Domains of analyticity for response solutions in strongly dissipative forced systems

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Abstract

We study the ordinary differential equation $\varepsilon\ddot{x}+\dot{x}+\varepsilon g(x)=\varepsilon f(\boldsymbol{\omega}t)$, where g and f are real-analytic functions, with f quasi-periodic in t with frequency vector $\boldsymbol{\omega}$. If $c_0\in\mathbb{R}$ is such that $g(c_0)$ equals the average of f and $g'(c_0)\neq 0$, under very mild assumptions on $\boldsymbol{\omega}$ there exists a quasi-periodic solution close to c_0 with frequency vector $\boldsymbol{\omega}$. We show that such a solution depends analytically on ε in a domain of the complex plane tangent more than quadratically to the imaginary axis at the origin.

1 Introduction

Consider the ordinary differential equation in $\mathbb{R}$

$$\varepsilon\ddot{x}+\dot{x}+\varepsilon\, g(x)=\varepsilon\, f(\boldsymbol{\omega}t), \tag{1.1}$$

where $\varepsilon\in\mathbb{R}$ is small and $\boldsymbol{\omega}\in\mathbb{R}^d$, with $d\in\mathbb{N}$, is assumed (without loss of generality) to have rationally independent components, i.e. $\boldsymbol{\omega}\cdot\boldsymbol{\nu}\neq 0\ \forall\boldsymbol{\nu}\in\mathbb{Z}^d_*:=\mathbb{Z}^d\setminus\{\mathbf{0}\}$. For $\varepsilon>0$ the equation describes a one-dimensional system with mechanical force g, subject to a quasi-periodic forcing term f with frequency vector $\boldsymbol{\omega}$ and in the presence of strong dissipation. We refer to [6] for some physical background. A quasi-periodic solution to (1.1) with the same frequency vector $\boldsymbol{\omega}$ as the forcing term will be called a *response solution*.

Hypothesis 1. *The functions $g\colon\mathbb{R}\to\mathbb{R}$ and $f\colon\mathbb{T}^d\to\mathbb{R}$ are real-analytic. There is $c_0\in\mathbb{R}$ such that $g(c_0)=f_{\mathbf{0}}$, where $f_{\mathbf{0}}$ is the average of f on $\mathbb{T}^d$, and $a:=g'(c_0)\neq 0$.*

In other words we assume that c_0 is a simple zero of the function $g(x)-f_{\mathbf{0}}$. Denote by $\Sigma_\xi:=\{\boldsymbol{\psi}=(\psi_1,\ldots,\psi_d)\in(\mathbb{C}/2\pi\mathbb{Z})^d : |\mathrm{Im}\,\psi_k|\leq\xi \text{ for } k=1,\ldots,d\}$, with $\xi>0$, the strip where f is analytic. By the analyticity assumptions one can write

$$f(\boldsymbol{\psi})=\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d}\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}}f_{\boldsymbol{\nu}},\qquad g(x)=\sum_{p=0}^{\infty}a_p\,(x-c_0)^p\,,$$

where

$$|f_{\boldsymbol{\nu}}|\leq\Phi\,\mathrm{e}^{-\xi|\boldsymbol{\nu}|},\qquad a_p:=\frac{1}{p!}\frac{\mathrm{d}^p g}{\mathrm{d}x^p}(c_0),\qquad |a_p|\leq\Gamma\,\rho^p,$$

for suitable constants Φ, Γ and ρ. Set $N(f) = N$ if f is a trigonometric polynomial of degree N and $N(f) = \infty$ otherwise, and define

$$\beta_n(\boldsymbol{\omega}) := \min\left\{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| : 0 < |\boldsymbol{\nu}| \le 2^n, |\boldsymbol{\nu}| \le N(f)\right\}, \qquad \varepsilon_n(\boldsymbol{\omega}) := \frac{1}{2^n}\log\frac{1}{\beta_n(\boldsymbol{\omega})},$$

$$\alpha_n(\boldsymbol{\omega}) := \min\left\{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| : 0 < |\boldsymbol{\nu}| \le 2^n\right\}, \qquad \mathfrak{B}(\boldsymbol{\omega}) := \sum_{n=0}^{\infty}\frac{1}{2^n}\log\frac{1}{\alpha_n(\boldsymbol{\omega})}.$$

Hypothesis 2. $\lim_{n\to\infty}\varepsilon_n(\boldsymbol{\omega}) = 0.$

In particular no assumption at all is required on $\boldsymbol{\omega}$ if f is a trigonometric polynomial, since $\beta_n(\boldsymbol{\omega})$ is eventually constant in that case. For fixed f a weaker f-dependent assumption could be required; see Section 4.

Before stating our results we need some more notations. We define the sets $C_R := \{\varepsilon \in \mathbb{C} : |\mathrm{Re}\,\varepsilon^{-1}| > (2R)^{-1}\}$ and $\Omega_{R,B} := \{\varepsilon \in \mathbb{C} : |\mathrm{Re}\,\varepsilon| \ge B\,(\mathrm{Im}\,\varepsilon)^2 \text{ and } 0 < |\varepsilon| < 2R\}$. C_R consists of two disks with radius R and centers $(R, 0)$ and $(-R, 0)$, while $\Omega_{R,B}$ is the intersection of the disk of center $(0, 0)$ and radius $2R$ with two parabolas with vertex at the origin: all such sets are tangent at the origin to the imaginary axis. Note that the smaller B, the more flattened are the parabolas. If $2RB < 1$ one has $C_R \subset \Omega_{R,B}$.

The following has been proved in [1].

Theorem 1.1. *Assume Hypotheses 1 and 2 for the system (1.1) and denote by Σ_ξ the strip of analyticity of f. Then there exist $\varepsilon_0 > 0$ and $B_0 > 0$ such that for all $B > B_0$ there is a response solution $x(t) = c_0 + u(\boldsymbol{\omega}t, \varepsilon)$ to (1.1), with $u(\boldsymbol{\psi}, \varepsilon) = O(\varepsilon)$ analytic in $\boldsymbol{\psi} \in \Sigma_{\xi'}$ and $\varepsilon \in \Omega_{\varepsilon_0,B}$, for some $\xi' < \xi$.*

In the theorem above ε_0 has to be small, while B_0 must be large enough. However, for B as close as wished to B_0 one can take $\bar\varepsilon < \varepsilon_0$ small enough for the condition $\bar\varepsilon B < 1$ to be satisfied, so as to obtain that $C_{\bar\varepsilon/2}$ is contained inside the analyticity domain. In this respect Theorem 1.1 extends previous results in the literature [6, 7], where analyticity in a pair of disks was obtained under stronger conditions on $\boldsymbol{\omega}$, such as the standard Diophantine condition

$$|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| > \frac{\gamma}{|\boldsymbol{\nu}|^\tau} \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^d_*, \tag{1.2}$$

or the Bryuno condition $\mathcal{B}(\boldsymbol{\omega}) < \infty$ If either $d = 1$ or $d = 2$ and $\boldsymbol{\omega}$ satisfies the standard Diophantine condition (1.2) with $\tau = 1$, the response solution is Borel-summable.

In the present letter we remove in Theorem 1.1 the condition for B to be large, by proving the following result.

Theorem 1.2. *Assume Hypotheses 1 and 2 for the system (1.1) and denote by Σ_ξ the strip of analyticity of f. Then for all $B > 0$ there exists $\varepsilon_0 > 0$ such that there is a response solution $x(t) = c_0 + u(\boldsymbol{\omega}t, \varepsilon)$ to (1.1), with $u(\boldsymbol{\psi}, \varepsilon) = O(\varepsilon)$ analytic in $\boldsymbol{\psi} \in \Sigma_{\xi'}$ and $\varepsilon \in \Omega_{\varepsilon_0,B}$, for some $\xi' < \xi$. The dependence of ε_0 on B is of the form $\varepsilon_0 = \varepsilon_1 B^\alpha$, for some $\alpha > 0$ and ε_1 independent of B.*

In Section 2 we introduce the main technical tools: we show that we can represent the solutions as a formal power series with coefficients that can be represented graphically in terms of trees; then in Section 3, by relying on the tree representation, we provide bounds on the coefficients which assure the convergence of the series. We anticipate that the series expansion is not a power series: indeed, the solution is not expected to be analytic in a neighbourhood of the origin; see [6, 7, 1] for further comments.

The proof of the theorem given in Section 3 yields the value $\alpha = 8$: such a value is non-optimal and could be improved by a more careful analysis. Thanks to Theorem 1.2 we can estimate the domain of analyticity by the union of the domains $\Omega_{\varepsilon_0,B}$, with $\varepsilon_0 = \varepsilon_1 B^\alpha$, by letting B varying in $(0,1]$. This provides a domain that near the origin has boundary of the form $|\mathrm{Re}\,\varepsilon| \approx \varepsilon_1^{-\beta} |\mathrm{Im}\,\varepsilon|^{2+\beta}$, where $\beta = 1/\alpha$.

As mentioned above, both Theorems 1.1 and 1.2 can be proved under a slightly weaker condition on $\boldsymbol{\omega}$, which, however, depends on the width of the strip of analyticity of f. Hypothesis 2, on the contrary, is independent of f. More comments are in Section 4.

2 Tree representation

We can rewrite (1.1) as

$$\varepsilon \ddot{x} + \dot{x} + \varepsilon\, a\,(x - c_0) + \mu\varepsilon \sum_{p=2}^{\infty} a_p (x - c_0)^p = \mu\varepsilon \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}} f_{\boldsymbol{\nu}}, \tag{2.1}$$

where $a := a_1$ and $\mu = 1$. However, we can consider μ as a free parameter and study (2.1) for $\varepsilon \in \mathbb{C}$ and $\mu \in \mathbb{R}$. Then we look for a quasi-periodic solution to (2.1) of the form

$$x(t,\varepsilon,\mu) = c_0 + u(\boldsymbol{\omega} t, \varepsilon, \mu), \qquad u(\boldsymbol{\psi},\varepsilon,\mu) = \sum_{k=1}^{\infty} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \mu^k \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}} u^{(k)}_{\boldsymbol{\nu}}(\varepsilon). \tag{2.2}$$

By inserting (2.2) into (2.1) we obtain a recursive definition for the coefficients $u^{(k)}_{\boldsymbol{\nu}}(\varepsilon)$, which admits a natural graphical representation in terms of trees. The discussion below is self-contained; however, the reader can be find useful, for details or pictures, to refer to [3, 4, 5] for a general introduction to the tree formalism and to [2] for its implementation in the same context as the present paper.

A *rooted tree* θ is a graph with no cycle, such that all the lines are oriented toward a unique point (*root*) which has only one incident line (root line). All the points in θ except the root are called *nodes*. The orientation of the lines in θ induces a partial ordering relation ($\preceq$) between the nodes. Given two nodes v and w, we shall write $w \prec v$ every time v is along the path (of lines) which connects w to the root. We shall write $w \prec \ell$ if $w \preceq v$, where v is the node which ℓ exits. For any node v denote by p_v the number of lines entering v: v is called and *end node* if $p_v = 0$ and an *internal node* if $p_v > 0$. We denote by $N(\theta)$ the set of nodes, by $E(\theta)$ the set of end nodes, by $V(\theta)$ the set of internal nodes and by $L(\theta)$ the set of lines; one has $N(\theta) = E(\theta) \amalg V(\theta)$.

We associate with each end node $v \in E(\theta)$ a *mode* label $\boldsymbol{\nu}_v \in \mathbb{Z}^d_*$ and with each internal node an *degree* label $d_v \in \{0,1\}$. With each line $\ell \in L(\theta)$ we associate a *momentum* $\boldsymbol{\nu}_\ell \in \mathbb{Z}^d$. We impose the following constraints on the labels:

1. $\boldsymbol{\nu}_\ell = \sum_{w\in E_\ell(\theta)} \boldsymbol{\nu}_w$, where $E_\ell(\theta) := \{w \in E(\theta) : w \prec \ell\}$;

2. $p_v \geq 2 \; \forall v \in V(\theta)$;

3. if $d_v = 0$ then the line ℓ exiting v has $\boldsymbol{\nu}_\ell = \mathbf{0}$.

We shall write $V(\theta) = V_0(\theta) \amalg V_1(\theta)$, where $V_0(\theta) := \{v \in V(\theta) : d_v = 0\}$. For any discrete set A we denote by $|A|$ its cardinality. Define the *degree* and the *order* of θ as $d(\theta) := |E(\theta)| + |V_1(\theta)|$ and $k(\theta) := |N(\theta)|$, respectively.

We call *equivalent* two labelled rooted trees which can be transformed into each other by continuously deforming the lines in such a way that they do not cross each other. In the following we shall consider only inequivalent labelled rooted trees, and we shall call them call trees *tout court*, for simplicity.

We associate with each node $v \in N(\theta)$ a *node factor* F_v and with each line $\ell \in L(\theta)$ a *propagator* $\mathcal{G}_\ell$, such that

$$F_v := \begin{cases} -\varepsilon^{d_v} \, a_{p_v}, & v \in V(\theta), \\ \varepsilon \, f_{\boldsymbol{\nu}_v}, & v \in E(\theta), \end{cases} \qquad \mathcal{G}_\ell := \begin{cases} 1/D(\varepsilon, \boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell), & \boldsymbol{\nu}_\ell \neq \mathbf{0}, \\ 1/a, & \boldsymbol{\nu}_\ell = \mathbf{0}, \end{cases}$$

where $D(\varepsilon, s) := -\varepsilon s^2 + \mathrm{i} s + \varepsilon \, a$. Then, by defining

$$\mathscr{V}(\theta, \varepsilon) := \left(\prod_{v \in N(\theta)} F_v \right) \left(\prod_{\ell \in L(\theta)} \mathcal{G}_\ell \right) \tag{2.3}$$

one has

$$u_{\boldsymbol{\nu}}^{(k)}(\varepsilon) = \sum_{\theta \in \mathcal{T}_{k,\boldsymbol{\nu}}} \mathscr{V}(\theta, \varepsilon), \quad \boldsymbol{\nu} \in \mathbb{Z}^d \tag{2.4}$$

where $\mathcal{T}_{k,\boldsymbol{\nu}}$ is the set of trees of order k and momentum $\boldsymbol{\nu}$ associated with the root line. Note that $u_{\mathbf{0}}^{(1)} = 0$ and $u_{\boldsymbol{\nu}}^{(2)} = 0$ for all $\boldsymbol{\nu} \in \mathbb{Z}^d$.

3 Proof of Theorem 1.2

We shall prove Theorem 1.2 in the case in which $N(f) = \infty$. The case of trigonometric polynomials is in fact easier and can be dealt with as shown in [2].

Lemma 3.1. *Set* $c_0 = \min\{1/8, B/18, B/8|a|, |a|/8, |a|B/4, \sqrt{|a|}/2\}$. *There exists* $\varepsilon_1 > 0$ *such that one has* $|D(\varepsilon, s)| \geq c_0 \max\{\min\{1, s^2\}, |\varepsilon|^2\}$ *for all* $s \in \mathbb{R}$ *and all* $\varepsilon \in \Omega_{B,\varepsilon_1}$.

Proof. Write $\varepsilon = x + \mathrm{i}y$, with $|x| \geq By^2$ and x small enough. By symmetry it is enough to study $y \geq 0$. One has $|D(\varepsilon, s)|^2 = (s + ya - ys^2)^2 + x^2(a - s^2)^2$. If $y = 0$ the bound is straightforward. If $y > 0$ denote by s_1 and s_2 the two roots of $s + ya - ys^2 = 0$: one has $s_1 = -ay + O(y^2)$ and $s_2 = 1/y + ay + O(y^2)$. Let ε_1 be so small that $|s_1 + ay| \leq |a|y/2$, $|s_2 - 1/y| \leq 1/6y$ and $18|a|y^2 \leq 1$ for $|\varepsilon| \leq \varepsilon_1$. The following inequalities are easily checked: (1) if $|s| < 2|a|y$, then $|x| \, |a - s^2| \geq |ax|/2 \geq |a|By^2/2 \geq Bs^2/8|a|$; (2) if $|s - s_2| < 1/2y$, then $|x| \, |a - s^2| \geq |x|s^2/2 \geq |x|/18y^2 \geq B/18$; (3) if $|s| \geq 2|a|y$ and $|s - s_2| \geq 1/2y$, then (3.1) $|s + ya - ys^2| \geq y|s - s_1| \, |s - s_2| \geq |a|y/4$, (3.2) $|s + ya - ys^2| \geq |s - s_1|/2 \geq |s|/8$, (3.3) if either $a < 0$ or $a > 0$ and $|a - s^2| > |a|/2$ one has $|x| \, |a - s^2| > |ax|/2$, while if $a > 0$ and $|a - s^2| \leq |a|/2$ one has $|s + ya - ys^2| \geq |s| - y|a - s^2| \geq \sqrt{a}/2$. By collecting together all the bounds the assertion follows. ∎

Lemma 3.2. *For any tree θ one has $|E(\theta)| \ge |V(\theta)| + 1$ and hence $2|E(\theta)| \ge k(\theta) + 1$.*

Proof. By induction on the order $k(\theta)$. ∎

For $v \in V_1(\theta)$ define $E(\theta, v) := \{w \in E(\theta) :$ the line exiting w enters $v\}$ and set $r_v := |E(\theta, v)|$, $s_v := p_v - r_v$, $\boldsymbol{\mu}_v := \sum_{w\in E(\theta,v)} \boldsymbol{\nu}_w$ and $\mu_v := |\boldsymbol{\mu}_v|$. Define $V_2(\theta) := \{v \in V(\theta) : s_v = 0\}$ and $V_3(\theta) := \{v \in V(\theta) : r_v = s_v = 1\}$. For $v \in V_2(\theta)$ call ℓ_v the line exiting v, and for $v \in V_3(\theta)$ call ℓ_v the line exiting v and ℓ'_v the line entering v which does not exits an end node. Define $\overline{V}_2(\theta) := \{v \in V_2(\theta) : \boldsymbol{\nu}_{\ell_v} \neq \mathbf{0}\}$ and $\overline{V}_3(\theta) := \{v \in V_3(\theta) : \boldsymbol{\nu}_{\ell_v} \neq \mathbf{0}$ and $\boldsymbol{\nu}_{\ell'_v} \neq \mathbf{0}\}$, and set $\overline{V}_1(\theta) = \overline{V}_2(\theta) \amalg \overline{V}_3(\theta)$. By construction one has $\overline{V}_1(\theta) \subset V_1(\theta)$.

Lemma 3.3. There exists $C_0 > 0$ such that $C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| \ge \mathrm{e}^{-\xi|\boldsymbol{\nu}|/16}$ $\forall \boldsymbol{\nu} \in \mathbb{Z}^d_*$.

Proof. It follows from Hypothesis 2 by using that $\beta_n(\boldsymbol{\omega}) = \alpha_n(\boldsymbol{\omega})$ if $N(f) = \infty$. ∎

Lemma 3.4. *One has $C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_v}| \ge \mathrm{e}^{-\xi\mu_v/16}$ for $v \in \overline{V}_2(\theta)$ and $2C_0 \max\{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_v}|, |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_v}|\} \ge \mathrm{e}^{-\xi\mu_v/16}$ for $v \in \overline{V}_3(\theta)$.*

Proof. For $v \in \overline{V}_2(\theta)$ one has $\boldsymbol{\nu}_{\ell_v} = \boldsymbol{\mu}_v$, so that one can use Lemma 3.3. For $v \in \overline{V}_3(\theta)$ one proceeds by contradiction. Suppose that the assertion is false: this would imply

$$\mathrm{e}^{-\xi\mu_v/16} > C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_v}| + C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_v}| \ge C_0|\boldsymbol{\omega}\cdot(\boldsymbol{\nu}_{\ell_v} - \boldsymbol{\nu}_{\ell'_v})| = C_0|\boldsymbol{\omega}\cdot\boldsymbol{\mu}_v| \ge \mathrm{e}^{-\xi\mu_v/16},$$

where we have used that $E(\theta, v)$ contains only one node w and hence $\boldsymbol{\mu}_v = \boldsymbol{\nu}_w \neq \mathbf{0}$. ∎

Define $L_1(\theta, v) := \{\ell_v\}$ for $v \in \overline{V}_2(\theta)$ and $L_1(\theta, v) := \{\ell \in \{\ell_v, \ell'_v\} : 2C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell| \ge \mathrm{e}^{-\xi\mu_v/16}\}$ for $v \in \overline{V}_3(\theta)$. Lemma 3.4 yields $L_1(\theta, v) \neq \emptyset$ for all $v \in \overline{V}_1(\theta)$. Set also $L_1(\theta) := \{\ell \in L(\theta) : \exists v \in \overline{V}_1(\theta)$ such that $\ell \in L_1(\theta, v)\}$, $L_{\rm int}(\theta) := \{\ell \in L(\theta) : \ell$ exits a node $v \in V_1(\theta)\}$ and $L_0(\theta) := L_{\rm int}(\theta) \setminus L_1(\theta)$.

Lemma 3.5. *For any tree θ one has $4\,|L_0(\theta)| \le 3|E(\theta)| - 4$.*

Proof. By induction on $V(\theta)$. If $|V(\theta)| = 1$ then either $V(\theta) = V_0(\theta)$ or $V(\theta) = \overline{V}_2(\theta)$ and hence $|L_0(\theta)| = 0$, so that the bound holds. If $|V(\theta)| \ge 2$ the root line ℓ_0 of θ exits a node $v_0 \in V(\theta)$ with $s_{v_0} + r_{v_0} \ge 2$ and $s_{v_0} \ge 1$. Call $\theta_1, \ldots, \theta_{s_{v_0}}$ the trees whose respective root lines $\ell_1, \ldots, \ell_{s_{v_0}}$ enter v_0: one has $|E(\theta)| = |E(\theta_1)| + \ldots + |E(\theta_{s_{v_0}})| + r_{v_0}$. If $\ell_0 \notin L_0(\theta)$ then $|L_0(\theta)| = |L_0(\theta_1)| + \ldots + |L_0(\theta_{s_{v_0}})|$ and the bound follows from the inductive hypothesis.

If $\ell_0 \in L_0(\theta)$ then one has $|L_0(\theta)| = 1 + |L_0(\theta_1)| + \ldots + |L_0(\theta_{s_{v_0}})|$, so that, again by the inductive hypothesis, $4|L_0(\theta)| \le 3|E(\theta)| - 3r_{v_0} - 4\,(s_{v_0} - 1)$. If either $r_{v_0} + s_{v_0} \ge 3$ or $r_{v_0} + s_{v_0} = 2$ and $s_{v_0} = 2$, the bound follows. If $r_{v_0} + s_{v_0} = 2$ and $s_{v_0} = 1$, then $v_0 \in V_3(\theta)$, so that either $\boldsymbol{\nu}_{\ell_1} = \mathbf{0}$ or $2C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1}| \ge \mathrm{e}^{-\xi\mu_{v_0}/16}$, by Lemma 3.4, because $\ell_0 \in L_0(\theta)$ and hence $2C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_0}| < \mathrm{e}^{-\xi\mu_{v_0}/16}$. Therefore $\ell_1 \notin L_0(\theta)$. If v_1 is the node which ℓ_1 exits, call $\theta'_1, \ldots, \theta'_{s_{v_1}}$ the trees whose root lines enter v_1: one has $|L_0(\theta)| = 1 + |L(\theta'_1)| + \ldots + |L_0(\theta'_{s_{v_1}})|$ and hence, by the inductive hypothesis, $4|L_0(\theta)| \le 3|E(\theta)| - 3r_{v_0} - 3r_{v_1} - 4\,(s_{v_1} - 1)$, where $3r_{v_0} + 3r_{v_1} + 4s_{v_1} - 4 \ge 5$, so that the bound follows in this case too. ∎

Lemma 3.6. *For any $k \ge 1$ and $\boldsymbol{\nu} \in \mathbb{Z}^d$ and any tree $\theta \in \mathfrak{T}_{k,\boldsymbol{\nu}}$ one has*

$$|\mathscr{V}(\theta, \varepsilon)| \le A_0^k c_0^{-k} |\varepsilon|^{1+\frac{k+1}{8}} \prod_{v\in E(\theta)} \mathrm{e}^{-5\xi|\boldsymbol{\nu}_v|/8},$$

with A_0 a positive constant depending on Φ, Γ and ρ, and c_0 as in Lemma 3.1.

Proof. One bounds (2.3) as

$$|\mathscr{V}(\theta,\varepsilon)| \le |\varepsilon|^{d(\theta)} \Bigg(\prod_{v\in V(\theta)} |a_{p_v}| \Bigg) \Bigg(\prod_{v\in E(\theta)} |f_{\boldsymbol{\nu}_v}| \Bigg) \Bigg(\prod_{\ell\in L(\theta)} |\mathcal{G}_\ell| \Bigg).$$

We deal with the propagators by using Lemma 3.1 as follows. If ℓ exits a node $v \in \overline{V}_2(\theta)$, then we have

$$|\mathcal{G}_\ell| \prod_{w\in E(\theta,v)} |f_{\boldsymbol{\nu}_w}|\, |\mathcal{G}_{\ell_w}| \le \frac{1}{c_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell|^2} \prod_{w\in E(\theta,v)} \frac{|f_{\boldsymbol{\nu}_w}|}{c_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_w|^2} \le c_0^{-1}C_0^2 (c_0^{-1}C_0^2\Phi)^{|E(\theta,v)|} \prod_{w\in E(\theta,v)} \mathrm{e}^{-3\xi|\boldsymbol{\nu}_w|/4},$$

where ℓ_w denotes the line exiting w. For the other lines in $L_1(\theta)$ we distinguish three cases: given a node $v \in V_3(\theta)$ and denoting by v' the node which the line ℓ'_v exits, (1) if either $\ell'_v \notin L_1(\theta,v)$ or $\ell'_v \in L_1(\theta,v')$, we proceed as for the nodes $v \in \overline{V}_2(\theta)$ with $\ell = \ell_v$ and obtain the same bound; (2) if $L_1(\theta,v) = \{\ell'_v\}$ and $\ell'_v \notin L_1(\theta,v')$, we proceed as for the nodes $v \in \overline{V}_2(\theta)$ with $\ell = \ell'_v$ and we obtain the same bound once more; (3) if both lines ℓ_v, ℓ'_v belong to $L_1(\theta,v)$ and $\ell'_v \notin L_1(\theta,w)$, we bound

$$\left|\mathcal{G}_{\ell_v}\mathcal{G}_{\ell'_v}\right| \prod_{w\in E(\theta,v)} |f_{\boldsymbol{\nu}_v}|\, |\mathcal{G}_{\ell_w}| \le c_0^{-2}C_0^4 (c_0^{-1}C_0^2\Phi)^{|E(\theta,v)|} \prod_{w\in E(\theta,v)} \mathrm{e}^{-5\xi|\boldsymbol{\nu}_w|/8}.$$

For all the other propagators we bound (1) $|\mathcal{G}_\ell| \le 1/|a|$ if ℓ exits a node $v \in V_0(\theta)$, (2) $|\mathcal{G}_\ell| \le c_0^{-1}|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell|^{-2}$ if ℓ exits an end node and has not been already used in the bounds above for the lines $\ell \in L_1(\theta)$, and (3) $|\mathcal{G}_\ell| \le c_0^{-1}|\varepsilon|^{-2}$ if $\ell \in L_0(\theta)$. Then we obtain

$$|\mathscr{V}(\theta,\varepsilon)| \le |\varepsilon|^{d(\theta)-2|L_0(\theta)|}\Gamma^{|V(\theta)|}\rho^{|N(\theta)|}(c_0^{-1}C_0^2)^{|V_1(\theta)|}(c_0^{-1}C_0^2\Phi)^{|E_1(\theta)|}|a|^{-|V_0(\theta)|}\mathrm{e}^{-5\xi|\boldsymbol{\nu}|/8},$$

where we can bound, by using Lemma 3.2 and Lemma 3.5, $d(\theta) - 2|L_0(\theta)| = |E(\theta)| + |V_1(\theta)| - 2|L_0(\theta)| \ge |E(\theta)| - |L_0(\theta)| \ge 1 + |E(\theta)|/4 \ge 1 + (k(\theta)+1)/8$, so that the assertion follows. ■

Lemma 3.7. *For any $k \ge 1$ and $\boldsymbol{\nu} \in \mathbb{Z}^d$ one has*

$$\left| u_{\boldsymbol{\nu}}^{(k)}(\varepsilon) \right| \le A_1^k c_0^{-k} \mathrm{e}^{-\xi|\boldsymbol{\nu}|/2} |\varepsilon|^{1+\frac{k+1}{8}},$$

with A_1 a positive constant C depending on Φ, Γ, ξ and ρ, and c_0 as in Lemma 3.1.

Proof. The coefficients $u_{\boldsymbol{\nu}}^{(k)}$ are given by (2.4). Each value $\mathscr{V}(\theta,\varepsilon)$ is bounded through Lemma 3.6. The sum over the Fourier labels is performed by using a factor $\mathrm{e}^{-\xi|\boldsymbol{\nu}_v|/8}$ for each end node $v \in E(\theta)$. The sum over the other labels is easily bounded by a constant to the power k. ■

Lemma 3.7 implies that for ε small enough the series (2.2) converges uniformly to a function analytic in $\boldsymbol{\psi} \in \Sigma_{\xi'}$, with $\xi' < \xi/2$. Moreover such a function is analytic in $\varepsilon \in \Omega_{\varepsilon_0,B}$, provided $A_1^8\varepsilon_0/c_0^8$ is small enough. This completes the proof of Theorem 1.2.

4 Comments on the assumption on the rotation vector

In [1], for f analytic in Σ_ξ, the existence of a response solution as in Theorem 1.1 is proved under the weaker condition that for some $C > 0$ and $\eta < \xi$,

$$|\boldsymbol{\omega} \cdot \boldsymbol{\nu}|^{-1} \leq C\mathrm{e}^{\eta|\boldsymbol{\nu}|/N} \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^d_*, \tag{4.1}$$

with $N = 2$. Indeed, Theorem 3 in [1] assumes the existence of an approximate solution of order $N \geq 2$ for the response solution to be proved to exist and an approximate solution of order N exists if the condition (4.1) is satisfied. Of course, the condition (4.1) is really needed not for all $\boldsymbol{\nu} \in \mathbb{Z}^d_*$, but only eventually. Similarly, one could still impose the condition on $\boldsymbol{\omega}$ in terms of the quantity $\varepsilon_n(\boldsymbol{\omega})$ introduced in Section 1, by requiring that $\varepsilon_n(\boldsymbol{\omega}) \to \eta/4$ with $\eta < \xi$. However, a condition that is optimal for fixed f is better expressed without introducing $\varepsilon_n(\boldsymbol{\omega})$, as the latter introduces a spurious dependence on the arbitrary scale 2. We also note that, in all cases, the closer η is to ξ, the smaller the domain of analyticity of the solution $u(\boldsymbol{\psi}, \varepsilon)$ in both $\boldsymbol{\psi}$ and ε. In particular, if we look for solutions which are analytic in $\Sigma_{\xi'}$ for any $\xi' < \xi/2$, we need $\eta < \xi/4$.

In the same way, if we only require for the solution to be C^∞ in ε, as in [2], we can allow $\varepsilon_n(\boldsymbol{\omega}) \to \eta/2$, with $\eta < \xi$, or $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}|^{-1} \leq C\mathrm{e}^{\eta|\boldsymbol{\nu}|}$ for some $C > 0$ and $\eta < \xi$ and all $|\boldsymbol{\nu}|$ large enough. To obtain analyticity in the domain $\Omega_{\varepsilon_0,B}$, with $\varepsilon_0 = \varepsilon_1 B^{1+1/8}$, as in Theorem 1.2, the condition $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}|^{-1} \leq C\mathrm{e}^{\eta|\boldsymbol{\nu}|/6}$ for some $C > 0$ and $\eta < \xi$, eventually in $\boldsymbol{\nu}$, would be enough. This would give $C_0|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \geq \mathrm{e}^{-\xi''|\boldsymbol{\nu}|}$, with $\xi'' < \xi/6$, in Lemma 3.3, but it is easy to realise that the analysis, from that Lemma on, could be adapted so as to obtain analyticity in a strip $\Sigma_{\xi'}$, with $\xi' > 0$. Again, in such a case, when η tends to ξ, the domains of analyticity shrink to zero, that is both ε_0 and ξ' vanish. If we want that the width of the strip of analyticity in $\boldsymbol{\psi}$ of the solution be $\xi/2$, we need a stronger condition, such as that required in Lemma 3.3, that is $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}|^{-1} \leq C\mathrm{e}^{\eta|\boldsymbol{\nu}|/16}$ for some $C > 0$ and $\eta < \xi$, eventually in $\boldsymbol{\nu}$.

We also mention that, if we allowed for the solutions to be less regular in $\boldsymbol{\psi}$, say only finitely differentiable, an even weaker condition could be assumed on $\boldsymbol{\omega}$. For instance, in order to obtain solutions C^∞ in ε, we could require

$$|\boldsymbol{\omega} \cdot \boldsymbol{\nu}|^{-1} \leq C\mathrm{e}^{\eta|\boldsymbol{\nu}|/N}|\boldsymbol{\nu}|^{-p} \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}, \tag{4.2}$$

with $N = 1$, $\eta = \xi$ and p large enough; we refer to [2] for details. Analogous considerations hold for solutions analytic in ε and finitely differentiable in $\boldsymbol{\psi}$.

However, we preferred to assume Hypothesis 2 because, even though the assumption is not optimal for fixed f, nevertheless it has the advantage to be f-independent and imply all the conditions on η considered so far. Of course, it is a challenging problem whether the existence of response solutions can be proved without any assumption on $\boldsymbol{\omega}$, like in the case of forcing terms which are trigonometric polynomials.

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