

Global Kolmogorov tori in the planetary N -body problem. Announcement of result*

Gabriella Pinzari

Dipartimento di Matematica ed Applicazioni “R. Caccioppoli”
Università di Napoli “Federico II”
Monte Sant’Angelo – Via Cinthia I-80126 Napoli (Italy)
gabriella.pinzari@unina.it

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Abstract

We improve a result in [8] by proving the existence of a positive measure set of $(3n-2)$ -dimensional quasi-periodic motions in the spacial, planetary $(1+n)$ -body problem away from co-planar, circular motions. We also prove that such quasi-periodic motions reach with continuity corresponding $(2n-1)$ -dimensional ones of the planar problem, once the mutual inclinations go to zero (this is related to a speculation in [1]). The main tool is a full reduction of the $SO(3)$ -symmetry, which retains symmetry by reflections and highlights a quasi-integrable structure, with a small remainder, independently of eccentricities and inclinations.

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1 Set Up and Background

In [1], V. I. Arnold, partly solving, but undoubtedly clarifying important mathematical settings of the more than centennial question (going back to the investigations by Sir Isaac Newton, in the XVII century) on the motions of the planetary system, asserted his “Theorem on the stability of planetary motions” as follows.

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Theorem 1.1 ([1, 19, 26, 15, 10, 22, 8]) *In the many-body problem there exists a set of initial conditions having a positive Lebesgue measure and such that, if the initial positions and velocities belong to this set, the distances of the bodies from each other will remain perpetually bounded, provided the masses of the planets, eccentricities and inclinations are sufficiently small.*

In this paper, we announce an improvement (Theorem 2.1 in the next section) of Theorem 1.1. To present it, we devote this section to a short survey of related techniques, referring the reader to the aforementioned literature, or to the review papers [11, 3, 6] or, finally, to the introduction of [23] for more details.

Consider $(1+n)$ masses in the configuration space $E^3 = \mathbb{R}^3$ interacting through gravity. Let such masses be denoted as $m_0, \mu m_1, \dots, \mu m_n$, where m_0 is a leading mass (“sun”, of “order one”), while $\mu m_1, \dots, \mu m_n$ are n smaller masses (“planets”, of “order μ ”, with μ a very small number). This problem, a sub-problem (usually referred to as “planetary” system) of the more general N -body problem, emulates the solar system; hence, the study of it has a relevant physical meaning. It is very natural to regard this system (which is Hamiltonian¹) as a small perturbation of the leading dynamical problem consisting into the gravitational interaction of the sun separately with each planet. This corresponds to what follows. After letting the system free of the invariance by translations (i.e., eliminating the motion of the sun), one can write the $3n$ -degrees of freedom Hamiltonian governing the motions of the planets as

$$\begin{aligned} H_{\text{hel}}(y, x) &= \sum_{i=1}^n h_{2\text{B}}^{(i)}(y^{(i)}, x^{(i)}) + \mu f_{\text{hel}}(y, x) \\ &= \sum_{i=1}^n \left(\frac{|y^{(i)}|^2}{2\mathbf{m}_i} - \frac{\mathbf{m}_i \mathfrak{M}_i}{|x^{(i)}|} \right) \\ &\quad + \mu \sum_{1 \leq i < j \leq n} \left(\frac{y^{(i)} \cdot y^{(j)}}{m_0} - \frac{m_i m_j}{|x^{(i)} - x^{(j)}|} \right) \end{aligned} \quad (1)$$

where $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) = q^{(i)} - q^{(0)}$ denote the “heliocentric distances”, $y^{(i)} = (y_1^{(i)}, y_2^{(i)}, y_3^{(i)})$ their generalized conjugated momenta and $\mathbf{m}_i := \frac{m_0 m_i}{m_0 + \mu m_i}$, $\mathfrak{M}_i := m_0 + \mu m_i$ the “reduced masses”.

In order to exploit the integrability of the “two-body terms”

$$h_{2\text{B}}^{(i)} := \frac{|y^{(i)}|^2}{2\mathbf{m}_i} - \frac{\mathbf{m}_i \mathfrak{M}_i}{|x^{(i)}|}$$

a natural approach is to put the system in Delaunay² coordinates. This is a system of canonical action–angle variables $((\Lambda, \Gamma, H, \ell, g, h) \in \mathbb{R}^{3n} \times \mathbb{T}^{3n})$, whose rôle is the one of transforming (via the Liouville–Arnold Theorem) $h_{2\text{B}}^{(i)}$ into “Kepler form”, i.e., a function of actions only. It is well known that, due to the too many integrals of $h_{2\text{B}}^{(i)}$, this integrated form

$$h_{\text{K}}^{(i)} = -\frac{\mathbf{m}_i^3 \mathfrak{M}_i^2}{2\Lambda_i^2} \quad (2)$$

¹I. e., its motions are described by equations of the form $\begin{cases} \dot{y}_j^{(i)} = -\partial_{x_j^{(i)}} H_{3+3n}(p, q) \\ x_j^{(i)} = \partial_{y_j^{(i)}} H_{3+3n}(p, q) \end{cases}$ where $(p^{(i)}, q^{(i)}) :=$

$(p_1^{(i)}, p_2^{(i)}, p_3^{(i)}, q_1^{(i)}, q_2^{(i)}, q_3^{(i)})$ are canonical coordinates of the point-mass i , and H_{3+3n} is a suitable $(3+3n)$ -degrees of freedom Hamilton function, depending on $(p, q) = (p^{(0)}, \dots, p^{(n)}, q^{(0)}, \dots, q^{(n)})$.

²Delaunay and (see below) Poincaré coordinates are widely described in the literature. A definition may be found, e.g., in [7, 12]. Note that $(H, h) \in \mathbb{R}^n \times \mathbb{T}^n$ are denoted as (Θ, θ) in [7]. Delaunay–Poincaré coordinates were used by several authors, including Arnold, Nekhoroshev, Herman, Laskar, Chenciner, Féjóz, Robutel, etc.. We shall see below that, due to the proper degeneracy, there is a certain freedom in choosing canonical coordinates for the planetary system. See the definition of *Kepler map* in §3.2.

exhibits a dramatic loss of degrees of freedom: two actions ($\Gamma_i := |x^{(i)} \times y^{(i)}|$ and $H_i := x_1^{(i)} y_2^{(i)} - x_2^{(i)} y_1^{(i)}$) disappear completely. This circumstance is usually called the “proper degeneracy”.

Let us denote as

$$H_{\text{Del}} = h_{\text{K}}(\Lambda) + \mu f_{\text{Del}}(\Lambda, \Gamma, H, \ell, g, h) \quad (3)$$

where

$$h_{\text{K}}(\Lambda_1, \dots, \Lambda_n) := - \sum_{1 \leq i \leq n} \frac{\mathfrak{m}_i^3 \mathfrak{M}_i^2}{2\Lambda_i^2} \quad (4)$$

the system (1) expressed in Delaunay coordinates. The purpose is to determine a positive measure set of quasi-periodic motions for this system.

In 1954 A.N. Kolmogorov [18] discovered a breakthrough property of quasi-integrable dynamical systems: for a regular, slightly perturbed system

$$H(I, \varphi) = h(I) + \mu f(I, \varphi) \quad (I, \varphi) \in A \times \mathbb{T}^\nu$$

where $A \subset \mathbb{R}^\nu$ is open, a great number of quasi-periodic motions $(I_0, \varphi_0) \rightarrow (I_0, \varphi_0 + \partial_I h(I_0)t)$ of the unperturbed system h may be continued in the dynamics of the perturbed system, provided the Hessian $\partial_I^2 h(I)$ does not vanish identically in A . Due to the proper degeneracy, for the planetary system expressed in Delaunay variables (3), taking $I := (\Lambda, \Gamma, H)$ and $\varphi := (\ell, g, h)$, Kolmogorov’s non-degeneracy assumption is clearly violated. Despite of this fact, the perturbing function has good parity properties: Arnold noticed that such parities help in determining a quasi-integrable structure in *all* the variables for the planetary system, as now we explain.

Following Poincaré, one switches from Delaunay coordinates to a new set of canonical coordinates $(\Lambda, \lambda, \eta, \xi, p, q)$. These are not in action-angle form, but are in mixed action-angle (the couples (Λ, λ)) and rectangular form (the $z := (\eta, \xi, p, q)$). The variables (Λ, λ) have roughly the same meaning of the (Λ, ℓ) ; the z are defined in a neighborhood of $z = 0 \in \mathbb{R}^{4n}$ and the vanishing of (η_i, ξ_i) or of (p_i, q_i) corresponds to the vanishing of the i^{th} eccentricity, inclination, respectively.

Let us denote as

$$H_{\text{P}} = h_{\text{K}}(\Lambda) + \mu f_{\text{P}}(\Lambda, \lambda, z) \quad z = (\eta, \xi, p, q) \quad (5)$$

the system (1) expressed in Poincaré variables.

Since the perturbation f_{hel} in (1) does not change under reflection

$$(y_1^{(i)}, y_2^{(i)}, y_3^{(i)}, x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) \rightarrow (r_1 y_1^{(i)}, r_2 y_2^{(i)}, r_3 y_3^{(i)}, r'_1 x_1^{(i)}, r'_2 x_2^{(i)}, r'_3 x_3^{(i)}) \quad r_i, r'_i = \pm 1 \quad (6)$$

and rotation transformations

$$(y_1^{(i)}, y_2^{(i)}, y_3^{(i)}, x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) \rightarrow (\mathbf{R}(y_1^{(i)}, y_2^{(i)}, y_3^{(i)}), \mathbf{R}'(x_1^{(i)}, x_2^{(i)}, x_3^{(i)})) \quad \mathbf{R}, \mathbf{R}' \in \text{SO}(3) \quad (7)$$

and due to the fact that the transformations (respectively, reflections with respect to the coordinate planes and rotation about the k -axis)

$$\begin{aligned} \mathcal{R}_1^- : \quad q^{(i)} &= (-x_1^{(i)}, x_2^{(i)}, x_3^{(i)}), & p^{(i)} &= (y_1^{(i)}, -y_2^{(i)}, -y_3^{(i)}) \\ \mathcal{R}_2^- : \quad q^{(i)} &= (x_1^{(i)}, -x_2^{(i)}, x_3^{(i)}), & p^{(i)} &= (-y_1^{(i)}, y_2^{(i)}, -y_3^{(i)}) \\ \mathcal{R}_3^- : \quad q^{(i)} &= (x_1^{(i)}, x_2^{(i)}, -x_3^{(i)}), & p^{(i)} &= (y_1^{(i)}, y_2^{(i)}, -y_3^{(i)}) \\ \mathcal{R}_g : \quad q^{(i)} &= (\mathcal{R}_g^{(3)}(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}), & p^{(i)} &= (\mathcal{R}_g^{(3)}(y_1^{(i)}, y_2^{(i)}, y_3^{(i)}) \end{aligned} \quad (8)$$

where

$$\mathcal{R}_g^{(3)} := \begin{pmatrix} \cos g & -\sin g \\ \sin g & \cos g \end{pmatrix} \quad g \in \mathbb{T}$$

have a nice expression in Poincaré variables, respectively,

$$\begin{aligned} \mathcal{R}_1^- : \quad & (\Lambda'_i, \lambda'_i, \eta'_i, \xi'_i, p'_i, q'_i) = (\Lambda_i, -\lambda_i, \eta_i, -\xi_i, -p_i, q_i) \\ \mathcal{R}_2^- : \quad & (\Lambda'_i, \lambda'_i, \eta'_i, \xi'_i, p'_i, q'_i) = (\Lambda_i, \pi - \lambda_i, -\eta_i, \xi_i, p_i, -q_i) \\ \mathcal{R}_3^- : \quad & (\Lambda'_i, \lambda'_i, \eta'_i, \xi'_i, p'_i, q'_i) = (\Lambda_i, \lambda_i, \eta_i, \xi_i, -p_i, -q_i) \\ \mathcal{R}_g : \quad & (\Lambda'_i, \lambda'_i, \eta'_i, \xi'_i, p'_i, q'_i) = (\Lambda_i, \lambda_i + g, \mathcal{R}_{-g}^{(3)}(\eta_i, \xi_i), \mathcal{R}_{-g}^{(3)}(p_i, q_i)) \end{aligned} \tag{9}$$

one then sees that the averaged (“secular”) perturbation

$$f_{\mathbb{P}}^{\text{av}}(\Lambda, \eta, \xi, p, q) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f_{\mathbb{P}}(\Lambda, \lambda, \eta, \xi, p, q) d\lambda$$

enjoys the following symmetries. If we denote

$$t_j := \frac{\eta_j - i\xi_j}{\sqrt{2}} \quad t_{j+n} := \frac{p_j - iq_j}{\sqrt{2}} \quad t_j^* := \frac{\eta_j + i\xi_j}{\sqrt{2}i} \quad t_{j+n}^* := \frac{p_j + iq_j}{\sqrt{2}i}$$

and

$$f_{\mathbb{P}}^{\text{av}}(\Lambda, t, t^*) = \sum_{a, a^* \in \mathbb{N}^n} \mathfrak{F}_{a, a^*}(\Lambda) t^\alpha t^{*\alpha^*}$$

the Taylor expansion of $f_{\mathbb{P}}^{\text{av}}$ in powers of t, t^* , we then have

Proposition 1.1 (D’Alembert rules)

$$f_{\mathbb{P}}^{\text{av}}(\Lambda, \eta, \xi, p, q) = \begin{cases} f_{\mathbb{P}}^{\text{av}}(\Lambda, \eta, -\xi, -p, q) \\ f_{\mathbb{P}}^{\text{av}}(\Lambda, -\eta, \xi, p, -q) \\ f_{\mathbb{P}}^{\text{av}}(\Lambda, \eta, \xi, -p, -q) \end{cases}$$

$$\mathfrak{F}_{a, a^*}(\Lambda) \neq 0 \iff |a|_1 = |a^*|_1, \tag{10}$$

where $|a|_1 := \sum_{i=1}^n a_i$.

By D’Alembert rules one has that the expansion of $f_{\mathbb{P}}^{\text{av}}$ around $z = 0$ contains only even monomials and starts with

$$\begin{aligned} f_{\mathbb{P}}^{\text{av}}(\Lambda, \eta, \xi, p, q) &= C_0(\Lambda) + \sum_{1 \leq i, j \leq n} \mathcal{Q}_{ij}^{(h)}(\Lambda) (\eta_i \eta_j + \xi_i \xi_j) \\ &+ \sum_{1 \leq i, j \leq n} \mathcal{Q}_{ij}^{(v)}(\Lambda) (p_i p_j + q_i q_j) + O(z^4) \end{aligned}$$

where $C_0(\Lambda)$, $\mathcal{Q}_{ij}^{(h)}(\Lambda)$ and $\mathcal{Q}_{ij}^{(v)}(\Lambda)$ are suitable coefficients, expressed in terms of Laplace coefficients, computed in [19, 15, 10]. This expansion shows that the point $z = (\eta, \xi, p, q) = 0$ is an *elliptic equilibrium point* for $f_{\mathbb{P}}^{\text{av}}(\Lambda, \eta, \xi, p, q)$. A natural question is whether, from here, it is also possible to transform $f_{\mathbb{P}}^{\text{av}}$ into

$$\check{H}_{\mathbb{P}}(\Lambda, \lambda, z) = h_K(\Lambda) + \mu \check{f}_{\mathbb{P}}(\Lambda, \lambda, z)$$

where \check{f}_P^{av} is in ‘‘Birkhoff normal form’’ (hereafter, BNF) of a suitable order (say, of order three). This means

$$\begin{aligned} \check{f}_P^{\text{av}} &= C_0(\Lambda) + \sum_{i=1}^n \sigma_i(\Lambda) w_i + \sum_{i=1}^n \varsigma_i(\Lambda) w_{i+n} + \sum_{r=2}^3 \sum_{1 \leq i_1 \dots i_r \leq 2n} \tau_{i_1 \dots i_r}(\Lambda) w_{i_1} \dots w_{i_r} \\ &+ O(z^7) \end{aligned} \quad (11)$$

where $\sigma_i(\Lambda)$, $\varsigma_i(\Lambda)$ are the eigenvalues of $\mathcal{Q}^{(h)}(\Lambda)$, $\mathcal{Q}^{(v)}(\Lambda)$ and, for $1 \leq i \leq n$, $w_i := \frac{\eta_i^2 + \xi_i^2}{2}$, $w_{i+n} := \frac{p_i^2 + q_i^2}{2}$. Then Arnold aims to solve the problem of the proper degeneracy (and hence to prove Theorem 1.1) by obtaining Kolmogorov full-dimensional tori bifurcating from the elliptic equilibrium $z = 0$, via the following abstract result.

Theorem 1.2 (The Fundamental Theorem, [1]) *Let*

$$H = h(I) + \mu f(I, \varphi, u, v) \quad (I, \varphi, u, v) \in A \times \mathbb{T}^\nu \times B \quad (12)$$

where $A \subset \mathbb{R}^\nu$, $B \subset \mathbb{R}^{2\ell}$ are open, $0 \in B$, $(I, \varphi) = (I_1, \dots, I_\nu, \varphi_1, \dots, \varphi_\nu)$, $(u, v) = (u_1, \dots, u_\ell, v_1, \dots, v_\ell)$ be real-analytic and

(i) $\det(\partial_I^2 h(I)) \neq 0$;

(ii) $f^{\text{av}} := \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} f(I, \varphi, u, v) d\varphi = \sum_{r=0}^3 \sum_{1 \leq i_1 \dots i_r \leq m} \beta_{i_1 \dots i_r}(I) w_{i_1} \dots w_{i_r} + O(u, v)^7$, where $w_i := \frac{u_i^2 + v_i^2}{2}$;

(iii) $\det(\beta_{ij}(I)) \neq 0$.

Then, for any $\kappa > 0$ one can find a number $\varepsilon_0 = \varepsilon_0(\kappa)$ such that, if $0 < \varepsilon < \varepsilon_0$ and $0 < \mu < \varepsilon^8$, the set $F_\varepsilon := A \times \mathbb{T}^\nu \times B_\varepsilon^{2\ell}(0)$ may be decomposed into a set F_ε^* which is invariant for the motions of H and a set f_ε the measure of which is smaller than κ . More precisely, F_ε^* foliates into $(\nu + \ell)$ -dimensional invariant manifolds $\{\mathcal{T}_\omega\}_\omega$ close to

$$I_i = I_i^*(\omega) \quad \varphi_i \in \mathbb{T} \quad u_i^2 + v_i^2 = \varepsilon^2 I_j^*(\omega)$$

where the motion is analytically conjugated to the linear flow

$$\theta \rightarrow \theta + \omega t \quad \theta \in \mathbb{T}^{\nu+\ell}.$$

Despite this brilliant strategy, Arnold applied Theorem 1.2 to the case of the planar three-body problem only, by explicitly checking assumptions (i)–(iii). For the general case, he was aware of some extra-difficulties, about which he gave just some vague³ indications.

A first problem is represented by the so-called ‘‘secular degeneracies’’: the ‘‘first order Birkhoff invariants’’ $\sigma_1, \dots, \sigma_n, \varsigma_1, \dots, \varsigma_n$ satisfy, identically⁴,

$$\varsigma_n \equiv 0 \quad \sum_{i=1}^n (\sigma_i + \varsigma_i) \equiv 0. \quad (13)$$

Such relations are in contrast with usual non-resonance requirements in order to construct BNF [16]. But the problem is only apparent. Indeed, it has been recently understood [21, 8] that, by

³We recall, at this respect, the aforementioned contributions by J. Laskar and P. Robutel for the spatial three-body case and by R. Herman and J. Féoz for the general case. Note that, while the strategy followed by Laskar and Robutel is intimately related to [1], the one by Herman and Féoz uses a different KAM scheme with respect to Theorem 1.2 and suitable procedures to bypass certain ‘‘degeneracies’’ recalled below.

⁴Arnold was aware only of the former relation in (13). The latter seems to have been noticed, in its full generality, by M. Herman, in the 90s.

the symmetry \mathcal{R}_g in (9), only resonances $\sum_{i=1}^n (\sigma_i(\Lambda)k_i + \varsigma_i(\Lambda)k_{i+n}) = 0$ with $\sum_{i=1}^{2n} k_i = 0$ are really important for the construction of BNF, while resonances (13) do not belong to this class. Moreover, in [10] it has been proved that they are the only ones to be *identically satisfied*; result next improved in [8], where, by direct computation, it has been seen that they are the only ones to be satisfied in an *open set*: compare item (v)–(a) of Theorem 1.3.

A much more serious problem is the following⁵

Proposition 1.2 (Rotational degeneracy [7]) *For the system (5), BNF can be constructed up to any prefixed⁶ order p but all the coefficient $\tau_{i_1 \dots i_r}(\Lambda)$ of the generic monomial $w_{i_1} \dots w_{i_r}$ with some of the i_k 's equal to $2n$ vanish identically.*

for which, in particular, the “torsion” matrix (the matrix of the second-order coefficients) $\tau = (\tau_{ij})$ has an identically vanishing row and column, hence,

$$\det \tau \equiv 0 .$$

This violates assumption (iii) of Theorem 1.2.

However, such negative result, understood only “a posteriori”, is just the counterpart of Theorem 1.3 below.

Theorem 1.3 ([22, 8, 7]) *It is possible to determine a global set of canonical coordinates⁷*

$$RPS = (\Lambda, \lambda, \eta, \xi, p, q) \tag{14}$$

which are related to Poincaré coordinates $(\Lambda, \lambda, \eta, \xi, p, q)$ by

$$\begin{aligned} \Lambda &= \Lambda , & \lambda &= \lambda + \varphi_1(\Lambda, z) & \eta_j + i\xi_j &= (\eta_j + i\xi_j)e^{i\varphi_2(\Lambda, z)} + O(z^3) \\ p &= U(\Lambda)p + O(z^3) & q &= U(\Lambda)q + O(z^3) \end{aligned} \quad (*)$$

where $U(\Lambda)$ is a $n \times n$ unitary matrix, i.e., verifying $U(\Lambda)U^t(\Lambda) = \text{id}$ and φ_1, φ_2 are suitable functions defined in a global neighborhood of $z = 0$, such that

- (i) (p_n, q_n) are integrals for f_{RPS} .
- (ii) D' Alembert rules (9) are preserved, and correspond to the reflections and the rotation in (8). In particular, denoting as

$$H_{RPS}(\Lambda, \lambda, \bar{z}) = h_K(\Lambda) + \mu f_{RPS}(\Lambda, \lambda, \bar{z})$$

the system (1) expressed in the RPS variables, where \bar{z} denotes z deprived of (p_n, q_n) , then

- (iii) The point $\bar{z} = 0 \in \mathbb{R}^{2n-1}$, which corresponds to the vanishing of all eccentricities and mutual inclinations, is an elliptic equilibrium point for $\bar{z} \rightarrow f_{RPS}^{\text{av}}(\Lambda, \bar{z})$.
- (iv) For any fixed $p \in \mathbb{N}$, $p \geq 2$, it is possible to conjugate H_{RPS} to

$$\check{H}_{RPS}(\Lambda, \check{\lambda}, \check{z}) = h_K(\Lambda) + \mu \check{f}_{RPS}(\Lambda, \check{\lambda}, \check{z})$$

where

$$\begin{aligned} \check{f}_{RPS}^{\text{av}}(\Lambda, \check{\lambda}, \check{z}) &= C_0(\Lambda) + \sum_{i=1}^n \sigma_i(\Lambda)\check{w}_i + \sum_{i=1}^{n-1} \varsigma_i(\Lambda)\check{w}_{i+n} \\ &+ \sum_{r=2}^p \sum_{1 \leq i, j \leq 2n-1} \tau_{i_1 \dots i_r}(\Lambda)\check{w}_{i_1} \dots \check{w}_{i_r} + O(\check{z}^{2p+1}) . \end{aligned}$$

⁵Proposition 1.2 answers, in particular, a question raised by M. R. Herman, who, in [15], declared not to know if the planetary torsion might vanish identically. More in general, Proposition 1.2 generalizes Laplace resonance in (13) to any order of BNF.

⁶Namely, with 3 replaced by p and $O(z^7)$ by $O(z^{2p+1})$ in (11).

⁷RPS stands for “Regular”, “Planetary” and “Symplectic”.

(v) More precisely, for any $p \in \mathbb{N}$, $a_+^{(1)} > 0$ if $a_+^{(n)} := \infty$, for any $1 \leq i \leq n-1$, it is possible to choose numbers $a_+^{(i+1)} > a_-^{(i+1)} \gg a_+^{(i)}$, such that, if $\mathcal{A} := \{\Lambda = (\Lambda_1, \dots, \Lambda_n) : a_-^{(i)} \leq a^{(i)}(\Lambda_i) \leq a_+^{(i)}\}$, then

- (a) $(\sigma(\Lambda), \bar{\zeta}(\Lambda)) \cdot k \neq 0$ for any $\Lambda \in \mathcal{A}$, $k \in \mathbb{Z}^{2n-1}$, $0 < |k|_1 \leq 2p$, $k \neq (1, \dots, 1)$;
- (b) $\det \tau(\Lambda) \neq 0$ for any $\Lambda \in \mathcal{A}$.

Clearly, Theorem 1.3 below and Theorem 1.2 (with $\nu := n$, $\ell := 2n-1$, $I := \Lambda$, $\varphi := \check{\lambda}$, $(u, v) := \check{z}$) suddenly imply Theorem 1.1, simply replacing⁸ “inclinations” with “mutual inclinations” in the statement. That (*) and (i) imply Proposition 1.2 follows by a classical unicity argument in BNF, suitably adapted to the properly–degenerate case; see [7].

We just mention that the variables (14) have been obtained via a suitable “Poincaré regularization” of a set of action–angle variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$, which we may call “planetary” Deprit variables⁹, since they are in turn easily related to a set of variables $(R, \Phi, \Psi, r, \varphi, \psi)$ studied in the 80s by F. Boigey and, in their full generality, by A. Deprit [2, 9]. The variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$ “unfold” and extend to any $n \geq 2$ a classical procedure of reduction of the number of degrees of freedom available only for the $n = 2$ case and known since the XIX century, after Jacobi [17, 25] (often referred to as “Jacobi reduction of the nodes”). For the relation between the “original” Deprit variables $(R, \Phi, \Psi, r, \varphi, \psi)$ and the planetary version $(\Lambda, \Gamma, \Psi, \lambda, \gamma, \psi)$ or the relation between the latter and Jacobi reduction of the nodes, see [23].

2 Result

Clearly, the *elliptic equilibrium point* of the secular perturbation at the origin plays a fundamental rôle in order to determine a quasi–integrable structure in the problem with respect to *all* of its degrees of freedom. As remarked, such equilibrium is determined by the symmetries of the system (i.e., relations (8)). Once the system is put in a set of coordinates such that SO(3)–symmetry is completely reduced, hence \mathcal{R}_g will not play its symmetrizing rôle anymore, the elliptic equilibrium, in general disappears. On the other hand, reducing completely the number of degrees of freedom has its advantages, since it clarifies the structure of phase space and lets the system free of extra–integrals. It is then natural to ask what is the destine of Kolmogorov tori, in such case.

In order to clarify this and other related questions, let us add some more comments.

- The variables (14) realize a *partial reduction* of the SO(3)–invariance: in such variables, the system has $(3n - 1)$ degrees of freedom, one over the minimum. As said, this is useful in order to describe with regularity the co–inclined, co–circular configuration and to keep the elliptic equilibrium for $\bar{z} = 0$. On the other hand, the fact of having one more degree of freedom than needed implies that possible $(3n - 1)$ –dimensional resonant tori corresponding to rotations in the invariable plane of non–resonant $(3n - 2)$ –dimensional tori are missed, with subsequent under–estimate ($\sim \varepsilon^{4n-2}$ instead of $\sim \varepsilon^{4n-4}$) of the measure of the invariant set F_ε^* mentioned in Theorem 1.2.
- In [8] a construction is shown that allows to switch to a “full reduction” to $(3n - 2)$ degrees of freedom. Such procedure is a bit involved, but allows, at the end to reduce completely

⁸Substantially, switching from Poincaré to RPS variables corresponds to replace the n inclinations of the planets with respect to a prefixed frame (i, j, k) , with $(n - 1)$ mutual inclinations among the planets plus the negligible inclination of the invariable plane with respect to k . Recall that the invariable plane is the plane orthogonal to the total angular momentum C .

⁹The variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$, in such “planetary form”, have been rediscovered by the author during her PhD. Note that, apart for few cases [20, 13] of application to the three–body problem, where they reduce to the variables of Jacobi reduction, Deprit variables seem to have remained un–noticed by most. See also [5] for the proof of the symplecticity of $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$ found in [22].

the number of degrees of freedom and, simultaneously, to deal with one only singularity. It generalizes the analogue singularity of Jacobi variables for $n = 2$, for which, the planar configuration is not allowed. Therefore, one has to discard a positive measure set in order to stay away from it. The measure of the invariant set F_ε^* is therefore estimated as $\sim (\varepsilon^{4n-4} - \varepsilon_0^{4n-4})$ with an arbitrary $0 < \varepsilon_0 < \varepsilon$.

- The completely reduced variables that are obtained via the full reduction of the previous item for the $n = 2$ case are analogue Jacobi's variables (they are not the same) and lead to the same BNF studied in [26]. Differently from what happens for the above discussed case $n = 2$, for $n \geq 3$, the full reduction studied in [8] loses (besides the \mathcal{R}_g -symmetry in (10)) also reflection symmetries and hence the elliptic equilibrium. Such equilibrium needs to be restored via an Implicit Function Theorem procedure, that is successful in the range of small eccentricities and inclinations.
- From the two previous items one has that, while a “continuity” (letting the inclinations to zero) between $(3n - 1)$ -dimensional Lagrangian tori of the partially reduced problem in space (whose existence has been discussed in [10, 8]) and $(2n)$ -dimensional Lagrangian tori of the unreduced planar problem follows from [8], instead, an analogous continuity between $(3n - 2)$ -dimensional Lagrangian tori of the fully reduced problem in space (again discussed in [10, 8]) and $(2n - 1)$ -dimensional Lagrangian tori of the fully planar problem (discussed in [7]) once inclinations go to zero *is naturally expected but, up to now, remains unproved*. Compare also the arguments in [26, 10] on this issue. As mentioned in the previous section, we recall, at this respect, that a controversial (indeed, erroneous) continuity argument between the planar Delaunay coordinates and the spacial planetary coordinates obtained via Jacobi reduction of the nodes was argued by Arnold [1] in order to infer non-degeneracy of BNF of the spacial three-body problem.
- Recall the definitions of $F_\varepsilon, F_\varepsilon^*$ in Theorem 1.2. In both the cases discussed above (partial and full reduction), the “density” of F_ε^* inside of F_ε , i.e., the ratio

$$d := \frac{\text{meas } F_\varepsilon^*}{\text{meas } F_\varepsilon}$$

goes to one as $\varepsilon \rightarrow 0$. That is, *one has to keep more and more close to the co-inclined, co-circular configuration, in order to encounter more and more tori*. In [4] it has been proved that one can take

$$d = 1 - \sqrt{\varepsilon} .$$

Note in fact that the perturbative technique which leads to Theorem 1.2 (or to its improvement discussed in [4]) is developed with respect to ε , rather than with respect to the initial parameter μ appearing in (1). This circumstance is an intrinsic consequence of the fact that the tori obtained via Theorem 1.2 bifurcate from the elliptic equilibrium and that, in general, the Birkhoff series (11) diverges.

- In [1] Arnold realized that, in the case of the planar three-body problem the series (11) is instead convergent (in this case $f_{\mathbb{P}}^{\text{av}}$ is integrable). This allows him to prove

$$d = 1 - \chi(\mu)$$

where $\chi(\mu) \rightarrow 0$ as $\mu \rightarrow 0$. For this particular case, the tori do not bifurcate from the elliptic equilibrium, but a different quasi-integrable structure is exploited in [1] (besides also a different perturbative technique at the place of Theorem 1.2). In [23], a slightly weaker result has been proved for the case of the spacial three-body problem and the planar general problem:

$$d = 1 - \chi(\mu, \alpha)$$

where α denotes the maximum semi axes ration and $\chi(\mu, \alpha) \rightarrow 0$ as $(\mu, \alpha) \rightarrow 0$. Note that for such cases $f_{\mathbb{P}}^{\text{av}}$ is not integrable.

- From the astronomical point of view, the investigation mentioned in the two last items is motivated by the fact that, for example, Asteroids or trans-Neptunian planets exhibit relatively large inclinations or eccentricities. From the theoretical point of view, the question is to understand wether it is possible to find different quasi-integrable structures in the planetary N-body problem besides the one determined by the elliptic equilibrium.

We prove the following result.

Theorem 2.1 *Assume that the semi-major axes of the planets are suitably spaced; let α denote the maximum of such ratios. If α is small enough and the mass ratio μ is small with respect to some power of α , one can find a number ε_0 and a positive measure set $F_{\alpha, \mu}^* \subset F := F_{\varepsilon_0}$ of Lagrangian, $(3n - 2)$ -dimensional, Diophantine tori, the density of which in F goes to one as $(\alpha, \mu) \rightarrow (0, 0)$. Letting the maximum of the mutual inclinations going to zero, such $(3n - 2)$ -dimensional tori are closer and closer to Lagrangian, $(2n - 1)$ -dimensional, Diophantine tori of the corresponding planar problem.*

In the next sections we provide the main ideas behind the proof of Theorem 2.1. Note that we shall not enter into the (technical) details of the estimate of the density of $F_{\alpha, \mu}^*$, for length reasons.

3 Tools and Sketch of Proof

The proof of Theorem 2.1 relies upon four tools.

3.1 A symmetric reduction of the $\text{SO}(3)$ -symmetry

The first tool is a new set of canonical action-angle coordinates which perform a reduction of the total angular momentum in the $(1 + n)$ -body problem, and, simultaneously, keep symmetry by reflection and are regular for planar motions. Their definition is as follows.

Let $a^{(i)} \in \mathbb{R}_+$, $P^{(i)} \in \mathbb{R}^3$, with $|P^{(i)}| = 1$, and $e^{(i)}$, denote, respectively, the *semi-major axis*, the *direction of the perihelion* and the eccentricity of the i^{th} instantaneous ellipse \mathfrak{E}_i through $(x^{(i)}, y^{(i)})$; let $\mathcal{A}^{(i)}$, with $0 \leq \mathcal{A}^{(i)} \leq \mathcal{A}_{\text{tot}}^{(i)} = \pi(a^{(i)})^2 \sqrt{1 - (e^{(i)})^2}$, the area spanned by $x^{(i)}$ on \mathfrak{E}_i with respect to $P^{(i)}$ and $C^{(i)} = x^{(i)} \times y^{(i)}$ the i^{th} angular momentum. Define the following partial sums

$$S^{(j)} := \sum_{k=j}^n C^{(k)} \quad 1 \leq j \leq n \quad (15)$$

so that $S^{(1)} := C$ is the *total angular momentum*, while $S^{(n)} = C^{(n)}$. Define, finally, the following n couples of P-nodes, $(\tilde{\nu}_j, \tilde{\eta}_j)_{1 \leq j \leq n}$

$$\tilde{\nu}_1 := k^{(3)} \times C, \quad \tilde{\eta}_j := S^{(j)} \times P^{(j)}, \quad \tilde{\nu}_{j+1} := P^{(j)} \times S^{(j+1)}, \quad \tilde{\eta}_n := P^{(n)} \quad (16)$$

with $1 \leq j \leq n - 1$. Then define the coordinates

$$P_* = (\Lambda, \chi, \Theta, \ell, \kappa, \vartheta) \quad (17)$$

where

$$\begin{aligned} \Lambda &= (\Lambda_1, \dots, \Lambda_n) \in \mathbb{R}^n & \ell &= (\ell_1, \dots, \ell_n) \in \mathbb{T}^n \\ \chi &= (\chi_0, \bar{\chi}) \in \mathbb{R} \times \mathbb{R}^{n-1} & \kappa &= (\kappa_0, \bar{\kappa}) \in \mathbb{T} \times \mathbb{T}^{n-1} \\ \Theta &= (\Theta_0, \bar{\Theta}) \in \mathbb{R} \times \mathbb{R}^{n-1} & \vartheta &= (\vartheta_0, \bar{\vartheta}) \in \mathbb{T} \times \mathbb{T}^{n-1} \end{aligned}$$

with $\bar{\chi} = (\chi_1, \dots, \chi_{n-1})$, $\bar{\kappa} = (\kappa_1, \dots, \kappa_{n-1})$, $\bar{\Theta} = (\Theta_1, \dots, \Theta_{n-1})$, $\bar{\vartheta} = (\vartheta_1, \dots, \vartheta_{n-1})$, via the following formulae.

$$\begin{aligned} \Theta_{j-1} &= \begin{cases} C_3 := C \cdot k^{(3)} & j = 1 \\ S^{(j)} \cdot P^{(j-1)} & 2 \leq j \leq n \end{cases} & \vartheta_{j-1} &= \begin{cases} \zeta := \alpha_{k^{(3)}}(k^{(1)}, \tilde{\nu}_1) & j = 1 \\ \alpha_{P^{(j-1)}}(\tilde{\mathfrak{n}}_{j-1}, \tilde{\nu}_j) & 2 \leq j \leq n \end{cases} \\ \chi_{j-1} &:= \begin{cases} G = |S^{(1)}| & j = 1 \\ |S^{(j)}| & 2 \leq j \leq n \end{cases} & \kappa_{j-1} &:= \begin{cases} \mathfrak{g} := \alpha_{S^{(1)}}(\tilde{\nu}_1, \tilde{\mathfrak{n}}_1) & j = 1 \\ \alpha_{S^{(j)}}(\tilde{\nu}_j, \tilde{\mathfrak{n}}_j) & 2 \leq j \leq n \end{cases} \quad (18) \\ \Lambda_i &:= \mathfrak{M}_i \sqrt{\mathfrak{m}_i a^{(i)}} & \ell_i &:= 2\pi \frac{\mathcal{A}^{(i)}}{\mathcal{A}_{\text{tot}}^{(i)}} := \text{mean anomaly of } x^{(i)} \text{ on } \mathfrak{E}_i \end{aligned}$$

Note that

- The variables (17) are very different from the planetary Deprit variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$ mentioned in the previous section. For example, they do not provide the Jacobi reduction of the nodes when $n = 2$. Indeed, the definition of (17) is based on $2n$ nodes (16), the nodes between the mutual planes orthogonal to $S^{(j)}$ and $P^{(j)}$ and $P^{(j)}$ and $S^{(j+1)}$. Deprit's reduction is instead based on n nodes, the nodes among the planes orthogonal to the $S^{(j)}$'s. Let us incidentally mention that, for the three-body case ($n = 2$), the variables (18) are trickily related to certain canonical variables introduced in §2.2 of [23]. This relation will be explained elsewhere.
- While, in the case of the variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$, inclinations among the $S^{(j)}$'s cannot be let to zero, it is not so for the variables (18), where the planar configuration can be reached with regularity. And in fact, in the planar case, the change between planar Delaunay variables $(\Lambda, \Gamma, \ell, g)$ and the planar version $(\Lambda, \chi, \ell, \kappa)$ of (18) reduces to

$$\begin{cases} \Lambda = \Lambda \\ \ell = \ell \end{cases} \quad \begin{cases} \chi_{i-1} = \sum_{j=i}^n \Gamma_j \\ \kappa_{i-1} = g_i - g_{i-1} \end{cases} \quad 1 \leq i \leq n$$

with $g_0 \equiv 0$. Note incidentally that the variables (18) are instead singular in correspondence of the vanishing of the inclinations about $P^{(j)}$ and $S^{(j)}$ or $S^{(j+1)}$ and $P^{(j)}$; configurations with no physical meaning.

- The variables (17) have in common with the variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$ and the Delaunay variables $(\Lambda, \Gamma, H, \ell, g, h)$ the fact of being singular for zero eccentricities (since in this case the perihelia are not defined). We however give up any attempt to regularize such vanishing eccentricities. The reason is that the Euclidean lengths of the $C^{(j)}$'s are not¹⁰ actions (apart for $\chi_{n-1} = |C^{(n)}|$) and hence the regularization does not seem¹¹ to be (if existing) easy. Note that, since we are interested to high eccentricities motions, we shall have to stay away from these singularities.
- Another remarkable property of the variables (17), besides the one of being regular for zero inclinations is that they retain the symmetry by reflections, as explained in Proposition 3.1 below. This does not happen for the variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$. As we shall explain better in the next section, such symmetry property plays a rôle in order to highlight a global¹² quasi-integrable structure of H_{χ_0} in (19) below and, especially, to have an explicit expression of it.

¹⁰Indeed, for $1 \leq j \leq n-1$, $|C^{(j)}|^2 = \chi_{j-1}^2 + \chi_j^2 - 2\Theta_j^2 + 2\sqrt{(\chi_j^2 - \Theta_j^2)(\chi_{j-1}^2 - \Theta_j^2)} \cos \vartheta_j$.

¹¹Recall that $e^{(j)} = 0$ corresponds to $|C^{(j)}| = \Lambda_j$.

¹²With a remainder independent of eccentricities and inclinations; compare Proposition 3.3.

Proposition 3.1 *The action–angle coordinates (17) are canonical. Moreover, letting H_{χ_0} the system (1) in these variables, $(\Theta_0, \vartheta_0, \chi_0)$ are integrals of motion for H_{χ_0} , which so takes the form*

$$H_{\chi_0} = h_K(\Lambda) + \mu f_{\chi_0}(\Lambda, \bar{\chi}, \bar{\Theta}, \ell, \bar{\kappa}, \bar{\vartheta}) . \quad (19)$$

Finally, in such variables, the reflection¹³ transformation

$$(y_1^{(i)}, y_2^{(i)}, y_3^{(i)}, x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) \rightarrow (y_1^{(i)}, -y_2^{(i)}, y_3^{(i)}, x_1^{(i)}, -x_2^{(i)}, x_3^{(i)}) \quad (20)$$

is

$$(\Lambda, \chi, \Theta, \ell, \kappa, \vartheta) \rightarrow (\Lambda, \chi, -\Theta, \ell, \kappa, 2\pi\mathbb{Z}^n - \vartheta) .$$

Therefore, any of the points

$$(\Theta, \vartheta) = (0, \pi k) \quad k \in \{0, 1\}^n \quad \vartheta \pmod{2\pi\mathbb{Z}^n} \quad (21)$$

which represents a¹⁴ planar configuration, is an equilibrium point for the function $(\bar{\Theta}, \bar{\vartheta}) \rightarrow f_{\chi_0}(\Lambda, \bar{\chi}, \bar{\Theta}, \ell, \bar{\chi}, \bar{\vartheta})$.

3.2 An integrability property

The second tool is an integrability property of the planetary system. To describe it, we generalize a bit the situation, introducing the concept of *Kepler map*.

- Given $2n$ positive “mass parameters” $\mathbf{m}_1, \dots, \mathbf{m}_n, \mathfrak{M}_1, \dots, \mathfrak{M}_n$, a set $\mathfrak{X} \subset \mathbb{R}^{5n}$ and a bijection

$$\begin{aligned} \tau: \quad \mathfrak{X} &\rightarrow \{(\mathfrak{E}_1, \dots, \mathfrak{E}_n) \in (E^3)^n, \mathfrak{E}_i: \text{ellipse}\} \\ \mathbf{X} \in \mathfrak{X} &\rightarrow (\mathfrak{E}_1(\mathbf{X}), \dots, \mathfrak{E}_n(\mathbf{X})) \end{aligned}$$

which assigns to any $\mathbf{X} \in \mathfrak{X}$ a n -plet of ellipses $(\mathfrak{E}_1, \dots, \mathfrak{E}_n)$ in the Euclidean space E^3 with strictly positive eccentricities and having a common focus S , we shall say that an injective map

$$\phi: \quad (\mathbf{X}, \ell) \in \mathcal{D}^{6n} := \mathfrak{X} \times \mathbb{T}^n \rightarrow (y_\phi(\mathbf{X}, \ell), x_\phi(\mathbf{X}, \ell)) \in (\mathbb{R}^3)^n \times (\mathbb{R}^3)^n$$

is a *Kepler map* if ϕ associates to $(\mathbf{X}, \ell) \in \mathfrak{X} \times \mathbb{T}^n$, with $\ell = (\ell_1, \dots, \ell_n)$ (*mean anomalies*) an element

$$(y_\phi(\mathbf{X}, \ell), x_\phi(\mathbf{X}, \ell)) = (y_\phi^{(1)}(\mathbf{X}, \ell_1), \dots, y_\phi^{(n)}(\mathbf{X}, \ell_n), x_\phi^{(1)}(\mathbf{X}, \ell_1), \dots, x_\phi^{(n)}(\mathbf{X}, \ell_n))$$

in the following way. Letting, respectively, $P_\phi^{(i)}(\mathbf{X}), a_\phi^{(i)}(\mathbf{X}), e_\phi^{(i)}(\mathbf{X})$ and $N_\phi^{(i)}(\mathbf{X})$ the direction from S to the perihelion, the semi–major axis, the eccentricity and a prefixed direction of the plane of $\mathfrak{E}_i(\mathbf{X})$, $x_\phi^{(i)}(\mathbf{X}, \ell_i)$ are the coordinates with respect to a prefixed orthonormal frame (i, j, k) centered in S of the point of $\mathfrak{E}_i(\mathbf{X})$ such that $\frac{1}{2}a_\phi^{(i)}\sqrt{1 - (e_\phi^{(i)})^2}\ell_i \pmod{\pi a_\phi^{(i)}\sqrt{1 - (e_\phi^{(i)})^2}}$ is the area spanned from $P_\phi^{(i)}(\mathbf{X})$ to $x_\phi^{(i)}(\mathbf{X}, \ell_i)$ relatively to the positive (counterclockwise) orientation determined by $N_\phi^{(i)}(\mathbf{X})$ and

$$y_\phi^{(i)}(\mathbf{X}, \ell_i) = \mathbf{m}_i \sqrt{\frac{\mathfrak{M}_i}{(a_\phi^{(i)})^3}} \partial_{\ell_i} x_\phi^{(i)}(\mathbf{X}, \ell_i) . \quad (22)$$

¹³Note that the reflection in (20) is slightly different from \mathcal{R}_2^- in (8). This is not important, since indeed in (6) the signs s_i and r_i may be chosen independently.

¹⁴Depending on the signs of the cosines of the mutual inclinations, there are 2^{n-1} planar configurations. The one with all the $C^{(i)}$ parallel and in the same verse corresponds, in the variables (18), to $(\Theta, \vartheta) = ((0, \dots, 0), (\pi, \dots, \pi))$.

- A Kepler map will be called *canonical* if any $X \in \mathfrak{X}$ has the form $X = (P, Q, \Lambda)$ where $\Lambda = (\Lambda_1, \dots, \Lambda_n) = (m_1 \sqrt{\mathfrak{M}_1 a_\phi^{(1)}}, \dots, m_n \sqrt{\mathfrak{M}_n a_\phi^{(n)}})$, $P = (P_1, \dots, P_{2n})$, $Q = (Q_1, \dots, Q_{2n})$ and the map

$$(\Lambda, \ell, P, Q) \rightarrow (y, x) = (y^{(1)}, \dots, y^{(n)}, x^{(1)}, \dots, x^{(n)})$$

preserves the standard 2-form:

$$\sum_{i=1}^n d\Lambda_i \wedge d\ell_i + \sum_{i=1}^{2n} dP_i \wedge dQ_i = \sum_{i=1}^n \sum_{j=1}^3 dy_j^{(i)} \wedge dx_j^{(i)} .$$

Examples of canonical Kepler maps are

- The map ϕ_{Del} which defines the Delaunay variables $(\Lambda, \Gamma, H, \ell, g, h)$;
- The map ϕ_{Dep} which defines the planetary Deprit variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$;
- The map ϕ_{P_*} which defines the variables $P_* = (\Lambda, \chi, \Theta, \ell, \kappa, \vartheta)$ in (18).

- The following classical relations then hold for (not necessarily canonical) Kepler maps

$$\frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\ell_i}{|x_\phi^{(i)}|} = \frac{1}{a_\phi^{(i)}} , \quad \frac{1}{2\pi} \int_{\mathbb{T}} y_\phi^{(i)} d\ell_i = 0 , \quad \frac{1}{2\pi} \int_{\mathbb{T}} \frac{x_\phi^{(i)}}{|x_\phi^{(i)}|^3} d\ell_i = 0 . \quad (23)$$

- Given a canonical Kepler map ϕ , put $H_\phi := H_{\text{hel}} \circ \phi$, where H_{hel} is as in (1). Then

$$H_\phi = h_K(\Lambda_1, \dots, \Lambda_n) + \mu f_\phi(X, \ell_1, \dots, \ell_n)$$

where h_K is as in (4) and

$$f_\phi(X, \ell_1, \dots, \ell_n) := \sum_{1 \leq i < j \leq n} \left(\frac{y_\phi^{(i)}(X, \ell_i) \cdot y_\phi^{(j)}(X, \ell_j)}{m_0} - \frac{m_i m_j}{|x_\phi^{(i)}(X, \ell_i) - x_\phi^{(j)}(X, \ell_j)|} \right)$$

is the perturbing function (1) expressed in the variables (Λ, ℓ, P, Q) . Imposing a suitable restriction of the the domain so as to exclude *orbit collision*, one has that the *secular ϕ -perturbing function*, i.e., the average

$$(f_\phi)_{\text{av}}(X) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f_\phi(X, \ell_1, \dots, \ell_n) d\ell_1 \cdots d\ell_n$$

is well defined. Due to (22), the “indirect” part of the perturbing function, i.e., the term $\sum_{1 \leq i < j \leq n} y_\phi^{(i)}(X, \ell_i) \cdot$

$y_\phi^{(j)}(X, \ell_j)/m_0$ has zero average and hence $(f_\phi)_{\text{av}}$ is just the average of the Newtonian (or “direct”) part:

$$(f_\phi)_{\text{av}}(X) = \sum_{1 \leq i < j \leq n} (f_\phi^{(ij)})_{\text{av}}$$

with

$$(f_\phi^{(ij)})_{\text{av}} := - \frac{m_i m_j}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{d\ell_i d\ell_j}{|x_\phi^{(i)}(X, \ell_i) - x_\phi^{(j)}(X, \ell_j)|} \quad i < j .$$

- If we consider the expansion

$$(f_\phi^{(ij)})_{\text{av}} = (f_\phi^{(ij)})_{\text{av}}^{(0)} + (f_\phi^{(ij)})_{\text{av}}^{(1)} + (f_\phi^{(ij)})_{\text{av}}^{(2)} + \dots$$

where

$$(f_\phi^{(ij)})_{\text{av}}^{(k)}(X) := -\frac{m_i m_j}{(2\pi)^2} \frac{1}{k!} \frac{d^k}{d\varepsilon^k} \int_{\mathbb{T}^2} \frac{d\ell_i d\ell_j}{|\varepsilon x_\phi^{(i)}(X, \ell_i) - x_\phi^{(j)}(X, \ell_j)|} \Big|_{\varepsilon=0}$$

we have that, in this expansion, the two first terms depend only on Λ_j . More precisely, due to (23),

$$(f_\phi^{(ij)})_{\text{av}}^{(0)} = -\frac{m_i m_j}{a^{(j)}} , \quad (f_\phi^{(ij)})_{\text{av}}^{(1)} = 0 .$$

Therefore, the term $(f_\phi^{(ij)})_{\text{av}}^{(2)}$ carries the first non-trivial information. In the case of the map $\phi = \phi_{\mathbb{P}_*}$, we have

Proposition 3.2 (i) *The functions $(f_{\phi_{\mathbb{P}_*}}^{(ij)})_{\text{av}}^{(2)}$ (more in general, $(f_{\phi_{\mathbb{P}_*}}^{(ij)})_{\text{av}}$) depend only on Λ_i , Λ_j , Θ_i , \dots , $\Theta_{j \wedge (n-1)}$, χ_{i-1} , \dots , $\chi_{j \wedge (n-1)}$, κ_i , \dots , κ_{j-1} , ϑ_i , \dots , $\vartheta_{j \wedge (n-1)}$ where $a \wedge b$ denotes the minimum of a and b .*

(ii) *In particular, for any $1 \leq i \leq n-1$, the nearest-neighbor terms $(f_{\phi_{\mathbb{P}_*}}^{(i, i+1)})_{\text{av}}^{(2)}$ (more, in general, $(f_{\phi_{\mathbb{P}_*}}^{(i, i+1)})_{\text{av}}$) depend only on Λ_i , Λ_j , χ_{i-1} , χ_i , $\chi_{(i+1) \wedge (n-1)}$, Θ_i , $\Theta_{(i+1) \wedge (n-1)}$, κ_i , ϑ_i , $\vartheta_{i+1 \wedge (n-1)}$.*

(iii) *The function $(f_{\phi_{\mathbb{P}_*}}^{(n-1, n)})_{\text{av}}^{(2)}$ depends only on Λ_{n-1} , Λ_n , χ_{n-2} , χ_{n-1} , Θ_{n-1} , ϑ_{n-1} , while it does not depend on κ_{n-1} . Then it is integrable.*

(iv) *$(f_{\phi_{\mathbb{P}_*}}^{(n-1, n)})_{\text{av}}^{(2)}$ is integrable in the sense of Arnold–Liouville sense: There exists a suitable global neighborhood B^2 of $0 \in \mathbb{R}^2$ (where 0 corresponds to $C^{(\nu-1)} \parallel C^{(n)}$), a set $A \subset \mathbb{R}^4$ and a real-analytic, canonical change of coordinates*

$$\begin{aligned} \phi_1 : & \left((\Lambda_{n-1}, \Lambda_n, \chi_{n-2}, \chi_{n-1}), (\tilde{\ell}_{n-1}, \tilde{\ell}_n, \tilde{\kappa}_{n-2}, \tilde{\kappa}_{n-1}), (p_{n-1}, q_{n-1}) \right) \\ & \rightarrow \left((\Lambda_{n-1}, \Lambda_n, \chi_{n-2}, \chi_{n-1}), (\ell_{n-1}, \ell_n, \kappa_{n-2}, \kappa_{n-1}), (\Theta_{n-1}, \vartheta_{n-1}) \right) \end{aligned}$$

defined on $A \times \mathbb{T}^4 \times B^2$ which transforms $(f_{\phi_{\mathbb{P}_}}^{(n-1, n)})_{\text{av}}^{(2)}$ into a function $h_{\chi_0}^{(2n+1)}$ depending only on Λ_{n-1} , Λ_n , χ_{n-2} , χ_{n-1} , $\frac{p_{n-1}^2 + q_{n-1}^2}{2}$.*

Note that

- The main point of Proposition 3.2 is that the action $\chi_{n-1} = |C^{(n)}|$ is an integral for $(f_{\phi_{\mathbb{P}_*}}^{(n-1, n)})_{\text{av}}^{(2)}$. Clearly, this is general: whatever is ϕ , $|C^{(j)}|$ is an integral for $(f_\phi^{(ij)})_{\text{av}}^{(2)}$. This fact has been observed firstly, for the case of the three-body problem, in [14], using Jacobi reduction of the nodes. In that case Harrington observed that $(f_{\phi_{\text{Jac}}^{(12)}}^{(12)})_{\text{av}}^{(2)}$ depends only on $(\Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2, G, \gamma_1)$ and the integrability is exhibited via the couple (Γ_1, γ_1) . As we already observed, in such case the planetary Deprit variables and the variables obtained by Jacobi reduction of the nodes are the same.
- An important issue that is used in the proof of Theorem 2.1 (precisely, in order to check certain non-degeneracy assumptions involved in Theorem 3.1 below) is the *effective integration* of $(f_{\phi_{\mathbb{P}_*}}^{(n-1, n)})_{\text{av}}^{(2)}$. Clearly, in principle, this could be achieved using any of the sets of variables mentioned in the two previous items: planetary Deprit variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$ or the variables (17). However, the integration using planetary Deprit variables carries considerable analytic difficulties and has been performed only qualitatively [20, 13]. Using the variables (17), such integration can be achieved by a suitable convergent Birkhoff series, exploiting the equilibrium points in (21). Compare also Proposition 3.3 and the comments below.

3.3 Global quasi-integrability of the planetary system

The third tool is the following

Proposition 3.3 *There exist natural numbers m, ν_1, \dots, ν_m , with $\nu_1 + \dots + \nu_m = 3n - 2 > m$ and a positive real number s such that, if the semi-major axes of the planets are suitably spaced and the maximum semi-axes ratio α is sufficiently small, for any positive number \bar{K} sufficiently small with respect to some positive power of α^{-1} and any μ small with respect to some power of α , one can find a number $\rho(\alpha, \bar{K})$ which goes to zero as a power law with respect to α and $\frac{1}{\bar{K}}$, positive numbers $\gamma_1, \dots, \gamma_m$ depending only on α and μ , a domain $D \subset \mathbb{R}^{3n-2}$ a global neighborhood $B^{2(n-1)}$ of $0 \in \mathbb{R}^{2(n-1)}$ and, if $C \subset \mathbb{R}^{2n-1}$ is as in Theorem 3.1 with $\nu = 3n - 2$ and $\ell = n - 1$, a real-analytic and symplectic transformation*

$$((\hat{\Lambda}, \hat{\chi}), (\hat{\ell}, \hat{\kappa}), (\hat{p}, \hat{q})) \in C_\rho \times \mathbb{T}_s^{2n-1} \times B_{\sqrt{2\rho}}^{2(n-1)} \rightarrow ((\Lambda, \bar{\chi}), (\ell, \bar{\kappa}), (\bar{\Theta}, \bar{\theta}))$$

which conjugates the Hamiltonian in (19) to

$$\hat{H}_{\chi_0} = \hat{h}_{\chi_0}(\hat{\Lambda}, \hat{\chi}, \hat{p}, \hat{q}) + \mu \hat{f}_{\chi_0}(\hat{\Lambda}, \hat{\chi}, \hat{\ell}, \hat{\kappa}, \hat{p}, \hat{q}) \quad (24)$$

where $\hat{h}_{\chi_0}(\hat{\Lambda}, \hat{\chi}, \hat{p}, \hat{q})$ depends on $(\hat{p}_i, \hat{q}_i)_{1 \leq i \leq n-1}$ only via $\hat{J}(\hat{p}, \hat{q}) := (\frac{\hat{p}_1^2 + \hat{q}_1^2}{2}, \dots, \frac{\hat{p}_{n-1}^2 + \hat{q}_{n-1}^2}{2})$ and letting ω the gradient of \hat{h}_{χ_0} with respect to $(\hat{\Lambda}, \hat{\chi}, \hat{J})$, then $D \supseteq \omega^{-1}(\mathcal{D}_{\gamma_1, \dots, \gamma_m, \tau}^{K, 3n-2}) \supset \emptyset$. Finally, the following holds. If $L, E, \hat{\rho}$ are as in Theorem 3.1, then one can take $\hat{\rho} = \rho$, $L = L_0(\alpha)/\mu$, $E = \mu E_0(\alpha)e^{-\bar{K}s}$, where $L_0(\alpha), E_0(\alpha)$ do not exceed some power of α^{-1} .

Here are some comments of the proof of Proposition 3.3.

- The function \hat{h}_{χ_0} is a sum

$$\hat{h}_{\chi_0} = \sum_{i=1}^{2n-1} \hat{h}_{\chi_0}^{(i)} \quad (25)$$

where

$$\hat{h}_{\chi_0}^{(1)}, \dots, \hat{h}_{\chi_0}^{(n)}$$

are close to the respective Keplerian terms

$$h_K^{(1)}, \dots, h_K^{(n)}$$

in (2), while

$$\hat{h}_{\chi_0}^{(n+1)}, \dots, \hat{h}_{\chi_0}^{(2n+1)}$$

are as follows. $\hat{h}_{\chi_0}^{(2n+1)}$ is close to the function $\mu h_{\chi_0}^{(2n-1)}$, where $h_{\chi_0}^{(2n-1)}$ is defined in the last item of Proposition 3.3. For $n \geq 3$ and $2n - 2 \geq i \geq n + 1$, inductively, $\hat{h}_{\chi_0}^{(i)}$ is as follows. Consider the “projection¹⁵ over normal modes” of $(f_{\phi_{P^*}}^{(i-n, i-n+1)})_{\text{av}}^{(2)} \circ \phi_1 \circ \dots \circ \phi_{2n-1-i}$ with respect to the variables (p_j, q_j) with $j \geq i - n + 1$ and $(\chi_i, \tilde{\kappa}_i)$ with $i \geq i - n$. This is a function of

$$\Lambda_{i-n}, \dots, \Lambda_n, \chi_{i-n-1}, \dots, \chi_{n-1}, \Theta_{i-n}, \vartheta_{i-n}, \frac{p_{i-n+1}^2 + q_{i-n+1}^2}{2}, \dots, \frac{p_{n-1}^2 + q_{n-1}^2}{2}$$

and is integrable in the sense of Liouville–Arnold: there exists ϕ_{2n-i} which lets this projection into a function $h_{\chi_0}^{(i)}$ of

$$\Lambda_{i-n}, \dots, \Lambda_n, \chi_{i-n-1}, \dots, \chi_{n-1}, \frac{p_{i-n}^2 + q_{i-n}^2}{2}, \dots, \frac{p_{n-1}^2 + q_{n-1}^2}{2}.$$

Then $\hat{h}_{\chi_0}^{(i)}$ is close to $\mu h_{\chi_0}^{(i)}$.

¹⁵By “projection over normal modes” of a given function $f(I, \varphi, p, q) = \sum_{(a,b) \in \mathbb{N}^m \times \mathbb{N}^m, k \in \mathbb{Z}^n} f_k(I) e^{kk \cdot \varphi} \prod_{i=1}^m (\frac{u_i - iv_i}{\sqrt{2}})^{a_i} (\frac{u_i + iv_i}{i\sqrt{2}})^{b_i}$ we mean the function $\sum_{a \in \mathbb{N}^m} f_0(I) \prod_{i=1}^m (\frac{u_i^2 + v_i^2}{2i})^{a_i}$.

- The exponential decay of E with respect to \bar{K} follows from a suitable averaging technique derived from [24], carefully adapted to our case.
- The functions in (25) are of different strengths, with respect to the mass parameter μ and the semi-major axes ratios $\alpha_i := \frac{a^{(i)}}{a^{(i+1)}}$. The first n ones, which are, as said, close to be Keplerian, are of order

$$\sim \frac{1}{a^{(1)}}, \dots, \frac{1}{a^{(n)}}.$$

The remaining $(n-1)$ ones are much smaller

$$\sim \mu \frac{(a^{(1)})^2}{(a^{(2)})^3}, \dots, \mu \frac{(a^{(n-1)})^2}{(a^{(n)})^3}$$

(they have the strength of $\mu(f_{\phi_{P_*}}^{(1,2)})_{\text{av}}^{(2)}, \dots, \mu(f_{\phi_{P_*}}^{(n-1,n)})_{\text{av}}^{(2)}$, which are so). Therefore in order to apply a KAM scheme to the Hamiltonian (24), we need a formulation suitably adapted to this case. This is given in the following section.

3.4 Multi-scale KAM theory

The fourth tool is a multi-scale KAM Theorem. To quote it, let us fix the following notations.

Given $m, \nu_1, \dots, \nu_m \in \mathbb{N}$, $\nu := \nu_1 + \dots + \nu_m$, let us decompose

$$\mathbb{Z}^\nu \setminus \{0\} = \bigcup_{i=1}^m \mathfrak{L}_i \setminus \mathfrak{L}_{i-1}$$

where

$$\mathbb{Z}^\nu =: \mathfrak{L}_0 \supset \mathfrak{L}_1 \supset \mathfrak{L}_2 \supset \dots \supset \mathfrak{L}_m = \{0\}$$

is a decreasing sequence of sub-lattices defined by

$$\mathfrak{L}_i := \{k = (k_1, \dots, k_m) \in \mathbb{Z}^\nu = \mathbb{Z}^{\nu_1} \times \dots \times \mathbb{Z}^{\nu_m} : k_1 = \dots = k_i = 0\}.$$

Next, given $\gamma, \gamma_1, \dots, \gamma_m, \tau \in \mathbb{R}_+$, define the “multi-scale Diophantine” number sets

$$\mathcal{D}_{\gamma_i; \tau}^{\nu, K, i} := \left\{ \omega \in \mathbb{R}^\nu : |\omega \cdot k| \geq \frac{\gamma}{|k|^\tau} \quad \forall k \in \mathfrak{L}_{i-1} \setminus \mathfrak{L}_i, |k|_1 \leq K \right\}$$

$$\mathcal{D}_{\gamma_1 \dots \gamma_m; \tau}^{\nu, K} := \bigcap_{i=1}^m \mathcal{D}_{\gamma_i; \tau}^{\nu, K, i} \quad \mathcal{D}_{\gamma_1 \dots \gamma_m; \tau}^\nu := \bigcap_{K \in \mathbb{N}} \mathcal{D}_{\gamma_1 \dots \gamma_m; \tau}^{\nu, K}$$

Explicitly, a number $\omega = (\omega_1, \dots, \omega_m) \in \mathbb{R}^\nu = \mathbb{R}^{\nu_1} \times \dots \times \mathbb{R}^{\nu_m}$ belongs to $\mathcal{D}_{\gamma_1 \dots \gamma_m; \tau}^\nu$ if, for any $k = (k_1, \dots, k_m) \in \mathbb{Z}^{\nu_1} \times \dots \times \mathbb{Z}^{\nu_m} \setminus \{0\}$,

$$\left| \sum_{j=1}^m \omega_j \cdot k_j \right| \geq \begin{cases} \frac{\gamma_1}{|k|^\tau} & \text{if } k_1 \neq 0; \\ \frac{\gamma_2}{|k|^\tau} & \text{if } k_1 = 0, \quad k_2 \neq 0; \\ \dots & \\ \frac{\gamma_m}{|k_m|^\tau} & \text{if } k_1 = \dots = k_{m-1} = 0, \quad \dots, \quad k_m \neq 0. \end{cases}$$

Note that the choice $m = 1$ gives the usual Diophantine set $\mathcal{D}_{\gamma_1, \tau}^\nu$. The $m = 2$ -case, with $\gamma_1 = O(1)$ and $\gamma_2 = O(\mu)$ has been considered in [1] (and [4]) for the proof of Theorem 1.2.

Theorem 3.1 (Multi-scale KAM Theorem) *Let $m, \ell, \nu_1, \dots, \nu_m \in \mathbb{N}$, with $\nu := \nu_1 + \dots + \nu_m \geq \ell$, $\tau_* > \nu$, $\gamma_1 \geq \dots \geq \gamma_m > 0$, $0 < 4s \leq \bar{s} < 1$, $\rho > 0$, $D \subset \mathbb{R}^{\nu-\ell} \times \mathbb{R}^\ell$, $A := D_\rho$, $B^{2\ell}$ a neighborhood (with possibly different radii) of $0 \in \mathbb{R}^{2\ell}$ such that, if $\bar{I}(u, v) := (\frac{u_1^2+v_1^2}{2}, \dots, \frac{u_\ell^2+v_\ell^2}{2})$, then $\Pi_{\mathbb{R}^\ell} D = \bar{I}(B^{2\ell})$, $C := \Pi_{\mathbb{R}^{\nu-\ell}} D$ and let*

$$H(I, \varphi, u, v) = h(I, u, v) + f(I, \varphi, u, v)$$

be real-analytic on $C_\rho \times \mathbb{T}_{\frac{\nu-\ell}{\bar{s}+s}}^{\nu-\ell} \times B_{\sqrt{2\rho}}^{2\ell}$, where h depends on (u, v) only via $\bar{I}(u, v)$. Assume that $\omega_0 := \partial_{(I, \bar{I})} h$ is a diffeomorphism of A with non singular Hessian matrix $U := \partial_{(I, \bar{I})}^2 h$ and let U_k denote the $(\nu_k + \dots + \nu_m) \times \nu$ submatrix of U , i.e., the matrix with entries $(U_k)_{ij} = U_{ij}$, for $\nu_1 + \dots + \nu_{k-1} + 1 \leq i \leq \nu$, $1 \leq j \leq \nu$, where $2 \leq k \leq m$. Let

$$M \geq \sup_A \|U\|, \quad M_k \geq \sup_A \|U_k\|, \quad \bar{M} \geq \sup_A \|U^{-1}\|, \quad E \geq \|f\|_{\rho, \bar{s}+s}$$

$$\bar{M}_k \geq \sup_A \|T_k\| \quad \text{if} \quad U^{-1} = \begin{pmatrix} T_1 \\ \vdots \\ T_m \end{pmatrix} \quad 1 \leq k \leq m.$$

Define

$$K := \frac{6}{s} \log_+ \left(\frac{EM_1^2 L}{\gamma_1^2} \right)^{-1} \quad \text{where} \quad \log_+ a := \max\{1, \log a\}$$

$$\hat{\rho}_k := \frac{\gamma_k}{3M_k K^{\tau_*+1}}, \quad \hat{\rho} := \min\{\hat{\rho}_1, \dots, \hat{\rho}_m, \rho\}$$

$$L := \max\{\bar{M}, M_1^{-1}, \dots, M_m^{-1}\}$$

$$\hat{E} := \frac{EL}{\hat{\rho}^2}$$

Then one can find two numbers $\hat{c}_\nu > c_\nu$ depending only on ν such that, if the perturbation f so small that the following ‘‘KAM condition’’ holds

$$\hat{c}_\nu \hat{E} < 1,$$

then, for any $\omega \in \Omega_ := \omega_0(D) \cap \mathcal{D}_{\gamma_1, \dots, \gamma_m, \tau_*}^\nu$, one can find a unique real-analytic embedding*

$$\begin{aligned} \phi_\omega : \quad \vartheta = (\hat{\vartheta}, \bar{\vartheta}) \in \mathbb{T}^\nu &\rightarrow (\hat{v}(\vartheta; \omega), \hat{\vartheta} + \hat{u}(\vartheta; \omega), \mathcal{R}_{\bar{\vartheta} + \bar{u}(\vartheta; \omega)} w_1, \dots, \mathcal{R}_{\bar{\vartheta} + \bar{u}(\vartheta; \omega)} w_\ell) \\ &\in \text{Re } C_r \times \mathbb{T}^{\nu-\ell} \times \text{Re } B_{\sqrt{2r}}^{2\ell} \end{aligned}$$

where $r := c_\nu \hat{E} \hat{\rho}$ such that $\mathbb{T}_\omega := \phi_\omega(\mathbb{T}^\nu)$ is a real-analytic ν -dimensional H -invariant torus, on which the H -flow is analytically conjugated to $\vartheta \rightarrow \vartheta + \omega t$.

Theorem 3.1 is essentially Proposition 3 of [4] suitably adapted to our case. Applying Theorem 3.1 to the Hamiltonian (24) (with $I := (\hat{\Lambda}, \hat{\chi})$, $\varphi := (\hat{\ell}, \hat{\kappa})$, $(u, v) := (\hat{p}, \hat{q})$, $\nu = 3n-2$, $\ell = n-1$, m, ν_1, \dots, ν_m as in Proposition 3.3) gives the proof of Theorem 2.1. More details will be published elsewhere. ■

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