{O}_0$, and, for $i > 0$,*

$$\begin{aligned} \mathcal{O}^*_i \;\; := \;\; & \Big\{\xi \in \mathcal{O}^*_{i-1} \,:\, |\omega^{(i-1)}(\xi)\cdot k + \Omega^{(i-1)}(\xi)\cdot l| \ge \frac{\gamma}{1+|k|^\tau}, \forall (k,l) \in \mathbf{I},\ |k| \le K_{i-1}\,, \\ & |\omega^{(i-1)}(\xi)\cdot k + p| \ge \frac{\gamma^{2/3}}{1+|k|^\tau}\,, \forall (k,p) \ne (0,0)\,,\ |k| \le K_{i-1},\ p \in \mathbb{Z}\Big\}\,. \end{aligned} \tag{5.64}$$

Moreover, $\forall\, 1 \le i \le \nu$, $H^i = H^{i-1}\circ\Phi^i$ where $\Phi^i : D_i \times \mathcal{O}^*_i \to D_{i-1}$ is a (Lipschitz) family (in $\xi \in \mathcal{O}^*_i$) of close-to-the-identity analytic symplectic maps. Define

$$\bar{\varepsilon}_i := \sum_{h=0}^{2} \varepsilon^{(h)}_i\,, \quad \varepsilon^{(h)}_i := \gamma^{-1}\|P^{(h)}_i\|^T_{s_i,r_i,N_i,\theta_i,\mu_i}\,, \quad \Theta_i := \gamma^{-1}\|P_i\|^T_{s_i,r_i,N_i,\theta_i,\mu_i}\,. \tag{5.65}$$

$(\mathbf{S2})_\nu$ *$\forall 0 \le i \le \nu - 1$, the $\varepsilon^{(0)}_i, \varepsilon^{(1)}_i, \varepsilon^{(2)}_i \in (0,1)$ satisfy (5.59) with $\mathtt{K} = 4^{2\bar{\tau}+1}$, $C_* = 4K_0^{2\bar{\tau}}$, $K_* = s_0 K_0/4$.*

$(\mathbf{S3})_\nu$ *$\forall 0 \le i \le \nu$, we have $\bar{\varepsilon}_i \le C_\star \bar{\varepsilon}_0 e^{-K_*\chi^i}$ and $\Theta_i \le 2\Theta_0$.*

$(\mathbf{S4})_\nu$ *$\forall 0 \le i \le \nu$ and $\forall\, \xi \in \mathcal{O}^*_i$, denote (recall (5.34))*

$$\hat{\omega}^{(i)} := \nabla_y \langle P_i(\xi)\rangle|_{y=0,z=\bar{z}=0} \qquad \text{and} \qquad \hat{\Omega}^{(i)}_j(\xi) := \partial^2_{z_j\bar{z}_j\,|\,y=0,z=0}\langle P_i(\xi)\rangle\,.$$

There exist constants $\hat{a}^{(i)}_\pm(\xi) \in \mathbb{R}$ such that

$$|\hat{\omega}^{(i)}(\xi)|\,,\ |\hat{\Omega}^{(i)}(\xi)|_\infty\,,\ |\hat{a}^{(i)}_\pm(\xi)| \le 2\gamma\bar{\varepsilon}_i\,, \quad |\hat{\Omega}^{(i)}_j(\xi) - \hat{a}^{(i)}_{\mathtt{s}(j)}(\xi)| \le 40\gamma\frac{\bar{\varepsilon}_i}{|j|}\,, \quad \forall |j| \ge 6(N_i+1)\,, \tag{5.66}$$

uniformly in $\xi \in \mathcal{O}^*_i$.

Proof. The statement $(\mathbf{S1})_0$ follows by the hypothesis. $(\mathbf{S2})_0$ is empty. $(\mathbf{S3})_0$ is trivial. $(\mathbf{S4})_0$ follows by Lemma 5.2 and (5.44). We then proceed by induction.

$(\mathbf{S1})_{\nu+1}$. We wish to apply the KAM step Proposition 5.2 with $\mathcal{N} = \mathcal{N}_\nu$, $P = P_\nu, N_0 = N_\nu, \theta = \theta_\nu \ldots$ and $N_0^+ = N_{\nu+1}, \theta_+ = \theta_{\nu+1}, \ldots$ Our definitions in (5.62) (and $\tau > 1/b$) imply that the conditions[2] (5.45)-(5.46) are satisfied, for all $\nu \in \mathbb{N}$, taking $K_0 >$ large enough. Moreover, since

$$\delta^+ = \delta_{\nu+1} := \min\Big\{1 - \frac{s_{\nu+1}}{s_\nu}, 1 - \frac{r_{\nu+1}}{r_\nu}\Big\} \quad \text{so that} \quad 2^{-\nu-2} \le \delta_{\nu+1} \le 2^{-\nu-1} , \tag{5.67}$$

and $(S3)_\nu$ the condition (5.47) is satisfied, for $\bar\varepsilon_0 \le \varepsilon_0$ small enough, $\forall \nu \in \mathbb{N}$. Finally, by $(S1)_\nu$, condition (5.9) holds for $|j| \ge \theta_\nu N_\nu$, and (5.12) and (5.17) hold (by definition) for all $\xi \in \mathcal{O}^*_{\nu+1}$. Hence Proposition 5.2 applies. For all $\xi \in \mathcal{O}^*_{\nu+1}$ the Hamiltonian flow $e^{\mathrm{ad}_{F_\nu}} : D_{\nu+1} \times \mathcal{O}^*_{\nu+1} \to D_\nu$ and we define

$$H^{\nu+1} := e^{\mathrm{ad}_{\mathrm{F}_\nu}} H^\nu = \mathcal{N}_{\nu+1} + P_{\nu+1} : D_{\nu+1} \times \mathcal{O}^*_{\nu+1} \to \mathbb{C} ,$$

where (recall (5.34))

$$\omega^{(\nu+1)} = \omega^{(\nu)} + \hat\omega^{(\nu)} , \quad \Omega^{(\nu+1)} = \Omega^{(\nu)} + \hat\Omega^{(\nu)} ,$$

and $\hat\omega^{(\nu)}, \hat\Omega^{(\nu)}$ are defined by $(S4)_\nu$. Let $a_\pm^{(\nu+1)} = a_\pm^{(\nu)} + \hat a_\pm^{(\nu)} = a^{(0)} + \sum_{i \le \nu} \hat a_\pm^{(i)}$. By $(\mathbf{S3})_\nu$- $(\mathbf{S4})_\nu$ we have that, by (4.2), (5.9) holds for $\Omega^{(\nu+1)}$ and for all $|j| > \theta_{\nu+1} N_{\nu+1} > 6(N_\nu + 1)$ for $\bar\varepsilon_0 \le \varepsilon_0$ small enough.

$(\mathbf{S2})_{\nu+1}$ follows by (5.48) and (5.62).

$(\mathbf{S3})_{\nu+1}$. By $(S2)_\nu$ we can apply Lemma 5.8 and (5.60) implies $\bar\varepsilon_{\nu+1} \le C_\star \bar\varepsilon_0 e^{-K_* \chi^{\nu+1}}$. Moreover, for ε_0 small enough,

$$\Theta_{\nu+1} \overset{(5.49)}{\le} \Theta_0 \Pi_{i=0}^\nu \Big(1 + C\delta_{i+1}^{-2} K_i^{2\bar\tau} \bar\varepsilon_i\Big) \overset{(5.67),(S3)_\nu}{\le} 2\Theta_0 .$$

$(\mathbf{S4})_{\nu+1}$ follows by Lemma 5.2 and $(S3)_\nu$. ■

The fundamental estimate (4.15) follows by (5.66) and the following corollary.

Corollary 5.1. *For all $\xi \in \cap_\nu \mathcal{O}^*_\nu$ the*

$$\hat\Omega^\infty := \sum_{\nu \ge 0} \hat\Omega^{(\nu)} , \quad \hat a_\pm^\infty := \sum_{\nu \ge 0} \hat a_\pm^{(\nu)} \quad \text{satisfy} \quad |\hat\Omega^\infty|_\infty , \ |\hat a_\pm^\infty| \lessdot \gamma \bar\varepsilon_0 \tag{5.68}$$

and

$$|\hat\Omega_j^\infty - \hat a^\infty_{\mathtt{s}(j)}| \lessdot \frac{K_0^{\tau+1}}{|j|} \bar\varepsilon_0 \, \gamma^{2/3} , \quad \forall |j| \ge 6(N_0 + 1) = 6(\gamma^{-1/3} \hat c K_0^{\tau+1} + 1) . \tag{5.69}$$

Proof. The bounds in (5.68) follow from (5.66) and $(S3)_\nu$. Let us prove (5.69) when $j > 0$ (the case $j < 0$ is analogous). For all $\forall \nu \ge 0$, $j \ge 6(N_\nu + 1)$, we have

$$\begin{aligned} |\hat\Omega_j^\infty - \hat a_+^\infty| &\le \sum_{n=0}^\nu |\hat\Omega_j^{(n)} - \hat a_+^{(n)}| + \sum_{n > \nu} |\hat\Omega_j^{(n)} - \hat a_+^{(n)}| \\ &\overset{(5.66)}{\le} \frac{40\gamma}{j} \sum_{n=0}^\nu \bar\varepsilon_n + \sum_{n>\nu} |\hat\Omega_j^{(n)}| + |\hat a_+^{(n)}| \overset{(5.66),(S3_\nu)}{\lessdot} \frac{\bar\varepsilon_0 \gamma}{j} + \gamma \sum_{n > \nu} \bar\varepsilon_n . \end{aligned}$$

Therefore, $\forall \nu \ge 0$, $6(N_\nu + 1) \le j < 6(N_{\nu+1} + 1)$,

$$|\hat\Omega_j^\infty - \hat a_+^\infty| \lessdot \frac{\bar\varepsilon_0 \gamma}{j} + \gamma \frac{N_{\nu+1}}{j} \sum_{n>\nu} \bar\varepsilon_n \overset{(5.62)}{\lessdot} \frac{\bar\varepsilon_0\gamma}{j} + \frac{\gamma}{j} \gamma^{-1/3} K_0^{\tau+1} 2^{\rho(\nu+1)} \sum_{n>\nu} \bar\varepsilon_n$$

and (5.69) follows by $(S3)_\nu$. ■

[2] For example the first inequality in (5.45) reads $N_{\nu+1} \ge \max\{N_\nu, \hat c \gamma^{-1/3} K_\nu^{\tau+1}, \bar N\}$.

PROOF OF THEOREM 4.1-II) We apply the iterative Lemma 5.9 to H^0 defined in (5.6). The symplectic transformation Φ in (4.10) is defined by

$$\Phi := \lim_{\nu\to\infty} \Phi_{00} \circ \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi^\nu$$

with Φ_{00} defined in (5.4). The final frequencies ω^∞ and Ω^∞ are

$$\omega^\infty = \lim_{\nu\to\infty} \omega^{(\nu)} = \omega + \sum_{\nu\geq 0} \hat{\omega}^{(\nu)}, \qquad \Omega^\infty = \lim_{\nu\to\infty} \Omega^{(\nu)} = \Omega + \hat{\Omega}^\infty = \Omega + \sum_{\nu\geq 0} \hat{\Omega}^{(\nu)}. \tag{5.70}$$

The KAM iteration procedure that we are using is the same as that of the abstract KAM theorem of [2]. To be more precise in [2] one solves the homological equation (5.13) for all ξ in a larger set where only the Melnikov conditions (5.12) hold (see Proposition 5.1-**I**), but not (5.17). Clearly, the solutions of the homological equation, the new perturbation P^+ and the new frequencies ω^+, Ω^+ in (5.34), coincide with those in [2] on this smaller set of parameters.

The procedure is completed by extending ω^+, Ω^+ to Lipschiz functions in the whole parameter set. By the Kirszbraun theorem (see e.g. [21]) the extended frequencies satisfy the bounds (5.36) for every $\xi \in \mathcal{O}_0$. In the set of ξ where (5.12) and (5.17) hold, the extended frequencies satisfy also (5.36)-(5.39).

Lemma 5.10. $\mathcal{O}^*_\infty \subset \cap_i \mathcal{O}^*_i$ *(see* (4.14) *and* (5.64)*).*

PROOF. If $\xi \in \mathcal{O}^*_\infty$ then, for all $|k| \leq K_i$, $|l| \leq 2$,

$$\begin{aligned} |\omega^{(i)}(\xi)\cdot k + \Omega^{(i)}(\xi)\cdot l| &\geq |\omega^\infty(\xi)\cdot k + \Omega^\infty(\xi)\cdot l| - |\omega^\infty - \omega^{(i)}||k| - 2|\Omega^\infty - \Omega^{(i)}|_\infty \\ &\geq \frac{2\gamma}{1+|k|^\tau} - K_i \sum_{\nu>i} |\hat{\omega}^{(\nu)}| - 2\sum_{\nu>i} |\hat{\Omega}^{(\nu)}|_\infty \geq \frac{\gamma}{1+|k|^\tau} \end{aligned}$$

by the definition of K_i in (5.62), $(S3)_\nu$ and (5.66). The other estimate is analogous. ∎

As a consequence, for $\xi \in \mathcal{O}^*_\infty$, Corollary 5.1 holds. Then (4.15) follows by (5.70), (4.2) and (5.69). This concludes the proof of Theorem 4.1.

6 Measure estimates: proof of Theorem 4.2

We have to estimate the measure of

$$\mathcal{O} \setminus \mathcal{O}^*_\infty = \bigcup_{(k,l)\in\Lambda_0\cup\Lambda_1\cup\Lambda_2^+\cup\Lambda_2^-} \mathcal{R}_{kl}(\gamma) \bigcup_{(k,p)\in\mathbb{Z}^{n+1}\setminus\{0\}} \tilde{\mathcal{R}}_{kp}(\gamma^{2/3}) \bigcap \omega^{-1}(\mathcal{D}_{\gamma^{2/3}}) \tag{6.1}$$

where

$$\mathcal{R}_{kl}(\gamma) := \mathcal{R}^\tau_{kl}(\gamma) := \Big\{\xi \in \mathcal{O} \,:\, |\omega^\infty(\xi)\cdot k + \Omega^\infty(\xi)\cdot l| < \frac{2\gamma}{1+|k|^\tau}\Big\} \tag{6.2}$$

$$\tilde{\mathcal{R}}_{kp}(\gamma^{2/3}) := \tilde{\mathcal{R}}^\tau_{kp}(\gamma^{2/3}) := \Big\{\xi \in \mathcal{O} \,:\, |\omega^\infty(\xi)\cdot k + p| < \frac{2\gamma^{2/3}}{1+|k|^\tau}\Big\}$$

and

$$\Lambda_h := \Big\{(k,l) \in \mathbf{I} \text{ (see (4.5))}\,,\, |l| = h\Big\}, \quad h = 0,1,2\,, \qquad \Lambda_2 = \Lambda_2^+ \cup \Lambda_2^-\,, \tag{6.3}$$

$$\Lambda_2^+ := \Big\{(k,l) \in \Lambda_2\,,\, l = \pm(e_i + e_j)\Big\}, \quad \Lambda_2^- := \Big\{(k,l) \in \Lambda_2\,,\, l = e_i - e_j\Big\}.$$

We first consider the most difficult case Λ_2^-. Setting $\mathcal{R}_{k,i,j}(\gamma) := \mathcal{R}_{k,e_i-e_j}(\gamma)$ we show that

$$\Big|\bigcup_{(k,l)\in\Lambda_2^-} \mathcal{R}_{k,l}(\gamma)\Big| = \Big|\bigcup_{(k,i,j)\in\,\mathtt{I}} \mathcal{R}_{k,i,j}(\gamma)\Big| \lessdot \gamma^{2/3}\rho^{n-1} \tag{6.4}$$

where

$$\mathtt{I} := \Big\{(k,i,j) \in \mathbb{Z}^n \times (\mathbb{Z} \setminus \mathcal{I})^2 \,:\, (k,i,j) \neq (0,i,i)\,,\ \mathtt{j} \cdot k + i - j = 0\Big\}\,. \tag{6.5}$$

Note that the indices in $\mathtt{I}$ satisfy

$$||i| - |j|| \leq \kappa\, |k| \quad \text{and} \quad k \neq 0\,. \tag{6.6}$$

Since the matrix A in (4.16) is invertible, the bound (4.9) implies, for ϵ small enough, that

$$\omega^\infty : \mathcal{O} \to \omega^\infty(\mathcal{O}) \ \text{ is invertible and } \ |(\omega^\infty)^{-1}|^{\rm lip} \leq 2\|A^{-1}\|\,. \tag{6.7}$$

Lemma 6.1. *For* $(k,i,j) \in \mathtt{I}$, $\eta \in (0,1)$, *we have*

$$|\mathcal{R}^\tau_{k,i,j}(\eta)| \lessdot \frac{\eta \rho^{n-1}}{1 + |k|^{\tau+1}}\,. \tag{6.8}$$

Proof. By (4.9) and (4.16)

$$\omega^\infty(\xi) \cdot k + \Omega_i^\infty(\xi) - \Omega_j^\infty(\xi) = \omega^\infty(\xi) \cdot k + \sqrt{i^2 + \mathrm{m}} - \sqrt{j^2 + \mathrm{m}} + r_{k,i,j}(\xi)$$

where

$$|r_{k,i,j}(\xi)| = O(\epsilon\gamma)\,,\ |r_{k,i,j}|^{\rm lip} = O(\epsilon)\,. \tag{6.9}$$

We introduce the final frequencies $\zeta := \omega^\infty(\xi)$ as parameters (see (6.7)), and we consider

$$f_{k,i,j}(\zeta) := \zeta \cdot k + \sqrt{i^2 + \mathrm{m}} - \sqrt{j^2 + \mathrm{m}} + \tilde r_{k,i,j}(\zeta)$$

where also $\tilde r_{k,i,j} := r_{k,i,j} \circ (\omega^\infty)^{-1}$ satisfies (6.9). In the direction $\zeta = sk|k|^{-1} + w$, $w \cdot k = 0$, the function $\tilde f_{k,i,j}(s) := f_{k,i,j}(sk|k|^{-1} + w)$ satisfies

$$\tilde f_{k,i,j}(s_2) - \tilde f_{k,i,j}(s_1) \overset{(6.9)}{\geq} (s_2 - s_1)(|k| - C\varepsilon) \geq (s_2 - s_1)|k|/2\,.$$

Since $|k| \geq 1$ (recall (6.6)), by Fubini theorem,

$$\Big|\Big\{\zeta \in \omega^\infty(\mathcal{O}) \,:\, |f_{k,i,j}(\zeta)| \leq \frac{2\eta}{1 + |k|^\tau}\Big\}\Big| \lessdot \frac{\eta\rho^{n-1}}{1 + |k|^{\tau+1}}\,.$$

By (6.7) the bound (6.8) follows. ■

We split

$$\mathtt{I} = \mathtt{I}_> \cup \mathtt{I}_< \quad \text{where} \quad \mathtt{I}_> := \Big\{(k,i,j) \in \mathtt{I} \,:\, \min\{|i|, |j|\} > C_\sharp \gamma^{-1/3}(1 + |k|^{\tau_0})\Big\} \tag{6.10}$$

where $C_\sharp > C_\star$ in (4.15) for $\tau_0 := n + 1$. We set $\mathtt{I}_< := \mathtt{I} \setminus \mathtt{I}_>$.

Lemma 6.2. *For all* $(k,i,j) \in \mathtt{I}_>$ *we have*

$$\mathcal{R}^{\tau_0}_{k,i,j}(\gamma^{2/3}) \subset \mathcal{R}^{\tau_0}_{k,i_0,j_0}(2\gamma^{2/3}) \tag{6.11}$$

(see (6.2)*),* $i_0, j_0 \in \mathbb{Z} \setminus \mathcal{I}$ *satisfy*

$$\mathtt{s}(i_0) = \mathtt{s}(i)\,,\ \mathtt{s}(j_0) = \mathtt{s}(j)\,,\ |i_0| - |j_0| = |i| - |j| \tag{6.12}$$

and

$$\min\{|j_0|, |i_0|\} = \Big[C_\sharp \gamma^{-1/3}(1 + |k|^{\tau_0})\Big]\,. \tag{6.13}$$

Proof. Since $|j| \geq \gamma^{-1/3} C_\star$, by (4.15) and (4.16) we have the frequency asymptotic

$$\Omega_j^\infty(\xi) = |j| + \frac{\mathtt{m}}{2|j|} + \vec{a} \cdot \xi + a^\infty_{\mathtt{s}(j)}(\xi) + O\Big(\frac{\mathtt{m}^2}{|j|^3}\Big) + O\Big(\varepsilon \frac{\gamma^{2/3}}{|j|}\Big) . \tag{6.14}$$

By (6.6) we have $||i|-|j|| = ||i_0|-|j_0|| \leq C|k|$, $|k| \geq 1$. If $\xi \in \mathcal{O} \setminus \mathcal{R}^{\tau_0}_{k,i_0,j_0}(2\gamma^{2/3})$, since $|i|, |j| \geq \mu_0 := \min\{|i_0|, |j_0|\}$ (recall (6.10) and (6.13)), we have

$$\begin{aligned}
|\omega^\infty(\xi) \cdot k + \Omega_i^\infty(\xi) - \Omega_j^\infty(\xi)| \quad &\geq \quad |\omega^\infty(\xi) \cdot k + \Omega_{i_0}^\infty(\xi) - \Omega_{j_0}^\infty(\xi)| \\
&\quad -|\Omega_i^\infty(\xi) - \Omega_{i_0}^\infty(\xi) - \Omega_j^\infty(\xi) + \Omega_{j_0}^\infty(\xi)| \\
&\stackrel{(6.14)}{\geq} \frac{4\gamma^{2/3}}{1+|k|^{\tau_0}} - ||i| - |i_0| - |j| + |j_0|| \\
&\quad -|a^\infty_{\mathtt{s}(i)} - a^\infty_{\mathtt{s}(i_0)} - a^\infty_{\mathtt{s}(j)} + a^\infty_{\mathtt{s}(j_0)}| \\
&\quad -C\varepsilon \frac{\gamma^{2/3}}{\mu_0} - C\frac{\mathtt{m}^2}{\mu_0^3} - \frac{\mathtt{m}}{2}\frac{||i|-|j||}{|i|\,|j|} - \frac{\mathtt{m}}{2}\frac{||i_0|-|j_0||}{|i_0|\,|j_0|} \\
&\stackrel{(6.12)}{\geq} \frac{4\gamma^{2/3}}{1+|k|^{\tau_0}} - C\varepsilon\frac{\gamma^{2/3}}{\mu_0} - C\frac{|k|}{\mu_0^2} \stackrel{(6.13)}{\geq} \frac{2\gamma^{2/3}}{1+|k|^{\tau_0}}
\end{aligned}$$

taking $C_\sharp$ in (6.13) large enough. Therefore $\xi \in \mathcal{O} \setminus \mathcal{R}^{\tau_0}_{k,i,j}(\gamma^{2/3})$ proving (6.11). ■

As a corollary we deduce:

Lemma 6.3. $\Big| \bigcup_{(k,i,j) \in \mathtt{I}_>} \mathcal{R}^\tau_{k,i,j}(\gamma) \Big| \lessdot \gamma^{2/3} \rho^{n-1}$.

Proof. Since $0 < \gamma \leq 1$ and $\tau \geq \tau_0$ (see (4.18)), we have (see (6.2)) $\mathcal{R}^\tau_{k,i,j}(\gamma) \subset \mathcal{R}^{\tau_0}_{k,i,j}(\gamma^{2/3})$. Then Lemma 6.2 and (6.8) imply that, for each $p \in \mathbb{Z}$,

$$\Big| \bigcup_{(k,i,j) \in \mathtt{I}_>,\, |i|-|j|=p} \mathcal{R}^\tau_{k,i,j}(\gamma) \Big| \lessdot \frac{\gamma^{2/3}\rho^{n-1}}{1+|k|^{\tau_0+1}} .$$

Therefore

$$\Big| \bigcup_{(k,i,j) \in \mathtt{I}_>} \mathcal{R}^\tau_{k,i,j}(\gamma) \Big| \lessdot \sum_{k, |p| \leq C|k|} \frac{\gamma^{2/3}\rho^{n-1}}{1+|k|^{\tau_0+1}} \lessdot \sum_k \frac{\gamma^{2/3}\rho^{n-1}}{1+|k|^{\tau_0}}$$

proving the lemma. ■

Lemma 6.4. $\Big| \bigcup_{(k,i,j) \in \mathtt{I}_<} \mathcal{R}^\tau_{k,i,j}(\gamma) \Big| \lessdot \gamma^{2/3} \rho^{n-1}$.

Proof. For all $(k,i,j) \in \mathtt{I}_<$ such that $\mathcal{R}^\tau_{k,i,j}(\gamma) \neq \emptyset$ we have (see (6.6))

$$\min\{|i|,|j|\} < C\gamma^{-1/3}(1+|k|^{\tau_0}) , \ ||i|-|j|| \leq C|k| \implies \max\{|i|,|j|\} < C'\gamma^{-1/3}(1+|k|^{\tau_0}) .$$

Therefore, using also Lemma 6.1 and (6.6)

$$\begin{aligned}
\Big| \bigcup_{(k,i,j) \in \mathtt{I}_<} \mathcal{R}^\tau_{k,i,j}(\gamma) \Big| \ &\lessdot \ \sum_k \sum_{|i| \leq C'\gamma^{-1/3}(1+|k|^{\tau_0})} \ \sum_{||i|-|j|| \leq C|k|} \frac{\gamma\rho^{n-1}}{1+|k|^{\tau+1}} \\
&\lessdot \ \sum_k \frac{\gamma^{2/3}\rho^{n-1}}{1+|k|^{\tau-\tau_0}}
\end{aligned}$$

which, by (4.18), gives the lemma. ■

Lemmata 6.3, 6.4 imply (6.4). This concludes the case $(k,l) \in \Lambda_2^-$. Let consider the other cases. The analogue of Lemma 6.1 is

Lemma 6.5. *For* $(k,l) \in \Lambda_0 \cup \Lambda_1 \cup \Lambda_2^+$, $\eta \in (0,1)$, *we have*

$$|\mathcal{R}_{kl}(\eta)| \lessdot \frac{\eta\rho^{n-1}}{1+|k|^\tau}\,. \tag{6.15}$$

Proof. We consider only the case $(k,l) \in \Lambda_2^+$, $l = e_i + e_j$, the other ones being analogous. By (4.9) and (4.16)

$$f_{k,l}(\xi) := \omega^\infty(\xi)\cdot k + \Omega_i^\infty(\xi) + \Omega_j^\infty(\xi) = \bar\omega\cdot k + \sqrt{i^2+\mathrm{m}} + \sqrt{j^2+\mathrm{m}} + (A^T k + 2\vec a)\cdot\xi + r_{k,l}(\xi) \tag{6.16}$$

where

$$|r_{k,l}(\xi)| = O(\epsilon\gamma)\,,\ |r_{k,l}|^{\mathrm{lip}} = O(\epsilon)\,. \tag{6.17}$$

If $k = \vec a = 0$ then the function in (6.16) is bigger than $\sqrt{\mathrm{m}}$ and $\mathcal{R}_{0l}(\eta) = \emptyset$. Otherwise, by (4.17), the vector

$$\tilde a := A^T k + 2\vec a = A^T\big(k + 2(A^{-1})^T\vec a\big) \quad \text{satisfies} \quad |\tilde a| \geq c = c(A,\vec a) > 0\,,\ \forall k \neq 0\,. \tag{6.18}$$

The function $\tilde f_{k,l}(s) := f_{k,l}(s\tilde a|\tilde a|^{-1} + w)$, $\tilde a\cdot w = 0$, satisfies

$$\tilde f_{k,l}(s_2) - \tilde f_{k,l}(s_1) \overset{(6.9)}{\geq} (s_2 - s_1)(|\tilde a| - C\varepsilon) \geq (s_2 - s_1)|\tilde a|/2\,.$$

Then (6.15) follows by (6.18) and Fubini theorem. ■

By Lemma 6.5 and standard arguments (as above)

$$\Big|\bigcup_{(k,l)\in\Lambda_0\cup\Lambda_1\cup\Lambda_2^+} \mathcal{R}_{kl}(\gamma)\Big| \lessdot \gamma\rho^{n-1}\,, \qquad \Big|\bigcup_{(k,p)\in\mathbb{Z}^{n+1}\setminus\{0\}} \tilde{\mathcal{R}}_{kp}(\gamma^{2/3})\Big| \lessdot \gamma^{2/3}\rho^{n-1} \tag{6.19}$$

and $|\omega^{-1}(\mathcal{D}_{\gamma^{2/3}})| \lessdot \gamma^{2/3}$. Finally (6.1), (6.4), (6.19) imply (4.19).

7 Application to DNLW

For $\vec\jmath = (j_1,\ldots,j_d) \in \mathbb{Z}^d$, $\vec\sigma = (\sigma_1,\ldots,\sigma_d) \in \{\pm\}^d$ we denote $\vec\sigma\cdot\vec\jmath := \sigma_1 j_1 + \ldots + \sigma_d j_d$, and, given $(u_j,\bar u_j)_{j\in\mathbb{Z}} = (u_j^+,u_j^-)_{j\in\mathbb{Z}}$, we define the monomial $u_{\vec\jmath}^{\vec\sigma} := u_{j_1}^{\sigma_1}\cdots u_{j_d}^{\sigma_d}$ (of degree d).

7.1 The partial Birkhoff normal form

We now consider the Hamiltonian (1.4) when $F(s) = s^4/4$ since terms of order five or more will not make any difference, see remark 7.1.

After a rescaling of the variables (and of the Hamiltonian) it becomes

$$\begin{aligned} H &= \sum_{j\in\mathbb{Z}} \lambda_j u_j^+ u_j^- + \sum_{\vec\jmath\in\mathbb{Z}^4,\vec\sigma\in\{\pm\}^4,\vec\sigma\cdot\vec\jmath=0} u_{\vec\jmath}^{\vec\sigma} =: N + G \qquad (7.1)\\ &= \sum_{j\in\mathbb{Z}} \lambda_j u_j \bar u_j + \sum_{|\alpha|+|\beta|=4,\,\pi(\alpha,\beta)=0} G_{\alpha,\beta} u^\alpha \bar u^\beta\,, \quad G_{\alpha,\beta} := \frac{(|\alpha|+|\beta|)!}{\alpha!\beta!} = \frac{4!}{\alpha!\beta!}\,, \end{aligned}$$

where $(u^+,u^-) = (u,\bar u) \in \ell^{a,p}\times\ell^{a,p}$ for some $a>0$, $p>1/2$, and the momentum is (see (2.85))

$$\pi(\alpha,\beta) = \sum_{j\in\mathbb{Z}} j(\alpha_j - \beta_j)\,.$$

Note that $0 \leq G_{\alpha,\beta} \leq 4!$ (recall $\alpha! = \Pi_{i\in\mathbb{Z}}\alpha_i!$)

Lemma 7.1. *For all $R > 0$, N_0 satisfying (3.1), the Hamiltonian G defined in (7.1) belongs to $\mathcal{Q}_R^T(N_0, 3/2, 4)$ and*

$$\|G\|_{R,N_0,3/2,4}^T = \|X_G\|_R \lessdot R^2 . \tag{7.2}$$

Proof. The Hamiltonian vector field $X_G := (-\mathrm{i}\partial_{\bar u} G, \mathrm{i}\partial_u G)$ has components

$$\mathrm{i}\sigma\partial_{u_l^\sigma} G = \mathrm{i}\sigma \sum_{|\alpha|+|\beta|=3, \pi(\alpha,\beta)=-\sigma l} G_{\alpha,\beta}^{l,\sigma} u^\alpha \bar u^\beta , \quad \sigma = \pm , \ l \in \mathbb{Z} ,$$

where

$$G_{\alpha,\beta}^{l,+} = (\alpha_l + 1) G_{\alpha+e_l,\beta} , \qquad G_{\alpha,\beta}^{l,-} = (\beta_l + 1) G_{\alpha,\beta+e_l} .$$

Note that $0 \le G_{\alpha,\beta}^{l,\sigma} \le 5!$ By Definitions 2.6, 2.8 and (2.2)

$$\|X_G\|_R = \frac{1}{R} \sup_{\|u\|_{a,p},\|\bar u\|_{a,p}<R} \Big(\sum_{l\in\mathbb{Z},\sigma=\pm} e^{2a|l|} \langle l\rangle^{2p} \Big(\sum_{|\alpha|+|\beta|=3\,,\pi(\alpha,\beta)=-\sigma l} G_{\alpha,\beta}^{l,\sigma} |u^\alpha||\bar u^\beta| \Big)^2 \Big)^{1/2} .$$

For each component

$$\begin{aligned} \sum_{|\alpha|+|\beta|=3\,,\pi(\alpha,\beta)=-\sigma l} G_{\alpha,\beta}^{l,\sigma} |u^\alpha||\bar u^\beta| \quad &\lessdot \sum_{\sigma_1 j_1+\sigma_2 j_2+\sigma_3 j_3=-\sigma l} |u_{j_1}^{\sigma_1}||u_{j_2}^{\sigma_2}||u_{j_3}^{\sigma_3}| \\ &\lessdot \big(\tilde u * \tilde u * \tilde u\big)_{-\sigma l} \end{aligned}$$

where $\tilde u := (\tilde u_l)_{l\in\mathbb{Z}}$, $\tilde u_j := |u_j| + |\bar u_j|$, and $*$ denotes the convolution of sequences. Note that $\|\tilde u\|_{a,p} \le \|u\|_{a,p} + \|\bar u\|_{a,p}$. Since $\ell^{a,p}$ is an Hilbert algebra, $\|\tilde u * \tilde u * \tilde u\|_{a,p} \lessdot \|\tilde u\|_{a,p}^3$, and

$$\begin{aligned} \|X_G\|_R \quad &\lessdot \quad R^{-1} \sup_{\|u\|_{a,p},\|\bar u\|_{a,p}<R} \Big(\sum_{l\in\mathbb{Z},\sigma=\pm} e^{2a|l|} \langle l\rangle^{2p} \big|(\tilde u * \tilde u * \tilde u)_{-\sigma l}\big|^2 \Big)^{1/2} \\ &\lessdot \quad R^{-1} \sup_{\|u\|_{a,p},\|\bar u\|_{a,p}<R} \|\tilde u * \tilde u * \tilde u\|_{a,p} \lessdot R^{-1} \sup_{\|u\|_{a,p},\|\bar u\|_{a,p}<R} \|\tilde u\|_{a,p}^3 \lessdot R^2 . \end{aligned} \tag{7.3}$$

Moreover $G \in \mathcal{H}_R^{\rm null}$, namely G Poisson commutes with the momentum $\mathcal{M} := \sum_{j\in\mathbb{Z}} j u_j \bar u_j$, because (see (2.80))

$$\{\mathcal{M}, u_{\vec j}^{\vec\sigma}\} = -\mathrm{i}\vec\sigma \cdot \vec j u_{\vec j}^{\vec\sigma} . \tag{7.4}$$

We now prove that, for all $N \ge N_0$, the projection $\Pi_{N,3/2,4} G \in \mathcal{T}_R(N, 3/2, 4)$. Hence (7.2) follows by (7.3) (see Definition 3.4). By Definition 3.2 (with $g \rightsquigarrow G$, no (x, y)-variables and $z = u$, $\bar z = \bar u$), in particular (3.12), (3.13), we get

$$\begin{aligned} \Pi_{N,3/2,4} G \quad &= \sum_{|m|,|n|>3N/2,\sigma,\sigma'=\pm} G_{m,n}^{\sigma,\sigma'}(w_L) u_m^\sigma \bar u_n^{\sigma'} \qquad \text{with} \\ G_{m,n}^{\sigma,\sigma'}(w_L) \quad &= \sum_{\substack{\sum_{j\in\mathbb{Z}} |j|(\alpha_j+\beta_j)<4N^L , \\ \pi(\alpha,\beta)=-\sigma m-\sigma' n}} G_{\alpha,\beta,m,n}^{\sigma,\sigma'} u^\alpha \bar u^\beta \qquad \text{and} \\ G_{\alpha,\beta,m,n}^{+,+} \quad &= \frac{1}{2-\delta_{mn}} G_{\alpha+e_m+e_n,\beta} = \frac{1}{2-\delta_{mn}} \frac{4!}{(1+\delta_{mn})!} = 12 = G_{\alpha,\beta,m,n}^{-,-} \\ G_{\alpha,\beta,m,n}^{+,-} \quad &= G_{\alpha+e_m,\beta+e_n} = 24 = G_{\alpha,\beta,m,n}^{-,+} . \end{aligned}$$

These coefficients trivially satisfy (3.15) (with $f \rightsquigarrow G$), so $\Pi_{N,3/2,4} G \in \mathcal{T}_R(N, 3/2, 4)$. ■

We now perform a Birkhoff semi–normal form on the *tangential sites*

$$\mathcal{I} := \{\mathtt{j}_1, \ldots, \mathtt{j}_n\} \subset \mathbb{Z} , \quad \mathtt{j}_1 < \cdots < \mathtt{j}_n , \tag{7.5}$$

recall (2.83). Let $\mathcal{I}^c := \mathbb{Z} \setminus \mathcal{I}$.

Set

$$\overline{G} := \frac{1}{2} \sum_{i \text{ or } j \in \mathcal{I}} \overline{G}_{ij} u_i^+ u_i^- u_j^+ u_j^- \,, \quad \overline{G}_{ij} := 12(2 - \delta_{ij}) \,, \quad \hat{G} := \sum_{\substack{\vec{\jmath} \in \mathbb{Z}^4,\, \vec{\sigma} \in \{+,-\}^4, \\ \vec{\sigma} \cdot \vec{\jmath} = 0,\, \vec{\jmath} \in (\mathcal{I}^c)^4}} u_{\vec{\jmath}}^{\vec{\sigma}} \,. \tag{7.6}$$

By (7.2) and noting that $\overline{G}$, $\hat{G}$ are projections of G, for $R > 0$, N_0 satisfying (3.1), we have

$$\|\overline{G}\|^T_{R,N_0,3/2,4} \,, \; \|\hat{G}\|^T_{R,N_0,3/2,4} \lessdot R^2 \,. \tag{7.7}$$

Proposition 7.1. (Birkhoff normal form) *For any $\mathcal{I} \subset \mathbb{Z}$ and $\mathrm{m} > 0$, there exists $R_0 > 0$ and a real analytic, symplectic change of variables*

$$\Gamma : B_{R/2} \times B_{R/2} \subset \ell^{a,p} \times \ell^{a,p} \;\to\; B_R \times B_R \subset \ell^{a,p} \times \ell^{a,p} \,, \quad 0 < R < R_0 \,,$$

that takes the Hamiltonian $H = N + G$ in (7.1) *into*

$$H_{\text{Birkhoff}} := H \circ \Gamma = N + \overline{G} + \hat{G} + K \tag{7.8}$$

where $\overline{G}, \hat{G}$ are defined in (7.6) *and*

$$K := \sum_{\substack{\vec{\jmath} \in \mathbb{Z}^{2d},\, \vec{\sigma} \in \{+,-\}^{2d}, \\ d \geq 3,\, \vec{\sigma} \cdot \vec{\jmath} = 0}} K_{\vec{\jmath},\vec{\sigma}} u_{\vec{\jmath}}^{\vec{\sigma}} \tag{7.9}$$

satisfies, for $N_0' := N_0'(\mathrm{m}, \mathcal{I}, L, b)$ large enough,

$$\|K\|^T_{R/2,N_0',2,3} \lessdot R^4 \,. \tag{7.10}$$

The rest of this subsection is devoted to the proof of Proposition 7.1. We start following the strategy of [25]. By (2.80) the Poisson bracket

$$\{N, u_{\vec{\jmath}}^{\vec{\sigma}}\} = -\mathrm{i} \vec{\sigma} \cdot \lambda_{\vec{\jmath}} u_{\vec{\jmath}}^{\vec{\sigma}} \tag{7.11}$$

where $\lambda_{\vec{\jmath}} := (\lambda_{j_1}, \ldots, \lambda_{j_d})$ and $\lambda_j := \lambda_j(\mathrm{m}) := \sqrt{j^2 + \mathrm{m}}$.

The following lemma extends Lemma 4 of [25].

Lemma 7.2. (Small divisors) *Let $\vec{\jmath} \in \mathbb{Z}^4$, $\vec{\sigma} \in \{\pm\}^4$ be such that $\vec{\sigma} \cdot \vec{\jmath} = 0$ and (up to permutation of the indexes)*

$$\vec{\jmath} = 0 \,, \sum_{i=1}^4 \sigma_i \neq 0 \,, \tag{7.12}$$

$$\text{or} \quad \vec{\jmath} = (0, 0, q, q) \,, \; q \neq 0 \,, \sigma_1 = \sigma_2 \,, \tag{7.13}$$

$$\text{or} \quad \vec{\jmath} = (p, p, -p, -p) \,, \; p \neq 0 \,, \sigma_1 = \sigma_2 \,, \tag{7.14}$$

$$\text{or} \quad \vec{\jmath} \neq (p, p, q, q) \,. \tag{7.15}$$

Then, there exists an absolute constant $c_* > 0$, such that, for every $\mathrm{m} \in (0, \infty)$,

$$|\vec{\sigma} \cdot \lambda_{\vec{\jmath}}(\mathrm{m})| \geq \frac{c_* \mathrm{m}}{(n_0^2 + \mathrm{m})^{3/2}} > 0 \quad \textit{where} \quad n_0 := \min\{\langle j_1 \rangle, \langle j_2 \rangle, \langle j_3 \rangle, \langle j_4 \rangle\} \,. \tag{7.16}$$

PROOF. In the Appendix. ■

The map $\Gamma := e^{\mathrm{ad}_F}$ is obtained as the time-1 flow generated by the Hamiltonian

$$F := - \sum_{\substack{\vec{\jmath}\cdot\vec{\sigma}=0\,,\vec{\sigma}\cdot\lambda_{\vec{\jmath}}\neq 0 \\ \text{and } \vec{\jmath}\notin(\mathcal{I}^c)^4}} \frac{\mathrm{i}}{\vec{\sigma}\cdot\lambda_{\vec{\jmath}}} u_{\vec{\jmath}}^{\vec{\sigma}} \tag{7.17}$$

We notice that the condition $\vec{\jmath}\cdot\vec{\sigma}=0\,,\vec{\sigma}\cdot\lambda_{\vec{\jmath}}\neq 0$ is equivalent to requiring that $\vec{\jmath}\cdot\vec{\sigma}=0$ and $\vec{\jmath},\vec{\sigma}$ satisfy (7.12)-(7.15). By Lemma 7.2 there is a constant $\bar{c}>0$ (depending only on m and $\mathcal{I}$) such that

$$\vec{\jmath}\cdot\vec{\sigma}=0\,,\vec{\sigma}\cdot\lambda_{\vec{\jmath}}\neq 0 \text{ and } \vec{\jmath}\notin(\mathcal{I}^c)^4 \quad\Longrightarrow\quad |\vec{\sigma}\cdot\lambda_{\vec{\jmath}}|\geq\bar{c}>0\,. \tag{7.18}$$

We have proved that the moduli of the small divisors in (7.17) are uniformly bounded away from zero. Hence F is well defined and, arguing as in Lemma 7.1, we get

$$\|X_F\|_R \lessdot R^2\,. \tag{7.19}$$

Moreover $F\in\mathcal{H}_R^{\mathrm{null}}$ because in (7.17) the sum is restricted to $\vec{\sigma}\cdot\vec{\jmath}=0$ (see also (7.4)).

Lemma 7.3. *F in (7.17) solves the homological equation*

$$\{N,F\}+G=\mathrm{ad}_F(N)+G=\overline{G}+\hat{G} \tag{7.20}$$

where $\overline{G}$, $\hat{G}$ are defined in (7.6).

PROOF. We claim that the only $\vec{\jmath}\in\mathbb{Z}^4$, $\vec{\sigma}\in\{\pm\}^4$ with $\vec{\jmath}\cdot\vec{\sigma}=0$ which do not satisfy (7.12)-(7.15) have the form

$$j_1=j_2\,,\ j_3=j_4\,,\ \sigma_1=-\sigma_2\,,\ \sigma_3=-\sigma_4 \text{ (or permutations of the indexes)}\,. \tag{7.21}$$

Indeed:

If $\vec{\jmath}=0$, $\sum_i\sigma_i=0$: the σ_i are pairwise equal and (7.21) holds.

If $\vec{\jmath}=(0,0,q,q)$, $q\neq 0$, and $\sigma_1=-\sigma_2$: by $\vec{\jmath}\cdot\vec{\sigma}=0$ we have also $\sigma_3=-\sigma_4$ and (7.21) holds.

If $\vec{\jmath}=(p,p,-p,-p)$, $p\neq 0$ and $\sigma_1=-\sigma_2$: by $\vec{\jmath}\cdot\vec{\sigma}=0$ we have also $\sigma_3=-\sigma_4$ and (7.21) holds.

If $j_1=j_2$, $j_3=j_4$, $j_1,j_3\neq 0$, $j_1\neq -j_3$:

CASE 1: $j_1\neq j_3$. Then $0=\vec{\sigma}\cdot\vec{\jmath}=(\sigma_1+\sigma_2)j_1+(\sigma_3+\sigma_4)j_3$ implies $\sigma_1=-\sigma_2$, $\sigma_3=-\sigma_4$.

CASE 2: $j_1=j_3$ and so $j_1=j_2=j_3=j_4\neq 0$. Hence $0=(\sigma_1+\sigma_2+\sigma_3+\sigma_4)j_1$ and (7.21) follows.

By (7.17) and (7.11) all the monomials in $\{N,F\}$ cancel the monomials of G in (7.1) except for those in $\hat{G}$ (see (7.6)) and those of the form $|u_p|^2|u_q|^2$, p or $q\in\mathcal{I}$, which contribute to $\overline{G}$. The expression in (7.6) of $\overline{G}$ follows by counting the multiplicities. ■

The Hamiltonian $F\in\mathcal{H}_R^{\mathrm{null}}$ in (7.17) is quasi-Töplitz:

Lemma 7.4. *Let $R>0$. If $N_0:=N_0(\mathrm{m},\mathcal{I},L,b)$ is large enough, then F defined in (7.17) belongs to $\mathcal{Q}_R^T(N_0,3/2,4)$ and*

$$\|F\|_{R,N_0,3/2,4}^T \lessdot R^2\,. \tag{7.22}$$

PROOF. We have to show that $F\in\mathcal{H}_R^{\mathrm{null}}$ verifies Definition 3.4. For all $N\geq N_0$, we compute, by (7.17) and Definition 3.2 (in particular (3.12)), the projection

$$\Pi_{N,3/2,4}F = \sum_{\substack{|n|,|m|>CN/4\,, \\ \sigma,\sigma'=\pm\,,|\sigma m+\sigma' n|<4N^L}} F_{m,n}^{\sigma,\sigma'}(w^L)u_m^\sigma u_n^{\sigma'} \tag{7.23}$$

where

$$F^{\sigma,\sigma'}_{m,n}(w^L) := -12\mathrm{i} \sum_{\substack{|i|+|j|<4N^L,\ i \text{ or } j\in\mathcal{I}\,,\\ \sigma_i i+\sigma_j j+\sigma m+\sigma' n=0,\ i\neq j \text{ if } m=n}} \frac{u_i^{\sigma_i} u_j^{\sigma_j}}{\sigma_i\lambda_i+\sigma_j\lambda_j+\sigma\lambda_m+\sigma'\lambda_n} \tag{7.24}$$

$$= \sum_{\substack{\sum_j |j|(\alpha_j+\beta_j)<4N^L,\ \sum_{j\in\mathcal{I}}(\alpha_j+\beta_j)>0\,,\\ \sigma m+\sigma' n=-\pi(\alpha,\beta),\ |\alpha|+|\beta|=2,\ \alpha\neq\beta \text{ if } m=n}} F^{\sigma,\sigma'}_{\alpha,\beta,m,n} u^\alpha \bar u^\beta \tag{7.25}$$

and

$$F^{\sigma,\sigma'}_{\alpha,\beta,m,n} := -\frac{24\mathrm{i}}{\alpha!\beta!}\frac{1}{\lambda_{\alpha,\beta}+\sigma\lambda_m+\sigma'\lambda_n}\,, \qquad \lambda_{\alpha,\beta} := \sum_h \lambda_h(\alpha_h-\beta_h)\,. \tag{7.26}$$

Notice that in (7.24) the restriction $i\neq j$ if $m=n$ is equivalent to requiring

$$\{(i,j,m,n),(\sigma_i,\sigma_j,\sigma,\sigma')\} \neq \{(i,i,m,m),(\sigma_i,-\sigma_i,\sigma,-\sigma)\}\,,$$

see Formula (7.17) and (7.21). Indeed if $m=n$, $|i|+|j|<4N^L$ and $|m|>CN/4$ then, by momentum conservation, we have a contribution to (7.24) only if $\sigma=-\sigma'$ and hence $|i|=|j|$.

We define the Töplitz approximation

$$\tilde F := \sum \tilde F^{\sigma,\sigma'}_{m,n}(w^L) u^\sigma_m u^{\sigma'}_n \quad \text{with} \quad \tilde F^{\sigma,\sigma'}_{m,n}(w^L) := \sum \tilde F^{\sigma,\sigma'}_{\alpha,\beta,m,n} u^\alpha \bar u^\beta \tag{7.27}$$

where the indexes in the two sums have the same restrictions as in (7.23), (7.25), respectively, and the coefficients are

$$\tilde F^{\sigma,-\sigma}_{\alpha,\beta,m,n} := -\frac{24\mathrm{i}}{\alpha!\beta!}\frac{1}{\lambda_{\alpha,\beta}+\sigma|m|-\sigma|n|}\,, \qquad \tilde F^{\sigma,\sigma}_{\alpha,\beta,m,n} := 0\,. \tag{7.28}$$

The coefficients in (7.28) are well defined for $N\geq N_0$ large enough, because

$$\begin{aligned} |\lambda_{\alpha,\beta}+\sigma|m|-\sigma|n|| &\geq |\lambda_{\alpha,\beta}+\sigma\lambda_m-\sigma\lambda_n| - |\lambda_m-|m|| - |\lambda_n-|n|| \\ &\overset{(7.18),(7.30)}{\geq} \bar c - \frac{\mathrm{m}}{2}\left(\frac{1}{|m|}+\frac{1}{|n|}\right) \geq \bar c - \frac{2}{3}\frac{\mathrm{m}}{N_0} \geq \frac{\bar c}{2}\,, \end{aligned} \tag{7.29}$$

($\bar c$ defined in (7.18)) having used the elementary inequality

$$|\sqrt{n^2+\mathrm{m}} - |n|| \leq \frac{1}{2|n|}\,. \tag{7.30}$$

Then (7.27), (7.28), (7.29) imply, arguing as in the proof of Lemma 7.1, that

$$\|X_{\tilde F}\|_R \lessdot R^2\,. \tag{7.31}$$

For proving that $\tilde F\in\mathcal{T}_R(N_0,3/2,4)$ we have to show (3.15) (with $f\rightsquigarrow\tilde F$), namely

$$\tilde F^{\sigma,\sigma'}_{\alpha,\beta,m,n} = \tilde F^{\sigma,\sigma'}_{\alpha,\beta}(\mathtt{s}(m),\sigma m+\sigma' n) \tag{7.32}$$

with

$$\tilde F^{\sigma,-\sigma}_{\alpha,\beta}(s,h) := -\frac{24\mathrm{i}}{\alpha!\beta!}\frac{1}{\lambda_{\alpha,\beta}+sh}\,, \qquad \tilde F^{\sigma,\sigma}_{\alpha,\beta}(s,h)=0\,, \quad s=\pm\,,\ h\in\mathbb{Z}\,.$$

Recalling (7.28), this is obvious when $\sigma'=\sigma$. When $\sigma'=-\sigma$ we first note that $\mathtt{s}(m)=\mathtt{s}(n)$. Indeed the restriction on the first sum in (7.27) is (recall (7.23)) $|m|,|n|>3N/2$, $|\sigma m-\sigma n|<4N^L$, which implies $\mathtt{s}(m)=\mathtt{s}(n)$ by (3.1). Then

$$\sigma|m|-\sigma|n| = \sigma\mathtt{s}(m)m - \sigma\mathtt{s}(n)n = \mathtt{s}(m)(\sigma m-\sigma n)$$

and (7.32) follows. We have proved that $\tilde{F} \in \mathcal{T}_R(N_0, 3/2, 4)$.

The Töplitz defect, defined by (3.29), is

$$\hat{F} := \sum \hat{F}^{\sigma,\sigma'}_{m,n}(w^L) u^\sigma_m u^{\sigma'}_n \quad \text{with} \quad \hat{F}^{\sigma,\sigma'}_{m,n}(w^L) := \sum \hat{F}^{\sigma,\sigma'}_{\alpha,\beta,m,n} u^\alpha \bar{u}^\beta \tag{7.33}$$

where the indexes in the two sums have the same restrictions as in (7.23)-(7.25), and

$$\hat{F}^{\sigma,\sigma}_{\alpha,\beta,m,n} = -\frac{24\mathrm{i}}{\alpha!\beta!} \frac{N}{\lambda_{\alpha,\beta} + \sigma\lambda_m + \sigma\lambda_n} \tag{7.34}$$

$$\begin{aligned} \hat{F}^{\sigma,-\sigma}_{\alpha,\beta,m,n} &= -N\frac{24\mathrm{i}}{\alpha!\beta!}\left(\frac{1}{\lambda_{\alpha,\beta} + \sigma\lambda_m - \sigma\lambda_n} - \frac{1}{\lambda_{\alpha,\beta} + \sigma|m| - \sigma|n|}\right) \\ &= \frac{24\mathrm{i}}{\alpha!\beta!} \frac{N\sigma(\lambda_m - |m| - \lambda_n + |n|)}{(\lambda_{\alpha,\beta} + \sigma\lambda_m - \sigma\lambda_n)(\lambda_{\alpha,\beta} + \sigma|m| - \sigma|n|)} \end{aligned} \tag{7.35}$$

We now proof that the coefficients in (7.34)-(7.35) are bounded by a constant independent of N.

The coefficients in (7.34) are bounded because

$$|\lambda_{\alpha,\beta}| \leq \sum_h \lambda_h(|\alpha_h| + |\beta_h|) \leq \sum_h |h|(|\alpha_h| + |\beta_h|) + \sqrt{\mathrm{m}}\sum_h(|\alpha_h| + |\beta_h|) \leq 4N^L + 2\sqrt{\mathrm{m}}$$

by (7.26)-(7.25) (note that $\lambda_h \leq |h| + \sqrt{\mathrm{m}}$) and

$$|\lambda_{\alpha,\beta} + \sigma\lambda_m + \sigma\lambda_n| \geq |\lambda_m + \lambda_n| - |\lambda_{\alpha,\beta}| \geq 3N - 4N^L - 2\sqrt{\mathrm{m}} \geq 3N/2$$

for $N \geq N_0$ large enough.

The coefficients in (7.35) are bounded by (7.18), (7.29), and

$$N|\lambda_m - |m| - \lambda_n + |n|| \overset{(7.30)}{\leq} N\frac{\mathrm{m}}{2}\Big(\frac{1}{|m|} + \frac{1}{|m|}\Big) \leq \frac{2}{3}\mathrm{m}\,.$$

Hence arguing as in the proof of Lemma 7.1 we get

$$\|X_{\hat{F}}\|_R \lessdot R^2\,. \tag{7.36}$$

In conclusion, (7.19), (7.31), (7.36) imply (7.22) (recall (3.30)). ■

Proof of Proposition 7.1 completed. We have

$$\begin{aligned} e^{\mathrm{ad}_F} H &= e^{\mathrm{ad}_F} N + e^{\mathrm{ad}_F} G = N + \{N, F\} + \sum_{i\geq 2}\frac{1}{i!}\mathrm{ad}^i_F(N) + G + \sum_{i\geq 1}\frac{1}{i!}\mathrm{ad}^i_F(G) \\ &\overset{(7.20)}{=} N + \overline{G} + \hat{G} + \sum_{i\geq 1}\frac{1}{(i+1)!}\mathrm{ad}^i_F\big(\mathrm{ad}_F(N)\big) + \sum_{i\geq 1}\frac{1}{i!}\mathrm{ad}^i_F(G) \\ &= N + \overline{G} + \hat{G} + K \end{aligned}$$

where, using again (7.20),

$$K := \sum_{i\geq 1}\frac{1}{(i+1)!}\mathrm{ad}^i_F(\overline{G} + \hat{G} - G) + \sum_{i\geq 1}\frac{1}{i!}\mathrm{ad}^i_F G =: K_1 + K_2\,. \tag{7.37}$$

Proof of (7.9). We claim that in the expansion of K in (7.37) there are only monomials $u^{\vec{\sigma}}_{\vec{\jmath}}$ with $\vec{\jmath} \in \mathbb{Z}^{2d}$, $\vec{\sigma} \in \{+,-\}^{2d}$, $d \geq 3$. Indeed $F, G, \overline{G}$, $\hat{G}$ contain only monomials of degree four and, for any monomial $\mathfrak{m}$, $\mathrm{ad}_F(\mathfrak{m})$ contains only monomials of degree equal to the $\deg(\mathfrak{m}) + 2$. The restriction $\vec{\sigma} \cdot \vec{\jmath} = 0$ follows by the Jacobi identity (2.81), since $F, G, \overline{G}, \hat{G}$ preserve momentum, i.e. Poisson commute with M.

PROOF OF (7.10). We apply Proposition 3.2 with (no (x, y) variables and)

$$f \rightsquigarrow F\,, \quad g \rightsquigarrow \begin{cases} \overline{G} + \hat{G} - G & \text{for } K_1\,, \\ G & \text{for } K_2\,, \end{cases} \quad r \rightsquigarrow R\,, \quad r' \rightsquigarrow R/2\,, \quad \delta \rightsquigarrow 1/2\,,$$

$$\theta \rightsquigarrow 3/2\,, \quad \theta' \rightsquigarrow 2\,, \quad \mu \rightsquigarrow 4\,, \quad \mu' \rightsquigarrow 3\,,$$

N_0 defined in Lemma 7.4 and $N_0' \geq N_0$ satisfying (3.64) and

$$\kappa (N_0')^{b-L} \ln N_0' \leq 1\,, \qquad (6+\kappa)(N_0')^{L-1} \ln N_0' \leq 1/2\,. \tag{7.38}$$

Note that (3.65) follows by (7.38). By (7.22), the assumption (3.63) is verified for every $0 < R < R_0$, with R_0 small enough. Then Proposition 3.2 applies and (7.10) follows by (3.67) (with $h \rightsquigarrow 1$), (7.2), (7.22) and (7.7). ■

7.2 Action–angle variables

We introduce action-angle variables on the tangential sites $\mathcal{I} := \{\mathtt{j}_1, \ldots, \mathtt{j}_n\}$ (see (7.5)) via the analytic and symplectic map

$$\Phi(x, y, z, \bar{z}; \xi) := (u, \bar{u}) \tag{7.39}$$

defined by

$$u_{\mathtt{j}_l} := \sqrt{\xi_l + y_l}\, e^{\mathrm{i} x_l},\ \bar{u}_{\mathtt{j}_l} := \sqrt{\xi_l + y_l}\, e^{-\mathrm{i} x_l},\ l = 1, \ldots, n\,, \quad u_j := z_j\,,\ \bar{u}_j := \bar{z}_j\,,\ j \in \mathbb{Z} \setminus \mathcal{I}\,. \tag{7.40}$$

Let

$$\mathcal{O}_\varrho := \Big\{ \xi \in \mathbb{R}^n \ :\ \frac{\varrho}{2} \leq \xi_l \leq \varrho\,,\ l = 1, \ldots, n \Big\}\,. \tag{7.41}$$

Lemma 7.5. (Domains) *Let $r, R, \rho > 0$ satisfy*

$$16 r^2 < \varrho\,, \qquad \varrho = C_* R^2 \quad \text{with} \quad C_*^{-1} := 48 n \kappa^{2p} e^{2(s + a\kappa)}\,. \tag{7.42}$$

Then, for all $\xi \in \mathcal{O}_\varrho \cup \mathcal{O}_{2\varrho}$, the map

$$\Phi(\,\cdot\,;\xi) : D(s, 2r) \to \mathcal{D}(R/2) := B_{R/2} \times B_{R/2} \subset \ell^{a,p} \times \ell^{a,p} \tag{7.43}$$

is well defined and analytic ($D(s, 2r)$ is defined in (2.5) and κ in (3.1)).

PROOF. Note first that for $(x, y, z, \bar{z}) \in D(s, 2r)$ we have (see (2.6)) that $|y_l| < 4r^2 \overset{(7.42)}{<} \rho/4 < \xi_l$, $\forall \xi \in \mathcal{O}_\rho \cup \mathcal{O}_{2\rho}$. Then the map $y_l \mapsto \sqrt{\xi_l + y_l}$ is well defined and analytic. Moreover, for $\xi_l \leq 2\rho$, $|\mathtt{j}_l| \leq \kappa$, $x \in \mathbb{T}^n_s$, $\|z\|_{a,p} < 2r$, we get

$$\begin{aligned} \|u(x, y, z, \bar{z}; \xi)\|^2_{a,p} &\overset{(7.39)}{=} \sum_{l=1}^{n} (\xi_l + y_l) |e^{2\mathrm{i} x_l}| |\mathtt{j}_l|^{2p} e^{2a|\mathtt{j}_l|} + \sum_{j \in \mathbb{Z} \setminus \mathcal{I}} |z_j|^2 \langle j \rangle^{2p} e^{2a|j|} \\ &\leq \quad n \Big(2\varrho + \frac{\varrho}{4} \Big) e^{2s} \kappa^{2p} e^{2a\kappa} + 4r^2 \overset{(7.42)}{<} R^2/4 \end{aligned}$$

proving (7.43) (the bound for $\bar{u}$ is the same). ■

Given a function $F : \mathcal{D}(R/2) \to \mathbb{C}$, the previous Lemma shows that the composite map $F \circ \Phi : D(s, 2r) \to \mathbb{C}$. The main result of this section is Proposition 7.2: if F is quasi-Töplitz in the variables $(u, \bar{u})$ then the composite $F \circ \Phi$ is quasi-Töplitz in the variables $(x, y, z, \bar{z})$ (see Definition 3.4).

We write

$$F = \sum_{\alpha, \beta} F_{\alpha,\beta} \mathfrak{m}_{\alpha,\beta}\,, \quad \mathfrak{m}_{\alpha,\beta} := (u^{(1)})^{\alpha^{(1)}} (\bar{u}^{(1)})^{\beta^{(1)}} (u^{(2)})^{\alpha^{(2)}} (\bar{u}^{(2)})^{\beta^{(2)}}\,, \tag{7.44}$$

where

$$u = (u^{(1)}, u^{(2)}), \quad u^{(1)} := \{u_j\}_{j\in\mathcal{I}}, \ u^{(2)} := \{u_j\}_{j\in\mathbb{Z}\setminus\mathcal{I}}, \quad \text{similarly for } \bar{u},$$

and

$$(\alpha,\beta) = (\alpha^{(1)}+\alpha^{(2)}, \beta^{(1)}+\beta^{(2)}), \ \ (\alpha^{(1)},\beta^{(1)}) := \{\alpha_j,\beta_j\}_{j\in\mathcal{I}}, \ \ (\alpha^{(2)},\beta^{(2)}) := \{\alpha_j,\beta_j\}_{j\in\mathbb{Z}\setminus\mathcal{I}}. \quad (7.45)$$

We define

$$\mathcal{H}^d_R := \Big\{ F \in \mathcal{H}_R \ : \ F = \sum_{|\alpha^{(2)}+\beta^{(2)}|\geq d} F_{\alpha,\beta} u^\alpha \bar{u}^\beta \Big\}. \quad (7.46)$$

Proposition 7.2. (Quasi–Töplitz) *Let N_0,θ,μ,μ' satisfying* (3.1) *and*

$$(\mu'-\mu)N_0^L > N_0^b, \qquad N_0 2^{-\frac{N_0^b}{2\kappa}+1} < 1. \quad (7.47)$$

If $F \in \mathcal{Q}^T_{R/2}(N_0,\theta,\mu') \cap \mathcal{H}^d_{R/2}$ with $d=0,1$, then $f := F\circ\Phi \in \mathcal{Q}^T_{s,r}(N_0,\theta,\mu)$ and

$$\|f\|^T_{s,r,N_0,\theta,\mu,\mathcal{O}_\varrho} \lessdot (8r/R)^{d-2} \|F\|^T_{R/2,N_0,\theta,\mu'}. \quad (7.48)$$

The rest of this section is devoted to the proof of Proposition 7.2. Introducing the action-angle variables (7.40) in (7.44), and using the Taylor expansion

$$(1+t)^\gamma = \sum_{h\geq 0} \binom{\gamma}{h} t^h, \quad \binom{\gamma}{0} := 1, \quad \binom{\gamma}{h} := \frac{\gamma(\gamma-1)\ldots(\gamma-h+1)}{h!}, \ h\geq 1, \quad (7.49)$$

we get

$$f := F\circ\Phi = \sum_{k,i,\alpha^{(2)},\beta^{(2)}} f_{k,i,\alpha^{(2)},\beta^{(2)}} e^{\mathrm{i}k\cdot x} y^i z^{\alpha^{(2)}} \bar{z}^{\beta^{(2)}} \quad (7.50)$$

with Taylor–Fourier coefficients

$$f_{k,i,\alpha^{(2)},\beta^{(2)}} := \sum_{\alpha^{(1)}-\beta^{(1)}=k} F_{\alpha,\beta} \prod_{l=1}^n \xi_l^{\frac{\alpha_l^{(1)}+\beta_l^{(1)}}{2}-i_l} \binom{\frac{\alpha_l^{(1)}+\beta_l^{(1)}}{2}}{i_l}. \quad (7.51)$$

We need an upper bound on the binomial coefficients.

Lemma 7.6. *For $|t|<1/2$ we have*

$$(i) \sum_{h\geq 0} |t|^h \Big| \binom{\frac{k}{2}}{h} \Big| \leq 2^k, \ \forall k\geq 0, \quad (ii) \sum_{h\geq 1} |t|^h \Big| \binom{\frac{k}{2}}{h} \Big| \leq 3^k |t|, \ \forall k\geq 1. \quad (7.52)$$

Proof. By (7.49) and the definition of majorant (see (2.11)) we have

$$\sum_{h\geq 0} \Big| \binom{\frac{k}{2}}{h} \Big| t^h = M(1+t)^{\frac{k}{2}} \overset{(2.39)}{\prec} (M(1+t)^{\frac12})^k = \Big(\sum_{h\geq 0} \Big| \binom{\frac12}{h} \Big| t^h \Big)^k \prec \Big(\sum_{h\geq 0} t^h \Big)^k \quad (7.53)$$

because $\Big|\binom{\frac12}{h}\Big| \leq 1$ by (7.49). For $|t|<1/2$ the bound (7.53) implies (7.52)-(i). Ne

$$\sum_{h\geq 1} |t|^h \Big| \binom{\frac{k}{2}}{h} \Big| \leq |t| \sum_{h\geq 0} |t|^h \Big| \binom{\frac{k}{2}}{h+1} \Big| \overset{(7.49)}{=} |t| \sum_{h\geq 0} |t|^h \Big| \binom{\frac{k}{2}}{h} \Big| \frac{|\frac{k}{2}-h|}{h+1} \leq k|t| \sum_{h\geq 0} |t|^h \Big| \binom{\frac{k}{2}}{h} \Big| \overset{(7.52)-(i)}{\leq} k 2^k |t|$$

which implies (7.52)-(ii) for $k\geq 1$. ∎

Lemma 7.7. **(M-regularity)** *If $F \in \mathcal{H}^d_{R/2}$ then $f := F \circ \Phi \in \mathcal{H}_{s,2r}$ and*

$$\|X_f\|_{s,2r,\mathcal{O}_\varrho \cup \mathcal{O}_{2\varrho}} \lessdot (8r/R)^{d-2} \|X_F\|_{R/2} \,. \tag{7.54}$$

Moreover if F preserves momentum then so does $F \circ \Phi$.

Proof. We first bound the majorant norm

$$\|f\|_{s,2r,\mathcal{O}_\varrho \cup \mathcal{O}_{2\varrho}} \overset{(7.50),(7.46)}{:=} \sup_{\xi \in \mathcal{O}_\varrho \cup \mathcal{O}_{2\varrho}} \sup_{(y,z,\bar z) \in D(2r)} \sum_{k,i,|\alpha^{(2)}+\beta^{(2)}| \geq d} |f_{k,i,\alpha^{(2)},\beta^{(2)}}| e^{|k|s} |y^i| |z^{\alpha^{(2)}}| |\bar z^{\beta^{(2)}}| \,. \tag{7.55}$$

Fix $\alpha^{(2)}, \beta^{(2)}$. Since for all $\xi \in \mathcal{O}_\varrho \cup \mathcal{O}_{2\varrho}$, $y \in B_{(2r)^2}$, we have $|y_l/\xi_l| < 1/2$ by (7.42), we have

$$\sum_k e^{|k|s} \sum_i |f_{k,i,\alpha^{(2)},\beta^{(2)}}| |y|^i \tag{7.56}$$

$$\overset{(7.51)}{\leq} \sum_{\alpha^{(1)},\beta^{(1)}} e^{s(|\alpha^{(1)}|+|\beta^{(1)}|)} |F_{\alpha,\beta}| \xi^{\frac{\alpha^{(1)}+\beta^{(1)}}{2}} \prod_{l=1}^n \sum_{i_l \geq 0} \left| \frac{y_l}{\xi_l} \right|^{i_l} \left| \binom{\frac{\alpha_l^{(1)}+\beta_l^{(1)}}{2}}{i_l} \right| \tag{7.57}$$

$$\overset{(7.52)}{\leq} \sum_{\alpha^{(1)},\beta^{(1)}} e^{s(|\alpha^{(1)}|+|\beta^{(1)}|)} |F_{\alpha,\beta}| \xi^{\frac{\alpha^{(1)}+\beta^{(1)}}{2}} \prod_{l=1}^n 2^{\alpha_l^{(1)}+\beta_l^{(1)}} \tag{7.58}$$

$$\leq \sum_{\alpha^{(1)},\beta^{(1)}} e^{s(|\alpha^{(1)}|+|\beta^{(1)}|)} |F_{\alpha,\beta}| (2\varrho)^{\frac{|\alpha^{(1)}|+|\beta^{(1)}|}{2}} 2^{|\alpha^{(1)}|+|\beta^{(1)}|} = \sum_{\alpha^{(1)},\beta^{(1)}} (2e^s\sqrt{2\varrho})^{|\alpha^{(1)}|+|\beta^{(1)}|} |F_{\alpha,\beta}| \,.$$

Then, substituting in (7.55),

$$\|f\|_{s,2r,\mathcal{O}_\varrho \cup \mathcal{O}_{2\varrho}} \leq \sup_{\|z\|_{a,p}, \|\bar z\|_{a,p} < 2r} G(z,\bar z) \qquad \text{where} \tag{7.59}$$

$$G(z,\bar z) := \sum_{|\alpha^{(2)}+\beta^{(2)}| \geq d} (2e^s\sqrt{2\varrho})^{|\alpha^{(1)}|+|\beta^{(1)}|} |F_{\alpha,\beta}| |z^{\alpha^{(2)}}| |\bar z^{\beta^{(2)}}| \,. \tag{7.60}$$

By (7.42), for all $\|z\|_{a,p}, \|\bar z\|_{a,p} < 2r$, the vector $(u^*, \bar u^*)$ defined by

$$u_j^* = \bar u_j^* := 2e^s\sqrt{2\varrho}\,, \quad j \in \mathcal{I}\,, \quad u_j^* := (R/(8r))|z_j|\,, \ \bar u_j^* := (R/(8r))|\bar z_j|\,, \quad j \in \mathbb{Z} \setminus \mathcal{I} \tag{7.61}$$

belongs to $B_{R/2} \times B_{R/2}$. Then, by (7.60), recalling (2.11), Definition 2.2 (and since $R/(8r) > 1$ by (7.42)),

$$G(z,\bar z) \leq (8r/R)^d (MF)(u^*, \bar u^*) \leq (8r/R)^d \|F\|_{R/2}\,, \quad \forall \, \|z\|_{a,p}, \|\bar z\|_{a,p} < 2r\,.$$

Hence by (7.59)

$$\|f\|_{s,2r,\mathcal{O}_\varrho \cup \mathcal{O}_{2\varrho}} \leq (8r/R)^d \|F\|_{R/2}\,. \tag{7.62}$$

This shows that f is M-regular. Similarly we get

$$\|\partial_z f\|_{s,2r,\mathcal{O}_\varrho \cup \mathcal{O}_{2\varrho}} \leq \|\partial_{u^{(2)}} F\|_{R/2} (8r/R)^{d-1}\,, \quad \text{same for } \partial_{\bar z}\,. \tag{7.63}$$

Moreover, by the chain rule, and (7.62)

$$\|\partial_{x_i} f\|_{s,2r,\mathcal{O}_\varrho \cup \mathcal{O}_{2\varrho}} \leq (\|\partial_{u_i^{(1)}} F\|_{R/2} + \|\partial_{\bar u_i^{(1)}} F\|_{R/2}) \sqrt{2\varrho + \varrho/4}\, e^s (8r/R)^d$$

$$\|\partial_{y_i} f\|_{s,2r,\mathcal{O}_\varrho \cup \mathcal{O}_{2\varrho}} \leq (\|\partial_{u_i^{(1)}} F\|_{R/2} + \|\partial_{\bar u_i^{(1)}} F\|_{R/2}) \frac{e^s}{\sqrt{\varrho/2 - \varrho/4}} (8r/R)^d\,.$$

Then (7.54) follows by (7.42) (recalling (2.2)). ■

Definition 7.1. *For a monomial* $\mathfrak{m}_{\alpha,\beta} := (u^{(1)})^{\alpha^{(1)}}(\bar u^{(1)})^{\beta^{(1)}}(u^{(2)})^{\alpha^{(2)}}(\bar u^{(2)})^{\beta^{(2)}}$ *(as in* (7.44)*) we set*

$$\mathfrak{p}(\mathfrak{m}_{\alpha,\beta}) := \sum_{l=1}^{n} \langle \mathtt{j}_l \rangle (\alpha^{(1)}_{\mathtt{j}_l} + \beta^{(1)}_{\mathtt{j}_l}) \,, \quad \langle j \rangle := \max\{1, |j|\} \,. \tag{7.64}$$

For any F *as in* (7.44)*,* $K \in \mathbb{N}$*, we define the projection*

$$\Pi_{\mathfrak{p}\geq K} F := \sum_{\mathfrak{p}(\mathfrak{m}_{\alpha,\beta})\geq K} F_{\alpha,\beta}\mathfrak{m}_{\alpha,\beta} \,, \quad \Pi_{\mathfrak{p}<K} := I - \Pi_{\mathfrak{p}\geq K} \,. \tag{7.65}$$

Lemma 7.8. *Let* $F \in \mathcal{H}_{R/2}$*. Then*

$$\| X_{(\Pi_{\mathfrak{p}\geq K} F)\circ \Phi} \|_{s,r,\mathcal{O}_\varrho} \leq 2^{-\frac{K}{2\kappa}+1} \| X_{F\circ\Phi} \|_{s,2r,\mathcal{O}_{2\varrho}} \,. \tag{7.66}$$

PROOF. For each monomial $\mathfrak{m}_{\alpha,\beta}$ as in (7.44) with $\mathfrak{p}(\mathfrak{m}_{\alpha,\beta}) \geq K$ we have

$$|\alpha^{(1)} + \beta^{(1)}| \overset{(7.45)}{=} \sum_{l=1}^{n} \alpha^{(1)}_{\mathtt{j}_l} + \beta^{(1)}_{\mathtt{j}_l} \overset{(3.1)}{\geq} \kappa^{-1} \sum_{l=1}^{n} \langle \mathtt{j}_l \rangle (\alpha^{(1)}_{\mathtt{j}_l} + \beta^{(1)}_{\mathtt{j}_l}) \overset{(7.64)}{=} \kappa^{-1}\mathfrak{p}(\mathfrak{m}_{\alpha,\beta}) \geq \kappa^{-1} K$$

and then, $\forall \xi \in \mathcal{O}_\varrho$, $y \in B_{r^2}$,

$$\begin{aligned} |(\mathfrak{m}_{\alpha,\beta}\circ\Phi)(x,y,z,\bar z;\xi)| &\overset{(7.40)}{=} |(\xi+y)^{\frac{\alpha^{(1)}+\beta^{(1)}}{2}} e^{\mathrm{i}(\alpha^{(1)}-\beta^{(1)})\cdot x} z^{\alpha^{(2)}} \bar z^{\beta^{(2)}}| \\ &= 2^{-\frac{|\alpha^{(1)}+\beta^{(1)}|}{2}} |(2\xi+2y)^{\frac{\alpha^{(1)}+\beta^{(1)}}{2}} e^{\mathrm{i}(\alpha^{(1)}-\beta^{(1)})\cdot x} z^{\alpha^{(2)}} \bar z^{\beta^{(2)}}| \\ &\leq 2^{-\frac{K}{2\kappa}} |(\mathfrak{m}_{\alpha,\beta}\circ\Phi)(x,2y,z,\bar z;2\xi)| \,. \end{aligned} \tag{7.67}$$

The bound (7.66) for the Hamiltonian vector field follows applying the above rescaling argument to each component, and noting that the derivatives with respect to y in the vector field decrease the degree in ξ by one. ■

Let N_0, θ, μ, μ' be as in Proposition 7.2. For $N \geq N_0$ and $F \in \mathcal{H}_{R/2}$ we set

$$f^* := \Pi_{N,\theta,\mu}\Big((F - \Pi_{N,\theta,\mu'} F)\circ\Phi \Big) \,. \tag{7.68}$$

Note that $\Pi_{N,\theta,\mu'}$ is the projection on the bilinear functions in the variables $u, \bar u$, while $\Pi_{N,\theta,\mu}$ in the variables $x, y, z, \bar z$.

Lemma 7.9. *We have*

$$\| X_{f^*} \|_{s,r,\mathcal{O}_\varrho} \leq 2^{-\frac{N^b}{2\kappa}+1} \| X_{F\circ\Phi} \|_{s,2r,\mathcal{O}_{2\varrho}} \,. \tag{7.69}$$

PROOF. We first claim that if $F = \mathfrak{m}_{\alpha,\beta}$ is a monomial as in (7.44) with $\mathfrak{p}(\mathfrak{m}_{\alpha,\beta}) < N^b$ then $f^* = 0$.

CASE 1: $\mathfrak{m}_{\alpha,\beta}$ is (N,θ,μ')–bilinear, see Definition 3.2. Then $\Pi_{N,\theta,\mu'}\mathfrak{m}_{\alpha,\beta} = \mathfrak{m}_{\alpha,\beta}$ and $f^* = 0$, see (7.68).

CASE 2: $\mathfrak{m}_{\alpha,\beta}$ is *not* (N,θ,μ')–bilinear. Then $\Pi_{N,\theta,\mu'}\mathfrak{m}_{\alpha,\beta} = 0$ and $f^* = \Pi_{N,\theta,\mu}(\mathfrak{m}_{\alpha,\beta}\circ\Phi)$, see (7.68). We claim that $\mathfrak{m}_{\alpha,\beta}\circ\Phi$ is not (N,θ,μ)–bilinear, and so $f^* = \Pi_{N,\theta,\mu}(\mathfrak{m}_{\alpha,\beta}\circ\Phi) = 0$. Indeed,

$$\mathfrak{m}_{\alpha,\beta}\circ\Phi = (\xi+y)^{\frac{\alpha^{(1)}+\beta^{(1)}}{2}} e^{\mathrm{i}(\alpha^{(1)}-\beta^{(1)})\cdot x} z^{\alpha^{(2)}} \bar z^{\beta^{(2)}} \tag{7.70}$$

is (N,θ,μ)–bilinear if and only if (see Definitions 3.2 and 3.1)

$$z^{\alpha^{(2)}}\bar z^{\beta^{(2)}} = z^{\tilde\alpha^{(2)}}\bar z^{\tilde\beta^{(2)}} z_m^{\sigma} z_n^{\sigma'} \,,$$

$$\sum_{j\in\mathbb{Z}\setminus\mathcal{I}} |j|(\tilde\alpha^{(2)}_j + \tilde\beta^{(2)}_j) < \mu N^L \,, \quad |m|, |n| > \theta N \,, \quad |\alpha^{(1)} - \beta^{(1)}| < N^b \,. \tag{7.71}$$

We deduce the contradiction that $\mathfrak{m}_{\alpha,\beta} = (u^{(1)})^{\alpha^{(1)}}(\bar u^{(1)})^{\beta^{(1)}}(u^{(2)})^{\tilde\alpha^{(2)}}(\bar u^{(2)})^{\tilde\beta^{(2)}} u_m^\sigma u_n^{\sigma'}$ is (N,θ,μ')-bilinear because (recall that we suppose $\mathfrak{p}(\mathfrak{m}_{\alpha,\beta}) < N^b$)

$$\sum_{l=1}^n |\mathtt{j}_l|(\alpha^{(1)}_{\mathtt{j}_l} + \beta^{(1)}_{\mathtt{j}_l}) + \sum_{j\in\mathbb{Z}\setminus\mathcal{I}} |j|(\tilde\alpha^{(2)}_j + \tilde\beta^{(2)}_j) \overset{(7.64),(7.71)}{<} \mathfrak{p}(\mathfrak{m}_{\alpha,\beta}) + \mu N^L < N^b + \mu N^L \overset{(7.47)}{<} \mu' N^L .$$

For the general case, we divide $F = \Pi_{\mathfrak{p}<N^b} F + \Pi_{\mathfrak{p}\geq N^b} F$. By the above claim

$$f^* = \Pi_{N,\theta,\mu}\Big(\big((Id - \Pi_{N,\theta,\mu'})\Pi_{\mathfrak{p}\geq N^b} F\big)\circ\Phi\Big) = \Pi_{N,\theta,\mu}\Big(\big(\Pi_{\mathfrak{p}\geq N^b}(Id - \Pi_{N,\theta,\mu'})F\big)\circ\Phi\Big) .$$

Finally, (7.69) follows by (2.79) and applying Lemma 7.8 to $\big(\Pi_{\mathfrak{p}\geq N^b}(Id - \Pi_{N,\theta,\mu'})F\big)\circ\Phi$. ■

Lemma 7.10. *Let $F \in \mathcal{T}_{R/2}(N,\theta,\mu')$ with $\Pi_{\mathfrak{p}\geq N^b} F = 0$. Then $F\circ\Phi(\cdot;\xi) \in \mathcal{T}_{s,2r}(N,\theta,\mu')$, $\forall \xi \in \mathcal{O}_\varrho \cup \mathcal{O}_{2\varrho}$.*

Proof. Recalling Definition 3.3 we have

$$F = \sum_{|m|,|n|>\theta N, \sigma,\sigma'=\pm} F^{\sigma,\sigma'}(\mathtt{s}(m), \sigma m + \sigma' n) u_m^\sigma u_n^{\sigma'} \quad \text{with} \quad F^{\sigma,\sigma'}(\varsigma, h) \in \mathcal{L}_{R/2}(N,\mu',h) .$$

Composing with the map Φ in (7.40), since $m,n\notin\mathcal{I}$, we get

$$F\circ\Phi = \sum_{\sigma,\sigma'=\pm\,,|m|,|n|>\theta N} F^{\sigma,\sigma'}(\mathtt{s}(m),\sigma m+\sigma' n)\circ\Phi\, z_m^\sigma z_n^{\sigma'} .$$

Each coefficient $F^{\sigma,\sigma'}(\mathtt{s}(m),\sigma m+\sigma' n)\circ\Phi$ depends on n,m,σ,σ' only through $\mathtt{s}(m), \sigma m+\sigma' n, \sigma, \sigma'$. Hence, in order to conclude that $F\circ\Phi \in \mathcal{T}_{s,2r}(N,\theta,\mu')$ it remains only to prove that $F^{\sigma,\sigma'}(\mathtt{s}(m),\sigma m+\sigma' n)\circ\Phi \in \mathcal{L}_{s,2r}(N,\mu',\sigma m+\sigma' n)$, see Definition 3.1. Each monomial $\mathfrak{m}_{\alpha,\beta}$ of $F^{\sigma,\sigma'}(\mathtt{s}(m),\sigma m+\sigma' n) \in \mathcal{L}_{R/2}(N,\mu',\sigma m+\sigma' n)$ satisfies

$$\sum_{l=1}^n (\alpha_{\mathtt{j}_l}+\beta_{\mathtt{j}_l})|\mathtt{j}_l| + \sum_{j\in\mathbb{Z}\setminus\mathcal{I}} (\alpha_j+\beta_j)|j| < \mu' N^L \quad \text{and} \quad \mathfrak{p}(\mathfrak{m}_{\alpha,\beta}) < N^b$$

by the hypothesis $\Pi_{\mathfrak{p}\geq N^b} F = 0$. Hence $\mathfrak{m}_{\alpha,\beta}\circ\Phi$ (see (7.70)) is (N,μ')-low momentum, in particular $|\alpha^{(1)} - \beta^{(1)}| \leq \mathfrak{p}(\mathfrak{m}_{\alpha,\beta}) < N^b$. ■

Proof of Proposition 7.2. Since $F \in \mathcal{Q}^T_{R/2}(N_0,\theta,\mu')$ (see Definition 3.4), for all $N\geq N_0$, there is a Töplitz approximation $\tilde F \in \mathcal{T}_{R/2}(N,\theta,\mu')$ of F, namely

$$\Pi_{N,\theta,\mu'} F = \tilde F + N^{-1}\hat F \quad \text{with} \quad \|X_F\|_{R/2}, \|X_{\tilde F}\|_{R/2}, \|X_{\hat F}\|_{R/2} < 2\|F\|^T_{R/2,N_0,\theta,\mu'} . \tag{7.72}$$

In order to prove that $f := F\circ\Phi \in \mathcal{Q}^T_{s,r}(N_0,\theta,\mu)$ we define its candidate Töplitz approximation

$$\tilde f := \Pi_{N,\theta,\mu}((\Pi_{\mathfrak{p}<N^b}\tilde F)\circ\Phi) , \tag{7.73}$$

see (7.65). Lemma 7.10 applied to $\Pi_{\mathfrak{p}<N^b}\tilde F \in \mathcal{T}_{R/2}(N,\theta,\mu')$ implies that $(\Pi_{\mathfrak{p}<N^b}\tilde F)\circ\Phi \in \mathcal{T}_{s,2r}(N,\theta,\mu')$ and then, applying the projection $\Pi_{N,\theta,\mu}$ we get $\tilde f \in \mathcal{T}_{s,2r}(N,\theta,\mu) \subset \mathcal{T}_{s,r}(N,\theta,\mu)$. Moreover, by (7.73) and applying Lemma 7.7 to $\Pi_{\mathfrak{p}<N^b}\tilde F$ (note that $\Pi_{\mathfrak{p}<N^b}\tilde F$ is either zero or it is in $\mathcal{H}^d_{R/2}$ with $d\geq 2$ because it is bilinear), we get

$$\begin{aligned} \|X_{\tilde f}\|_{s,r,\mathcal{O}_\varrho} \overset{(2.79)}{\leq} \|X_{(\Pi_{\mathfrak{p}<N^b}\tilde F)\circ\Phi}\|_{s,r,\mathcal{O}_\varrho} &\overset{(7.54)}{\lessdot} (8r/R)^{d-2}\|X_{\Pi_{\mathfrak{p}<N^b}\tilde F}\|_{R/2} \\ &\overset{(2.79),(7.72)}{\lessdot} (8r/R)^{d-2}\|F\|^T_{R/2,N_0,\theta,\mu'} . \end{aligned} \tag{7.74}$$

Moreover the Töplitz defect is

$$\begin{aligned}
\hat f \quad &:= \quad N(\Pi_{N,\theta,\mu} f - \tilde f) \stackrel{(7.73)}{=} N\,\Pi_{N,\theta,\mu}\big((F - \Pi_{\mathfrak p < N^b}\tilde F)\circ\Phi\big)\\
&= \quad N\Pi_{N,\theta,\mu}\big((F-\tilde F)\circ\Phi\big) + N\Pi_{N,\theta,\mu}\big((\tilde F - \Pi_{\mathfrak p<N^b}\tilde F)\circ\Phi\big)\\
&\stackrel{(7.72),(7.65)}{=} \quad \Pi_{N,\theta,\mu}(\hat F\circ\Phi) + N\Pi_{N,\theta,\mu}\Big(\big(F - \Pi_{N,\theta,\mu'}F\big)\circ\Phi\Big) + N\Pi_{N,\theta,\mu}\big((\Pi_{\mathfrak p\geq N^b}\tilde F)\circ\Phi\big)\\
&\stackrel{(7.68)}{=} \quad \Pi_{N,\theta,\mu}(\hat F\circ\Phi) + Nf^* + N\Pi_{N,\theta,\mu}\big((\Pi_{\mathfrak p\geq N^b}\tilde F)\circ\Phi\big)\,.
\end{aligned}$$

Using (2.79), Lemmata 7.8 and 7.9 imply that, since $N2^{-\frac{N^b}{2\kappa}+1}\leq 1$, $\forall\, N\geq N_0$ by (7.47),

$$\begin{aligned}
\|X_{\hat f}\|_{s,r,\mathcal O_\varrho} \quad &\leq \quad \|X_{\hat F\circ\Phi}\|_{s,r,\mathcal O_\varrho} + N2^{-\frac{N^b}{2\kappa}+1}(\|X_{F\circ\Phi}\|_{s,2r,\mathcal O_{2\varrho}} + \|X_{\tilde F\circ\Phi}\|_{s,2r,\mathcal O_{2\varrho}})\\
&\lessdot \quad \|X_{\hat F\circ\Phi}\|_{s,2r,\mathcal O_\varrho} + \|X_{F\circ\Phi}\|_{s,2r,\mathcal O_{2\varrho}} + \|X_{\tilde F\circ\Phi}\|_{s,2r,\mathcal O_{2\varrho}}\\
&\stackrel{(7.54)}{\lessdot} \quad (8r/R)^{d-2}(\|X_{\hat F}\|_{R/2} + \|X_F\|_{R/2} + \|X_{\tilde F}\|_{R/2}) \qquad (7.75)\\
&\stackrel{(7.72)}{\lessdot} \quad (8r/R)^{d-2}\|F\|^T_{R/2,N_0,\theta,\mu'} \qquad (7.76)
\end{aligned}$$

(to get (7.75) we also note that $F,\hat F,\tilde F\in\mathcal H^d_{R/2}$ with $d=0,1$, unless are zero).
The bound (7.48) follows by (7.54), (7.74), (7.76). ■

We conclude this subsection with a lemma, similar to Lemma 7.7, used in Lemma 7.12 (see (7.90)).

Lemma 7.11. *Let* $F\in\mathcal H_{R/2}$, $f:=F\circ\Phi$ *and* $\tilde f(x,y):=f(x,y,0,0)-f(x,0,0,0)$. *Then, assuming* (7.42),

$$\|X_{\tilde f}\|_{s,2r,\mathcal O_\varrho\cup\mathcal O_{2\varrho}}\lessdot\|X_F\|_{R/2}\,. \qquad (7.77)$$

Moreover if F *preserves momentum then so does* $\tilde f$.

Proof. We proceed as in Lemma 7.7. The main difference is that here there are no $(z,\bar z)$-variables and the sum in (7.56) runs over $i\neq 0$. Then in the product in (7.57) (at least) one of the sums is on $i_l\geq 1$. Therefore we can use the second estimate in (7.52) gaining a factor[3] $8r^2/\varrho$ (since $|y_l|/|\xi_l|\leq 8r^2/\varrho$ by (7.41)). Continuing as in the proof of Lemma 7.7 we get (recall (7.54) with $d=0$)

$$\|X_{\tilde f}\|_{s,2r,\mathcal O_\varrho\cup\mathcal O_{2\varrho}}\lessdot(r^2/\varrho)(r/R)^{-2}\|X_F\|_{R/2}\stackrel{(7.42)}{\lessdot}\|X_F\|_{R/2}$$

proving (7.77). ■

7.3 Proof of Theorem 1.1

We now introduce the action-angle variables (7.40) (via the map (7.39)) in the Birkhoff normal form Hamiltonian (7.8). Hence we obtain the parameter dependent family of Hamiltonians

$$H' := H_{\rm Birkhoff}\circ\Phi = \mathcal N + P \qquad (7.78)$$

where (up to a constant), by (7.6),

$$\mathcal N := \omega(\xi)\cdot y + \Omega(\xi)z\bar z\,,\quad P := \frac12 Ay\cdot y + By\cdot z\bar z + \hat G(z,\bar z) + K'(x,y,z,\bar z;\xi)\,, \qquad (7.79)$$

$$\omega(\xi) := \bar\omega + A\xi\,,\ \bar\omega := (\lambda_{\mathtt j_1},\ldots,\lambda_{\mathtt j_n})\,,\quad \Omega(\xi) := \bar\Omega + B\xi\,,\ \bar\Omega := (\lambda_j)_{j\in\mathbb Z\setminus\mathcal I}\,, \qquad (7.80)$$

$$A = (A_{lh})_{1\leq l,h\leq n}\,,\ A_{lh} := 12(2-\delta_{lh})\,,\ B = (B_{jl})_{j\in\mathbb Z\setminus\mathcal I,1\leq l\leq n}\,,\ B_{jl} := 24\,,\ K' := K\circ\Phi\,. \qquad (7.81)$$

The parameters ξ stay in the set $\mathcal O_\varrho$ defined in (7.41) with $\rho = C_*R^2$ as in (7.42). As in (4.7) we decompose the perturbation

$$P = P_{00} + \bar P \quad \text{where} \quad P_{00}(x;\xi) := K'(x,0,0,0;\xi)\,,\quad \bar P := P - P_{00}\,. \qquad (7.82)$$

[3]Actually we have the constant 3 instead of 2 in (7.58) and $3e^s$ instead of $2e^s$ in (7.59) and (7.61).

Lemma 7.12. *Let $s, r > 0$ as in (7.42) and N large enough (w.r.t. $\mathrm{m}, \mathcal{I}, L, b$). Then*

$$\|X_{P_{00}}\|_{s,r} \lessdot R^6 r^{-2}\,, \qquad \|\bar{P}\|^T_{s,r,N,2,2} \lessdot r^2 + R^5 r^{-1} \tag{7.83}$$

and, for $\lambda > 0$,

$$|X_{P_{00}}|^{\lambda}_{s,r} \lessdot (1 + \lambda/\varrho) R^6 r^{-2}\,, \qquad |X_{\bar{P}}|^{\lambda}_{s,r} \lessdot (1 + \lambda/\varrho)(r^2 + R^5 r^{-1})\,, \tag{7.84}$$

for ξ belonging to

$$\mathcal{O} := \mathcal{O}(\varrho) := \Big\{ \xi \in \mathbb{R}^n \ : \ \frac{2}{3}\varrho \le \xi_l \le \frac{3}{4}\varrho\,, \quad l = 1, \ldots, n \Big\} \subset \mathcal{O}_\rho\,. \tag{7.85}$$

PROOF. By the definition (7.82) we have

$$\begin{aligned}\|X_{P_{00}}\|_{s,r} \overset{(2.55)}{\le} \|X_{K'}\|_{s,r} \overset{(3.32)}{\le} \|K'\|^T_{s,r,N,2,2} &\overset{(7.81)}{=} \|K \circ \Phi\|^T_{s,r,N,2,2} \\ &\overset{(7.48)}{\lessdot} \Big(\frac{r}{R}\Big)^{-2} \|K\|^T_{R/2,N,2,2}\end{aligned} \tag{7.86}$$

(applying (7.48) with $d \rightsquigarrow 0$, $N_0 \rightsquigarrow N$, $\theta \rightsquigarrow 2$, $\mu \rightsquigarrow 2$, $\mu' \rightsquigarrow 3$) and taking N large enough so that (7.47) holds. Take also $N \ge N_0'$ defined in Proposition 7.1. Then by (7.86) we get

$$\|X_{P_{00}}\|_{s,r} \overset{(3.35)}{\lessdot} \Big(\frac{r}{R}\Big)^{-2} \|K\|^T_{R/2,N_0',2,2} \overset{(7.10)}{\lessdot} \Big(\frac{r}{R}\Big)^{-2} R^4 \lessdot \frac{R^6}{r^2}$$

proving the first estimate in (7.83). Let us prove the second bound. By (7.82) and (7.79) we write

$$\bar{P} = \frac{1}{2} A y \cdot y + B y \cdot z\bar{z} + \hat{G}(z, \bar{z}) + K_1 + K_2 \tag{7.87}$$

where

$$K_1 := K'(x, y, z, \bar{z}; \xi) - K'(x, y, 0, 0; \xi)\,, \quad K_2 := K'(x, y, 0, 0; \xi) - K'(x, 0, 0, 0; \xi)\,.$$

Using (7.7) (note that $r < R$ by (7.42)) for $N \ge N_0$ large enough to fulfill (3.1), we have by (3.35)

$$\Big\| \frac{1}{2} A y \cdot y + B y \cdot z\bar{z} + \hat{G}(z, \bar{z}) \Big\|^T_{s,r,N,2,2} \lessdot r^2\,. \tag{7.88}$$

By (7.48) (with $d \rightsquigarrow 1$, $N_0 \rightsquigarrow N$, $\mu \rightsquigarrow 2$, $\mu' \rightsquigarrow 3$), for $N \ge N_0(\mathrm{m}, \mathcal{I}, L, b)$ large enough, we get

$$\|K_1\|^T_{s,r,N,2,2} \lessdot \Big(\frac{r}{R}\Big)^{-1} R^4 \lessdot \frac{R^5}{r}\,. \tag{7.89}$$

Moreover, since K_2 does not depend on $(z, \bar{z})$, we have

$$\|K_2\|^T_{s,r,N,2,2} \overset{(3.34)}{=} \|X_{K_2}\|_{s,r} \overset{(7.77)}{\lessdot} \|X_K\|_{R/2} \overset{(3.32)}{\lessdot} \|K\|^T_{R/2,N_0',2,3} \overset{(7.10)}{\lessdot} R^4\,. \tag{7.90}$$

In conclusion, (7.87), (7.88), (7.89), (7.90) imply the second estimate in (7.83):

$$\|\bar{P}\|^T_{s,r,N,2,2} \lessdot r^2 + \frac{R^5}{r} + R^4 \overset{(7.42)}{\lessdot} r^2 + \frac{R^5}{r}\,.$$

Let us prove the estimates (7.84) for the Lipschitz norm defined in (2.87) (which involves only the sup-norm of the vector fields). First

$$|X_{P_{00}}|_{s,r} \overset{(2.61)}{\le} \|X_{P_{00}}\|_{s,r} \overset{(7.83)}{\lessdot} R^6 r^{-2}\,, \quad |X_{\bar{P}}|_{s,r} \overset{(2.61)}{\le} \|X_{\bar{P}}\|_{s,r} \overset{(3.32)}{\le} \|\bar{P}\|^T_{s,r,N,2,2} \overset{(7.83)}{\lessdot} r^2 + R^5 r^{-1}$$

Next, since the vector fields $X_{P_{00}}$, $X_{\bar P}$ are *analytic* in the parameters $\xi \in \mathcal{O}_\varrho$, Cauchy estimates in the domain $\mathcal{O} \subset \mathcal{O}_\varrho$ (see (7.85)) imply

$$|X_{P_{00}}|^{\rm lip}_{s,r,\mathcal{O}} \lessdot \rho^{-1} |X_{P_{00}}|_{s,r,\mathcal{O}_\rho} \lessdot R^6 r^{-2}, \quad |X_{\bar P}|^{\rm lip}_{s,r,\mathcal{O}} \lessdot \rho^{-1} |X_{\bar P}|_{s,r,\mathcal{O}_\rho} \lessdot r^2 + R^5 r^{-1}$$

and (7.84) are proved. ■

All the assumptions of Theorems 4.1-4.2 are fulfilled by H' in (7.78) with parameters $\xi \in \mathcal{O}$ defined in (7.85). The hypothesis (A1)-(A2) follow from (7.80), (7.81) with

$$a(\xi) = 24 \sum_{l=1,\ldots,n} \xi_l \,, \qquad \text{and} \qquad M_1 = 24 + \|A\| \,.$$

Then (A3)-(A4) and the quantitative bounds (4.8), (4.13) follow by (7.83)-(7.84), choosing

$$s = 1, \ r = R^{1+\frac34}\,, \ \varrho = C_* R^2 \text{ as in (7.42)}, \ N \text{ as in Lemma 7.12}, \ \theta = 2, \ \mu = 2, \ \gamma = R^{3+\frac15}\,, \qquad (7.91)$$

and taking R *small enough*. Hence Theorem 4.1 applies.

Let us verify that also the assumptions of Theorem 4.2 are fulfilled. Indeed (4.16) follows by (7.80), (7.81) with $\vec a = 24(1, \ldots, 1) \in \mathbb{R}^n$. The matrix A defined in (7.81) is invertible and

$$A^{-1} = (A^{-1}_{lh})_{1 \le l,h \le n}\,, \quad A^{-1}_{lh} = \frac{1}{12}\Big(\frac{2}{2n-1} - \delta_{lh} \Big)\,.$$

Finally the non-degeneracy assumption (4.17) is satisfied because $A = A^T$ and

$$2A^{-1}\vec a = \frac{4}{2n-1}(1, \ldots, 1) \notin \mathbb{Z}^n \setminus 0\,.$$

We deduce that the Cantor set of parameters $\mathcal{O}^*_\infty \subset \mathcal{O}$ in (4.14) has asymptotically full density because

$$\frac{|\mathcal{O} \setminus \mathcal{O}^*_\infty|}{|\mathcal{O}|} \overset{(4.19)}{\lessdot} \rho^{-1}\gamma^{2/3} \overset{(7.91)}{\lessdot} R^{-2} R^{\frac23 (3+\frac15)} = R^{\frac{2}{15}} \to 0\,.$$

The proof of Theorem 1.1 is now completed.

Remark 7.1. *The terms* $\sum_{k \ge 5} f_k s^k$ *in* (1.2) *contribute to the Hamiltonian* (7.1) *with monomials of order 6 or more and* (7.8) *holds (with a possibly different* K *satisfying* (7.10)*). On the contrary, the term* $f_4 s^4$ *in* (1.2) *would add monomials of order* 5 *to the Hamiltonian in* (7.1)*. Hence* (7.10) *holds with* R^3 *instead of* R^4*. This estimate is not sufficient. These* 5*-th order terms should be removed by a Birkhoff normal form. For simplicity, we did not pursue this point.*

8 Appendix

Proof of Lemma 2.14. We need some notation, we write

$$E = \oplus_{j=1}^4 E_j\,, \quad E_1 := (\mathbb{C}^n, |\ |_\infty)\,, \quad E_2 := (\mathbb{C}^n, |\ |_1)\,, \quad E_3 := E_4 := \ell^{a,p}_{\mathcal{I}}$$

so that a vector $v = (x, y, z, \bar z) \in E$ can be expressed by its four components $v^{(j)} \in E_j$, $v^{(1)} := x$, $v^{(2)} := y$, $v^{(3)} := z, v^{(4)} := \bar z$, and the norm (2.2) is

$$\|v\|_{E,s,r} := \sum_{j=1}^4 \frac{|v^{(j)}|_{E_j}}{\rho_j}\,, \quad \text{where } \rho_1 = s\,, \quad \rho_2 = r^2\,, \quad \rho_3 = \rho_4 = r\,. \qquad (8.1)$$

We are now ready to prove (2.64). By definition

$$
\begin{aligned}
\|dX(v)\|_{\mathcal{L}((E,s,r);(E,s',r'))} &:= \sup_{\|Y\|_{E,s,r}\le 1} \|dX(v)[Y]\|_{E,s',r'} \stackrel{(8.1)}{=} \sup_{\|Y\|_{E,s,r}\le 1} \sum_{i=1}^{4} \frac{|dX^{(i)}(v)[Y]|_{E_i}}{\rho'_i} \\
&= \sup_{\|Y\|_{E,s,r}\le 1} \sum_{i=1}^{4} \frac{|\sum_{j=1}^{4} d_{v^{(j)}} X^{(i)}(v) Y^{(j)}|_{E_i}}{\rho'_i} \\
&\le \sup_{\|Y\|_{E,s,r}\le 1} \sum_{i,j=1}^{4} \frac{|d_{v^{(j)}} X^{(i)}(v) Y^{(j)}|_{E_i}}{\rho'_i} \\
&\le \sup_{\|Y\|_{E,s,r}\le 1} \sum_{i,j=1}^{4} \frac{1}{\rho'_i} \|d_{v^{(j)}} X^{(i)}(v)\|_{\mathcal{L}(E_j,E_i)} |Y^{(j)}|_{E_j} \\
&\le \sup_{\|Y\|_{E,s,r}\le 1} \sup_{\tilde v\in D(s,r)} \sum_{i,j=1}^{4} \frac{1}{\rho'_i} \frac{|X^{(i)}(\tilde v)|_{E_i}}{(\rho_j-\rho'_j)} |Y^{(j)}|_{E_j}
\end{aligned}
$$

by the Cauchy estimates in Banach spaces. Then

$$
\begin{aligned}
\|dX(v)\|_{\mathcal{L}((E,s,r);(E,s',r'))} &\le \sup_{\tilde v\in D(s,r)} \sum_{i=1}^{4} \frac{\rho_i}{\rho'_i} \frac{|X^{(i)}(\tilde v)|_{E_i}}{\rho_i} \sup_{\|Y\|_{E,s,r}\le 1} \sum_{j=1}^{4} \Big(1-\frac{\rho'_j}{\rho_j}\Big)^{-1} \frac{|Y^{(j)}|_{E_j}}{\rho_j} \\
&\stackrel{(8.1)}{\le} \max_{i=1,\dots,4} \frac{\rho_i}{\rho'_i} \max_{j=1,\dots,4} \Big(1-\frac{\rho'_j}{\rho_j}\Big)^{-1} \sup_{\tilde v\in D(s,r)} \|X(\tilde v)\|_{E,s,r} \le 4\delta^{-1}|X|_{s,r}
\end{aligned}
$$

by (2.53), (2.65). This proves (2.64). ■

Proof of Lemma 7.2. We first extend Lemma 4 of [25] proving that:

Lemma 8.1. *If $0 \le i \le j \le k \le l$ with $i \pm j \pm k \pm l = 0$ for* SOME *combination of plus and minus signs and $(i,j,k,l) \neq (p,p,q,q)$ for $p,q \in \mathbb{N}$, then, there exists an absolute constant $c > 0$, such that*

$$
| \pm \lambda_i(\mathrm{m}) \pm \lambda_j(\mathrm{m}) \pm \lambda_k(\mathrm{m}) \pm \lambda_l(\mathrm{m})| \ge c\mathrm{m}(i^2+\mathrm{m})^{-3/2} \tag{8.2}
$$

for ALL *possible combinations of plus and minus signs*

Proof. When $i > 0$ it is a reformulation of the statement of Lemma 4 of [25]. Let us prove it also for $i = 0$. Then $j \pm k \pm l = 0$ for some combination of plus and minus signs. Since $(i,j,k,l) \neq (0,0,q,q)$, the only possibility is $l = j + k$ with $j \ge 1$ (otherwise $i = j = 0$ and $k = l$). We have to study

$$
\delta(\mathrm{m}) := \pm\lambda_0(\mathrm{m}) \pm \lambda_j(\mathrm{m}) \pm \lambda_k(\mathrm{m}) \pm \lambda_l(\mathrm{m})
$$

for all possible combinations of plus and minus signs. To this end, we distinguish them according to their number of plus and minus signs. To shorten notation we let, for example, $\delta_{++-+} = \lambda_0 + \lambda_j - \lambda_k + \lambda_l$, similarly for the other combinations. The only interesting cases are when there are one or two minus signs. The case when there are no (or four) minus signs is trivial. When there are 3 minus signs we reduce to the case with one minus sign by a global sign change.

One minus sign. Since $\delta_{++-+}, \delta_{+-++}, \delta_{-+++} \ge \delta_{+++-} := \delta$ we study only the last case. We have

$$
\delta(0) = j + k - l = 0\,, \qquad \delta'(\mathrm{m}) = \frac{1}{2}\left(\frac{1}{\lambda_0} + \frac{1}{\lambda_j} + \frac{1}{\lambda_k} - \frac{1}{\lambda_l}\right) \ge \frac{1}{2\lambda_0} = \frac{1}{2\sqrt{\mathrm{m}}}\,.
$$

Therefore $\delta(\mathrm{m}) \ge \sqrt{\mathrm{m}} \ge c\mathrm{m}(1+\mathrm{m})^{-3/2}$ for an absolute constant $c > 0$.

Two minus signs. Now we have $\delta_{-+-+}, \delta_{--++} \ge \delta_{+--+}$ and all other cases reduce to these ones by inverting signs. So we consider only $\delta = \delta_{+--+}$. Since the function $f(t) := \sqrt{t^2 + \mathrm{m}}$ is monotone increasing and convex for $t \geq 0$, we have the estimate

$$\lambda_l - \lambda_k \geq \lambda_{l-p} - \lambda_{k-p}, \quad \forall\, 0 \leq p \leq k. \tag{8.3}$$

Hence $\lambda_l - \lambda_k \geq \lambda_{j+1} - \lambda_1$ and $\lambda_{j+1} - \lambda_j \geq \lambda_2 - \lambda_1$ (using $j = l - k \geq 1$). Therefore

$$\delta = \lambda_0 - \lambda_j - \lambda_k + \lambda_l \geq \lambda_0 - \lambda_j - \lambda_1 + \lambda_{j+1} \geq \lambda_2 - 2\lambda_1 + \lambda_0 \geq \mathrm{m}(4+\mathrm{m})^{-3/2}.$$

The last inequality follows since $f''(t) = \mathrm{m}(t^2+\mathrm{m})^{-3/2}$ is decreasing and

$$\lambda_2 - 2\lambda_1 + \lambda_0 = f(2) - 2f(1) + f(0) = f''(\xi) \geq f''(2)$$

for some $\xi \in (0, 2)$. ■

We complete the proof of Lemma 7.2. We first consider the trivial cases (7.12)-(7.14).

CASE (7.12). Since $\sum_i \sigma_i \neq 0$ is even, (7.16) follows by

$$|\sigma \cdot \lambda_{\vec{\jmath}}| = |\sum_i \sigma_i \lambda_{\vec{0}}| \geq 2\lambda_{\vec{0}} = 2\sqrt{\mathrm{m}} \geq \mathrm{m}(1+\mathrm{m})^{-3/2}.$$

CASE (7.13). By $\vec{\sigma} \cdot \vec{\jmath} = (\sigma_3 + \sigma_4)q = 0$, $q \neq 0$, we deduce $\sigma_3 = -\sigma_4$. Hence (7.16) follows by

$$|\sigma \cdot \lambda_{\vec{\jmath}}| = |(\sigma_1 + \sigma_2)\lambda_0| = 2\sqrt{\mathrm{m}} \geq \mathrm{m}(1+\mathrm{m})^{-3/2}$$

CASE (7.14). Since $\vec{\jmath} = (p, p, -p, -p)$ and $\sigma_1 = \sigma_2$ then to achieve $\vec{\sigma} \cdot \vec{\jmath} = 0$ we must have $\sigma_3 = \sigma_4 = \sigma_2$ and

$$|\sigma \cdot \lambda_{\vec{\jmath}}| = |4\lambda_p| = 4\sqrt{p^2 + \mathrm{m}} \geq \mathrm{m}(p^2+\mathrm{m})^{-3/2}.$$

CASE (7.15). Set $|j_1| =: i$, $|j_2| =: j$, $|j_3| =: k$, $|j_4| =: l$. After reordering we can assume $0 \leq i \leq j \leq k \leq l$. Since, by assumption, $\vec{\sigma} \cdot \vec{\jmath} = 0$, the following combination of plus and minus signs gives

$$\mathtt{s}(j_1)\sigma_1 i + \mathtt{s}(j_2)\sigma_2 j + \mathtt{s}(j_3)\sigma_3 k + \mathtt{s}(j_4)\sigma_4 l = 0.$$

Hence Lemma 8.1 implies (7.16) for every $\vec{\jmath}$ except when $|j_1| = |j_2|$ and $|j_3| = |j_4|$ (in this case $i = j$ and $k = l$ and Lemma 8.1 does not apply). We now prove that (7.16) holds also in these cases.

We have that

$$\vec{\sigma} \cdot \lambda_{\vec{\jmath}} = (\sigma_1 + \sigma_2)\lambda_{j_1} + (\sigma_3 + \sigma_4)\lambda_{j_3}$$

where $\sigma_a + \sigma_b = 0, \pm 2$ so that (7.16) holds trivially unless $\sigma_1 + \sigma_2 = -(\sigma_3 + \sigma_4)$.

We consider this last case. If $\sigma_1 + \sigma_2 = -(\sigma_3 + \sigma_4) = 0$ then the equality

$$\vec{\sigma} \cdot \vec{\jmath} = \sigma_1(j_1 - j_2) + \sigma_3(j_3 - j_4) = 0$$

implies that $j_1, \ldots, j_4$ are pairwise equal, contrary to our hypothesis.

If $\sigma_1 + \sigma_2 = \pm 2$ and $i := |j_1| < k := |j_3|$ then

$$|\vec{\sigma} \cdot \lambda_{\vec{\jmath}}| \geq 2\lambda_{j_3} - 2\lambda_{j_1} = 2\lambda_k - 2\lambda_i \overset{(8.3)}{\geq} 2\lambda_{k-i} - 2\lambda_0 \overset{(k>i)}{\geq} 2\lambda_1 - 2\lambda_0 \geq 1/\sqrt{1+\mathrm{m}}$$

giving (7.16). If $|j_1| = |j_2| = |j_3| = |j_4|$ and $\sigma_1 + \sigma_2 = -(\sigma_3 + \sigma_4) = \pm 2$ then the relation $\vec{\sigma} \cdot \vec{\jmath} = \sigma_1(j_1 + j_2 - j_3 - j_4) = 0$ implies that the $j_1, \ldots, j_4$ are pairwise equal, contrary to the hypothesis. ■

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Massimiliano Berti, Michela Procesi, Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli Federico II, Via Cintia, Monte S. Angelo, I-80126, Napoli, Italy, m.berti@unina.it, michela.procesi@unina.it

Luca Biasco, Dipartimento di Matematica, Università di Roma 3, Largo San Leonardo Murialdo, I-00146, Roma, Italy, biasco@mat.uniroma3.it.

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