Commun. Math. Phys. 312, 501–557 (2012)
Digital Object Identifier (DOI) 10.1007/s00220-012-1483-2

Communications in
**Mathematical
Physics**

A Normal Form for the Schrödinger Equation with Analytic Non-linearities

M. Procesi[1,⋆], **C. Procesi**[2]

[1] Department of Mathematics, University of Naples Federico II, Naples 80138, Italy
[2] Department of Mathematics, University of Rome, La Sapienza, Rome 00185, Italy.
E-mail: procesi@mat.uniroma1.it

Received: 15 January 2011 / Accepted: 29 November 2011
Published online: 25 April 2012 –

Abstract: We discuss a class of normal forms of the completely resonant non-linear Schrödinger equation on a torus. We stress the geometric and combinatorial constructions arising from this study.

Contents

⋆ Supported by ERC grant "New connections between dynamical systems and Hamiltonian PDEs" and partially by the PRIN2009 grant "Critical Point Theory and Perturbative Methods for Nonlinear Differential Equations".

1. Introduction

In this paper we exhibit a normal form, with remarkable integrability properties, for the completely resonant non-linear Schrödinger equation on the torus $\mathbb{T}^n$, $n \in \mathbb{N}$ (NLS for brevity):

$$- iu_t + \Delta u = \kappa |u|^{2q} u + \partial_{\bar{u}} G(|u|^2), \quad q \geq 1 \in \mathbb{N}. \tag{1}$$

where $u := u(t, \varphi), \varphi \in \mathbb{T}^n$ and $G(a)$ is a real analytic function whose Taylor series starts from degree $q + 2$. The case $q = 1$ is of particular interest and is usually referred to as the *cubic NLS*.

It is well known that Eq. 1, the NLS, can be written as an infinite dimensional Hamiltonian dynamical system.

It has the *energy* $H = \int_{\mathbb{T}^n} (|\nabla(u)|^2 + \kappa(q+1)^{-1}|u|^{2(q+1)} + G(|u|^2)) \frac{d\phi}{(2\pi)^n}$, the *momentum* $M = \int_{\mathbb{T}^n} \bar{u}(\varphi)\nabla u(\varphi) \frac{d\phi}{(2\pi)^n}$ and the *mass* $L = \int_{\mathbb{T}^n} |u(\varphi)|^2 \frac{d\phi}{(2\pi)^n}$, as integrals of motion.

Passing to the Fourier representation

$$u(t, \varphi) := \sum_{k \in \mathbb{Z}^n} u_k(t) e^{\mathrm{i}(k,\varphi)}, \tag{2}$$

we have, up to a rescaling of u and of time, in coordinates:

$$H := \sum_{k \in \mathbb{Z}^n} |k|^2 u_k \bar{u}_k \pm \sum_{k_i \in \mathbb{Z}^n : \sum_{i=1}^{2q+2} (-1)^i k_i = 0} u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4} \dots u_{k_{2q+1}} \bar{u}_{k_{2q+2}} + \int_{\mathbb{T}^n} G(|u|^2) \frac{d\phi}{(2\pi)^n}. \tag{3}$$

We fix the sign to be + since in our treatment it does not play any particular role.

1.1. Preliminaries. By Formula (3), we can write Eq. (1) as an infinite dimensional Hamiltonian dynamical system, where the quadratic term consists of infinitely many independent oscillators with rational frequencies and hence completely resonant (all the bounded solutions are periodic). In order to study resonant systems a standard instrument is the "Resonant Birkhoff normal form". In Formula (3) denote by $K := \sum_{k\in\mathbb{Z}^n} |k|^2 u_k \bar{u}_k$.

The first step of "resonant Birkhoff normal form" is the sympletic change of variables which reduces the Hamiltonian H to

$$H = H_{\text{Res}} + H^{(2q+4)}; \quad H_{\text{Res}} = K + H^{(2q+2)}_{res}(u, \bar{u}),$$

where $H^{(2q+4)}$ is an analytic function of degree at least $2q+4$ while $H^{(2q+2)}_{res}$ is of degree $2q+2$ and consists exactly of the degree $2q+2$ terms of (3) which Poisson commute with K. Then one wishes to treat the *truncated system* $H_{\text{Res}} = K + H^{(2q+2)}_{res}(u, \bar{u})$, as the new unperturbed Hamiltonian and $H^{(2q+4)}$ as a small perturbation. An ideal situation is when the truncated system is integrable, this is the case for the cubic NLS in dimension 1, as shown by Kuksin and Pöschel in [14]. However the special degenerations of the truncated system used by these authors are not valid in the case of the non-cubic NLS, already in dimension one, nor for the cubic case in dimension higher than one.

Although the truncated system appears to be very complicated (see formula (8)) we show that it admits infinitely many invariant subspaces (cf. §2.1.1), defined by requiring $u_k = 0$ for all $k \notin S$, where $S = \{v_1, \dots, v_m\}$, *tangential sites*, is some (arbitrarily large) subset of $\mathbb{Z}^n$ satisfying the *completeness condition* (see Proposition 1).

The dynamics on these subspaces depends in a subtle way on the geometric properties of S, we show – in Proposition 1 *ii)*– that for generic choices of S the behavior is integrable and that all the $|u_{v_i}|$ are constants of motion. Suitable non-generic choices of S lead also to interesting non-integrable behavior as for instance in the paper [7].

By momentum conservation, it is easily seen that for any set $S \subset \mathbb{Z}^n$, the subspace $u_k = 0$ for all $k \notin \text{Span}(S)$ is invariant. We restrict to this subspace[1] and denote by $S^c := \text{Span}(S)\setminus S$ the *normal sites*. We collect in H_{Res} the terms by the degree (which we denote by $\#S^c$) in the variables $u_k,\ \bar{u}_k,\ k \in S^c$ we have

$$H_{Res} = H_S + H_{\#S^c=1} + H_{\#S^c=2} + H_{\#S^c>2},$$

by definition the completeness is equivalent to the fact that the term of degree one is zero, i.e. $H_{\#S^c=1} = 0$.

We show that the term $H_{\#S^c>2}$ is negligible and we give an explicit formula for $H_{\#S^c=2}$ described by an infinite dimensional matrix (cf. Formula (30)) with coefficients depending on the "tangential angles". This is done explicitly by 1) putting the tangential variables in action–angle coordinates and then 2) introducing parameters for the actions and finally 3) isolating the terms of the Hamiltonian H_{Res} of degree ≤ 2. The resulting Hamiltonian is what we call the *normal form*, it is quadratic and explicitly described by a matrix which depends on the "tangential angles". Hence the dynamics of this quadratic Hamiltonian is apparently non-integrable and given by an infinite set of coupled linear equations with *non-constant* coefficients.

It is natural at this point to try to reduce the normal form to constant coefficients, exploiting the fact that $H^{(2q+2)}_{res}$ is *smaller* than K. However the quadratic term K is very degenerate and does not satisfy the *second Melnikov condition*, hence the perturbative

[1] Notice that this subspace is invariant not only for H_{Res} but also for the full Hamiltonian H.

methods, see for instance [9], fail. In finite dimensional systems one can still approximately reduce (to constant coefficients) matrices whose diagonal part does not satisfy the second Melnikov condition, see [6]. This is done via a change of variables which is not close–to–identity and hence must be constructed explicitly. In our infinite dimensional setting however these kinds of results are *not* applicable, since in general the change of variables suggested by the finite dimensional analog is not analytic.

1.2. The object of this paper. The main contribution of this paper is to construct, for generic S, an explicit analytic symplectic change of variables which removes the dependence from the tangential angles so that the Normal form is block–diagonal (with blocks of dimension $\leq 2n$) and integrable, see Theorem 1 for a precise statement. Notice that this symplectic transformation is not close–to–identity. It is given by explicit algebraic formulas (Formula (70)) and not constructed through a recursive algorithm. This is due to the fact that we can achieve a complete control on the diagonal blocks of the normal form. In turn this is done by codifying the corresponding matrix in terms of graphs, see Definition 9, and describing the possible blocks which may appear in the normal form, depending on the choice of the tangential sites, combinatorially using *finitely* many graphs.

Then we find *optimal* constraints on the tangential sites, given by a finite list of polynomial inequalities on the coordinates of S. If S satisfies these inequalities we say that it is *generic* and then, these constraints make the normal form *as simple as possible*.

We organize our constraints in 6 different requirements, summarized in Definition 22. Under these constraints the normal form is block–diagonal with blocks of dimension bounded by $n + 1$, except finitely many exceptional blocks of size bounded by $2n$. The diagonal blocks are explicitly described as functions of the average tangential actions ξ and angles x.

Then, for these infinitely many choices of the tangential sites S, we exhibit[2] a symplectic change of variables (cf. Formula (70)) which makes the normal form with constant coefficients and still block–diagonal.

Finally we show that, in dimension one and two, the normal form has both stable and unstable regions, namely there are open sets for the parameters ξ where the normal form is completely elliptic–hence its Hamiltonian flow is stable. For all the remaining values of the parameters ξ there are a finite number of unstable directions. In the stable region one may perform a further analytic change of variables which reduces the normal form to the standard elliptic one $(\omega(\xi), y) + \sum_k \bar{\Omega}_k |z_k|^2$ (cf. Corollary 1).

1.3. Some related literature. The idea of choosing an appropriate set of tangential sites S was first used by Bourgain in [4] in a slightly different context. He studied the cubic NLS in dimension two and proved the existence of quasi–periodic solutions with two frequencies by using a combination of Lyapunov-Schmidt reduction techniques and a Nash–Moser algorithm to solve the small divisor problem (the so–called *Craig–Wayne–Bourgain* approach, see [4,8] and for a recent generalization also [2]). In [4] it is shown that, for appropriate choices of the tangential sites, one may find simple solutions for the bifurcation equation where only the Fourier indexes of the tangential sites are excited.

[2] In general, in order to construct a change of variables one solves a Hamilton–Jacobi equation, finding a generating function for the change of variables. In our case however we do not use this procedure, indeed the change of variables was *guessed* directly.

This strategy was generalized by Wang in [17] to study the NLS on a torus $\mathbb{T}^n$ and prove existence of quasi periodic solutions with n frequencies. A similar idea was exploited in [12] and [13] to look for "wave packet" periodic solutions (i.e. periodic solutions which at leading order excite an arbitrarily large number of "tangential sites") of the cubic NLS in any dimension both in the case of periodic and Dirichlet boundary conditions.

In the context of KAM theory and normal form, this idea was used by Geng in [10] for the NLS in dimension one with the nonlinearity $|u|^4 u$.

A similar strategy is used by Geng-You and Xu in [11], to study the cubic NLS in dimension two. In that paper the authors show that one may give constraints on the tangential sites so that the normal form is non-integrable (i.e. it depends explicitly on the angle variables) but block diagonal with blocks of dimension 2. They apply this result to perform a KAM algorithm and prove existence (but not stability) of quasi–periodic solutions. We also mention the paper [16], which studies the non-local NLS and the beam equation both for periodic and Dirichlet boundary conditions. The main result of that paper is that, by only requiring very simple constraints on the tangential sites, the leading order of the normal form Hamiltonian is quadratic and block diagonal, with blocks of uniformly bounded dimension.

Finally we mention the preprints by Wang [18] and [19], which use the *Craig–Wayne–Bourgain* approach to study quasi–periodic solutions for the NLS (1) in any dimension.

1.4. Description of the paper. In Sect. 2 we introduce some necessary Hamiltonian formalism, we perform the Birkhoff change of variables and study the *truncated system H_{Res}*. In particular we study invariant subspaces and in Propositions 1 and 2 we give conditions for their *completeness* and *integrability*. Finally we pass to the elliptic–action angle variables and define the functional domains in which we work. All the results and techniques of this section are pretty standard so we try to review them concisely.

Having introduced the relevant notations, in Sect. 3 we give the notion of *generic* tangential set S and state our main results Theorem 1 and Corollary 1.

In Sect. 4 we impose Constraint 1 on the tangential sites S; this enables us to define our normal form N – see Proposition 4–and prove that N satisfies non-degeneracy in the action variables– see Proposition 5. Finally we discuss the perturbation P and estimate its size– see Proposition 6.

In Sect. 5 we define two spaces $V^{0,1}$ and $F^{0,1}$ on which we study the linear operator $ad(N) := \{N, \cdot\}$. This gives two matrix descriptions of N.

In Sect. 6 we describe the two matrices in terms of two graphs $\tilde{\Gamma}_S$ and Λ_S with vertices respectively the basis elements of $V^{0,1}$ and $F^{0,1}$ and edges connecting those couples of elements which have a non-zero matrix coefficient. This is a standard way to display infinite matrices, in particular one easily sees that the connected components of the graph correspond to block–diagonal terms in the matrix.

From these graphs we deduce a more abstract *geometric graph* Γ_S which still contains all the information necessary to compute the matrix entries of $ad(N)$.

In 6.1 we define a graph Γ_S^{geo} with vertices on $\mathbb{R}^n$ which contains Γ_S but is easier to study. With these notations we prove– Proposition 7– a first rough bound on the dimension of the block–diagonal blocks in $ad(N)$.

Finally in 6.2– Theorems 2 and 3– we state our main results on the connected components of Γ_S^{geo} and Γ_S, this is the core of the paper. It is interesting to notice that these results hold independently of the number of tangential sites and hence remain true also if one excites infinitely many tangential sites.

It is possible that our constraints may be improved (the best possible result is that the geometric constraints are sufficient to bound the dimension of all the blocks by $n + 1$). This is actually true in low dimensions $n = 1, 2$ for all q. For $q = 1$ we believe it to be true in any dimension, this will be discussed in a separate paper.

In Sect. 7 we formalize our graphs as subgraphs of a Cayley graph (we group the relevant definitions and properties in the Appendix). This is the content of Proposition 9 and enables us to endow our graphs with a group action that simplifies the combinatorial analysis.

In Sect. 8 we impose Constraints 2 and 3. This enables us to identify the connected components of Λ_S with those of Γ_S– see Proposition 10 and Corollary 4. The isomorphism between the connected components of the two graphs is the key point which allows us to construct the change of variables which sends N to constant coefficients, as can be seen in Example 14.

In Sect. 9 we define a finite set of connected graphs, the *possible combinatorial graphs*. To a graph $\mathcal{A}$ of this set, with k vertices, we associate a list of $k-1$ linear and quadratic equations in n variables, given by Formula (61). Then in Proposition 11 we show that $\mathcal{A}$ is isomorphic to a connected subgraph of Γ_S if and only if its equations have solutions in S^c (the solutions are identified with a special vertex in Γ_S, called the root). This enables us to describe the infinite connected components of Γ_S via a finite set of graphs.

To a possible combinatorial graph $\mathcal{A}$ we associate its Eqs. (61), which have as coefficients linear and quadratic functions of the tangential sites. If these equations do not have real solutions for generic choices of S then $\mathcal{A}$ cannot be isomorphic to a connected subgraph of Γ_S for generic S. This is a geometric condition from which one expects to be able to rule out the connected components of Γ_S as soon as $k-1 \geq n+1$ by imposing that those overdetermined systems of equations be generically incompatible. However this simple idea does not cover various pathological cases. We try to give an idea of the main problems.

Given a graph $\mathcal{A}$ with k vertices its Eqs. (61) may not be of maximal rank for particular choices of $S \subset \mathbb{Z}^{k-1}$, this can be avoided by introducing appropriate genericity constraints, as Constraint 5. Unfortunately it may well be, see Example 9, that Eqs. (61) are linearly dependent for all choices of S, independently of the dimension n such that $S \in \mathbb{Z}^n$. In this case one is faced with a compatibility problem, namely one can try to exclude these graphs by requiring that the equations are incompatible for generic choices of S, see Example 9 and Constraint 4.

This does not conclude the analysis since it is possible that the equations be always compatible, see Remark 22. So it is possible that one has a graph with $k > n+1$ vertices but still with rank $\leq n$, this is the reason of our bound $k \leq 2n$. To simplify the problem we introduce the notion of *colored rank*, see Definition 20; we have Theorem 4.

In Sect. 10, using Theorem 4, we discuss *possible combinatorial graphs* $\mathcal{A}$ with rank $r = n + 1$, when $S \subset \mathbb{Z}^n$.

We prove that if their equations are always compatible then their (unique) solution must be a point in S. This means that $\mathcal{A}$ cannot be isomorphic to a connected subgraph of Γ_S (which has vertices in S^c).

This enables us to prove Theorems 2 and 3.

In Sect. 11 we prove Theorem 1 by exhibiting in Formula 70 the change of variables which reduces the normal form N to constant coefficients. We also give explicit formulæ which allow to compute N in this new set of variables, via the *combinatorial graphs*.

In Sect. 12 we prove Proposition 3 and Corollary 1. The most relevant notion is that of arithmetic constraint. Roughly speaking we want to ensure that if a combinatorial graph $\mathcal{A}$ is such that its equations have a *unique solution* in $\mathbb{R}^2$, then this solution is not integer valued.

This result enables us to prove the existence of stable regions for the parameters ξ, where N is purely elliptic. Corollary 1 follows from the theory of Quadratic hamiltonians and from Proposition 3.

2. Hamiltonian Formalism

We work on the scale of complex Hilbert spaces

$$\bar{\boldsymbol{\ell}}^{(a,p)} := \{u = \{u_k\}_{k\in\mathbb{Z}^n} \mid |u_0|^2 + \sum_{k\in\mathbb{Z}^n} |u_k|^2 e^{2a|k|}|k|^{2p} := ||u||^2_{a,p} < \infty\}, \qquad (4)$$
$$a > 0,\;\; p > n/2\,,$$

equipped with the symplectic structure $\mathrm{i}\sum_{k\in\mathbb{Z}^n} du_k \wedge d\bar{u}_k$.

These choices are rather standard in the literature and consist in requiring that the functions $u(\varphi)$ extend to analytic functions in the complex domain $|Im(\varphi)| < a$, with Sobolev regularity on the boundary, the condition $p > n/2$ ensures that our function spaces are Hilbert algebras.

Remark 1. It is not necessary to assume that the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. The theory works and in fact we shall apply it, also if $\mathbb{T}^n = \mathbb{R}^n/\Lambda$, where Λ is a lattice generated by a not necessarily orthonormal basis.

We may write, for any d,

$$[u]^{2d} := \sum_{k_i\in\mathbb{Z}^n} u_{k_1}\bar{u}_{k_2}u_{k_3}\bar{u}_{k_4}\ldots u_{k_{2d-1}}\bar{u}_{k_{2d}} = \sum_{\substack{\alpha,\beta\in(\mathbb{Z}^n)^{\mathbb{N}}:\\ |\alpha|=|\beta|=d}} \binom{d}{\alpha}\binom{d}{\beta} u^\alpha \bar{u}^\beta, \qquad (5)$$

where $\alpha : k \mapsto \alpha_k \in \mathbb{N}$ and $u^\alpha = \prod_k u_k^{\alpha_k}$, the same for β. It is easily seen that for any d the function $[u]^{2d}$ is an analytic function of $u, \bar{u}$. Moreover $[u]^{2d}$ is *regular*, namely its Hamiltonian vector field is an analytic function from $\bar{\boldsymbol{\ell}}^{(a,p)} \times \bar{\boldsymbol{\ell}}^{(a,p)}$ to itself.

In formula (3) we may expand G in Taylor series obtaining a totally convergent sum of terms $[u]^{2d}$; this shows that our Hamiltonian is analytic and regular.

The torus $\mathbb{T}^n$ acts on itself by translations leaving invariant the symplectic form, in fact it gives rise in this way to a *moment map* in the sense of symplectic Geometry or a *momentum vector* in the language of Mechanics. The Hamiltonian is invariant under translation so by Noether's Theorem it Poisson commutes with momentum.

We thus will systematically apply the fact that our Hamiltonian H (see Formula (3)) has $n+1$ conserved quantities: the n–vector *momentum* $M = \sum k|u_k|^2$ and the scalar *mass* $L = \sum_k |u_k|^2$, with

$$\{M, u_h\} = \mathrm{i}hu_h, \quad \{M, \bar{u}_h\} = -\mathrm{i}h\bar{u}_h, \quad \{L, u_h\} = \mathrm{i}u_h, \quad \{L, \bar{u}_h\} = -\mathrm{i}\bar{u}_h. \qquad (6)$$

The terms in Eq. (5) commute with L. The conservation of momentum selects the terms with $\sum_k(\alpha_k - \beta_k)k = 0$. A first useful consequence of the conservation of momentum is that given any set $S \subset \mathbb{Z}^n$, setting

$$\bar{\ell}^{(a,p)}_S := \{u \in \bar{\ell}^{(a,p)} :\; u_k = 0,\; \forall k \notin\; \mathrm{Span}(S)\},$$

$\bar{\ell}^{(a,p)}_S \times \bar{\ell}^{(a,p)}_S$ is an invariant subspace for the dynamics.

Remark 2. This has the following geometric interpretation, the lattice $\Lambda := \mathrm{Span}(S) \subset \mathbb{Z}^n$ is of some rank k and it is the character group of a torus $\bar{T} = \mathbb{R}^k/\Lambda$ with a natural map $\pi : T \to \bar{T}$. Under this map a simple variant of the space $\bar{\ell}^{(a,p)}$ for the torus $\bar{T}$ is identified to $\bar{\ell}_S^{(a,p)}$.

2.1. One step of Birkhoff normal form. A monomial $u^\alpha \bar{u}^\beta$ is an eigenvector of $ad(K) := \{K, -\}$ of eigenvalue $\sum_k (\alpha_k - \beta_k)|k|^2$, where K is the quadratic part

$$K := \sum_{k \in \mathbb{Z}^n} |k|^2 u_k \bar{u}_k \quad \text{quadratic energy.} \tag{7}$$

We apply a step of the Birkhoff normal form (cf. [1,4,5]), by which we cancel all the terms of degree $2(q+1)$ which do not Poisson commute with K. This is done by applying a well known analytic change of variables, with generating function

$$A := \sum_{\substack{\alpha,\beta \in (\mathbb{Z}^n)^{\mathbb{N}} : |\alpha| = |\beta| = q+1 \\ \sum_k (\alpha_k - \beta_k) k = 0,\ \sum_k (\alpha_k - \beta_k)|k|^2 \neq 0}} \binom{q+1}{\alpha}\binom{q+1}{\beta} \frac{u^\alpha \bar{u}^\beta}{\sum_k (\alpha_k - \beta_k)|k|^2}.$$

We denote the change of variables by $\Psi^{(1)} := e^{ad(A)}$ and notice that it is well defined and analytic: $B_{\epsilon_0} \times B_{\epsilon_0} \to B_{2\epsilon_0} \times B_{2\epsilon_0}$, with $\epsilon_0 = (2c_{a,p})^{-1}$ (here B_r denotes the open ball of radius r and $c_{a,p}$ is the algebra constant of the space[3] $\bar{\ell}^{(a,p)}$).

By construction $\Psi^{(1)}$ brings (3) to the form $H = H_{\mathrm{Res}} + P^{2(q+2)}$ where:

$$H_{\mathrm{Res}} := \sum_{k \in \mathbb{Z}^n} |k|^2 u_k \bar{u}_k + \sum_{\substack{\alpha,\beta \in (\mathbb{Z}^n)^{\mathbb{N}} : |\alpha| = |\beta| = q+1 \\ \sum_k (\alpha_k - \beta_k) k = 0,\ \sum_k (\alpha_k - \beta_k)|k|^2 = 0}} \binom{q+1}{\alpha}\binom{q+1}{\beta} u^\alpha \bar{u}^\beta, \tag{8}$$

$P^{2(q+2)}(u)$ has degree at least $2(q+2)$ in u, it is analytic and regular and satisfies the bound:

$$\sup_{(u,\bar{u}) \in B_\varepsilon \times B_\varepsilon} \|X_{P^{2(q+2)}}\|_{a,p} \le \mathrm{cost}\, \varepsilon^{2q+3}, \quad \forall \varepsilon < \epsilon_0,$$

where cost denotes a universal constant (depending only on q, $c_{a,p}$ and the function G).

Remark 3. The three constraints in the second summand of formula (8) express the conservation of L, M and the quadratic energy K.

Definition 1. *We say that a list $k_1, \dots, k_{2d}$ of vectors in $\mathbb{Z}^n$ is* **resonant** *if, up to reordering, we have*

$$k_1 + k_3 \cdots + k_{2d-1} = k_2 + \cdots + k_{2d}, \quad |k_1|^2 + \cdots + |k_{2d-1}|^2 = |k_2|^2 + \cdots + |k_{2d}|^2.$$

We say that the list is **integrable** *if furthermore, up to reordering, we have $k_{2i-1} = k_{2i}$, $i = 1, \dots, d$.*

[3] Notice that the unperturbed Hamiltonian K is completely resonant so A does not have small divisors. Since $\bar{\ell}^{(a,p)}$ is a Hilbert algebra, this implies that the change of variables does not lose regularity.

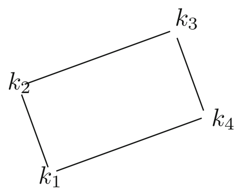


Fig. 1. A resonant quadruple k_1, k_2, k_3, k_4

The resonant lists with $d = q + 1$ describe the *resonant monomials*, that is those monomials which Poisson commute with K, which appear in $H_{\rm Res}$. The integrable lists describe the monomials in $|u_h|^2$.

Example 1. $[q = 1]$

$$k_1 + k_3 = k_2 + k_4, \qquad |k_1|^2 + |k_3|^2 = |k_2|^2 + |k_4|^2$$

is equivalent to

$$k_1 + k_3 = k_2 + k_4, \qquad (k_1 - k_2, k_3 - k_2) = 0 \tag{9}$$

Notice that a quadruple k_1, k_2, k_3, k_4 is resonant if these points are the vertices of a rectangle; it is integrable if and only if the corresponding rectangle is degenerate (Fig. 1).

2.1.1. Invariant subspaces. In view of Remark 2 we wish to study the Hamiltonian $H_{\rm Res}$ on the invariant subspaces $\ell_S^{a,p}$ for suitable choices of S. We want to characterize those subsets $S \subset \mathbb{Z}^n$, such that the Hamiltonian vector field $X_{H_{\rm Res}}$ is tangent to the subspace of equation

$$u_k = 0 = \bar{u}_k\,, \quad \forall k \in S^c := {\rm Span}(S) \backslash S,$$

this of course implies that this subspace is stable under the dynamics, a set S with this property is called *complete*. We denote by H_S the Hamiltonian $H_{\rm Res}$ restricted to such a subspace; naturally H_S depends only on $u_k, \bar{u}_k$ with $k \in S$.

The next statement follows immediately from the definitions:

Proposition 1. *i) S is complete if and only if, for any choice of $2q + 1$ vectors $v_i \in S$ the following holds:*
If there exists a further vector $w \in \mathbb{Z}^n$ such that the list $v_1, \dots, v_{2q+1}, w$ is resonant then $w \in S$.
ii) If all the lists in S of $2q + 2$ elements which are resonant are also integrable, then H_S depends only on the elements $|u_h|^2$ with $h \in S$.

Remark 4. A sufficient condition for S to be integrable is the following: set $S = \{v_1, \dots, v_m\}$, introduce variables e_i with $i = 1, \dots, m$. For any choice of $2q+2$ elements $e_{i_1}, \dots e_{i_{2q+2}}$ if the expression

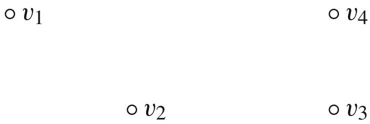
$$e_{i_1} + \cdots + e_{i_{q+1}} - (e_{i_{q+2}} + \cdots + e_{i_{2q+2}})$$

is not zero then

$$v_{i_1} + \cdots + v_{i_{q+1}} - (v_{i_{q+2}} + \cdots + v_{i_{2q+2}}) \neq 0.$$

We have thus shown that completeness and integrability are a *genericity* condition on S, the first of many which we will impose.

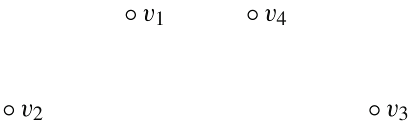
Example 2. $q = 1$, $n = 2, m = 4$ Four vectors v_1, v_2, v_3, v_4 in the plane are not complete if they form a picture of type



that is we have a right triangle which is not completed to a rectangle.

The list

is complete but not integrable, and finally



is complete and integrable.

When we partition

$$\mathrm{Span}(S) = S \cup S^c, \quad S := (v_1, \ldots, v_m),$$

where S is complete, we call the elements of S *tangential sites* and of S^c the *normal sites*. Of course the word tangential is justified by the fact that the Hamiltonian vector field is tangent to the subspace parametrized by the coordinates in S.

We introduce

$$A_r(\xi_1, \ldots, \xi_m) = \sum_{\sum_i k_i = r} \binom{r}{k_1, \ldots, k_m}^2 \prod_i \xi_i^{k_i}. \tag{10}$$

Proposition 2. *If S is complete and integrable the restricted Hamiltonian is*

$$\begin{aligned} H_S &= \sum_{i=1}^m |v_i|^2 |u_{v_i}|^2 + A_{q+1}(|u_{v_1}|^2, \ldots, |u_{v_m}|^2) \\ &= \sum_{i=1}^m |v_i|^2 |u_{v_i}|^2 + \sum_{\sum_i k_i = q+1} \binom{q+1}{k_1, \ldots, k_m}^2 \prod_i |u_{v_i}|^{2k_i}. \end{aligned}$$

Proof. This follows immediately from Formula (8) and the definitions. □

2.2. Tangential sites in action–angle coordinates. We set

$$u_k := z_k \text{ for } k \in S^c\,, \quad u_{v_i} := \sqrt{\xi_i + y_i}\,e^{\mathrm{i}x_i} = \sqrt{\xi_i}\,(1 + \frac{y_i}{2\xi_i} + \ldots)e^{\mathrm{i}x_i} \text{ for } i = 1, \ldots m, \tag{11}$$

considering the $\xi_i > 0$ as parameters $|y_i| < \xi_i$ while $y, x, w := (z, \bar{z})$ are dynamical variables.

Definition 2. *We denote by $\ell^{(\mathbf{a},\mathbf{p})}$ the subspace of $\bar{\ell}^{(\mathbf{a},\mathbf{p})} \times \bar{\ell}^{(\mathbf{a},\mathbf{p})}$ generated by the indices in S^c with coordinates $w = (z, \bar{z})$.*

For all $\varepsilon > 0$ and for all,

$$\xi \in A_{\varepsilon^2} := \{\xi \;:\; \frac{1}{2}\varepsilon^2 \le \xi_i \le \varepsilon^2\}. \tag{12}$$

Formula (11) is a well known analytic and symplectic change of variables $\Psi^{(2)}_{\xi}$ in the domain

$$\begin{aligned} D_{a,p}(s,r) = D(s,r) &:= \{x, y, w \;:\; x \in \mathbb{T}^m_s\,,\ |y| \le r^2\,,\ \|w\|_{a,p} \le r\} \\ &\subset \mathbb{T}^m_s \times \mathbb{C}^m \times \ell^{(a,p)}. \end{aligned} \tag{13}$$

Here $\varepsilon > 0, s > 0$ and $0 < r < \varepsilon/2$ are auxiliary parameters. $\mathbb{T}^m_s$ denotes the open subset of the complex torus $\mathbb{T}^m_{\mathbb{C}} := \mathbb{C}^m/2\pi\mathbb{Z}^m$, where $x \in \mathbb{C}^m$, $|\mathrm{Im}(x)| < s$. Moreover if

$$\sqrt{2m}(\max(|v_i|)^p e^{(s+a\max(|v_i|))}\varepsilon < \epsilon_0\,, \tag{14}$$

the change of variables sends $D(r,s) \to B_{\epsilon_0}$ so we can apply it to our Hamiltonian.

We thus assume that the parameters $\varepsilon,\ r,\ s$ satisfy (14).

Formula (11) puts in action angle variables $(y; x) = (y_1, \ldots, y_m; x_1, \ldots, x_m)$ the tangential sites, close to the action $\xi = \xi_1, \ldots, \xi_m$, which are parameters for the system. The symplectic form is now $dy \wedge dx + i\sum_{k\in S^c} dz_k \wedge d\bar{z}_k$.

Following [15] we study *regular* functions $F : A_{\varepsilon^2} \times D_{a,p}(s,r) \to \mathbb{C}$, that is whose Hamiltonian vector field X_F is analytic from $D(s,r) \to \mathbb{C}^m \times \mathbb{C}^m \times \ell_{a,p}$. In the variables ξ we require Lipschitz regularity. We use the weighted norm:

$$\|X_F\|^{\lambda}_{s,r} = \sup_{A_{\varepsilon^2}\times D(s,r)} \|X_F\|_{s,r} + \lambda \sup_{\xi\ne\eta\in A_{\varepsilon^2}\,,\ (x,y,w)\in D(s,r)} \frac{\|X_F(\eta) - X_F(\xi)\|_{s,r}}{|\eta - \xi|}, \tag{15}$$

where $\lambda = \varepsilon^2$ and

$$\|X_F\|_{s,r} := r^{-2}|\partial_x F| + s^{-1}|\partial_y F| + r^{-1}\|\partial_w F\|_{a,p}.$$

The different weights ensure that if $\|X_F\|^{\lambda}_{s,r} < \frac{1}{2}$ then F generates a close–to–identity symplectic change of variables from $D(r/2, s/2) \to D(r,s)$.

2.2.1. Quadratic Hamiltonians. We have the rules of Poisson bracket

$$\{y_i, y_j\} = \{x_i, x_j\} = 0, \quad \{y_i, x_j\} = \delta^i_j, \quad \{z_h, z_k\} = \{\bar{z}_h, \bar{z}_k\} = 0, \quad \{\bar{z}_h, z_k\} = \mathrm{i}\delta^h_k. \tag{16}$$

If we define w as the infinite row vector w with coordinates z_h and then $\bar{z}_h$ and J the standard skew symmetric matrix $J := \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$ we have the Poisson bracket[4] $\{w^t, w\} = \mathrm{i}J$.

[4] The apex t is the transpose.

Thus a quadratic Hamiltonian $\mathcal{Q}(w)$ in the elements of w represents by Poisson bracket a linear transformation on the space with basis w. If $\mathcal{Q}(w)$ is real, the matrix of this linear transformation is purely imaginary; thus it is convenient to denote it by $\mathrm{i}Q$ and write $ad(\mathcal{Q}) := \{\mathcal{Q}(w), w^t\} = \mathrm{i}Qw^t$. The equations of motion are $\dot{w} = \mathrm{i}wQ^t$. The matrix Q is related to the quadratic expression by[5]

$$\mathcal{Q}(w) = \frac{1}{2}(w, wJQ^t) = -\frac{1}{2}wQJw^t. \tag{17}$$

Quadratic Hamiltonians are closed under Poisson bracket and, by Jacobi's identity, if $\mathcal{Q}_1(w), \mathcal{Q}_2(w)$ correspond to matrices Q_1, Q_2, then $\{\mathcal{Q}_1(w), \mathcal{Q}_2(w)\}$ corresponds to $[Q_1, Q_2]$. Moreover, a quadratic Hamiltonian $\mathcal{Q}$ has $\|X_{\mathcal{Q}}\|_{r,s} < \infty$ if and only if its matrix Q is such that QJ is a continuous symmetric linear operator from $\ell_{a,p}$ to itself.

3. Main Dynamical Results

3.0.2. Genericity conditions. Our theorems hold under some constraints on S such as those of Remark 4. These constraints are expressed by the condition that the list of vectors S, thought of as a point in $\mathbb{Z}^{mn}$, does not lie in any of the varieties defined by a finite list of polynomial equations, called the *avoidable resonances*.

In order to explain this let us establish some simple language.

Definition 3. *Given a list $\mathcal{R} := \{P_1(\zeta), \dots, P_U(\zeta)\}$ of polynomials in d vector variables ζ_i, called* **resonance polynomials***, we say that a list of vectors $S = \{v_1, \dots, v_m\}$, $v_i \in \mathbb{C}^n$ is* **generic** *relative to $\mathcal{R}$ if, for any list $A = \{u_1, \dots, u_d\}$ such that $u_i \in S,\ \forall i,\ u_i \neq u_j$ if $i \neq j$, the evaluation of the resonance polynomials at $\zeta_i = u_i$ is non-zero.*

If m is finite this condition is equivalent to requiring that S (considered as a point in $\mathbb{C}^{nm}$) does not belong to the algebraic variety where at least one of the resonance polynomials is zero.

In our specific case the condition of being generic for the tangential sites S is expressed by a finite list of non-zero polynomials with integer coefficients depending on $d = 4q(n+1)$ vector variables $\zeta = (\zeta_1, \dots, \zeta_d)$ with $\zeta_i = (\zeta_i^1, \dots, \zeta_i^n)$. The explicit list of these resonances (see Definition 22) depends on some non trivial combinatorics, nevertheless it is easy to give a (highly) redundant list of inequalities out of which the resonances appear. There is a constant $C > 0$ depending only on q, n so that we can take as resonances the non-zero polynomials of the form:

i) *Linear inequalities.* For all non-zero vectors $(a_1, \dots, a_{4q(n+1)})$ with $a_i \in \mathbb{Z},\ |a_i| \le C$, we require that

$$\sum_{i=1}^{4q(n+1)} a_i \zeta_i \neq 0,$$

ii) *Quadratic inequalities.* Let $(\zeta_i, \zeta_j) = \sum_{h=1}^n \zeta_i^h \zeta_j^h$ be the *scalar products*. For all non zero matrices $\{a_{i,j}\}_{i,j=1}^{4q(n+1)}$ with $a_{i,j} \in \mathbb{Z},\ |a_{i,j}| \le C$, we require

$$\sum_{i,j=1}^{4q(n+1)} a_{i,j}(\zeta_i, \zeta_j) \neq 0.$$

[5] The parentheses represent the scalar product in $\mathbb{R}$.

iii) *Determinantal inequalities.* Consider n linear combinations u_h out of the list of elements $\mathcal{L} := \{\sum_{i=1}^{4q(n+1)} a_{h,i}\zeta_i,\ a_{h,i} \in \mathbb{Z},\ |a_{h,i}| \le C\}$.

The determinantal resonances are contained in the list of the formally non-zero expressions of type $\det(u_1, \dots, u_n),\ u_i \in \mathcal{L}$.

Given any $m \in \mathbb{N}$, let $S = \{v_1, \dots, v_m\} \in \mathbb{Z}^{nm}$ be a *generic* choice of the tangential sites.

Theorem 1. *For all r, s, ε satisfying (14) and for all $\xi \in A_{\varepsilon^2}$, there exists an analytic symplectic change of variables:*

$$\Phi_\xi : (y, x) \times (z, \bar{z}) \to (u, \bar{u})$$

from $D(s, r/2) \to B_{2\epsilon_0}$ such that the Hamiltonian (3) in the new variables is analytic and has the form

$$H \circ \Phi_\xi = (\omega(\xi), y) + \sum_{k \in S^c} \tilde{\Omega}_k |z_k|^2 + \tilde{\mathcal{Q}}(\xi, w) + \tilde{P}(\xi, y, x, w),$$

where

- **i) Non-degeneracy:** *$\omega_i(\xi) - |v_i|^2$ is homogeneous of degree q.*
  The map $(\xi_1, \dots, \xi_m) \mapsto (\omega_1(\xi), \dots, \omega_m(\xi))$ is a diffeomorphism for ξ outside a real algebraic hypersurface.
- **ii) Asymptotic of the normal frequencies:** *We have $\tilde{\Omega}_k = |k|^2 + \sum_{i=1}^m |v_i|^2 L^{(i)}(k)$, where $L^{(i)}(k) \in \mathbb{Z}$ satisfy $|L^{(i)}(k)| \le 4nq$.*
- **iii) Reducibility:** *The matrix $\tilde{Q}(\xi)$ which represents the quadratic form $\tilde{\mathcal{Q}}(\xi, w)$(see formula (17)) depends only on the variables ξ and all its entries are homogeneous of degree q in these variables. It is block–diagonal and satisfies the following properties:*
  All of the blocks except a finite number are self adjoint and of dimension $\le n+1$; the remaining finite number of blocks are of dimension $\le 2n$.
  All the (infinitely many) blocks are chosen from a finite list of matrices $\mathcal{M}(\xi)$.
- **iv) Smallness:** *If $\varepsilon^3 < r < \varepsilon/2$, the perturbation $\tilde{P}$ is small, more precisely we have the bounds:*

$$\|X_{\tilde{P}}\|_{s,r}^{\lambda} \le C(\varepsilon^{2q-1} r + \varepsilon^{2q+3} r^{-1}), \tag{18}$$

  where C is independent of r, ε.

Proof. See §13. □

Remark 5. At first inspection it may seem that the estimate on X_P is *too small* to be possible. Indeed P should contain terms from $P^{(2q+4)}$, which should contribute to X_P a term of order $\varepsilon^{2q+4} r^{-2}$. In fact for a *generic* choice of S these terms are constant so they do not enter in the vector field.

Remark 6. The list of matrices $\mathcal{M}(\xi)$ is constructed in Sect. 11, cf. Definition 24.

It contains at most $2n \cdot (2q)^{m-1}!$ matrices distributed in at most $2n \cdot (2q)^{4nq}!$ orbits under the group of permutations of the variables ξ_i.

In Example 15 we exhibit $\mathcal{M}(\xi)$ in the case $q = 1, n = 2$.

3.0.3. Stable regions for the normal form. An interesting issue is to see if one can use arithmetic constraints such as those of [11], to simplify those matrices in $\mathcal{M}$ which are not self–adjoint. In Proposition 15 we show that, for $n \leq 2$ and all q it is possible to choose the tangential sites so that the matrices reduce to only 2×2 matrices independently of m. This requires a notion of generic, of arithmetic (or probabilistic) nature, which we call the *x–constraints* discussed in §12.1. One may deduce the following very useful property, proved in §12.1.1:

Proposition 3. *Under the hypotheses of Proposition 15, there exists an open region $O_{\varepsilon^2} \subset \mathcal{A}_{\varepsilon^2}$ where all the non self–adjoint matrices in $\mathcal{M}$ have real and distinct eigenvalues.*

Proof. The region is the one where all discriminants are strictly positive; we show in §12.1.1 that it is a non empty open cone. □

It then follows fairly easily from this result and Theorem 1:

Corollary 1. *There exists an algebraic hypersurface $\mathcal{A}$ such that on the open region $A_{\varepsilon^2} \setminus \mathcal{A}$ there is an analytic symplectic change of coordinates taking $\mathcal{Q}$ into a diagonal form with constant coefficients plus a form $\bar{\mathcal{Q}}$ with constant coefficients depending only on finitely many variables $z_k, \bar{z}_k,\ k \in A$.*

The change of variables does not depend on (x, y), it is linear in w and analytic in ξ. The Hamiltonian is then

$$H_{\text{fin}} = (\omega(\xi), y) + \sum_{k \in S^c} \bar{\Omega}_k |z_k|^2 + \bar{\mathcal{Q}} + P(\xi, x, y, w), \tag{19}$$

where

$$\bar{\Omega}_k = \begin{cases} \tilde{\Omega}_k + \lambda_k(\xi)\,, & \forall k \in S^c \setminus A, \\ \tilde{\Omega}_k, & k \in A. \end{cases}$$

i) The correction $\lambda_k(\xi)$ is chosen in a finite list, say

$$\lambda_k(\xi) \in \{\lambda^{(1)}(\xi), \dots, \lambda^{(K)}(\xi)\}\,, \quad K := K(n, m), \tag{20}$$

of different (real) analytic functions of ξ.

ii) The functions $\lambda^{(i)}(\xi)$ are homogeneous of degree q in ξ. Let $\mathfrak{A}_{\varepsilon^2}$ be a tubular neighborhood of $\mathcal{A}$ with radius of order ε^2. For $\xi \in A_{\varepsilon^2} \setminus \mathfrak{A}_{\varepsilon^2}$ the $\lambda^{(i)}(\xi)$ satisfy the bounds

$$|\lambda^{(i)}(\xi)| \leq C\varepsilon^{2q}\,, \quad c\varepsilon^{2q} \leq |\lambda^{(i)}(\xi) \pm \lambda^{(j)}(\xi)| \leq C\varepsilon^{2q}\,, \quad |\nabla_\xi \lambda^{(i)}(\xi)| \leq C\varepsilon^{2q-2}. \tag{21}$$

iii) For $\xi \in A_{\varepsilon^2} \setminus \mathfrak{A}_\varepsilon$ item iv) of Theorem 1 holds.

iv) $\bar{\mathcal{Q}}$ is a quadratic Hamiltonian with constant coefficients in finitely many of the variables $z_k, \bar{z}_k,\ k \in S^c$.

v) For $n = 1, 2$ and all q it is possible to choose the tangential sites so that $\bar{\mathcal{Q}}$ is formed by 2×2 blocks which (outside the hypersurface $\mathcal{A}$) are semisimple with distinct eigenvalues. The region in which these eigenvalues are real is open non *empty and on this region the real eigenvalues are given by analytic functions* $\mu_k(\xi)$ *so that we may write*

$$H_{\text{fin}} = (\omega(\xi), y) + \sum_{k\in S^c} \bar{\Omega}_k |z_k|^2 + P(\xi, x, y, w), \tag{22}$$

with $\bar{\Omega}_k = \tilde{\Omega}_k + \mu_k(\xi)\,, \quad \forall k \in A.$

Proof. See Sect. 12.1.1. □

4. A Normal Form

In this section we make a preliminary study of the Hamiltonian H_{res} and introduce some simple constraints on the tangential sites S; this enables us to define our normal form.

Definition 4. *We call* x, y, w **dynamical variables**. *We give degree* 0 *to the angles* x, 2 *to* y *and* 1 *to* w.

We use the degree only for handling dynamical variables, as follows. We develop in the Taylor expansion, in particular since y is small with respect to ξ we develop $\sqrt{\xi_i + y_i} = \sqrt{\xi_i}(1 + \frac{y_i}{2\xi_i} + \ldots)$, as a series in $\frac{y_i}{\xi_i}$ we then develop the entire Hamiltonians H, H_{Res} as a series in y, w.

Definition 5 (Normal form). *We separate* $H_{\text{Res}} + P^{2(q+2)}(u) = H = N + P$, *where the* **normal form** N *collects all the terms of* H_{Res} *(as series in* y, w*) of degree* ≤ 2 *in the variables* y, w.

The series P *collects all terms of* $P^{2(q+2)}(u)$ *and all the terms of* H_{Res} *of degree* > 2 *in the variables* y, w.

It is easily seen that H, hence P, depend analytically on all the variables ξ, y, x, w in the domain $A_{\varepsilon^2} \times D(r, s)$.

In the new variables:

$$\begin{gathered} M = \sum_i \xi_i v_i + \sum_i y_i v_i + \sum_{k\in S^c} k|z_k|^2, \quad L = \sum_i \xi_i + \sum_i y_i + \sum_{k\in S^c} |z_k|^2, \\ \sum_{k\in\mathbb{Z}^n} |k|^2 u_k \bar{u}_k = K = (\omega_0, \xi + y) + \sum_{k\in S^c} |k|^2 |z_k|^2\,, \quad \omega_0 = (|v_1|^2, \ldots, |v_m|^2). \end{gathered} \tag{23}$$

Remark 7. The terms $\sum_i \xi_i$, $\sum_i \xi_i v_i$ and $\sum_i \xi_i |v_i|^2$ are constant and can be dropped, renormalizing the three quantities M, L, K (momentum, mass and quadratic energy).

We summarize the commutation rules:

$$\begin{gathered} \{M, y_h\}=\{L, y_h\}=\{K, y_h\}=0,\ \{M, x_h\} = v_h x_h,\ \{L, x_h\} = x_h,\ \{K, x_h\} = |v_h|^2, \\ \{M, z_k\} = \mathrm{i}k z_k,\ \{L, z_k\} = \mathrm{i} z_k,\ \{K, z_k\} = \mathrm{i}|k|^2 z_k, \\ \{M, \bar{z}_k\} = -\mathrm{i}k\bar{z}_k,\ \{L, \bar{z}_k\} = -\mathrm{i}\bar{z}_k,\ \{K, \bar{z}_k\} = -\mathrm{i}|k|^2\bar{z}_k. \end{gathered} \tag{24}$$

We formalize the momentum and mass by two linear maps.

$$\pi : \mathbb{Z}^m \to \text{Span}(S),\ \pi(e_i) = v_i, \quad \text{momentum}; \quad \eta : \mathbb{Z}^m \to \mathbb{Z},\ \eta(e_i) = 1 \quad \text{mass}, \tag{25}$$

where $e_1, \ldots, e_m$ are the elements of the standard basis of the lattice $\mathbb{Z}^m$.

Lemma 1. *Each monomial $e^{\mathrm{i}(\nu,x)}y^l z^\alpha \bar{z}^\beta$ is an eigenvector of the linear operators*[6] *$ad(M)$ and $ad(L)$ with eigenvalues (i.e. with momentum and mass) given by*

$$\pi(\nu) + \sum_{k\in S^c}(\alpha_k - \beta_k)k\,, \qquad \eta(\nu) + \sum_{k\in S^c}(\alpha_k - \beta_k). \tag{26}$$

Proof. This follows computing $\{M, e^{\mathrm{i}(\nu,x)}y^l z^\alpha \bar{z}^\beta\}$ and $\{L, e^{\mathrm{i}(\nu,x)}y^l z^\alpha \bar{z}^\beta\}$ using Formulas (23) and the rules of Poisson bracket. □

Remark 8. A monomial Poisson commutes with M and L if and only if the momentum and mass are zero, that is $\pi(\nu) = -\sum_{k\in S^c}(\alpha_k - \beta_k)k\,,\ \eta(\nu) = -\sum_{k\in S^c}(\alpha_k - \beta_k)$.

4.0.4. The normal form N. Our next task is to describe the Hamiltonian N of Definition 5, provided that S satisfies some basic constraints. This is done in Proposition 4.

N is described in terms of a list of vectors, called *edges* since they will appear as edges of a graph describing the non-diagonal elements in $ad(N)$.

Definition 6 (Edges). *Consider the elements*

$$X_q := \{\ell := \sum_{j=1}^{2q} \pm e_{i_j} = \sum_{i=1}^{m} \ell_i e_i, \quad \ell \neq 0, -2e_i\ \forall i\,, \quad \eta(\ell) \in \{0, -2\}\}. \tag{27}$$

Notice the **mass constraint** $\sum_i \ell_i = \eta(\ell) \in \{0, -2\}$. *We call all these elements respectively the* **black,** $\eta(\ell) = 0$ *and* **red** $\eta(\ell) = -2$ **edges** *and denote them by X_q^0, X_q^{-2} respectively.*

Example 3. For $q = 1$ we have $e_i - e_j, -(e_i + e_j),\ i \neq j$. For $q = 2$ we have all the terms for $q = 1$ and $e_i - e_j - e_a - e_b,,\ 2e_i - 2e_j,\ -3e_i + e_j,\ i \neq j, a, b$.

We start to impose a list of linear and quadratic inequalities on the choice of S which will be justified in Proposition 4.

Constraint 1. i) *We assume that $\sum_{j=1}^m n_j v_j \neq 0$ for all $n_i \in \mathbb{Z}$, $\sum_i n_i = 0,\ 1 < \sum_i |n_i| \le 2q + 2$.*

ii) $|\sum_i n_i v_i|^2 - \sum_i n_i |v_i|^2 \neq 0$ *when $n_i \in \mathbb{Z}$, $\sum_i n_i = 1,\ 1 < \sum_i |n_i| \le 2q + 1$.*

iii) *We assume that $\sum_{j=1}^m \ell_j v_j \neq 0$, when $u := \sum_{j=1}^m \ell_j e_j$ is either an edge or a sum or difference of two distinct edges.*

iv) $2\sum_{j=1}^m \ell_j |v_j|^2 + |\sum_{j=1}^m \ell_j v_j|^2 \neq 0$ *for all edges $\ell = \sum_{j=1}^m \ell_j e_j$ in X_q^{-2}.*

Lemma 2. *Constraint 1i) is an* **integrability** *constraint. Constraint 1ii) is the* **completeness** *constraint. Constraint iii) means that an edge $\ell = \sum_{j=1}^m \ell_j e_j$ is determined by the associated vector $\pi(\ell) = \sum_{j=1}^m \ell_j v_j$.*

Proof. i) The first statement follows from Remark 4.

ii) Using Proposition 1 it is enough to show that, under Constraint ii), we cannot find $2q + 1$ elements $u_j = v_{i_j}$ for which there is a further vector w in $\mathbb{Z}^m$ with $u_1, \ldots, u_{2q+1}, w$ resonant. Otherwise $w = \sum_i n_i v_i$ is a linear combination with ± 1 coefficients of the v_i, hence it is a vector satisfying the hypotheses of item ii), but the quadratic condition in the same item implies that the list is non resonant.

iii) is clear. □

[6] Given a polynomial P, we denote by $ad(P)$ the linear operator that associates to each polynomial A the polynomial $\{P, A\}$.

Constraint iv) will be used in the next proposition, we shall see that it excludes quadratic terms of type z_h^2 or $\bar{z}_h^2$ in H_{Res}.

For $q = 1$ this constraint means only that $-2|v_i|^2 - 2|v_j|^2 + |v_i + v_j|^2 \neq 0, i \neq j$ and this just means $v_i \neq v_j$.

Proposition 4. *Under all the previous constraints we have*

$$N := (\omega(\xi), y) + \sum_{k \in S^c} |k|^2 |z_k|^2 + \mathcal{Q}(x, w), \tag{28}$$

where (cf. Formula (23)) the coefficient

$$\omega(\xi) = \omega_0 + \nabla_\xi A_{q+1}(\xi) - (q+1)^2 A_q(\xi) \underline{1}, \tag{29}$$

does not depend on the dynamical variables. Here $\underline{1} \in \mathbb{N}^m$ denotes the vector with all coordinates equal to 1.

The term $\mathcal{Q}(x, w)$ is quadratic:

$$\mathcal{Q} = \sum_{\ell \in X_q^0} c(\ell) e^{\mathrm{i}(\ell, x)} \sum_{(h,k) \in \mathcal{P}_\ell} z_h \bar{z}_k + \sum_{\ell \in X_q^{-2}} c(\ell) \sum_{\{h,k\} \in \mathcal{P}_\ell} [e^{\mathrm{i}(\ell, x)} z_h z_k + e^{-\mathrm{i}(\ell, x)} \bar{z}_h \bar{z}_k], \tag{30}$$

where, given an edge ℓ, we set $\ell = \ell^+ - \ell^-$ and define:

$$c_q(\ell) \equiv c(\ell) := \begin{cases} (q+1)^2 \xi^{\frac{\ell^+ + \ell^-}{2}} \sum_{\substack{\alpha \in \mathbb{N}^m \\ |\alpha + \ell^+|_1 = q}} \binom{q}{\ell^+ + \alpha} \binom{q}{\ell^- + \alpha} \xi_i^\alpha & \ell \in X_q^0 \\ (q+1) q \xi^{\frac{\ell^+ + \ell^-}{2}} \sum_{\substack{\alpha \in \mathbb{N}^m \\ |\alpha + \ell^+|_1 = q-1}} \binom{q+1}{\ell^- + \alpha} \binom{q-1}{\ell^+ + \alpha} \xi_i^\alpha & \ell \in X_q^{-2} \\ c_q(\ell) = c_q(-\ell) & \ell \in X_q^2 \end{cases} \tag{31}$$

for the definition of $\mathcal{P}_\ell$ see Definition 7.

Example 4. Let us discuss $q = 1$, *the cubic NLS*. We have

$$\omega_i(\xi) := |v_i|^2 - 2\xi_i, \tag{32}$$

finally the quadratic form is

$$\mathcal{Q}(w) = 4 \sum_{\substack{1 \leq i \neq j \leq m \\ h,k \in S^c}}^{*} \sqrt{\xi_i \xi_j} e^{\mathrm{i}(x_i - x_j)} z_h \bar{z}_k$$
$$+ 2 \sum_{\substack{1 \leq i < j \leq m \\ h,k \in S^c}}^{**} \sqrt{\xi_i \xi_j} e^{-\mathrm{i}(x_i + x_j)} z_h z_k + 2 \sum_{\substack{1 \leq i < j \leq m \\ h,k \in S^c}}^{**} \sqrt{\xi_i \xi_j} e^{\mathrm{i}(x_i + x_j)} \bar{z}_h \bar{z}_k. \tag{33}$$

Notice that in the sums $\sum^{**}$ each term appears twice.

Here $\sum^*$ denotes that (h, k, v_i, v_j) satisfy:

$$\{(h, k, v_i, v_j) \mid h + v_i = k + v_j,\ |h|^2 + |v_i|^2 = |k|^2 + |v_j|^2\},$$

and $\sum^{**}$, that (h, v_i, k, v_j) satisfy:

$$\{(h, v_i, k, v_j) \mid h + k = v_i + v_j,\ |h|^2 + |k|^2 = |v_i|^2 + |v_j|^2\}.$$

Proof of Proposition 4. By definition the normal form N collects all the terms of H_{Res} (as a series in y, w) of degree ≤ 2 in the variables y, w. In turn H_{Res} is the sum of the quadratic term $K = \sum_k |k|^2|u_k|^2$ and of the terms of degree $2q + 2$ in the original variables $u, \bar{u}$.

From Remark 7 the quadratic term K contributes to N the terms

$$(\omega_0, y) + \sum_{k \in S^c} |k|^2 |z_k|^2.$$

The remaining terms $u_{k_1}\bar{u}_{k_2}u_{k_3}\bar{u}_{k_4} \dots u_{k_{2q+1}}\bar{u}_{k_{2q+2}}$ satisfy the constraints

$$\sum_i (-1)^i k_i = 0, \quad \sum_i (-1)^i |k_i|^2 = 0. \tag{34}$$

These terms may contribute to terms of N only if they are of total degree ≤ 2 in y, w.

We analyze the three possible cases, of degree 0, 1, 2 in w.

- *degree 0.* If all the k_i are in S the momentum $\sum_i (-1)^i k_i$ is a linear expression $\sum_j m_j v_j$. From momentum conservation and Constraint 1 i) we must have $m_j = 0,\ \forall j$. This implies that we can pair the even and odd $k's$ and, as shown in Proposition 2, this gives a contribution $A_{q+1}(\xi + y)$. In this expression the terms of degree ≤ 2 give a constant (which we ignore) and the term $(\nabla_\xi A_{q+1}(\xi), y)$.
- *degree 1.* One and only one of the $k_i = k \in S^c$. Formula (34) becomes

$$k - \sum_i n_i v_i = 0\,, \quad |k|^2 - \sum_i n_i |v_i|^2 = 0,$$

  where $\sum_i n_i v_i$ satisfies the hypotheses of Constraint 1 ii). Thus these terms do not occur and S is complete.
- *degree 2.* Given $h, k \in S^c$ we compute the coefficients of $z_h \bar{z}_k$ or $z_h z_k$ or $\bar{z}_h \bar{z}_k$. These terms are obtained when all but two of the k_i are in S. Each k_i in S contributes $\sqrt{\xi_i + y_i}e^{\pm x_i}$, giving a coefficient $\sqrt{\prod_{j=1}^m \xi_j^{\ell_j}}\, e^{\mathrm{i}(\ell, x)}$, whenever

$$(z_h \bar{z}_k): \quad \sum_{j=1}^m \ell_j v_j + h - k = 0, \quad \sum_{j=1}^m \ell_j |v_j|^2 + |h|^2 - |k|^2 = 0, \qquad \ell \in X_q^0, \tag{35}$$

$$(z_h z_k): \quad \sum_{j=1}^m \ell_j v_j + h + k = 0, \quad \sum_{j=1}^m \ell_j |v_j|^2 + |h|^2 + |k|^2 = 0, \qquad \ell \in X_q^{-2}, \tag{36}$$

$$(\bar{z}_h \bar{z}_k): \quad \sum_{j=1}^m \ell_j v_j - h - k = 0, \quad \sum_{j=1}^m \ell_j |v_j|^2 - |h|^2 - |k|^2 = 0, \qquad \ell \in X_q^2. \tag{37}$$

Definition 7. *Given $\ell \in X_q^{(0)}$ denote by $\mathcal{P}_\ell$ the set of pairs h, k satisfying Formula (35). If $\ell \in X_q^{(-2)}$ we denote by $\mathcal{P}_\ell$ the set of unordered pairs $\{h, k\}$ satisfying Formula (36).*

Constraint 1 iii), where u is the sum or difference of two edges, implies that h, k fix ℓ uniquely. In Formulas (36), (37) we see that we cannot have $\ell = \mp 2e_i$ since the equations in these formulas have the only solution $h = k = v_i \in S$. This explains why in Definition 6 we have excluded $\pm 2e_i$ as edges. Constraint 1 iv) implies that $h \neq k$ in Formulas (36), (37). By Constraint 1 iii) where u is an edge, in (35) $k = h$ implies $\ell = 0$. This contributes a term $(q+1)^2 A_q(\xi) \sum_{k \in S^c} |z_k|^2$. It is convenient to write

$$\sum_k (q+1)^2 A_q(\xi)|z_k|^2 = (q+1)^2 A_q(\xi)(\sum_k |z_k|^2 + \sum_i y_i) - (q+1)^2 A_q(\xi)(\sum_i y_i),$$

and notice that $(q+1)^2 A_q(\xi)(\sum_k |z_k|^2 + \sum_i y_i)$ is a mass term (hence a constant of motion for the whole Hamiltonian) and can be dropped from the Hamiltonian, so we change N into

$$N = K + (\nabla_\xi A_{q+1}(\xi) - (q+1)^2 A_q(\xi)\underline{1}, y) + \mathcal{Q}(x, w), \quad K = (\omega_0, y) + \sum_k |k|^2 |z_k|^2. \tag{38}$$

Recall that $\underline{1} \in \mathbb{N}^m$ denotes the vector with all coordinates equal to 1.

Let us now compute $\mathcal{Q}(x, w)$, given an edge ℓ set $\ell = \ell^+ - \ell^-$ formula (31) comes from the expansion

$$c_q(\ell) := \begin{cases} (q+1)^2 \displaystyle\sum_{e_{h_1} - e_{k_1} + e_{h_2} + \dots + e_{h_q} - e_{k_q} = \ell} \prod_{i=1}^q (\xi_{h_i} \xi_{k_i})^{1/2} & \ell \in X_q^0 \\ (q+1)q \displaystyle\sum_{e_{h_1} - e_{k_1} + e_{h_2} + \dots + e_{h_{q-1}} - e_{k_{q-1}} - e_{h_q} - e_{k_q} = \ell} \prod_{i=1}^q (\xi_{h_i} \xi_{k_i})^{1/2} & \ell \in X_q^{-2} \\ c_q(-\ell) = c_q(\ell). \end{cases}$$

□

It is interesting to notice a point essential for the KAM algorithm, since it gives a locally invertible change of coordinates $\omega_i = \omega_i(\xi)$ expressing

$$(\nabla_\xi A_{q+1}(\xi) - (q+1)^2 A_q(\xi)\underline{1}, y) = \sum_{i=1}^m \omega_i y_i.$$

Proposition 5. *For every r, the Hessian of $A_r(\xi)$ is a non degenerate matrix as polynomial in ξ.*

Proof. Let $r = p^s t$ with p prime and $p \not| t$. It is well known and elementary that, if p does not divide $\binom{r}{\ell}$, then p^s divides the vector ℓ. The coefficients of $\partial_{\xi_1} \partial_{\xi_2} A_r(\xi)$ are

$$\ell_1 \ell_2 \binom{r}{\ell}^2 = r(r-1) \binom{r-2}{\ell_1 - 1, \ell_2 - 1, \dots, \ell_m} \binom{r}{\ell}.$$

We claim that they are divisible by $p^s r(r-1)$. Indeed if p does not divide $\binom{r}{\ell}$ we have seen that p^{2s} divides $\ell_1\ell_2$ while p^{s+1} does not divide $r(r-1)$. The coefficients of $\partial^2_{\xi_1} A_r(\xi)$ are

$$\ell_1(\ell_1-1)\binom{r}{\ell} = r(r-1)\binom{r-2}{\ell_1-2, \ell_2, \dots, \ell_m}.$$

It follows that the Hessian is divisible by $r(r-1)$, the off diagonal terms are divisible by $p^s r(r-1)$ while the diagonal contains the term $r(r-1)\mathrm{diag}(\xi_i^{r-2})$. Therefore, once we divide by $r(r-1)$ we have a matrix which, modulo p, is diagonal with non zero entries. □

From Proposition 5 we have the coordinate change:

Corollary 2. *The map $\xi \to \nabla_\xi A_{q+1}(\xi) - (q+1)^2 A_q(\xi)\underline{1}$ is a local diffeomorphism outside a real algebraic hypersurface.*

Proof. The Jacobian of this map is a matrix with entries polynomials in ξ with integer coefficients. Reasoning as in Proposition 5 we see that this Jacobian matrix is of the form $J = q(q+1)A - (q+1)^2 B$, where $q(q+1)A$ is the Hessian of $A_{q+1}(\xi)$ while B has as entries the derivatives of $A_q(\xi)$, therefore $B = qC$ has coefficients divisible by q. Thus when we divide J by $q(q+1)$ we have a matrix $A - (q+1)C$ with entries polynomials with integer coefficients. Modulo a prime p dividing $q+1$ we have only the contribution from A which gives the diagonal matrix with non zero entries discussed in the proof of Proposition 5. It follows that the determinant of the Jacobian J is a non-zero polynomial. □

4.0.5. The perturbation P. Remark that $P(x, y, w)$ is *regular* in the sense of §2.2. Indeed in (3) all the terms of degree > 2 are regular and the Birkhoff normal form and elliptic-action angle variables preserve this property by the chain rule.

We say that P is *of order* $\varepsilon^a r^b$ for some integers a, b if its norm is smaller than $C\varepsilon^a r^b$ for some ε, r independent constant C. This implies, since P is regular, that X_P is of order $\varepsilon^a r^{b-2}$ (i.e. $\|X_P\|^\lambda_{s,r}$ is bounded by $C'\varepsilon^a r^{b-2}$).

According to Definition 5, P comes from two types of terms. In a term– denote it by $P^{(3)}$– we collect all the terms of degree $2i + j > 2$ coming from the resonant terms $\prod_{i=1}^{q+1} u_{k_i}\bar{u}_{h_i}$ (of H_{Res}). In $P^{(2q+4)}$ we collect all the terms coming from products $\prod_{i=1}^{d} u_{k_i}\bar{u}_{h_i}$, with $d \geq q+2$ of $P^{2(q+2)}$.

Recall that $u_{v_i} = \sqrt{\xi_i + y_i}e^{\mathrm{i}x_i} = \sqrt{\xi_i}(1 + \frac{y_i}{2\xi_i} + \dots)e^{\mathrm{i}x_i}$ is of order ε, while z_k is of order r. Then the dominant term in $P^{(3)}$ is given by the dominant terms of the monomials of degree $2i + j = 3$. Hence all the other $2q-1$ variables are tangential and computed at $y = 0$. The order is hence $r^3\varepsilon^{2q-1}$

The order of $P^{(2q+4)}$ is clearly ε^{2q+4} and comes from a term depending only on ξ and possibly on x. However, by hypothesis all its Fourier coefficients must respect momentum conservation. Reasoning as in Proposition 4, by Constraint 1 iii) such a term is necessarily constant in the dynamical variables, hence we drop it, since it does not contribute to the vector field. Hence the order is either ε^{2q+6} or $\varepsilon^{2q+3}r$. For $r > \varepsilon^3$, the leading term is $\varepsilon^{2q+3}r$. Passing to the vector field, under our constraints:

Proposition 6. *The order of $X_{P^{(3)}}$ is $r\varepsilon^{2q-1}$.*

The order of $X_{P^{(2q+4)}}$ is $\varepsilon^{2q+3}r^{-1}$.

Remark 9. It is possible to improve the estimate $r > \varepsilon^3$ to $r > \varepsilon^{2q+1}$ by noticing that with one step of the Birkhoff normal form one can remove all the non-resonant terms in H of degree $< 4q + 2$, then we repeat the analysis as above. This procedure only changes ω and $\mathcal{Q}$ in a trivial way.

5. Matrix Description of $ad(N)$

5.1. The spaces $V^{i,j}$ and $F^{0,1}$.

Definition 8. *We denote by $V^{i,j}$ the space of functions spanned by elements of total degree i in y and j in w and $V^h = \sum_{i+j=h} V^{i,j}$, $V^\infty = \sum_{i,j} V^{i,j}$.*

The space $V^{0,1}$ has a basis over $\mathbb{C}$ given by the elements $\{e^{\mathrm{i}\sum_j \nu_j x_j} z_k, e^{-\mathrm{i}\sum_j \nu_j x_j} \bar{z}_k\}$, where $\nu \in \mathbb{Z}^m$, $k \in S^c$.

It can be also viewed as a *free module*[7] with basis the elements $z_k, \bar{z}_k$, over the algebra $\mathcal{F}$ of the finite Fourier series. This is useful since, by Formula (30), the function $\mathcal{Q}$ commutes with $\mathcal{F}$ and thus it can be described by a matrix, with entries in $\mathcal{F}$, in the basis $z_k, \bar{z}_k$.

We now impose the restrictions of momentum and mass conservation. Denote by $F^{0,1}$ the subspace of $V^{0,1}$ commuting with momentum. By formula (24) we see that $F^{0,1}$ has as basis, which we call the *frequency basis*, the set F_B of elements (cf. (25)),

$$F_B = \{e^{\mathrm{i}\sum_j \nu_j x_j} z_k, e^{-\mathrm{i}\sum_j \nu_j x_j} \bar{z}_k\}; \quad \sum_j \nu_j v_j + k = \pi(\nu) + k = 0\,, \quad k \in S^c. \tag{39}$$

An element of F_B is completely determined by the value of ν and the fact that the z variable may or may not be conjugated, thus sometimes we refer to $e^{\mathrm{i}\sum_j \nu_j x_j} z_{-\pi(\nu)}$ as $(\nu, +)$ and to $e^{-\mathrm{i}\sum_j \nu_j x_j} \bar{z}_{-\pi(\nu)}$ as $(\nu, -)$. By construction $\nu \in \mathbb{Z}^m_c$, where

$$\mathbb{Z}^m_c := \{\mu \in \mathbb{Z}^m \mid -\pi(\mu) \in S^c\}\,, \tag{40}$$

and we may identify F_B with $\mathbb{Z}^m_c \times \mathbb{Z}/(2)$.

We can further decompose the space $F^{0,1} = \oplus F^{0,1}_\ell$ by the eigenspaces of the mass operator $ad(L)$. Notice that the *mass* of $e^{\mathrm{i}\sum_j \nu_j x_j} z_k$ is $\ell = \sum_i \nu_i + 1$, thus on the subspace commuting with L we have $-1 = \sum_i \nu_i$ for $(\nu, \pm)$.

5.1.1. The action of $ad(N)$. In order to study the action of $ad(N)$ on the two spaces $F^{0,1}$ and $V^{0,1}$ we notice that:

Remark 10. i) The terms $\sum_k |k|^2 |z_k|^2 + \mathcal{Q}(x, w)$ Poisson commute with the algebra $\mathcal{F}$ of Fourier series in x.
ii) $ad(\sum_k |k|^2 |z_k|^2)$ is a diagonal matrix in the geometric basis $z_k, \bar{z}_k$.
iii) $ad((\omega(\xi), y) + \sum_k |k|^2 |z_k|^2)$ is a diagonal matrix in the frequency basis F_B.

Hence, in order to describe $ad(N)$, we only need to understand the action of $ad(\mathcal{Q})$ on the two spaces $F^{0,1}$ and $V^{0,1}$. We then have two matrix descriptions. One, denoted $\mathrm{i}Q(x)$, with respect to the basis w and with finite Fourier series as entries, the other $\mathrm{i}Q$ with respect to the frequency basis and with constant coefficients. Of course each can be deduced from the other in a simple way.

[7] A free module is the vector space of linear combinations of a basis with coefficients in an algebra.

6. Graph Representation

A matrix $(a_{i,j})$ over a basis indexed by a set I is conveniently displayed graphically by a graph with vertices the elements of I. Two vertices i, j are joined by an edge if $a_{i,j} \neq 0$, in this case it is also convenient to orient the edge and mark it with the entry of the matrix as

$$i \xleftarrow{a_{i,j}} j \ .$$

The usefulness of this construction lies in the fact that the connected components of the graph correspond to the diagonal indecomposable blocks into which the matrix can be decomposed.

We thus associate to the matrices $Q(x)$, Q two graphs $\tilde{\Gamma}_S$, Λ_S encoding the information of the non-zero off diagonal entries in the respective bases.

Definition 9. *The graph $\tilde{\Gamma}_S$ has as vertices the geometric basis, i.e. the variables $z_k, \bar{z}_h$, and edges corresponding to the nonzero entries of the matrix $Q(x)$ in this basis.*

The graph Λ_S has as vertices the elements of $F_B = \mathbb{Z}_c^m \times \mathbb{Z}/(2)$, and edges corresponding to the nonzero entries of the matrix Q in this frequency basis.

Remark 11. We could also introduce the graph describing the matrix Q on the entire space $V^{0,1}$ in its corresponding basis. It is just obtained as infinitely many copies of Λ_S (for all values of momentum) by multiplying with all possible exponentials $e^{\mathrm{i}\sum_j \nu_j x_j}$.

6.0.2. The rules. The rules to construct the graph are the Formulas (35), (36), (37).

To be explicit in our case, if $a_{i,j} \neq 0$ also $a_{j,i} \neq 0$ so we should connect each connected pair of vertices with two edges. In fact it is clear that both edges and their markings are uniquely determined by a single edge ℓ. We discuss the simple choices that we make in order to be explicit.

In case of an ordered pair (h, k) satisfying (35) for the edge $\ell \in X_q^0$, we display:

$$z_h \underset{c(\ell)e^{-\mathrm{i}\ell\cdot x}}{\overset{c(\ell)e^{\mathrm{i}\ell\cdot x}}{\leftrightarrows}} z_k \equiv z_h \underset{-\ell}{\overset{\ell}{\leftrightarrows}} z_k \ , \quad \bar{z}_h \underset{-c(\ell)e^{\mathrm{i}\ell\cdot x}}{\overset{-c(\ell)e^{-\mathrm{i}\ell\cdot x}}{\leftrightarrows}} \bar{z}_k \equiv \bar{z}_h \underset{\ell}{\overset{-\ell}{\leftrightarrows}} \bar{z}_k \ .$$

Of course, if $\ell \in X_q^0$ then also $-\ell \in X_q^0$. We choose one representative of the pair $\ell, -\ell$ (for instance using lexicographic ordering) and drop one of the arrows.

Similarly for $(a, \sigma), (b, \rho) \in \mathbb{Z}^m \times \mathbb{Z}/(2)$ such that $b = a + \ell, h = -\pi(b), k = -\pi(a), \sigma = \rho$ (and h, k as above) we have

$$(b, +) \underset{c(\ell)}{\overset{c(\ell)}{\leftrightarrows}} (a, +) \equiv (b, +) \xleftarrow{\ell} (a, +) \ ,$$

$$(b, -) \underset{-c(\ell)}{\overset{-c(\ell)}{\leftrightarrows}} (a, -) \equiv (b, -) \xleftarrow{\ell} (a, -) \, .$$

Notice that our convention in describing the basis F_B, implies that the arrow joining $(a, -)$ to $(b, -)$ has the opposite direction to that joining $\bar{z}_h$ to $\bar{z}_k$.

In case of an unordered pair (h, k) satisfying (36) for the edge $\ell \in X_q^{-2}$ we display:

$$z_h \underset{c(\ell)e^{-\mathrm{i}\ell\cdot x}}{\overset{-c(\ell)e^{\mathrm{i}\ell\cdot x}}{\rightleftarrows}} \bar{z}_k \equiv z_h \overset{\ell}{\text{———}} \bar{z}_k\ , \quad (a,+) \underset{c(\ell)}{\overset{-c(\ell)}{\rightleftarrows}} (b,-) \equiv (a,+) \overset{\ell}{\text{———}} (b,-)\ ,$$

where $-\pi(a) = h$, $-\pi(b) = k$ and $a + b = \ell$.

Remark 12. Since K commutes with $\mathcal{Q}$, a block for $\mathcal{Q}$ is contained in an eigenspace of K with fixed eigenvalue κ. We have

$$\begin{aligned} \{K, e^{\mathrm{i}\mu.x} z_k\} &= \mathrm{i}(\sum_i \mu_i |v_i|^2 + |k|^2) e^{\mathrm{i}\mu.x} z_k, \\ \{K, e^{-\mathrm{i}\mu.x} \bar{z}_k\} &= -\mathrm{i}(\sum_i \mu_i |v_i|^2 + |k|^2) e^{-\mathrm{i}\mu.x} \bar{z}_k. \end{aligned} \tag{41}$$

The eigenspace of K where $\sum_i \mu_i |v_i|^2 + |k|^2 = \kappa$ in general is an infinite block which has to be further reduced, by the explicit description of $\mathcal{Q}$, into the direct sum of infinitely many finite blocks.

While the graph $\tilde{\Gamma}_S$ appears naturally in the description of $Q(x)$, we find it convenient to forget the conjugate variables getting a purely *geometric graph* Γ_S with vertices in S^c and *colored edges*.

Definition 10. *Two points $h, k \in S^c$ are connected by a* **black edge** *if z_h, z_k are connected in $\tilde{\Gamma}_S$, the edge has the same orientation as the one joining z_h, z_k and mark the edge by $-\pi(\ell)$. Similarly, $h, k \in S^c$ are connected by a* **red edge** *if $z_h, \bar{z}_k$ are connected in $\tilde{\Gamma}_S$, the marking is again $-\pi(\ell)$.*

Example 5 ($q = 1$). Suppose we have in $\tilde{\Gamma}_S$ the connected component containing z_{k_1}:

$$\tilde{A}_{k_1,+} = \begin{tikzcd} & \bar{z}_{k_4} \arrow[d, dash, "-e_2-e_1"] & \\ z_{k_1} \arrow[r, "e_3-e_1"] & z_{k_2} \arrow[r, "e_3-e_2"] & z_{k_3} \end{tikzcd} \qquad \text{where} \quad \begin{cases} k_2 - k_1 + v_3 - v_1 = 0 \\ |k_2|^2 - |k_1|^2 + |v_3|^2 - |v_1|^2 = 0 \\ k_3 - k_2 + v_3 - v_2 = 0 \\ |k_3|^2 - |k_2|^2 + |v_3|^2 - |v_2|^2 = 0 \\ k_4 + k_2 - v_2 - v_1 = 0 \\ |k_4|^2 + |k_2|^2 - |v_2|^2 - |v_1|^2 = 0 \end{cases} \tag{42}$$

with $k_1 \neq k_2 \neq k_3 \neq k_4 \in S^c$. Then the block of the matrix $Q(x)$ corresponding to this graph is

$$4 \begin{pmatrix} 0 & \sqrt{\xi_1\xi_3} e^{-\mathrm{i}(x_3-x_1)} & 0 & 0 \\ \sqrt{\xi_1\xi_3} e^{\mathrm{i}(x_3-x_1)} & 0 & \sqrt{\xi_2\xi_3} e^{-\mathrm{i}(x_3-x_2)} & -\sqrt{\xi_1\xi_2} e^{-\mathrm{i}(x_1+x_2)} \\ 0 & \sqrt{\xi_2\xi_3} e^{\mathrm{i}(x_3-x_2)} & 0 & 0 \\ 0 & \sqrt{\xi_1\xi_2} e^{\mathrm{i}(x_1+x_2)} & 0 & 0 \end{pmatrix},$$

where we have arbitrarily chosen the ordering $z_{k_1}, z_{k_2}, z_{k_3}, \bar{z}_{k_4}$.

By the reality condition we also have the connected component:

$$\tilde{A}_{k_1,-} = \begin{tikzcd} & z_{k_4} \arrow[d, dash, "-e_2-e_1"] & \\ \bar{z}_{k_1} & \bar{z}_{k_2} \arrow[l, "e_3-e_1"'] & \bar{z}_{k_3}, \arrow[l, "e_3-e_2"'] \end{tikzcd}$$

which we think of as a *conjugated block*.

In conclusion the contribution of these two components to $\mathcal{Q}$ is:

$$\begin{aligned} &4\sqrt{\xi_1\xi_3}e^{\mathrm{i}(x_1-x_3)}z_{k_1}\bar{z}_{k_2} + 4\sqrt{\xi_2\xi_3}e^{\mathrm{i}(x_2-x_3)}z_{k_2}\bar{z}_{k_3} + 4\sqrt{\xi_1\xi_2}e^{-\mathrm{i}(x_1+x_2)}z_{k_2}z_{k_4} \\ &\quad + 4\sqrt{\xi_1\xi_3}e^{-\mathrm{i}(x_1-x_3)}\bar{z}_{k_1}z_{k_2} + 4\sqrt{\xi_2\xi_3}e^{-\mathrm{i}(x_2-x_3)}\bar{z}_{k_2}z_{k_3} + 4\sqrt{\xi_1\xi_2}e^{\mathrm{i}(x_1+x_2)}\bar{z}_{k_2}\bar{z}_{k_4}. \end{aligned} \tag{43}$$

Let now $a \in \mathbb{Z}^m$ be any vector such that $-\pi(a) = k_1$, then the graph Λ_S has the two components

$$\begin{aligned} \mathcal{A}_{(a,+)} = & \begin{tikzcd} & (-e_2-e_3-a,-) \arrow[d, dash, "-e_2-e_1"] & \\ (a,+) \arrow[r, "e_3-e_1"] & (a+e_3-e_1,+) \arrow[r, "e_3-e_2"] & (a-e_1-e_2+2e_3,+), \end{tikzcd} \\ \mathcal{A}_{(a,-)} = & \begin{tikzcd} & (-e_2-e_3-a,+) \arrow[d, dash, "-e_2-e_1"] & \\ (a,-) \arrow[r, "e_3-e_1"] & (a+e_3-e_1,-) \arrow[r, "e_3-e_2"] & (a-e_1-e_2+2e_3,-). \end{tikzcd} \end{aligned} \tag{44}$$

Finally the geometric graph corresponding to (42) is [8]

$$A_{k_1} = \begin{tikzcd} & k_4 \arrow[d, dash, double, "v_2+v_1"] & \\ k_1 \arrow[r, "v_1-v_3"] & k_2 \arrow[r, "v_2-v_3"] & k_3, \end{tikzcd} \tag{45}$$

The vectors appearing as vertices must satisfy the linear and quadratic constraints appearing in (42). Notice that we can deduce the list of equations associated to a geometric graph by looking at its vertices, indeed if k_i, k_j are connected by an edge then this arises from an ℓ (see Formulas (35)–(37)) which is uniquely determined.

Remark 13. All the connected components which we have described in this simple example are isomorphic (as marked graphs), this is a fundamental issue since it enables us to define the change of variables which reduces the Hamiltonian to constant coefficients. The geometric graph probably gives the clearer picture since it encodes in the simplest way the list of equations which the k_i must fulfill.

It is useful to notice that, as soon as $m > n$, corresponding to the components $\tilde{A}_{k_1,\pm} \in \tilde{\Gamma}_S$, there are infinitely many components $\mathcal{A}_{(a,\pm)} \in \Lambda_S$, one for each of the points $(a, \pm)$ such that $-\pi(a) = k$. These points are infinitely many if $m > n$, since the vectors v_i cannot be independent and so π must have a kernel $\ker(\pi)$, these components are obtained from a given one by translations by elements of $\ker(\pi)$.

[8] We shall denote by $=\!=\!=$ a red edge.

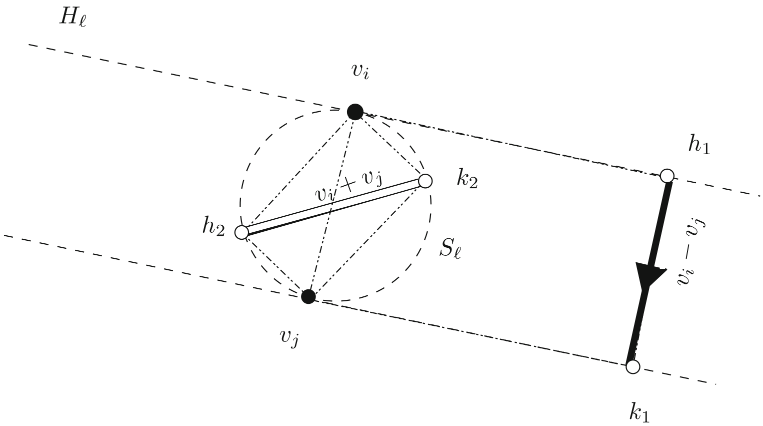


Fig. 2. The plane H_ℓ with $\ell = e_j - e_i$ and the sphere S_ℓ with $\ell = -e_i - e_j$. The points h_1, k_1, v_j, v_i form the vertices of a rectangle. Same for the points h_2, v_i, k_2, v_j

6.1. Geometric graph Γ_S^{geo}. We define a graph on $\mathbb{R}^n$ using the formulas (35) and (36).

Definition 11. *An edge $\ell \in X_q^{-2}$ defines a sphere S_ℓ through the relation:*

$$|x|^2 + (x, \sum_i \ell_i v_i) = -\frac{1}{2}(|\sum_i \ell_i v_i|^2 + \sum_i \ell_i |v_i|^2), \tag{46}$$

An edge $\ell \in X_q^0$ defines a plane H_ℓ through the relation

$$(x, \sum_i \ell_i v_i) = \frac{1}{2}(|\sum_i \ell_i v_i|^2 + \sum_i \ell_i |v_i|^2). \tag{47}$$

Definition 12. *Each $x \in S_\ell$ is joined by a* **red** *unoriented edge to $-x - \sum_i \ell_i v_i \in S_\ell$. Each $x \in H_\ell$ is joined by a* **black** *oriented edge to $x - \sum_i \ell_i v_i \in H_{-\ell}$. We construct the graph Γ_S^{geo} with vertices all the points of $\mathbb{R}^n$ and edges the black and red edges described in Fig. 2.*

It is convenient to mark each edge of the graph with the element $-\pi(\ell)$ from which it comes from. Notice that Constraint 1 implies that the edge ℓ is uniquely determined by the vector $\pi(\ell)$.

Remark 14. The points in H_ℓ are the initial vertices of an edge $\ell \in X_q^{(0)}$ ending in the parallel hyperplane $H_\ell - \sum_i \ell_i v_i = H_{-\ell}$.

The points in S_ℓ are the initial vertices of an edge of type $\ell \in X_q^{(2)}$ which is a diameter of the sphere.

Remark 15. The completeness Constraint 1 *ii)* on S implies that the vectors $v_1, \dots, v_m$ form a component of the graph Γ_S^{geo}. In this component every two vertices are joined by a red and by a black edge marked respectively $v_i + v_j$ and $v_i - v_j$.

This is independent of the choices of q, m, n.

Definition 13. *The component $v_1, \dots, v_m$ is called the* **special component** *of the graph Γ_S^{geo}.*

We want to understand the other connected components of the graph Γ_S^{geo}, which contain a purely geometric description of the possible components of Γ_S. Naturally most of the components of the graph Γ_S^{geo} will not be formed by integral vectors.

6.1.1. A rough estimate. Before we start a fine analysis we may recall first a simple result, which is proved in [16]:

Lemma 3. *i) The number of vertices which may be adjacent to a red edge is finite and bounded by some constant $N(q, n, \max_{i=1}^m(|v_i|)$.*

ii) For generic choices of S each path in $P \in \Gamma_S^{geo}$ containing only black edges cannot contain two distinct edges marked with the same $\ell \in X_q^{(0)}$.

Proof. i) Each $\ell \in X_q^{(-2)}$ defines a sphere and on each sphere there are only a finite number of points, at most R_ℓ^{n-1}, where R_ℓ is the radius of the sphere. If a vertex k is adjacent to ℓ by definition $k \in S_\ell$; the statement follows.

ii) In a minimal counterexample we suppose that an edge ℓ appears twice and all others appear at most once. Let x_1, x_2 be the two distinct vertices out of which ℓ exits and consider a path $P(x_1, x_2)$ joining them. By applying the linear equations in (35) to the vertices in $P(x_1, x_2)$ one may conclude that $x_2 = x_1 + \sum n_i v_i$, where the n_i are integers which depend on $P(x_1, x_2)$. Since $P(x_1, x_2)$ does not contain any other edge more than once, then $|n_i| \le (2q)^{m+1}$. We now write the condition that $x_1, x_2 \in H_\ell$, using (47):

$$2(x_1, \sum_i \ell_i v_i) = \sum_i \ell_i |v_i|^2 + |\sum_i \ell_i v_i|^2,$$

$$2(x_1 + \sum n_i v_i, \sum_i \ell_i v_i) = \sum_i \ell_i |v_i|^2 + |\sum_i \ell_i v_i|^2,$$

and this may be excluded by requiring

$$(\sum n_i v_i, \sum_i \ell_i v_i) \neq 0, \quad \forall \sum n_i v_i \neq 0, |n_i| \le (2q)^{m+1}$$

which is a genericity condition. □

This lemma immediately implies:

Proposition 7. *For generic choices of S there is a uniform bound on the number of vertices in each connected component of Γ_S^{geo}.*

Proof. By Lemma 3 *ii)* a path made of black edges has a finite (and uniformly bounded) length since each edge may appear at most once. So the connected components containing black edges have a uniform bound on the number of vertices. By Lemma 3 *i)* we may form a finite block with all the points adjacent to a red edge and all the vertices connected to them. Indeed if a vertex is connected to a vertex in a sphere by a path made of black edges then this path has finite length. □

This bound is clearly very rough, however to prove optimal bounds one must work much harder and this we shall do in the rest of the paper.

6.2. Geometric results. Our goal. We want to decompose the graph Γ_S^{geo} into simple blocks, as for instance that of (45). The fact that this may be possible with blocks of at most $n+1$ vertices is suggested by a simple count of parameters, indeed one sees in (42) that a tree with e edges occurs when the $e+1$ vertices (corresponding to $(e+1)n$ incognitæ) satisfy a set of $e(n+1)$ equations.

Indeed, this bound can be achieved for all blocks consisting only of black edges under all geometric constraints.

The core of the paper is to prove Theorems 2 and 3 by imposing a finite number of non-zero polynomial constraints on S; Constraints 1 are the beginning of this analysis. The full list of the explicit geometric constraints is summarized in Definition 22.

Theorem 2. *For a generic choice of the v_i as in Def. 22 we have:*

- *i) All connected components of the graph Γ_S^{geo} consisting only of black edges have at most $n+1$ vertices.*
- *ii) There are finitely many components in Γ_S containing red edges, each can contain at most $2n$ vertices.*
- *iii) The connected components of Γ_S^{geo} consisting only of black edges are divided into a finite number of families.*
- *iv) Each family in Γ_S^{geo} is formed by all the graphs obtained from a given one G, with $k+1$ affinely independent vertices, under translation by all the points of the $n-k$ dimensional subspace orthogonal to the span of G, minus a union A of a finite number of lower dimensional affine subspaces.*
  The translates $G+a,\ a\in A$ are contained in strictly larger connected components of Γ_S^{geo}.
  Moreover
- *v) All connected components of the graph Λ_S have at most $2n$ vertices. The vertices with the same sign are affinely independent. There may be complicated dependencies between vertices with different signs.*

Proof. See §10.1. □

The next result relates the three graphs $\Lambda_S, \tilde{\Gamma}_S, \Gamma_S$. Take a frequency $\mu\in\mathbb{Z}_c^m$, let $\mathcal{A}_{(\mu,+)}$ be the component in Λ_S of $(\mu,+)$ and set $k=-\pi(\mu)$. From Formula (39) the associated component in $\tilde{\Gamma}_S$ is the one of the element z_k and will be denoted by $\tilde{A}_{(k,+)}$. Finally in the geometric graph Γ_S we have the component of the element k which will be denoted by A_k with a similar description for $(\mu,-)$.

Theorem 3. *For a generic choice of S the map $-\pi$ establishes a graph isomorphism between $\mathcal{A}_{(\mu,\pm)}$ and $\tilde{A}_{(k,\pm)}$, which is also mapped isomorphically to A_k. All these maps are compatible with the markings.*

Proof. See §10.2. □

In particular the space spanned by all transforms of $e^{\mathrm{i}\mu.x}z_k$ applying the operator $ad(N)$ has a basis extracted from the frequency basis in correspondence, under $-\pi$, with the vertices of A_k.

All other connected components of Λ_S lying over A_k are obtained from $\mathcal{A}_{(\mu,\pm)}$ by adding all the elements ν such that $\pi(\nu)=0$.

7. A Formalization of the Graphs

The rules (35), (36), (37) determine the edges of the three graphs $\Lambda_S, \tilde{\Gamma}_S, \Gamma_S$ that we have introduced in §6.2. These rules consist of a linear and a quadratic constraint which encode respectively the conservation of momentum and of quadratic energy (i.e. the fact that we have kept only resonant terms). We want to see first that, if we implement only the linear rules, the graphs we construct are contained in some Cayley graphs (see the Appendix for the relevant definitions). Next we show that the quadratic rules select, inside the large Cayley graphs, the graphs of our interest.

7.1. The linear momentum constraints. Denote by $\mathbb{Z}^m := \{\sum_{i=1}^m a_i e_i,\ a_i \in \mathbb{Z}\}$ the lattice with basis the elements e_i.

We consider the group $G := \mathbb{Z}^m \rtimes \mathbb{Z}/(2)$[9] of couples (a, σ) with $a \in \mathbb{Z}^m \sigma = \pm$. The group structure is given by the rules

$$(a,+)(b,+) = (a+b,+)\,, \quad (a,-)(b,+) = (a-b,-)\,,$$
$$(a,+)(b,-) = (a+b,-)\,, \quad (a,-)(b,-) = (a-b,+).$$

It will be notationally convenient to identify by a the operator of left multiplication by $(a,+)$ and by τ the operator of left multiplication by $(0,-)$ so that

$$(a,+) = a(0,+)\,, \quad (a,-) = a\tau(0,+).$$

Note the commutation rules $a\tau = \tau(-a)$. Sometimes we refer to the elements $a = (a,+)$ as *black* and $a\tau = (a,-)$ as *red.*

Recall we defined the mass in Formula (25) by $\eta : \mathbb{Z}^m \to \mathbb{Z},\ \eta(e_i) := 1$. If $p \in \mathbb{Z}$ it is easily seen that the set $G_p := \{a\ :\ \eta(a) = 0,\ a\tau\ :\ \eta(a) = p\}$ form a subgroup. In particular G_{-2} is generated by the elements $e_i - e_j, (-e_i - e_j)\tau$.

The group G has also a simple geometric interpretation: for any $a \in \mathbb{Z}^m$ the element a acts on $\mathbb{Z}^m$ as the translation $t_a : x \mapsto x + a$, while the element τ is the sign change $\tau : x \to -x$, so $a\tau$ acts by $x \mapsto a - x$.

Remark 16. In our dynamical setting, we have chosen a list of vectors v_i and defined (cf. Formula (25)) $\pi : \mathbb{Z}^m \to \mathbb{R}^n$ by $\pi : e_i \mapsto v_i$.

We can think of G also as linear operators on $\mathbb{R}^n$ by setting

$$ak := -\pi(a) + k,\ k \in \mathbb{R}^n,\ a \in \mathbb{Z}^m\,, \quad \tau k := -k. \tag{48}$$

For each $q = 1, 2, \ldots$ we consider the Cayley graphs in $G, \mathbb{Z}^m, \mathbb{R}^n$ associated to the set $X = \{X_q^0 = (X_q^0, +),\ X_q^{-2}\tau = (X_q^{-2}, -)\}$ (cf. Formula (27)). Notice that, for all $a \in \mathbb{Z}^m$, we have $(a,-)^2 = (0,+) = Id$, the identity of G. In particular $X = X^{-1}$.

We take two elements $(a,\sigma),\ (b,\rho) \in G$ $(\sigma, \rho = \pm)$. We have

$$(b,\rho)(a,\sigma)^{-1} = b(0,\rho)(0,\sigma)(-a) = \begin{cases} b-a & \text{if } \rho = \sigma \\ (a+b)\tau & \text{if } \rho \neq \sigma \end{cases}. \tag{49}$$

[9] The notation $\rtimes$ stands for *semidirect product*

Therefore $(a,\sigma),\ (b,\rho)$ are joined by an oriented edge marked with $\ell \in X^0$ if $\sigma = \rho$ and $b - a = \ell$, while $(a,\sigma),\ (b,\rho)$ are joined by an edge marked with $\ell\tau,\ \ell \in X^{-2}$ if $\sigma = -\rho$ and $a + b = \ell$. Graphically

$$(b,+) \xleftarrow{\ \ell\ } (a,+)\ ,\quad (b,-) \xleftarrow{\ \ell\ } (a,-)\ ,\quad (a,+) \overset{\ell\tau}{\longrule} (b,-)\,.$$

We have obtained the same picture as in §6.0.2, only now we are not imposing the *quadratic* constraint $h = -\pi(a), k = -\pi(b)$, where (h,k) satisfy (35) or (36). We usually drop the τ in the marking of the unoriented edges associated to $\ell \in X_q^{-2}$.

The significance of this choice is:

Proposition 8. *i) The elements $X_q^0,\ X_q^{-2}\tau$ generate G_{-2}.*

ii) The Cayley graph $\mathbb{R}^n_X$ contains the geometric graph Γ_S^{geo} of Definition 10. We identify the basis F_B of Formula (39) with $\mathbb{Z}^m_c \times \mathbb{Z}/(2) \subset G$ then:

iii) The graph Λ_S (cf. 9) is a subgraph of the Cayley graph G_X.

iv) Each connected component of Λ_S is a full[10] subgraph of the Cayley graph G_X.

In view of the previous proposition we set a:

Definition 14. *A* **complete marked graph***, on a set $A \subset \mathbb{Z}^m \rtimes \mathbb{Z}/(2)$ is the full sub–graph generated by the vertices in A.*

In fact using conservation of mass and the action of G on $\mathbb{Z}^m$, it is even better to consider Λ_S lying in the orbit of G_{-2} in $\mathbb{Z}^m$ formed of elements $a \in \mathbb{Z}^m \mid \eta(a) = 0, -2$. This identification is not canonical but depends on the choice of a *root* $r \in \Lambda_S$ that corresponds to 0.

There are symmetries in the graph. The symmetric group S_m of the $m!$ permutations of the elements e_i preserves the graph. By Lemma 13 we have the right action of G, on the graph:

$$(b,\sigma) \mapsto (b,\sigma)\tau = b\sigma\tau,\quad (b,\sigma) \mapsto (b,\sigma)a = (b + \sigma a, \sigma),\ \forall a, b \in \mathbb{Z}^m. \tag{50}$$

Up to the G action any subgraph can be translated to one containing $(0,+)$; in this way we keep only the combinatorial information.

7.2. The quadratic energy constraints. We consider $\mathbb{R}^m$ with the standard scalar product.

Given a list S of m vectors $v_i \in \mathbb{R}^n$, we have defined the linear map $\pi : e_i \mapsto v_i$.

Let $S^2[\mathbb{Z}^m] := \{\sum_{i,j=1}^m a_{i,j} e_i e_j\},\ a_{i,j} \in \mathbb{Z}$ be the polynomials of degree 2 in the e_i with integer coefficients. We extend the map π and introduce a linear map $L^{(2)} : a \mapsto a^{(2)}$ as:

$$\pi(e_i) = v_i,\quad \pi(e_i e_j) := (v_i, v_j),\quad L^{(2)} : \mathbb{Z}^m \to S^2(\mathbb{Z}^m),$$
$$a = \sum a_i e_i \mapsto a^{(2)} := \sum a_i e_i^2.$$

We have $\pi(AB) = (\pi(A), \pi(B)), \forall A, B \in \mathbb{Z}^m$.

Remark 17. Notice that we have $a^{(2)} = a^2$ if and only if a equals 0 or one of the variables e_i.

[10] A full subgraph of a graph Γ consists of a subset of the vertices and all the edges in Γ between these chosen vertices.

Definition 15. *Given an element $u = (a, \sigma) = (\sum_i a_i e_i, \sigma) \in G$ set*

$$C(u) := \frac{\sigma}{2}(a^2 + a^{(2)}), \quad \frac{1}{2}K(u) := \pi(C(u)) = \frac{\sigma}{2}(|\sum_i a_i v_i|^2 + \sum_i a_i |v_i|^2). \quad (51)$$

We call $K(u)$ the **energy** *of u; this is exactly the eigenvalue of K given by Formula (41).*

Notice that $C(u)$ has integer coefficients.

For $u = (a, \sigma)$ and $g = (\sum_i n_i e_i, \rho)$ consider $gu = (b, \sigma\rho),\ b = \sum_i n_i e_i + \rho a$. We have

$$C(gu) = \sigma C(g) + C(u) + (\rho - 1)\frac{\sigma}{2}a^2 + \sigma(\sum_i n_i e_i)a. \quad (52)$$

From (52) we see that $K(gu) = K(u)$ if and only if:

$$0 = K(g) + (\rho - 1)|\pi(a)|^2 + 2(\sum_i n_i v_i, \pi(a)). \quad (53)$$

Definition 16. *Given an edge $u \xrightarrow{x} v$, $u = (a, \sigma), v = (b, \rho) = xu,\ x \in X_q$, we say that the edge is* **compatible** *with S or π if $K(u) = K(v)$.*

As in the previous section we identify the basis F_B of Formula (39) with $\mathbb{Z}^m_c \rtimes \mathbb{Z}/(2) \subset G$.

Proposition 9. *The graph Λ_S of Definition 9 is the subgraph of G_X in which we only keep the compatible edges and the vertices in $\mathbb{Z}^m_c \rtimes \mathbb{Z}/(2)$.*

Proof. i) If we have $a \in \mathbb{Z}^m$ and $b = (\ell, 1)a = \ell + a$, set $k := -\pi(b),\ h := -\pi(a)$, we have $k + \pi(\ell) = h$. The condition $K(a) = K(b)$ is given by formula (53) with $g = (\ell, 1)$. This in turn gives formula (47) with $x = h$, i.e. implies the fact that $h \in H_\ell$ or equivalently that h, k satisfy Eqs. (35).

Similarly if $b = (\ell, \tau)a = (\ell - a, \tau)$ we have $\pi(\ell) + h + k = 0$, the condition $K(a) = K(b)$ is given by (53) with $g = (\ell, \tau)$; this gives formula (46) with $x = h$ or $x = k$, i.e. implies the fact that $h, k \in S_\ell$ or equivalently that h, k satisfy Eqs. (36). □

Example 6. In our Example 5 consider the component $\mathcal{A}_{(a,+)}$ in (44). By construction the edges appear in the Cayley graph, moreover the condition that all the vertices have the same energy are the equations in (42). The projection of $\mathcal{A}_{(a,+)}$ with the map $-\pi$ gives A_{k_1} in (45).

This proposition is the combinatorial counterpart of *conservation of the quadratic energy K* computed in Formula (41) and summarized as:

- If u, v are in the same connected component of Λ_S we have $K(u) = K(v)$.
- Under the map $-\pi$, the component A maps to a connected component C of Γ_S.

Corollary 3. *A connected component A of Λ_S is a complete subgraph (cf. Definition 14) of G_X.*

Proof. Fix an element u of which we want to find the component. Consider the set of all elements v with the same energy as u. They determine a complete (or full) subgraph of the graph G_X, and an edge in this subgraph is by construction compatible, thus the component passing through u of this graph is the required one. □

8. Graph Isomorphism

We wish to identify the connected components of Λ_S with those of Γ_S.

Proposition 10. *i) Under the map $(a,\sigma)\mapsto -\pi(a)$ the graph Λ_S maps surjectively to the geometric graph Γ_S. The image of an edge in Λ_S is an edge in Γ_S.*

ii) The preimage of an edge in Γ_S is a set of edges in Λ_S which are simply permuted by right translations under the subgroup $\ker(\pi)$ *of $\mathbb{Z}^m$.*

Proof. *i)* This is just the definition of Γ_S since we have shown that a compatible edge $(a,\pm)\xleftarrow{\ \ell\ }(b,\pm)$ is such that setting $h=-\pi(a), k:=-\pi(b)$ one has that h,k respect (35).

ii) Given a compatible edge $(a,\pm)\xleftarrow{\ \ell\ }(b,\pm)$ let $h=-\pi(a)$, $k=-\pi(b)$. Consider now a' such that $a-a'\in\ker(\pi)$ and set $b'=a'+\ell$ so that by definition $(a',\pm)$ is connected to $(b',\pm)$ in G_X. We notice that $\pi(a)=\pi(a')$ so that $K(a)=K(a')$, the same holds for b' and we may conclude that $K(a')=K(b')$. This shows that $(a',\pm)\xleftarrow{\ \ell\ }(b',\pm)$ is a compatible edge. We follow the same reasoning in the case of a red edge (ℓ,τ). □

We now want to invert Proposition 10 and thus lift a connected component C of Γ_S to a connected component of Λ_S. In our Example 5, one can easily see that A_{k_1} is isomorphic to $\mathcal{A}_{a,\pm}$ and hence can be lifted. However this is not always the case unless we impose some further constraints. Indeed consider a connected graph in $\mathcal{A}\in\Lambda_S$ and let A be its projection on Γ_S. As seen in Corollary 7, the two graphs are isomorphic if and only if every circuit in A is also a circuit in $\mathcal{A}$.

There can be two cases: 1. the circuit in A contains an even number of *red edges*. 2. the circuit in A contains an odd number of *red edges*.

Example (Case 1). Suppose that the geometric graph contains a component

$$\begin{tikzcd} & k_3 \arrow[dl, "v_2-v_4"'] & \\ k_1 \arrow[rr, "v_2-v_1"] & & k_2 \arrow[ul, "v_2-v_3"'] \end{tikzcd}$$

which is the case provided that

$$v_1-3v_2+v_3+v_4=0\,,\qquad \begin{cases} 2(k_1,v_2-v_1)=|v_2-v_1|^2+|v_2|^2-|v_1|^2\\ 2(k_1,v_4-v_2)=|v_4-v_2|^2+|v_4|^2-|v_2|^2. \end{cases}$$

Let us choose any $a\in\mathbb{Z}^m$ such that $-\pi(a)=k_1$. We easily verify that the tree

$$(a,+)\xrightarrow{e_1-e_2}(a+e_1-e_2,+)\xrightarrow{e_3-e_2}(a+e_1-2e_2+e_3,+)\xrightarrow{e_4-e_2}(a+e_1-3e_2+e_3+e_4,+)$$

is contained in Λ_S. Let us call $v=e_1-3e_2+e_3+e_4$, by hypothesis $\pi(v)=0$, so that we have $-\pi(a+\alpha v)=k_1$ for all integers α. This implies that the connected component of $(a,+)$ has infinitely many vertices:

$$\begin{tikzcd} (a,+) \arrow[r, "e_1-e_2"] & (a+e_1-e_2,+) \arrow[r, "e_3-e_2"] & (a+e_1-2e_2+e_3,+) \arrow[r, "e_4-e_2"] & (a+v,+) \arrow[d, "e_1-e_2"] \\ \cdots & (a+2v,+) \arrow[l, "e_1-e_2"'] & (a+v+e_1-2e_2+e_3,+) \arrow[l, "e_4-e_2"'] & (a+v+e_1-e_2,+). \arrow[l, "e_3-e_2"'] \end{tikzcd}$$

To avoid this pathology we simply require that $v_1 - 3v_2 + v_3 + v_4 \neq 0$ so that this geometric graph does not have a realization.

Example (Case 2). Suppose that the geometric graph contains a component

$$\begin{tikzcd} & k_3 \arrow[dl, "v_2-v_4"'] \arrow[dr, equal, "v_2+v_3"] & \\ k_1 \arrow[rr, "v_2-v_1"] & & k_2 \end{tikzcd}$$

which is the case provided that

$$k_2 + k_3 = k_1 + v_2 - v_1 + k_1 + v_4 - v_2 = v_2 + v_3,$$

$$2k_1 = v_1 + v_2 + v_3 - v_4\,, \quad \begin{cases} 2(k_1, v_2 - v_1) = |v_2 - v_1|^2 + |v_2|^2 - |v_1|^2 \\ 2(k_1, v_4 - v_2) = |v_4 - v_2|^2 + |v_4|^2 - |v_2|^2; \end{cases}$$

we substitute $2k_1$ in one of the linear equations and obtain that this geometric graph does not have realization if

$$(v_1 + v_2 + v_3 - v_4, v_4 - v_2) \neq |v_4 - v_2|^2 + |v_4|^2 - |v_2|^2.$$

To repeat this reasoning in the general case we need the following trivial fact:

Lemma 4. *If $a = \sum_i n_i e_i \in \mathbb{Z}^m$ resp. (a, τ) is a product of d elements in X_q we have that $\sum_i |n_i| \leq 2dq$.*

It should be clear at this point that in order to *lift* the components of Γ_S with at most d vertices we must impose as many linear/quadratic inequalities on S as the number of loops which may appear in a component. Thus if we wish to impose only a finite number of constraints we cannot lift arbitrarily large components. Our strategy is the following: first we fix $d = 2n + 2$ and impose constraints to ensure that all components with at most d vertices can be lifted. Then we show that there are no compatible graphs in Γ_S^{geo} with d vertices; this excludes the existence of graphs C in Γ_S with d or more vertices. Otherwise we would be able to lift some subgraph of C with d vertices to a compatible graph in Λ_S. This means that the mapping $-\pi$ gives an isomorphism from each connected component of Λ_S to its image in Γ_S.

We impose

Constraint 2. *We assume $\sum_i \ell_i v_i \neq 0$, for all choices of the ℓ_i such that $\sum_i \ell_i = 0$, $\sum_i |\ell_i| \leq 4q(n+1)$ and $\sum_i \ell_i e_i \neq 0$.*

Under this constraint take an element $g = \sum_i n_i e_i$ which is a product of $d \leq 2n + 2$ elements in X. We have then $\sum_i |n_i| \leq 4q(n+1)$ so if $g \neq 0$ we have $\pi(g) = \sum_i n_i v_i \neq 0$. Then for all $k \in \mathbb{Z}^n$ $gk = \pi(g) + k \neq k,\ \forall k$, hence Case 1 may not occur.

For Case 2 let $g = (a, \tau),\ a = \sum_i n_i e_i,\ \sum_i n_i = -2$ be such that $gk = k$ for some $k \in \mathbb{Z}^n$. This is possible if and only if $\pi(a) = \sum_i n_i v_i = 2k$. Since we are assuming that there is a non-trivial odd loop starting from k, changing if necessary the starting point, the first step of the loop tells us that k lies in a sphere S_ℓ for some initial edge $\ell \in X$.

This implies that $k = -1/2 \sum_i n_i v_i$ satisfies a relation of type

$$|\sum_i n_i v_i|^2 - 2(\sum_i n_i v_i, \pi(\ell)) = 2K(\ell), \tag{54}$$

where $\ell = (\sum_i \ell_i e_i) \in X_q^{(-2)}$. This formula vanishes identically if $a^2 - 2a\ell = 2C(\ell) = -2(\ell^2 + \ell^{(2)})$. Thus

$$(a-\ell)^2 = -\ell^2 - 2\ell^{(2)}.$$

This implies that all coefficients of ℓ must be -1, and $\ell = -e_i - e_j$.
This implies $a - \ell = \pm(e_i - e_j)$ hence $a = -2e_i, -2e_j$ and $k = v_i, v_j$.
We impose

Constraint 3. *We assume that for all choices of the n_i such that $\sum_i n_i = -2$, $\sum_i |n_i| \leq 4q(n+1)$ all Eqs.* (54) *are not satisfied.*

If C is any marked graph which has at most d vertices, a minimal loop in C has at most d edges, thus:

Corollary 4. *Under the previous constraints if $C \subset \Gamma_S$ is a connected graph with at most $2n+2$ vertices then C can be lifted.*

Proof. By Corollary 7 we only need to prove that, under the previous hypotheses, it is not possible that a non-trivial element g, which is a product of at most $2n+2$ elements of X, fixes an element $k \in C$.

By the constraints that we have imposed this may happen if and only if this element generates a trivial constraint, that is an identity for all choices of v_i. If $g = a \in \mathbb{Z}^m$ this is excluded by Constraint 2 and for $g = a\tau$ it is excluded by Constraint 3. $\square$

9. The Equations Defining a Connected Component of Γ_S

Take a connected subgraph A of Γ_S which can be lifted (in particular this will be the case if A has at most $2n+2$ vertices by the previous constraints). Choose a root $x \in A$; we lift $x = -\pi(a)$, and this lifts A to the component $\mathcal{A}_{(a,+)}$ through a in Λ_S. For $h \in A$ we have an element $g_h \in G$ obtained by lifting a path in A from x to h and such that $h = g_h x$. We set

$$g_h := (L(h), \sigma(h)), \quad L(h) \in \mathbb{Z}^m,\ \sigma(h) \in \{1, \tau\} \implies h = -\pi(L(h)) + \sigma(h)x. \tag{55}$$

We then can deduce that:

Lemma 5. *For each $h \in A$ we have:*

$$\begin{cases} (x, \pi(g_h)) = \frac{1}{2}K(g_h) & \text{if } \sigma(h) = 1 \\ |x|^2 + (x, \pi(g_h)) = \frac{1}{2}K(g_h) & \text{if } \sigma(h) = \tau \end{cases} . \tag{56}$$

Proof. We use Formula (53) which implies that:

$$0 = K(g_h) + (\sigma(h) - 1)|x|^2 - 2(\pi(g_h), x). \tag{57}$$

To be explicit if $L(h) = \sum_i m_i e_i$ by (51):

$$\pi(g_h) = \sum_i m_i v_i, \quad K(g_h) = \sigma(h)(|\sum_i m_i v_i|^2 + \sum_i m_i |v_i|^2). \tag{58}$$

$\square$

The equations on x given in Formula (56) are a complete set of conditions for the existence of a graph A inside some connected component (which could also properly contain A) of Γ_S^{geo}. The reader should notice that these equations are completely analogous to the ones of Definition 11, given only for edges.

Definition 17. *Let $\mathcal{A} \subset G_X$ be the graph with vertices the elements g_h (and $g_x = (0, +) = Id$), this is called the* **combinatorial graph** *associated to A and the* **root** *x.*

Example 7. We explicitly compute the combinatorial graph associated to A_{k_1} of Example 5. We choose k_1 as the root.

$$\begin{tikzcd} \mathcal{A} = & & (-e_2 - e_3, -) \arrow[d, dash, "-e_2-e_1"] & \\ & (0,+) \arrow[r, "e_3-e_1"] & (e_3 - e_1, +) \arrow[r, "e_3-e_2"] & (-e_1 - e_2 + 2e_3, +). \end{tikzcd} \tag{59}$$

The system of equations associated to this graph is

$$\begin{cases} (x, v_3 - v_1) = |v_3|^2 - (v_1, v_3) \\ (x, -v_1 - v_2 + 2v_3) = 3|v_3|^2 - 2(v_1 + v_2, v_3) + (v_1, v_2). \\ |x|^2 - (x, v_2 + v_3) = -(v_2, v_3) \end{cases} \tag{60}$$

Notice that this graph **does not** belong to Λ_S but if $a \in \mathbb{Z}^m$ is such that $-\pi(a) = k_1$, then the right translation by $(a, +)$, i.e. $\mathcal{A}(a, +)$ gives $\mathcal{A}_{(a,+)} \in \Lambda_S$.

Remark 18. Notice that the map which associates to each $h \in A$ the element $g_h = (L(h), \sigma(h))$ is well defined only if A can be lifted. The construction of the $L(k)$ is in turn the key to the reducibility as can be seen in Example 14.

Consider now a complete subgraph (cf. Definition 14) of G_X which contains $(0, +)$. We associate to each vertex $g \neq (0, +)$ of the graph an equation:

$$\begin{cases} (x, \pi(a)) = \frac{1}{2} K(a) \quad \text{if } g = (a, +) \\ |x|^2 + (x, \pi(a)) = \frac{1}{2} K(a) \quad \text{if } g = (a, -) \end{cases} . \tag{61}$$

We think of this system of equations as associated to the graph.

Definition 18. *We call the set of complete subgraphs of G_X which contain $(0, +)$ and have at most $2n + 2$ vertices the set of* **possible combinatorial graphs***. We say that a possible combinatorial graph $\mathcal{A}$ has a geometric realization (in Γ_S^{geo}) if the equations associated to the graph have real solutions.*

The following statement holds.

Proposition 11. *A possible combinatorial graph $\mathcal{A}$ is a combinatorial graph if and only if Eqs. (61) have solutions in S^c.*

Remark 19. Notice that in a possible combinatorial graph one may deduce the color of each vertex by computing its mass. Indeed all vertices $(a, +)$ must have $\eta(a) = 0$ while $(a, -)$ corresponds to $\eta(a) = -2$.

We have reduced our problem to that of understanding which possible combinatorial graphs have a geometric realization. Naturally for given S this amounts to checking whether the equations associated to the graph are independent and– if they are not– to verify their compatibility.

Definition 19. *We say that two possible combinatorial graphs are equivalent if one is obtained from the other by right translation by an element of G (see formula (50)).*

Remark 20. It should be clear that if $\mathcal{A}$ has a geometric realization then so has any other equivalent possible combinatorial graph. Moreover the two identify the same components of Γ_S^{geo} with a different choice of the root.

Example 8. The following combinatorial graph is equivalent to $\mathcal{A}$ of Example 7:

$$\mathcal{A}' = \begin{tikzcd} & (-e_2 - e_1, -) \arrow[d, dash, "-e_2-e_1"] & \\ (e_1 - e_3, +) \arrow[r, "e_3-e_1"] & (0, +) \arrow[r, "e_3-e_2"] & (e_3 - e_2, +). \end{tikzcd} \tag{62}$$

Indeed it is obtained by right translation with the element $(e_1 - e_3, +)$. The equations are

$$\begin{cases} (x, v_1 - v_3) = |v_1|^2 - (v_1, v_3) \\ (x, v_3 - v_2) = |v_3|^2 - (v_2, v_3) \\ |x|^2 - (x, v_2 + v_1) = -(v_2, v_1); \end{cases} \tag{63}$$

it is easily seen that these equations are equivalent to the system given by formula (60), and they still identify the geometric graph A_{k_1} of Example 5 only now the root is in k_2.

9.1. Relations. Take a *possible combinatorial graph* $\mathcal{A}$

Definition 20. • *If $\mathcal{A}$ has $k+1$ vertices is said to be of* **dimension** *k.*
- *The dimension of the lattice generated by the vertices of $\mathcal{A}$ is the* **rank***,* rk $\mathcal{A}$*, of the graph $\mathcal{A}$. The dimension of the lattice generated by the black vertices $(a, +)$ (resp. red) is called the black (resp. red) rank of $\mathcal{A}$.*
- *If the rank of $\mathcal{A}$ is strictly less than the dimension of $\mathcal{A}$ we say that $\mathcal{A}$* **is degenerate***.*

Take a connected component A of Γ_S and choose a root $x \in A$. Assume that A can be lifted. Let $\mathcal{A} = \{g_a,\ a \in A\}$ be the combinatorial graph of which A is a geometric realization.

Lemma 6. *The rank of $\mathcal{A}$ does not depend on the choice of the root but only on A.*

Proof. If we change the root from x to another y we can stress in the notation $g_{a,x} = (L_x(a), \sigma_x(a))$ and have

$$g_{a,x} = g_{a,y} g_{y,x}, \implies L_x(a) = L_y(a) + \sigma_y(a) L_x(y),\ \sigma_x(a) = \sigma_y(a)\sigma_x(y). \tag{64}$$

This shows that the notion of rank is independent of the root. □

Notice that when we change the root in A we have a simple way of changing the colors and the ranks of the vertices of $\mathcal{A}$ that we leave to the reader.

If $\mathcal{A}$ is degenerate then there are non trivial relations, $\sum_a n_a a = 0,\ n_a \in \mathbb{Z}$, where the sum runs among the vertices $a \in \mathcal{A}$.

Remark 21. It is also useful to choose a maximal tree T in $\mathcal{A}$. There is a triangular change of coordinates from the vertices to the markings of T. Hence the relation can be also expressed as a relation between these markings.

We must have by linearity, for every relation $\sum_a n_a a = 0,\ n_a \in \mathbb{Z}$ that $0 = \sum_a n_a a^{(2)}$, where we recall that if $a = \sum a_i e_i$ we have that $a^{(2)} = \sum a_i e_i^2$. Finally we have $0 = \sum_a n_a \pi(a)$ and $\sum_a n_a \eta(a) = 0$.

Recalling that $\eta(a) = 0, -2$ (resp. if a is black or red), we have :

$$0 = \sum_{a \mid \eta(a)=-2} n_a. \tag{65}$$

Applying Formula (56) we deduce that, in order to ensure that the equations of $\mathcal{A}$ are compatible, we must have

$$\sum_a n_a K(a) = 2(x, \sum_a n_a \pi(a)) + [\sum_{a \mid \eta(a)=-2} n_a](x)^2 = 2(x, \sum_a n_a \pi(a)) = 0. \tag{66}$$

The expression $\sum_a n_a \frac{K(a)}{2} = \pi(\sum_a n_a C(a))$ is a linear combination with integer coefficients of the scalar products (v_i, v_j).

Given a *possible combinatorial graph* $\mathcal{A}$ with a relation:

Definition 21. *If* $\sum_a n_a C(a) \neq 0$ *we say that the graph has an* **avoidable resonance**.

Lemma 7. *A degenerate possible combinatorial graph $\mathcal{A}$ with an avoidable resonance has no geometric realization for a generic choice of the $S := \{v_i\}$.*

Proof. The graph has a realization only if $\sum_a n_a K(a) = 0$ but this polynomial, by definition, is not identically zero. □

Example 9. Consider the possible degenerate combinatorial graph

$$\begin{tikzcd}
\mathcal{A} = e_1 - e_2 & 0 \arrow[l, "e_1-e_2"'] \arrow[r, equal, "-e_1-e_3"] \arrow[d, equal, "-e_1-e_2"'] & -e_1 - e_3 \arrow[r, "e_1-e_3"] & -2e_3 . \\
& -e_1 - e_2 & &
\end{tikzcd}$$

The relation is $(e_1 - e_2) + 2(-e_1 - e_3) - (-2e_3) - (-e_1 - e_2) = 0$.

We may write the value of $C(a)$ of each vertex a; we get

$$\begin{tikzcd}
e_1^2 - e_1 e_2 \arrow[r, no head] & 0 \arrow[r, equal] \arrow[d, equal] & e_1 e_3 \arrow[r, no head] & -e_3^2 . \\
& e_1 e_2 & &
\end{tikzcd}$$

We have

$$\sum_a n_a C(a) = e_1^2 - e_1 e_2 + 2e_1 e_3 + e_3^2 - e_1 e_2,$$

so the equations of this graph are incompatible if $\pi(e_1^2 - e_1 e_2 + 2e_1 e_3 + e_3^2 - e_1 e_2) \neq 0$; this is a genericity condition.

We arrive now at the main theorem of the section:

Theorem 4. *Given a possible combinatorial graph of rank k for a given color, then either it has exactly k vertices of that color or it produces an avoidable resonance.*

Proof. Assume by contradiction that we can choose $k+1$ vertices $(a_0, a_1, \dots, a_k)$, different from the root of the given color so that we have a non trivial relation $\sum_i n_i a_i = 0$ with $n_0 \neq 0$ and the vertices $a_i,\ i = 1, \dots, k$ are linearly independent. We compute the resonance relation

$$2\sum_i n_i C(a_i) = \sum_i n_i \sigma(a_i)(a_i^2 + a_i^{(2)}) = \pm \sum_i n_i (a_i^2 + a_i^{(2)}),$$

since all the vertices a_i have the same color. By linearity we have $\sum_i n_i a_i^{(2)} = 0$. We deduce that $\sum_i n_i C(a_i) = \pm \sum_i n_i a_i^2$.

We consider the elements a_i with $i = 1, \dots, k$ as *independent variables* and write the relations as

$$0 = n_0 a_0 + \sum_{i=1}^k n_i a_i, \implies (\sum_{i=1}^k n_i a_i)^2 + n_0 \sum_{i=1}^k n_i a_i^2 = 0.$$

Now $\sum_{i=1}^k n_i a_i^2$ does not contain any mixed terms $a_h a_k,\ h \neq k$, therefore this equation can be verified if and only if the sum $\sum_{i=1}^k n_i a_i$ is reduced to a single term $n_i a_i$, and then we have $n_0 = -n_i$ and $a_0 = a_i$, a contradiction. $\square$

Constraint 4. *We impose that the vectors v_i are generic for all resonances arising from degenerate possible combinatorial graphs with at most $n+1$ elements of a given color.*

Remark 22. It is essential that we introduce the notion of colored rank, otherwise our statement is false as can be seen with the following graph:

$$\begin{array}{ccc} (-e_2 + e_1) & \text{=====} & (-2e_1) \\ | & & | \\ 0 & \text{=====} & (-e_2 - e_1). \end{array} \tag{67}$$

Relation is $(-e_2 + e_1) - (-e_2 - e_1) + (-2e_1) = 0$; we have

$$C(-e_2 + e_1) = e_1^2 - e_1 e_2, \quad C(-e_2 - e_1) = -e_1 e_2, \quad C(-2e_1) = -e_1^2,$$
$$e_1^2 - e_1 e_2 - (-e_1 e_2) - e_1^2 = 0.$$

Actually this graph does not really pose any problem since its only geometric realization is in S (hence it is **not** a true combinatorial graph). However we are not able to exclude the existence of more complicated graphs of this form which may have realization in S^c.

We are reduced to considering possible combinatorial graphs with at most $2n + 2$ vertices and such that the vertices of the same color are linearly independent. We call these graphs colored-non-degenerate.

We now look at Eqs. (61) associated to the graph by a choice of S. Consider a possible combinatorial graph $\mathcal{A}$ of black rank h and red rank k. If $h \leq n$ then we can require that the images of the black vertices $a \in \mathcal{A}$ through the map $-\pi$ are independent. Then the linear equations (61) associated to these vertices are independent and have solutions. The same holds for the red vertices, only Eqs. (61) associated to these vertices are quadratic and so the solutions need not be real.

Given a colored-non-degenerate possible combinatorial graph $\mathcal{A}$ with ranks $h, k \leq n$ for dimension n we associate to it the $n \times h$ matrix $M^+(\mathcal{A})$ with columns the vectors $\pi(a)$, where a runs over the black vertices, with the same for the e $n \times k$ matrix $M^-(\mathcal{A})$.

Constraint 5. *For any colored-non-degenerate possible combinatorial graph $\mathcal{A}$ with red and black rank $\leq n$ we require that:*[11]

$$\wedge^h(M^+(\mathcal{A})) \neq 0\,, \quad \wedge^k(M^-(\mathcal{A})) \neq 0.$$

If one of the colored ranks is $k = n + 1$, then any choice of S must lead to a relation between the vectors $\pi(a_i)$, where the a_i are the vertices of the same color. We will use this to show that either the equations are generically incompatible or they give a solution in the special component. This is the content of the next section.

10. Geometric Realization

Consider a possible–combinatorial graph $\mathcal{A}$ with $\leq 2n+2$ vertices and suppose that it has rank $n + 1$. By Theorem 4, the vertices of each color are linearly independent. We want to study its geometric realizations in dimension exactly n. For this we can consider the variety $R_{\mathcal{A}}$ *of realizations of the graph*, i.e. the set of points $(x, v_1, \dots, v_m) \in \mathbb{C}^{(m+1)n}$ which satisfy Eqs. (61) associated to $\mathcal{A}$.

Call $\theta : R_{\mathcal{A}} \to \mathbb{C}^{mn}$ the projection map $(x, v_1, \dots, v_m) \to (v_1, \dots, v_m)$. We say that a graph is *not* realizable for generic v_i if $\theta(R_{\mathcal{A}})$ is an algebraic variety of codimension at least one.[12]

Suppose that we have $n+1$ black vertices (different from the root). If we choose n of them (discarding say a_1) by Theorem 4, we can require that for generic S the $\pi(a_i)$ with $i = 2, \dots, n+1$, are independent. This we do by choosing S so that the determinant of the matrix M_1 having $\pi(a_i)$ as rows is non-zero. Then the system of equations is incompatible if the $n + 1 \times n + 1$ matrix obtained by adding the row $\pi(a_1)$ and the column of inhomogeneous terms has non-zero determinant. We compute this determinant which is a polynomial in the v_i and *if it is not identically zero* we impose it as a genericity constraint and $\mathcal{A}$ is not generically realizable.

If it is identically zero then the equations have a solution, which we can compute by Cramer's rule by discarding the first equation. Hence $\mathcal{A}$ is generically realizable.

In the same way suppose we have $n + 1$ red vertices. We choose one of them, say a_1, and subtract the equation for a_1 to the remaining Eqs. (61). We obtain a system of n linear equations $M_1x = b$ which, by Theorem 4, are generically independent. We

[11] If A is a $a \times b$ matrix and $h \leq \min(a, b)$ we denote by $\wedge^h A$ the matrix with entries the determinants of the $h \times h$ minors.

[12] In this discussion we ignore the delicate issues of whether a realization may be integral, real or imaginary.

impose as a genericity constraint $\det(M_1) \neq 0$ and solve the equations by Cramers rule. We obtain a solution $x_{\mathcal{A}}$ which is a rational function in the v_i with $\det(M_1)$ at the denominator. The graph has realization for all the S for which $x_{\mathcal{A}}$ solves the quadratic equation associated to a_1. We substitute $x_{\mathcal{A}}$ and rationalize. If the numerator is a non-zero polynomial then we impose it as a generiticity constraint and $\mathcal{A}$ is not generically realizable. Summarizing, we impose

Constraint 6. *For any colored-non-degenerate possible combinatorial graph $\mathcal{A}$ with red and/or black rank $n+1$, we impose that the vectors v_i are generic for all resonances described above.*

Example 10. We consider the combinatorial graph of example 8 in dimension $n = 2$. We impose

$$d = (v_{1,1} - v_{3,1})(v_{3,2} - v_{2,2}) - (v_{1,2} - v_{3,2})(v_{3,1} - v_{2,1}) \neq 0,$$

solve the first two Eqs. (63) by Cramer's rule and obtain the solution $x = (x_1, x_2)$:

$$x_1 = (|v_1|^2 - (v_1, v_3))(v_{3,2} - v_{2,2}) - (v_{1,2} - v_{3,2})(|v_2|^2 - (v_2, v_3))/d\,,$$
$$x_2 = (v_{1,1} - v_{3,1})(|v_2|^2 - (v_2, v_3)) - (|v_1|^2 - (v_1, v_3))(v_{3,1} - v_{2,1})/d.$$

We substitute in the last equation, rationalize and obtain that a realization exists only if

$$\begin{aligned}
&\Big((v_1, v_2) - (v_1, v_3) + |v_3|^2 - (v_2, v_3)\Big) \cdot \Big(v_{1,1}{}^3\, v_{2,1} + v_{1,1}\, v_{1,2}{}^2\, v_{2,1} + v_{1,2}{}^2\, v_{2,1}{}^2 \\
&+ v_{1,1}{}^2\, v_{1,2}\, v_{2,2} + v_{1,2}{}^3\, v_{2,2} - 2\, v_{1,1}\, v_{1,2}\, v_{2,1}\, v_{2,2} \\
&+ v_{1,1}{}^2\, v_{2,2}{}^2 - v_{1,1}{}^3\, v_{3,1} - v_{1,1}\, v_{1,2}{}^2\, v_{3,1} - 3\, v_{1,1}{}^2\, v_{2,1}\, v_{3,1} - 3\, v_{1,2}{}^2\, v_{2,1}\, v_{3,1} \\
&+ 2\, v_{1,2}\, v_{2,1}\, v_{2,2}\, v_{3,1} - 2\, v_{1,1}\, v_{2,2}{}^2\, v_{3,1} + 3\, v_{1,1}{}^2\, v_{3,1}{}^2 + 2\, v_{1,2}{}^2\, v_{3,1}{}^2 + 3\, v_{1,1}\, v_{2,1}\, v_{3,1}{}^2 \\
&- v_{1,2}\, v_{2,2}\, v_{3,1}{}^2 + v_{2,2}{}^2\, v_{3,1}{}^2 - 3\, v_{1,1}\, v_{3,1}{}^3 - v_{2,1}\, v_{3,1}{}^3 + v_{3,1}{}^4 \\
&- v_{1,1}{}^2\, v_{1,2}\, v_{3,2} - v_{1,2}{}^3\, v_{3,2} - 2\, v_{1,2}\, v_{2,1}{}^2\, v_{3,2} - 3\, v_{1,1}{}^2\, v_{2,2}\, v_{3,2} - 3\, v_{1,2}{}^2\, v_{2,2}\, v_{3,2} \\
&+ 2\, v_{1,1}\, v_{2,1}\, v_{2,2}\, v_{3,2} + 2\, v_{1,1}\, v_{1,2}\, v_{3,1}\, v_{3,2} + 4\, v_{1,2}\, v_{2,1}\, v_{3,1}\, v_{3,2} + 4\, v_{1,1}\, v_{2,2}\, v_{3,1}\, v_{3,2} \\
&- 2\, v_{2,1}\, v_{2,2}\, v_{3,1}\, v_{3,2} - 3\, v_{1,2}\, v_{3,1}{}^2\, v_{3,2} - v_{2,2}\, v_{3,1}{}^2\, v_{3,2} \\
&+ 2\, v_{1,1}{}^2\, v_{3,2}{}^2 + 3\, v_{1,2}{}^2\, v_{3,2}{}^2 - v_{1,1}\, v_{2,1}\, v_{3,2}{}^2 + v_{2,1}{}^2\, v_{3,2}{}^2 + 3\, v_{1,2}\, v_{2,2}\, v_{3,2}{}^2 \\
&- 3\, v_{1,1}\, v_{3,1}\, v_{3,2}{}^2 - v_{2,1}\, v_{3,1}\, v_{3,2}{}^2 + 2\, v_{3,1}{}^2\, v_{3,2}{}^2 - 3\, v_{1,2}\, v_{3,2}{}^3 - v_{2,2}\, v_{3,2}{}^3 + v_{3,2}{}^4\Big) = 0.
\end{aligned}$$

We thus have the final definition of generic for tangential sites S.

Definition 22. *We say that the tangential sites are* **generic** *if they do not vanish for any of the polynomials given by Constraints 1 through 6.*

Remark 23. Each of the constraints involves at most $2n+2$ edges, thus at most $4q(n+1)$ indices which have to be taken up to symmetry by S_m, hence can be taken in correspondence with the vector variables $y_1, \ldots, y_{4q(n+1)}$.

We have ensured that for generic choices of S only those graphs which are generically realizable are realized.

Example 11. Consider the possible combinatorial graph:

$$\begin{tikzcd}
 & (-e_3-e_4,-) \arrow[dl, dash, "-e_1-e_4"'] \arrow[d, dash, "-e_3-e_4"] \arrow[dr, dash, "-e_2-e_4"] & \\
(e_3-e_1,+) & (0,+) \arrow[l, "e_3-e_1"'] \arrow[r, "e_3-e_2"] & (e_3-e_2,+).
\end{tikzcd}$$

It is easily seen that in dimension $n = 2$ this graph is generically realizable, and its equations have the unique solution $x = v_3$.

We now want to study those graphs of rank $n + 1$ which are generically realizable in dimension n. As we have seen, on a Zariski open set of the space $v_1, \ldots, v_m$ we have a unique realization given by solving a system of linear equations and thus given by a vector x whoose coordinates are rational functions in the vectors v_i. We call this function the *generic realization*.

Theorem 5. *If $\mathcal{A}$ is a possible combinatorial graph of rank $n+1$ which has a realization for generic v_i's, then its generic realization is in the special component (the solution x belongs to the set S).*

The proof is based on two points. A graph which has a generic geometric realization in the special component is called *special*. It is easy to describe the special graphs, up to translation they correspond to combinatorial graphs with vertices in the set $-e_i, -e_j\tau, i, j = 1, \ldots, m$.

Lemma 8. *If for a non-degenerate graph of dimension $n > 1$ the solution to the associated system, in dimension n, is given by a polynomial, then the graph is special and the polynomial is of the form v_i for some i.*

For $n = 1$ the same result is true for a nondegenerate graph with 2 edges.

Proof. See Appendix B. □

Let $\mathcal{A}$ be a graph of rank $\geq n + 1$; consider as before the variety $R_{\mathcal{A}}$ of realizations of the graph, with its map $\theta : R_{\mathcal{A}} \to \mathbb{C}^{mn}$.

Proposition 12. *There is an irreducible hypersurface W of $\mathbb{C}^{mn}$ such that the map θ has an inverse on $\mathbb{C}^{mn} \setminus W$. The inverse is a polynomial map.*

Proof. Black edges. We have $n + 1$ linear equations $(x, \pi(a_i)) = b_i$ which are generically compatible. We solve them by Cramer's rule choosing an index j and discarding Eq. (61) associated to a vertex a_j. Since the equations are always compatible we must obtain, generically, the same solution for all choices of a_j. Consider the matrix M_j with rows the $\pi(a_i), i = 1, \ldots, n + 1 i \neq j$. The solution is a rational function of the v_i having as denominator the determinant of M_j. This reasoning defines the solution of our equations for all S for which there exists a j such that $\det(M_j) \neq 0$. We claim that each of these determinants is an irreducible polynomial so it defines an irreducible hypersurface H_j.

In fact a choice of n rows gives by assumption a surjective linear map $\mathbb{C}^{mn} \to \mathbb{C}^{n^2}$. Any surjective linear map can be considered (in appropriate coordinates) as a projection on the first mn coordinates. Hence an irreducible polynomial remains irreducible by composition. The claim follows since it is well known that the determinant is an irreducible polynomial of the matrix elements.

We claim that these hypersurfaces are not all equal. By hypothesis the matrix $B = (a_{ij})$ has rank $n+1$. All the matrices obtained by B dropping one row define the various determinantal varieties, H_j. The fact that these varieties are not equal is discussed in Appendix C. It depends on the fact that the matrices cannot have all the same kernel (otherwise the rank of B is $\leq n$). Then the result follows by Proposition 16.

Hence our solution is well defined outside a subvariety of codimension at least 2. This implies immediately that it is given by a polynomial using the following standard fact (which follows immediately from the unique factorization property of polynomial algebras): *Let W be a subvariety of $\mathbb{C}^N$ of codimension ≥ 2, let F be a rational function on $\mathbb{C}^N$ which is holomorphic on $\mathbb{C}^N \setminus W$, then F is a polynomial.*

Red edges. When we also have red edges we select $n+1$ linear and quadratic equations associated to the $n + 1$ vertices which are formally independent. We see that Eqs. (61) (for these vertices) are clearly equivalent to a system on n linear equations associated to formally linearly independent markings, plus a quadratic equation chosen arbitrarily among the ones appearing in (56). Thus a realization is obtained by solving this system and, by hypothesis, such solution satisfies the quadratic equation identically.

Let P be the space of functions $\sum_{i=1}^m c_i v_i,\ c_i \in \mathbb{R}$ and (P, P) their scalar products. By assumption we have a list of n equations $\sum_{j=1}^m a_{ij}(x, v_j) = (x, t_i) = b_i$ with the $t_i = \sum_{j=1}^m a_{ij} v_j$ linearly independent in the space P and $b_i = \sum_{h,k} a^i_{h,k}(v_h, v_k) \in (P, P)$.

Solve these equations by Cramer's rule considering the v_i as parameters. Write $x_i = f_i/d$, where $d(v) := \det(A(v))$ is the determinant of the matrix $A(v)$ with rows t_i, $f_i(v)$ is also a determinant of another matrix $B_i(v)$ both depending polynomially on the v_i. We have thus expressed the coordinates x_i as rational functions of the coordinates of the v_i. The denominator is an irreducible polynomial vanishing exactly on the determinantal variety of the v_i for which the matrix of rows $t_j,\ j = 1, \dots, n$ is degenerate.

Lemma 9. *Given $x = (x_1, \dots, x_n) = (f_1/d, \dots, f_n/d)$, let $(x)^2 = \sum_i x_i^2$. Assume there are two elements $a \in P, b \in (P, P)$ such that $(x)^2 + (x, a) + b = 0$ holds identically (in the parameters v_i); then x is a polynomial in the v_i.*

Proof. Substitute $x_i = f_i/d$ in the quadratic equation and get

$$d^{-2}(\sum_i f_i^2) + d^{-1} \sum_i f_i a_i + b = 0, \implies \sum_i f_i^2 + d \sum_i f_i a_i + d^2 b = 0.$$

Since $d = d(v) = \det(A(v))$ is irreducible this implies that d divides $\sum_i f_i^2$.

Since the f_i are real, for those $v := (v_1, \dots, v_m) \in \mathbb{R}^{mn}$ for which $d(A(v)) = 0$, we have $f_i(v) = 0, \forall i$; so f_i vanishes on all real solutions of $d(A(v)) = 0$. These solutions are Zariski dense, by Lemma 17, in the determinantal variety $d(A(v)) = 0$. In other words $f_i(v)$ vanishes on all the v solutions of $d(A(v)) = 0$ and thus $d(v)$ divides $f_i(v)$ for all i, hence x is a polynomial. □

Proof of Theorem 5. Once we fix a root we have that the variety R_A is the set of solutions of a system of $\geq n+1$ linear and quadratic equations in the variables x, v_i. We are assuming, by Proposition 12, that we have a solution $x = F(v)$ which is a polynomial in $v_1, \dots, v_m$. We now can apply Lemma 8. □

10.1. Proof of Theorem 2.

i) Assume by contradiction that there is a connected subgraph A of the graph Γ_S^{geo} with $n + 2$ vertices and all black edges. Then A is the geometric realization of a possible combinatorial graph $\mathcal{A}$ with $n+1$ non-zero black vertices which, by Theorem 4 must be independent. By Theorem 5 we have that A is contained in the special component and we have a contradiction.

ii) Such a component must contain an integral point in one of the spheres S_ℓ. Suppose by contradiction that there is a connected subgraph of this component with $2n+1$ vertices, denote it by A. By hypothesis A can be lifted, let $\mathcal{A}$ be its combinatorial graph. We can have at most n black and n red vertices otherwise the graph has rank $\geq n+1$, but if we have exactly n black vertices and at least one red vertex we also have rank $\geq n+1$ since the red vectors are linearly independent of the black similarly for red. So we can have at most $n-1$ black and n red vertices for a total (including the root) of $2n$ vertices.

iii) We put in the same family two components whose combinatorial graphs are equivalent.
There are only finitely many possible combinatorial graphs with at most $2n$ vertices. A family is formed by the geometric realization of one representative for each equivalence class of equivalent combinatorial graphs. This automatically chooses a root.

iv) Take a marked graph $\mathcal{A}$ with $k+1 < n+1$ vertices and all black edges. Call A a realization of $\mathcal{A}$ and let x be the root. Any other realization A' has a corresponding root x' such that $x-x'$ is a vector orthogonal to all the $\pi(a_i)$, where a_i are the non-zero vertices. By Costraint 5, the $\pi(a_i)$ are all independent. Conversely if x' is a point in $\mathbb{R}^n$ such that $x'-x$ is a vector orthogonal to all the $\pi(a_i)$, then it solves the same equations as x. Hence it is a vertex of a connected component which contains a translate of A and can only be bigger. If the component corresponds to a bigger graph, the point x' solves some further independent linear equations, with respect to x, and hence x' belongs to a lower dimensional affine subspace determined by the bigger graph; since these graphs are finitely many this completes the picture.

v) This is the content of Theorems 4 and 5.

Definition 23. *Let $\mathcal{B}_n = \mathcal{B}_{n,m,q}$ be the set of non-equivalent combinatorial graphs for a given dimension n.*

Each $\mathcal{A} \in \mathcal{B}_n$ has realizations in Γ_S and the choice of a representative in the equivalence class fixes a root in each component of Γ_S.

Example 12. The set $\mathcal{B}_1$ is simply the set of graphs with a single edge: $(0,+)$, $(\ell, 1+\eta(\ell))$.

For $n=2$ we have $\mathcal{B}_1$ and the complete graphs with three vertices $(0,+)$, $(\ell, 1+\eta(\ell))$, $(\ell', 1+\eta(\ell'))$. We should consider graphs with up to $2n=4$ vertices which are of rank ≤ 2 and such that the non-zero vertices of different colors are dependent. A direct computation shows that no such graphs exist.

Remark 24. One might think at this point that for any n the set $\mathcal{B}_n$ is only made of graphs with at most $n+1$ vertices which are affinely independent. However this conjecture seems quite hard to prove; it is true but requires a lengthy proof for $q=1$, and for general q it seems quite hard to verify even in dimension $n=3$ and indeed it may not be true.

10.2. Proof of Theorem 3. Once we have ensured that no graphs with more than $2n$ vertices exist we can apply Proposition 10 and Corollary 4. This gives an isomorphism between the components of Γ_S and those of Λ_S. More precisely for each family of components we choose one $A \in \Gamma_S$ and we also choose a root.

By translation this also determines a root for all other components in the same family. With these choices we can associate to A the combinatorial graph $\mathcal{A} \in \mathcal{B}_n$ of which it is a realization, see Definition 17. Let x be the vertex associated to (0, +). Then we obtain all the components in Λ_S over A by right translation with all the elements $(a, \pm)$ such that $x = -\pi(a)$.

To establish the isomorphism with the components of $\tilde{\Gamma}_S$ we make sure that two conjugate blocks are disjoint, i.e. that a pair $z_k, \bar{z}_k$ is never in the same block of $ad(N)$. This would correspond to a loop in the geometric graph which is not a loop in Λ_S, which is excluded by Constraint 3.

Corollary 5. *If the v_i are generic, in the projection map $\tilde{\Gamma}_S \to \Gamma_S$ the preimage of a connected component of Γ_S is the union of two disjoint and conjugate connected components of $\tilde{\Gamma}_S$.*

Each $\mathcal{A} \in \mathcal{B}_n$ has realizations in Γ_S and the choice of a representative in the equivalence class fixes a root in each component of Γ_S. For all $k \in S^c$ set $x(k) := x(A)$ to be the root of the component A of Γ_S to which k belongs. By Corollary 4 and Formula (55):

Lemma 10. *Each component A can be lifted defining in a compatible way elements $g(k)$ so that $k = g(k)x(A),\ g(k) = (L(k), \sigma(k))$, and if k_1, k_2 are joined by an edge marked $\ell \in G$ we have $g(k_2) = \ell g(k_1)$.*

Clearly if A is a realization of $\mathcal{A}$ then $(L(k), \sigma(k))$ is just the vertex of $\mathcal{A}$ which is identified with k in the isomorphism between A and $\mathcal{A}$.

Example 13. We consider the component A_{k_1} of Example 5 (which exists provided that $n > 2$). This component is the realization of the combinatorial graph $\mathcal{A}$ of Example 7. Hence:

$$\begin{aligned} g_{k_1} &= (0, +)\,, \quad g_{k_2} = (e_3 - e_1, +)\,, \quad g_{k_3} = (-e_1 - e_2 - 2e_3, +)\,, \\ g_{k_4} &= (-e_1 - e_2, -). \end{aligned} \tag{68}$$

11. Proof of Theorem 1

11.0.1. Reduction to constant coefficients. We think of $y = (y_1, \ldots, y_m)$, $x = (x_1, \ldots, x_m)$ as vectors so that, given $a = \sum_i n_i e_i \in \mathbb{Z}^m$, we have $a \cdot x := \sum_i n_i x_i$, $a \cdot dx := \sum_i n_i dx_i = d(a \cdot x)$. Furthermore $dy \wedge dx = \sum_i dy_i \wedge dx_i$.

Before proving the theorem in general we show how to reduce to constant coefficient a single block. As usual we use the graphs in Example 5.

Example 14. Consider for $q = 1$, the Hamiltonian:

$$\begin{aligned} &(\omega(\xi), y) + \sum_{i=1}^{4} |k_i|^2 |z_{k_i}|^2 + 4\sqrt{\xi_1\xi_3}e^{\mathrm{i}(x_1 - x_3)} z_{k_1} \bar{z}_{k_2} + 4\sqrt{\xi_2\xi_3}e^{\mathrm{i}(x_2 - x_3)} z_{k_2} \bar{z}_{k_3} \\ &+4\sqrt{\xi_1\xi_2}e^{-\mathrm{i}(x_1 + x_2)} z_{k_2} z_{k_4} + 4\sqrt{\xi_1\xi_3}e^{-\mathrm{i}(x_1 - x_3)} \bar{z}_{k_1} z_{k_2} + 4\sqrt{\xi_2\xi_3}e^{-\mathrm{i}(x_2 - x_3)} \bar{z}_{k_2} z_{k_3} \\ &+4\sqrt{\xi_1\xi_2}e^{\mathrm{i}(x_1 + x_2)} \bar{z}_{k_2} \bar{z}_{k_4}, \end{aligned}$$

the terms depending on $z, \bar{z}$ are those of Formula (43). We apply the change of variables:

$$z_{k_i} = e^{-\mathrm{i}L(k_i).x} z'_{k_i}, \quad y = y' + \sum_{i=1}^{4} L(k_i)|z'_{k_i}|^2, \quad x = x',$$

where $L(k_i)$ are defined in Lemma 10 and given in formula (68): $L(k_1) = 0$, $L(k_2) = e_3 - e_1$, $L(k_3) = -e_1 - e_2 - 2e_3$, $L(k_4) = -e_1 - e_2$. A direct check shows that this change of variables is symplectic and that the Hamiltonian in the new variables is:

$$(\omega(\xi), y') + \sum_{i=1}^{4} \Big((\omega_0, L(k_i)) + |k_i|^2 \Big) |z'_{k_i}|^2 + \tilde{\mathcal{Q}}, \tag{69}$$

where $\omega(\xi) = \omega_0 - 2\xi$, and:

$$\begin{aligned}\tilde{\mathcal{Q}} = &-2\sum_{i=1}^{4}(\xi, L(k_i))|z'_{k_i}|^2 + 4\sqrt{\xi_1\xi_3} z'_{k_1}\bar{z}'_{k_2} + 4\sqrt{\xi_2\xi_3} z'_{k_2}\bar{z}'_{k_3} \\ &+4\sqrt{\xi_1\xi_2} z'_{k_2}z'_{k_4} + 4\sqrt{\xi_1\xi_3}\bar{z}'_{k_1}z'_{k_2} + 4\sqrt{\xi_2\xi_3}\bar{z}'_{k_2}z'_{k_3} + 4\sqrt{\xi_1\xi_2}\bar{z}'_{k_2}\bar{z}'_{k_4}.\end{aligned}$$

Theorem 1 is contained in the following, more precise, proposition:

Proposition 13. *i) The equations*

$$z_k = e^{-\mathrm{i}L(k).x} z'_k, \quad y = y' + \sum_{k \in S^c} L(k)|z'_k|^2, \; x = x' \tag{70}$$

define a symplectic change of variables $\Psi^{(3)} : D(s, r/2) \to D(s, r)$, which preserves the spaces V^i.
We denote by $W = diag(\{e^{\mathrm{i}L(k).x}\}_{k \in S^c}, \{e^{-\mathrm{i}L(k).x}\}_{k \in S^c})$, the matrix of $\Psi^{(3)}$ on w.

ii) The Hamiltonian H in the new variables is $N' + P \circ \Psi^{(3)}$, where

$$N' = N \circ \Psi^{(3)} := (\omega(\xi), y') + \sum_{k \in S^c} \Big(|k|^2 + \big(\omega(\xi), L(k)\big) \Big) |z'_k|^2 + \mathcal{Q}'(w'), \tag{71}$$

and

$$\begin{aligned}\mathcal{Q}'(w') &:= \mathcal{Q}(x, w'W) \equiv \mathcal{Q} \circ \Psi^{(3)} \\ &= \sum_{\ell \in X_q^0} c(\ell) \sum_{(h,k) \in \mathcal{P}_\ell} z'_h \bar{z}'_k + \sum_{\ell \in X_q^{-2}} c(\ell) \sum_{\{h,k\} \in \mathcal{P}_\ell} [z'_h z'_k + \bar{z}'_h \bar{z}'_k],\end{aligned} \tag{72}$$

is independent of x.

iii) $\tilde{P} := P \circ \Psi^{(3)}$ satisfies the bounds of Theorem 1, iv).

Proof. *i)* By definition $|L(k)| \leq 4nq$ for all k. Since

$$\sup_{D(s,r/2)} |w'|_{a,p} \leq e^{Cs}|w|_{a,p} \leq e^{Cs} r/2 \leq r$$

for s small enough the transformation is well defined from $D(s, r/2)$ to $D(s, r)$. It is symplectic because:

$$dy \wedge dx + \mathrm{i}dz \wedge d\bar{z} = dy' \wedge dx' - \sum_k (L(k) \cdot dx') \wedge d(|z_k|^2)$$
$$+\mathrm{i}dz' \wedge d\bar{z}' + \sum_k d(L(k).x') \wedge (z'_k d\bar{z}'_k + \bar{z}'_k dz'_k) = dy' \wedge dx' + \mathrm{i}dz' \wedge d\bar{z}'.$$

Finally it preserves the spaces V^i since it is linear in the variables w which have degree 1 and in $y, |z_k|^2$ of degree 2. In fact it maps a space $V^{i,j}$ into $\sum_{h=0}^{i} V^{i-h,j+2h}$.

ii) We simply substitute the new variables in the Hamiltonian; we obtain that

$$(\omega(\xi), y) + \sum_k |k|^2|z_k|^2 = (\omega(\xi), y') + \sum_k (\omega(\xi), L(k))|z'_k|^2 + \sum_k |k|^2|z'_k|^2. \tag{73}$$

By definition of W we have $\mathcal{Q}(x, w) \circ \Psi^{(3)} = \mathcal{Q}(x, w'W)$. Formula (72) follows from Lemma 10. In fact we substitute in Formula (30) $z_k = e^{-\mathrm{i}L(k).x} z'_k$, then if $\ell \in X_q^0$, $(h, k) \in \mathcal{P}_\ell$ we have

$$e^{\mathrm{i}(\ell,x)} z_h \bar{z}_k = e^{\mathrm{i}(\ell,x)} e^{-\mathrm{i}L(h).x} z'_h e^{\mathrm{i}L(k).x} \bar{z}'_k$$

and, by Formula (49) we have $\ell - L(h) + L(k) = 0$. Similarly when $\ell \in X_q^{-2}$, $\{h, k\} \in \mathcal{P}_\ell$ we have

$$e^{\mathrm{i}(\ell,x)} z_h z_k = e^{\mathrm{i}(\ell,x)} e^{-\mathrm{i}L(h).x} z'_h e^{-\mathrm{i}L(k).x} z'_k$$

and, by Formula (49) we have $\ell - L(h) - L(k) = 0$.

iii) Let us prove the bounds. We notice that the total degree $2i + j$ is the same in the two sets of variables. Moreover $\Psi^{(3)}$ is independent of ξ, hence $P \circ \Psi^{(3)}$ has the same order as P, see Sect. 4.0.5, and the bounds follow by Proposition 6. □

Remark 25. It is possible to choose also infinite sets of v_i so that the change of variables is still convergent in a ball. For this it is enough to impose a reasonable growth to $|v_i|$ as $i \to \infty$.

11.1. Definitions of $\tilde{\Omega}_k$, $\tilde{\mathcal{Q}}$. From Formula (38) we have

$$N' = K' + (\nabla_\xi A_{q+1}(\xi) - (q+1)^2 A_q(\xi)\underline{1}, y' + \sum_k L(k)|z'_k|^2) + \mathcal{Q}'(w'),$$
$$K' = (\omega_0, y') + \sum_k ((\omega_0, L(k)) + |k|^2)|z'_k|^2. \tag{74}$$

We set

$$\tilde{\Omega}_k = (\omega_0, L(k)) + |k|^2,$$
$$\tilde{\mathcal{Q}} := \mathcal{Q}'(w') + \sum_k (\nabla_\xi A_{q+1}(\xi) - (q+1)^2 A_q(\xi)\underline{1}, L(k))|z'_k|^2, \tag{75}$$

and remark that the $\tilde{\Omega}_k$ are integers and the coefficients of the quadratic form $\tilde{\mathcal{Q}}$ are homogeneous in the variables ξ of degree q.

We can group $\tilde{\mathcal{Q}} = \sum_A \tilde{\mathcal{Q}}_A$, where the sum runs over all blocks $A \in \Gamma_S$ and $\tilde{\mathcal{Q}}_A$ involves only the variables $z'_k, \bar{z}'_k$ with k appearing in the block. We now use the graph language. Having made the change of variables we should really introduce a new graph $\tilde{\Gamma}'_S$ expressing the non-zero entries of Q in the basis z'. In fact by Remark 11 this is just a subgraph of that larger graph but it is also clearly isomorphic to $\tilde{\Gamma}_S$ although the matrix entries have changed, so by abuse of notation we still denote it by the same symbol $\tilde{\Gamma}_S$.

Take a block $A \in \Gamma_S$ and let $\tilde{A}_\pm$ be the corresponding disjoint conjugate components in $\tilde{\Gamma}_S$ (by convention, in $\tilde{A}_+$ the root x corresponds to z_x, while in $\tilde{A}_-$ the root x corresponds to $\bar{z}_x$).

Remark 26. 1. $\tilde{\mathcal{Q}}_A$ is a Hamiltonian on the symplectic space W_A with basis $(z'_k, \bar{z}'_k)$, k running over the vertices of A.

2. We denote the vertices in each $\tilde{A}_\pm$ by Z_A and $\bar{Z}_A$ resp. The variables Z_A and $\bar{Z}_A$ form the bases of two Lagrangian subspaces[13] decomposing W_A.
3. Both K' and $(\nabla_\xi A_{q+1}(\xi) - (q+1)^2 A_q(\xi)\underline{1}, y')$ in N' commute with $\mathcal{Q}(x, w'W)$, hence $ad(K' + (\nabla_\xi A_{q+1}(\xi) - (q+1)^2 A_q(\xi)\underline{1}, y'))$ takes a scalar value on any given block Z_A.

11.1.1. The matrix blocks of $\tilde{\mathcal{Q}}$. Set $\mathrm{i}Q'_A$ to be the matrix of $ad(\mathcal{Q}'_A)$ and $\mathrm{i}D'_A$ to be the (diagonal) matrix of

$$ad(\sum_k (\nabla_\xi A_{q+1}(\xi) - (q+1)^2 A_q(\xi)\underline{1}, L(k))|z'_k|^2)$$

in the geometric basis $z'_k, \bar{z}'_k$ with $k \in S^c$. Clearly the matrix representation of $\tilde{\mathcal{Q}}_A$ is $\tilde{Q}_A = Q'_A + D'_A$. Moreover, by definition of $\tilde{\Gamma}'_S$, we have $Q'_A = Q'_{\tilde{A}_+} \oplus Q'_{\tilde{A}_-}$.

Given two vertices $a \neq b \in \tilde{A}_+$, we have, from Formula (30), that the matrix element $Q'_{a,b}$ is non-zero if and only if they are joined by an edge ℓ and then $Q'_{a,b} = c(\ell)$ if $b = z'_k$ or $Q'_{a,b} = -c(\ell)$ if $b = \bar{z}'_k$. The element $c(\ell)$ is described in Formula (30). On the diagonal we have $Q'_{a,a} = 0$, while the $D'_{a,a} = (\nabla_\xi A_{q+1}(\xi) - (q+1)^2 A_q(\xi)\underline{1}, L(k))$ if $a = z'_k$ or $-(\nabla_\xi A_{q+1}(\xi) - (q+1)^2 A_q(\xi)\underline{1}, L(k))$ if $a = \bar{z}'_k$ (cf. (16)). The same rules hold for vertices $a \neq b \in \tilde{A}_-$ and one easily verifies that

$$\tilde{Q}_{\tilde{A}_-} = -\tilde{Q}_{\tilde{A}_+}. \tag{76}$$

By definition $L(k)$ depends only on the combinatorial graph $\mathcal{A}$ of which A is a realization, therefore the matrix $\tilde{Q}_A$ depends only on the combinatorial block $\mathcal{A}$.

In order to stress this point we write $\tilde{Q}_A \equiv C_{\mathcal{A}} = C_{\mathcal{A},+} \oplus C_{\mathcal{A},-}$, with $C_{\mathcal{A},-} = -C_{\mathcal{A},+}$.

Lemma 11. *For all combinatorial blocks $\mathcal{A}$ which do not contain red edges, the matrix $C_{\mathcal{A},+}$ is self–adjoint for all $\xi \in A_{\varepsilon^2}$. If $\mathcal{A}$ contains red edges then each vertex has a sign. This defines a diagonal matrix of signs $\Sigma_{\mathcal{A}}$, and $C_{\mathcal{A},+}$ is self–adjoint with respect to the indefinite form defined by $\Sigma_{\mathcal{A}}$.*

[13] Notice that the two bases $\mathrm{i}Z_A$ and $\bar{Z}_A$ are dual bases only when A does not contain red edges. Indeed, in general, the duality matrix is diagonal with elements ± 1.

Proof. This is essentially a consequence of the fact that H is real, but it follows immediately from the definition of $c(\ell)$ and the explicit formulas for $C_{\mathcal{A},+}$ given above. $\square$

Definition 24. *We denote by $\mathcal{M}$ the finite list of matrices $\{C_{\mathcal{A}}\}_{\mathcal{A}}$, where $\mathcal{A}$ runs over all the non-equivalent combinatorial blocks with chosen root at (0,+) which contain only the indices $1, \dots, 4qn$. We denote by $\mathcal{M}(\xi)$ the finite (and independent of m) list of matrices $\{C_{\mathcal{A}}\}_{\mathcal{A}}$, where $\mathcal{A}$ runs over all the non-equivalent combinatorial blocks with chosen root at 0.*

Corollary 6. *The matrices $\mathcal{M}(\xi)$ are obtained from the matrices $\mathcal{M}$ by permuting the variables $\{\xi_1, \dots, \xi_m\}$.*

Proof. In fact we have finitely many graphs with at most $2n$ vertices, the indices appearing in the edges are the ones appearing on a maximal tree with at most $2n$ edges. On each edge they involve at most $2q$ indices and so we have the a priori bound number $4qn$ of indices which, up to symmetry, can be taken to be $1, \dots, 4qn$. $\square$

Example 15. We describe the block $C_{\mathcal{A},+}$ for the graph $\mathcal{A}$ consisting of a unique edge ℓ. Recall that $\eta(\ell) = \sum_i \ell_i$ is either 0 or -2 so $1 + \eta(\ell) = 1, -1$ respectively. Set

$$c(\ell) = (q+1)a(\ell)\,, \quad \nabla_\xi A_{q+1}(\xi).\ell - (q+1)^2 A_q(\xi)\eta(\ell) = (q+1)(1+\eta(\ell))b(\ell)\,,$$

we then have:

$$C_{\mathcal{A},+} = (q+1)\begin{vmatrix} 0 & (1+\eta(\ell))a(\ell) \\ a(\ell) & b(\ell) \end{vmatrix}. \tag{77}$$

For $q = 1$ one gets

$$\begin{gathered} \mathcal{A}_1 \;=\; (0,+) \xrightarrow{1,2} (e_2 - e_1, +) \qquad \mathcal{A}_2 = (0,+) \overset{1,2}{=\!=\!=} (-e_1 - e_2, -)\,, \\ C_{\mathcal{A}_1,+} = 2\begin{vmatrix} 0 & 2\sqrt{\xi_1\xi_2} \\ 2\sqrt{\xi_1\xi_2} & \xi_1 - \xi_2 \end{vmatrix}, \quad C_{\mathcal{A}_2,+} = 2\begin{vmatrix} 0 & -2\sqrt{\xi_1\xi_2} \\ 2\sqrt{\xi_1\xi_2} & -\xi_1 - \xi_2 \end{vmatrix}. \end{gathered} \tag{78}$$

For $q = 1$ consider the component $\mathcal{A}$ of Formula 59, we obtain

$$C_{\mathcal{A},+} = 2\begin{pmatrix} 0 & 2\sqrt{\xi_1\xi_3} & 0 & 0 \\ 2\sqrt{\xi_1\xi_3} & \xi_1 - \xi_3 & 2\sqrt{\xi_2\xi_3} & -2\sqrt{\xi_1\xi_2} \\ 0 & 2\sqrt{\xi_2\xi_3} & \xi_1 + \xi_2 - 2\xi_3 & 0 \\ 0 & 2\sqrt{\xi_1\xi_2} & 0 & -\xi_1 - \xi_2 \end{pmatrix};$$

the reader can easily verify that in the Hamiltonian (69) $\tilde{\mathcal{Q}}$ is represented by the matrix above.

Proof of Theorem 1. The change of variables $\Phi_\xi = \Psi^{(1)} \circ \Psi^{(2)} \circ \Psi^{(3)}$. Item i) follows from Corollary 2. Item ii) follows from the corresponding item of Proposition 13. Item iii) also follows by item ii) of 13. The set of matrices $\mathcal{M}$ is defined in Definition 24. iv) follows from the same statement for P. $\square$

12. Proof of Proposition 3 and Corollary 1

12.1. The arithmetic constraints. We want to show now that in some special cases, on the integer points of the geometric graph we may impose much stronger conditions.

Proposition 14. *(i) For $n = 1$ and for generic choices of S, all the connected components of Γ_S are either vertices or single edges.*

(ii) For $n = 2$, and for every m there exist infinitely many choices of generic tangential sites $S = \{v_1, \dots, v_m\}$ such that, if A is a connect component of the geometric graph Γ_S, then A is either a vertex or a single edge.

Proof. (i) It is proven in Example 12.

(ii) This statement is proved in [11] for $q = 1$ by a very direct and lengthy computation. Here we give a more conceptual proof based on estimates of integral points on algebraic curves, valid for all q.

The simplest of such estimates is that, for all $0 < \delta < 1$ one can estimate the number of integral points of a circle of radius R by $\ll R^\delta$ as $R \to \infty$.

In general Bombieri and Pila prove, in [3] Theorem 5, that if C is a real absolutely irreducible algebraic curve of degree d and if $N > e^d$, the number of integral points in C in the square $[0, N] \times [0, N]$, is bounded by

$$N^{1/d} \exp(12\sqrt{\log(N) \log\log(N)}).$$

In particular for any $\delta > 0$ and N large we have a bound $N^{1/d+\delta}$.

We need a less fine estimate, if the curve is not necessarily absolutely irreducible but contains no lines we still get, by looking at its irreducible factors an estimate of type $N^{1/2+\delta}$ for N large. We want to use these bounds for our estimates.

Let us first characterize the sets $x, v_1, \dots, v_m$ such that there is an edge marked ℓ with vertex in x. We can interpret Formulas (47)–(46) by saying that $x, v_1, \dots, v_m$ satisfy an equation which is the equation for a sphere in either x (red edge) or one of the v_j's–here we consider the other variables as parameters. Suppose now that x is a vertex of a graph U with two different edges ℓ_1, ℓ_2. Hence x satisfies the two equations given by these edges.

Case $q = 1$. We know that there is an index $i = 1, \dots, m$ such that e_i appears in ℓ_1 but not in ℓ_2 (otherwise we would have $\ell_1 = e_i - e_j$ and $\ell_2 = -e_i - e_j$ which does not have a realization in Γ_S).

Suppose now that ℓ_2 is red. We next claim that if $|v_i| \le R$ for all i then $|x| < CR$ (where C is a universal constant). Indeed since one of the edges is red then x belongs to the circle of diameter v_1, v_2 (we are assuming without loss of generality that the red edge is $\ell_2 = -e_1 - e_2$).

Consider the set

$$A_U : \{v_1, \dots, v_m, x\} \subset \mathbb{Z}^{2m+2},\ |v_i| \le R, \quad x \quad \text{solves the equations given by } U := \ell_1, \ell_2.$$

We claim that $|A_U| \ll R^{2m-1+\delta}$. Without loss of generality we may suppose that ℓ_1 depends non trivially on e_3.

We first use the equation given by ℓ_1 to express $v_{3,1}$ in terms of the other parameters. This of course gives at most two solutions. Then the equation for ℓ_2 is a circle in x with diameter $\le 2R$.

Thus $\cup_U A_U$ has $\ll \binom{m}{2}^2 R^{2m-1+\delta}$ elements. When R is large $\ll \binom{m}{2}^2 R^{2m-1+\delta} < R^{2m}$, thus the projection of this set on the first m coordinates is not surjective and thus any point outside this projection is a set of tangential sites satisfying the condition of our proposition.

To treat the case of ℓ_1, ℓ_2 both black we need to ensure that $|x| < CR$ provided that the $|v_i| < R$. We compute x by Cramer's rule, the denominator is $\pi(\ell_1) \wedge \pi(\ell_2)$ while the numerator is bounded proportionally to $|\pi(\ell_1)||\pi(\ell_2)|R$. To obtain the desired bound we restrict $v_1, \dots, v_m$ to the sector where $|\pi(\ell_1) \wedge \pi(\ell_2)| \geq c(m)|\pi(\ell_1)||\pi(\ell_2)|$ for all choices of $\ell_1, \ell_2 \in X_1^0$; here $c(m)$ is some constant depending only on m. The set of v_i's which satisfy this constraint and have $|v_i| < R$ is still of the order of R^{2m}.

As done before, we use the equation given by ℓ_1 to express $v_{3,1}$ in terms of the other parameters. Then the equation for ℓ_2 is a circle in one of the variables v_1, v_2 with diameter $\leq 2CR$.

Case $q > 1$. It is no longer always true that there exists an index i such that e_i appears in ℓ_1 but not in ℓ_2. If this restriction is satisfied then the previous proof applies, otherwise we claim that we can apply Theorem 5 in the paper of Bombieri and Pila [3]. In fact look at an equation

$$|(x, y)|^2 + ((x, y), \sum_i m_i v_i) = -1/2(|\sum_i m_i v_i|^2 + \sum_i m_i |v_i|^2)$$

or equivalently

$$|2(x, y) + \sum_i m_i v_i|^2 := |(x', y')|^2 = -(|\sum_i m_i v_i|^2 + 2\sum_i m_i |v_i|^2),$$

since $\sum_i m_i = -2$, either $\sum_i m_i v_i = -e_a - e_b$ (where the previous arguments apply) or there is then an index j with $m_j > 0$, write the equation in terms of $(x', y'), z = v_{j,1}$ considering the other coordinates as parameters. This defines an ellipsoid

$$(x')^2 + (y')^2 + (m_j^2 + 2m_j)z^2 + az + b = 0$$

which, if the remaining coordinates of the v_j are bounded by some R, is contained in a cube $[-CR, CR]^3$ with C some fixed integer depending on the m_i. We now intersect with the other equation and claim that we have an absolutely irreducible curve; to its projection on one of the coordinate planes we apply the theorem of Bombieri and Pila. The other equation is of the form $(\vec{x}, \sum_i n_i v_i) = 1/2(|\sum_i n_i v_i|^2 + \sum_i n_i |v_i|^2)$ if the other edge is black or $(\vec{x}, \sum_i (n_i - m_i) v_i) = -1/2(|\sum_i n_i v_i|^2 + \sum_i n_i |v_i|^2) + 1/2(|\sum_i m_i v_i|^2 + \sum_i m_i |v_i|^2)$ if the edge is red. The equation is of the form $n_j x' z + cz + dy' + ex' + fz^2 + g = 0$, where $d = \sum_i n_i v_{i,2}$. If $d \neq 0$ we solve it for y and see that we have a plane quartic, otherwise we project it to the plane y', z still getting a plane quartic. In either case the quartic does not contain a real line since its real points are bounded; the estimate on its integral points follows from Theorem of Bombieri and Pila.

Two black edges. $\sum_i m_i e_i, \sum_j n_j e_j,\ \sum_i m_i = \sum_j n_j = 0$. The equations are

$$(\vec{x}, \sum_i m_i v_i) = 1/2(|\sum_i m_i v_i|^2 + \sum_i m_i |v_i|^2),$$
$$(\vec{x}, \sum_i n_i v_i) = 1/2(|\sum_i n_i v_i|^2 + \sum_i n_i |v_i|^2).$$

Say that $m_1 \neq n_1$, consider all the v_i, $i > 1$ and y as parameters, let z, w denote the coordinates of v_1 which we consider as variables, so write the equations as

$$x(a + m_1 z) = 1/2(m_1^2 + m_1)[z^2 + w^2] + m_1 zw + bz + cw + d,$$
$$x(a' + n_1 z) = 1/2(n_1^2 + n_1)[z^2 + w^2] + n_1 zw + b'z + c'w + d'.$$

We may assume $n_1 \neq 0$, otherwise we are in the previous case of an index appearing in ℓ_1 and not in ℓ_2. Project to the z, w plane and we see that we obtain the cubic

$$[1/2(m_1^2 + m_1)[z^2 + w^2] + m_1 zw + bz + cw + d](a' + n_1 z)$$
$$= (a + m_1 z)[1/2(n_1^2 + n_1)[z^2 + w^2] + n_1 zw + b'z + c'w + d']$$

of equation

$$Az[z^2 + w^2] + Bz^2 + Cw^2 + Dzw + Ez + Fw + G = 0$$

with $A = 1/2n_1 m_1(m_1 - n_1) \neq 0$. Let us show that this is absolutely irreducible. Otherwise it factors through a linear and a quadratic term, and we can always assume that the linear term is defined over $\mathbb{R}$ since with any factor we also have the conjugate factor. This implies that if there is a factorization it is of the form

$$(Az + K)(z^2 + w^2 + Mz + Nw + P)$$

which implies that

$$Az(Mz + Nw + P) + K(z^2 + w^2 + Mz + Nw + P) = Bz^2 + Cw^2 + Dzw + Ez + Fw + G$$
$$AM + K = B,\ K = C,\ AN = D,\ AP + KM = E,\ KN = F,\ KP = G$$

in particular $C = K$. Now $C = (m_1^2 + m_1)a' - (n_1^2 + n_1)a$, where $a = \sum_{i>1} m_i v_{i,1}$, $a' = \sum_{i>1} n_i v_{i,1}$.

We have $AF = CD$ and we claim that this imposes a non-trivial restriction to the parameters, thus for a large set of parameters we can apply the method.

We have $F = ca' - c'a$ and $c = -m_1 y + m_1 \sum_{j>1} m_j v_{j,2}$, $c' = -n_1 y + n_1 \sum_{j>1} n_j v_{j,2}$, while $D = a'm_1 + cn_1 - an_1 - c'm_1$. We see that in D the variable y disappears while in F it appears linearly with coefficient $-m_1 a' + n_1 a = \sum_{i>1}(n_1 m_i - m_1 n_i)v_{i,1}$. We cannot have $(n_1 m_i - m_1 n_i) = 0$ for all $i > 1$ unless ℓ_2 is a multiple of ℓ_1. This case, though, we have excluded in Theorem 4.

We conclude that, for any $\delta > 0$, the number of integral points are less than a constant (dependent only on δ) times $R^{1/2+\delta}$. At this point the proof is identical to the previous argument. □

We denote the sets S which do not contribute to any A_U as *arithmetically generic* and think of the condition $\nexists x \in \mathbb{Z}^2 : (v_1, \dots, v_m, x) \in \cup_U A_U$ as an *arithmetic* constraint.

Proposition 15. *Under the geometric and arithmetic constraint for $n = 1$ or $n = 2$ all the non-diagonal blocks in $\tilde{Q}$ are two by two and given by Formula* (77).

Remark 27. It is unclear what happens in higher dimension. One can use the same argument to exclude graphs of rank equal to the dimension, so Dimension 3 could still behave in a special way. On the other hand, for $q = 1$ there is a different method using the second Melnikov condition which we shall discuss elsewhere.

12.1.1. Real roots We ask if there are regions in the parameters ξ where all blocks have real roots. The issue is only for graphs containing red edges and the region is described by a finite set of inequalities given by Sylvester's Theory.

We discuss here the case in which all graphs containing a red edge reduce to this edge. As remarked in §12.1, one can have this case in dimension $n = 1, 2$ for all q.

The matrix associated to the graph $\mathcal{A}$ consisting of a unique red edge ℓ is given by Formula (77) where:

$$a(\ell) = q\xi^{\frac{\ell^+ + \ell^-}{2}} \sum_{\substack{\alpha\in\mathbb{N}^m \\ |\alpha+\ell^+|_1 = q-1}} \binom{q+1}{\ell^- + \alpha}\binom{q-1}{\ell^+ + \alpha}\xi_i^{\alpha},$$
$$\nabla_\xi A_{q+1}(\xi).\ell - (q+1)^2 A_q(\xi)\eta(\ell) = -(q+1)b(\ell). \tag{79}$$

The characteristic polynomial of $C_{\mathcal{A}}/(q+1)$ is $t(t - b(\ell)) + a(\ell)^2$ with discriminant $b(\ell)^2 - 4a(\ell)^2$. We want to show that there is a non empty open region in our parameter space where the roots of all these polynomials are distinct real, that is where all these discriminants are strictly positive. For this, using the usual lexicographical order, let us compute the leading terms of all these polynomials. Apply Formula (79) letting $d(\ell) := q - 1 - |\ell^+|_1$, we see that the leading monomial of $-4a(\ell)^2$ is $\xi_1^{d(\ell)}\xi^{\ell^+ + \ell^-}$. As for $b(\ell)$ it has the monomial ξ_1^q appearing with the following coefficient. In $\frac{\partial A_{q+1}(\xi)}{\partial \xi_1}$ the coefficient of ξ_1^q comes from $\frac{\partial \xi_1^{q+1}}{\partial \xi_1} = (q+1)\xi_1^q$ while for $i > 1$ in $\frac{\partial A_{q+1}(\xi)}{\partial \xi_i}$ the coefficient of ξ_1^q comes from $\frac{\partial (q+1)^2 \xi_i \xi_1^q}{\partial \xi_i} = (q+1)^2\xi_1^q$. If $\ell = (\ell_1, \dots, \ell_m)$ the coefficient of ξ_1^q in $b(\ell)$ is $\ell_1 + (q+1)\sum_{i>1}\ell_i$. Since $\sum_i \ell_i = -2$ we finally have $\ell_1 + (q+1)\sum_{i>1}\ell_i = \ell_1 - (q+1)\ell_1 - 2(q+1) = -q\ell_1 - 2(q+1) = -q(\ell_1 + 2) - 2$.

In the term $-(q+1)A_q(\xi)\eta(\ell) = 2(q+1)A_q(\xi)$ the monomial appears with coefficient $2(q+1)$. Thus we get a total contribution of $-q\ell_1$. Thus if $\ell_1 \neq 0$ the leading monomial of the discriminant is ξ_1^{2q} with positive coefficient. If $\ell_1 = 0$ let us look at the coefficient of ξ_1^{q-1} in $\frac{\partial A_{q+1}(\xi)}{\partial \xi_i}$, $i \neq 1$. This comes from the terms $\frac{q(q+1)}{2}\xi_1^{q-1}\xi_i^2$ giving $q(q+1)\xi_1^{q-1}\xi_i$ and $q(q+1)\xi_1^{q-1}\xi_i\xi_j$, $i \neq j \neq 1$ giving $q(q+1)\xi_1^{q-1}\sum_{j\neq 1,i}\xi_j$. Together we get $q(q+1)\xi_1^{q-1}\sum_{j\neq 1}\xi_j$. When we take the scalar product with ℓ we get thus a total contribution of $-2q(q+1)\xi_1^{q-1}\sum_{j\neq 1}\xi_j$. From $2(q+1)A_q(\xi)$ we get the term $2q^2(q+1)\xi_1^{q-1}\sum_{j\neq 1}\xi_j$. Thus we get a leading term of type $2(q^2-q)(q+1)\xi_1^{q-1}\xi_2$ unless $q = 1$; in this case we need to do a different argument.

The leading term of $b(\ell)^2$ is thus $\xi_1^{2q-2}\xi_2^2$ with positive coefficient. This gives the leading term in the discriminant unless $\ell^+ = 0$, hence $d(\ell) = 0$ and $\ell = -2e_2$, but this is not possible by one of our first constraints.

Finally for $q = 1$ the discriminants are all of type $\xi_i^2 + \xi_j^2 - 14\xi_i\xi_j$, so we see that in all cases we can apply the following lemma.

Given $j \in \mathbb{N}$ consider the list $\mathcal{M}_j$ of monomials of degree $\leq j$ in the variables ξ_i, $i = 1, \dots, m$ ordered lexicographically, denote by $A \prec B$ this ordering.

Given a positive constant D, set

$$\mathcal{A}_D := \{\xi \mid \xi_i > 0,\ A(\xi) > DB(\xi)\},\ \forall B \prec A,\ A, B \in \mathcal{M}_j\}.$$

Lemma 12. *i) For every $D > 0$ the open set $\mathcal{A}_D$ is non empty.*

ii) Consider a list of polynomials $f_i(\xi)$ of degree $\leq j$ and with real coefficients. If, for each i, the coefficient in f_i of the leading monomial is strictly positive, then there is a positive constant D so that in the region $\mathcal{A}_D$ we have $f_i(\xi) > 0$ for all i.

iii) Under the hypotheses of ii), if all the f_i are homogeneous, the non empty open set where $f_i(\xi) > 0$ for all i is a cone.

Proof. *i)* Consider the curve $\xi_i := t^{(j+1)^{m+1-i}}$, if $M = \prod_{i=1}^m \xi_i^{h_i}$, we have that on this curve $M(t) = t^{\sum_i h_i (j+1)^{m+1-i}}$. It is clear that $B = \prod_{i=1}^m \xi_i^{k_i} \prec A = \prod_{i=1}^m \xi_i^{h_i}$ if and only if the sequence $(k_1, \dots k_m) \prec (h_1, \dots h_m)$. But if $\sum_i k_i \leq j, \sum_i h_i \leq j, \quad (k_1, \dots k_m) \prec (h_1, \dots h_m)$ we have $\sum_i h_i (j+1)^{m+1-i} > \sum_i k_i (j+1)^{m+1-i}$ so that $\lim_{t\to\infty} A(t)/B(t) = \infty$. For any $D > 0$, for large t the curve lies in $\mathcal{A}_D$.

ii) The leading monomial is the maximum in the lexicographic order. Take a polynomial $f = aM + \sum_i^k a_i M_i$ with M leading monomial and $a > 0$. We have, in the quadrant $\xi_i > 0$, that $aM + \sum_i^k a_i M_i \geq aM - \sum_i^k |a_i| M_i$. If $\deg(f) \leq j$, in $\mathcal{A}_D$ we further have:

$$aM - \sum_i^k |a_i| M_i \geq \sum_i^k (\frac{a}{k} D - |a_i|) M_i .$$

Since $a > 0$ it is enough to choose $D > \max |a_i| \frac{k}{a}$.

iii) The set where an homogeneous polynomial is positive is a cone. □

Proof of Theorem 3. This now follows from the previous lemma and the discussion of the discriminants that we have performed. □

Proof of Corollary 1. We use the notations and results of Remark 26. We divide S^c in the connected components of Γ_S and apply the standard theory of quadratic Hamiltonians to each geometric block A:

$$H_A := \sum_{k\in A} \tilde{\Omega}_k |z_k'|^2 + \tilde{\mathcal{Q}}_A(w'),$$

where now all the $\tilde{\Omega}_k$ in the block are equal in the case of A black while they are opposite in the case of a red edge so that $ad(\sum_{k\in A} \tilde{\Omega}_k |z_k'|^2)$ acts as a scalar matrix on the variables Z_A.

If i $ad(\tilde{\mathcal{Q}}_A(w))$ is semi–simple with real eigenvalues, it is a standard fact that there exists a real linear symplectic change of variables ψ_A under which

$$H_A \circ \psi_A = \sum_{k\in A} \bar{\Omega}_k |z_k|^2,$$

where $\pm\bar{\Omega}_k$ are the eigenvalues of i$ad(H_A)$.

In particular for all geometric blocks A which do not contain a red edge, this property holds for all positive ξ by using Lemma 11. Formula (19) follows with $\bar{\mathcal{Q}} = \sum_{A\in \mathrm{red}} \tilde{\mathcal{Q}}_A$, here $A \in$ red means that A contains red edges, and we have seen that this is a finite set. We have proved ii).

The change of variables $\psi_A := w_A \to \mathcal{L}_A(\xi) w_A$ where $w_A = (z_k', \bar{z}_k')_{k\in A}$ and $\mathcal{L}_A$ is a matrix which diagonalizes $\tilde{Q}_A = \mathrm{i}ad(\tilde{\mathcal{Q}}_A)$. Since there are only a finite number of distinct matrices $\tilde{Q}_A$, we only need a finite number of distinct ψ_A.

We have $\bar{\Omega}_k = \tilde{\Omega}_k + \lambda_k(\tilde{Q}_A)$, where $\lambda_k(\tilde{Q}_A)$ runs over the eigenvalues of $\tilde{Q}_A$, this proves i).

Each change of variables ψ_A is locally analytic (real algebraic) in ξ for those ξ where the eigenvalues which are not identically equal are distinct. This condition identifies an algebraic hypersurface (where the non-identically equal eigenvalues of a combinatorial block coincide) which we have to remove. The algebraic hypersurface that we remove is a cone, so we can remove a conic neighborhood of this hypersurface arbitrarily determined by its intersection with the unit sphere. Since the choice of the neighborhood on the unit sphere is arbitrary, we easily see that in the domain A_{ε^2} this is equivalent to removing a tubular neighborhood of order ε^2. The bounds iii) follow by homogeneity of the functions.

To be more explicit, from Theorem 1 we have decomposed our space as an infinite direct orthogonal sum of symplectic spaces, each decomposed explicitly as the direct sum of two Lagrangian subspaces in duality, each stable under $ad(N)$.

It is a standard fact that, if for a given symplectic block the matrix is semisimple with real eigenvalues, then on that block there is a symplectic change of variables which makes it diagonal. In fact if the matrix preserves the decomposition into two Lagrangian subspaces, as in our case, we can take the change of variables preserving the two subspaces.

Under the geometric constraint all blocks relative to only black edges give rise to symmetric matrices for a positive quadratic form which thus are semisimple with real eigenvalues and can be diagonalized.

Consider next the cases in which, under the arithmetic constraint, we have the remaining 2×2 blocks associated to red edges. For these we can apply Theorem 3. It remains to prove that the global symplectic transformation defined as direct sum of all the ones diagonalizing each block is indeed continuous and preserves the domain. This follows from the fact that, up to a scalar summand, we have only finitely many types of blocks in $ad(\mathcal{Q})$. □

Acknowledgements. We wish to thank Nguyen Bich Van for correcting some formulas, Dorian Goldfeld and Jonathan Pila for introducing us to the paper [3] and finally Luca Biasco and Massimiliano Berti for several useful suggestions and discussions.

Appendix A. Marked Graphs

This section is independent of the previous parts of the paper. Its only purpose is to establish the correct algebraic language.This standard material in Group Theory.

A.1. The Cayley graphs. Let G be a group and $X = X^{-1} \subset G$ a subset.

A.1.1. Marked graphs.

Definition 25. *An X–marked graph is an oriented graph $\mathcal{A}$ such that each oriented edge is marked with an element $x \in X$.*

$$a \xrightarrow{x} b \qquad\qquad a \xleftarrow{x^{-1}} b$$

We mark the same edge, with opposite orientation, with x^{-1}.

A morphism of marked graphs $j : \mathcal{A}_1 \to \mathcal{A}_2$ is a map between the vertices, which preserves the oriented edges and their markings.

A morphism which is also injective is called an **embedding**.

Recall that

Definition 26. *i) A* **path** *p of length f, from a vertex a to a vertex b in a graph is a sequence of vertices $p = \{a = a_0, a_1, \dots, a_f = b\}$ such that a_{i-1}, a_i form an edge for all $i = 1, \dots, f$.*
The vertex a is called the **source** *and b the* **target** *of the path.*

ii) A **circuit** *is a path from a vertex a to itself.*
We always exclude the presence in a path of **trivial steps** *that is $a_{i-1} = a_{i+1}$.*

iii) A graph without circuits is called a **tree**.

iv) If we have an oriented path $p := \{a_0, a_1, \dots, a_f\}$ marked $a_{i-1} \xrightarrow{g_i} a_i$, $i = 1, \dots, f$ in an X–marked graph, then we set $g(p) := g_f g_{f-1} \dots g_1$.

v) If $g^2 = 1$ then an edge marked g has both orientations so we consider it as **unoriented**.

A.1.2. Cayley graphs. A typical way to construct an X–marked graph is the following. Consider an action $G \times A \to A$ of G on a set A, we then define.

Definition 27 (Cayley graph). *The graph A_X has as vertices the elements of A and, given $a, b \in A$ we join them by an oriented edge $a \xrightarrow{x} b$, marked x, if $b = xa$, $x \in X$.*

If G acts on two sets A_1 and A_2 and $\pi : A_1 \to A_2$ is a map compatible with the G action, then π is also a morphism of marked graphs.

A special case is obtained when G acts on itself by left (resp. right) multiplication and we have the Cayley graph G^l_X (resp. G^r_X). We concentrate on G^l_X which we just denote by G_X. One then immediately sees that

Lemma 13. *If G acts on a set A and $a \in A$ the* **orbit map** *$g \mapsto ga$ is compatible with the graph structure.*

The graph G_X is preserved by right multiplication by elements of G, that is if a, b are joined by an edge marked g then also ah, bh are so joined, for all $h \in G$.

The graphs G^l_X, G^r_X are isomorphic with opposite orientations under the map $g \mapsto g^{-1}$.

The graph G_X is connected if and only if X generates G, otherwise its connected components are the right cosets in G of the subgroup H generated by X.

Definition 28. *Given an X–marked graph $\mathcal{A}$. We say that $\mathcal{A}$ is compatible with G_X if it can be embedded $j : \mathcal{A} \to G_X$ in G_X.*

Note. Two embeddings of $\mathcal{A}$ in G_X differ by a right multiplication by an element of G.

Let us understand the conditions under which a connected graph $\mathcal{A}$ is compatible. Take two vertices h, k in $\mathcal{A}$ and join them by a path $p := k = k_0, k_1, \dots, k_t = h$. Assume that k_{i-1}, k_i, $i = 1, \dots, t$ is marked by the element $g_i \in X$. Then define $g(p) := g_t g_{t-1} \dots g_1$. We can fix an element $r \in \mathcal{A}$ which we call *the root* and lift it for instance to 1. Given any other element $h \in \mathcal{A}$, choose a path p from r to h and set $g_h := g(p)$. In order for this to be well defined we need that if h is joined by two distinct paths p_1, p_2 then $g(p_1) = g(p_2)$. In other words

Lemma 14. *$\mathcal{A}$ is compatible if and only if given any circuit p from r to r we have $g(p) = Id$.*

If this condition is fulfilled we have the special *lift* $j : a \mapsto g_a$ under which $r \mapsto 1$.

In particular suppose that G acts on a set A and $\mathcal{A} \subset A$ is a connected subgraph of A_X with f vertices. Then

Corollary 7. *A sufficient condition for $\mathcal{A}$ to be embedded in G_X is that, for any $a \in \mathcal{A}$, if an element $g \in G$ is a product $g = x_1 x_2 \dots x_d$ of $d \le f$ elements we have that $ga = a$ implies $g = 1$.*

Appendix B. Proof of Lemma 8

Proof. Consider a graph with $r + 1$ vertices and of rank $r \ge 2$. We distinguish the elements a_i, $i = 1, \dots, u$ corresponding to black vertices from the b_j, $j = 1, \dots, v$ of red vertices, we are assuming that both colors appear. We have $a_i(\underline{1}) = 0$, $b_j(\underline{1}) = -2$.

We have the equations

$$(x, \pi(a_i)) = K(a_i), \quad |x|^2 + (x, \pi(b_j)) = K(b_j).$$

If the solution x is polynomial in the v_i, it is linear by a simple degree computation. Since it is also equivariant under the orthogonal group, it follows that it has the form $x = \sum_s c_s v_s$ for some numbers c_s. Let now $a = -\sum_s c_s e_s$ so $x = -\pi(a)$. The fact that the given system of equations is satisfied for all v_i (this since they are now polynomials) is equivalent to the equations.

$$-aa_i = C(a_i), \; a^2 - ab_j = C(b_j). \tag{A.1}$$

By changing root if necessary we may always assume that there are black vertices different from the root. For such a vertex $a_i \ne 0$ we have an equation $-2aa_i = a_i^2 + a_i^{(2)}$ which implies that a_i divides $a_i^{(2)}$.

If $a_i = \sum_j p_j e_j$ we have that $a_i^{(2)} = \sum_j p_j e_j^2$ is an irreducible polynomial unless $a_i = p(e_h - e_k)$ (recall that $\sum_j p_j = 0$). Then $-2a - a_i = (e_h + e_k)$ which implies $-2a = (1 + p)e_h + (1 - p)e_k$, namely $a = \alpha e_h + (-\alpha - 1)e_k$, for some α.

Now we exploit the fact that a satisfies also all the other equations. If it satisfies another black equation–say with a vertex a_j– by linear independence of the vertices we must have $\alpha = 0$ or $\alpha = -1$ and $a = -e_h$ for some h. Hence the only case to exclude is 1 black and one or more red equations. For a red equation we have:

$$2a^2 - 2ab_j = -b_j^2 - b_j^{(2)} \iff a^2 + (a - b_j)^2 = -b_j^{(2)}.$$

By comparing the coefficients of the quadratic terms we see that $b_j = \sum_l q_l e_l$ cannot have any positive coefficient q_l, since $\eta(b_j) = -2$. Hence we must have $b_j = -e_h - e_k$ for the same h, k appearing in a. Now substitute in the equation

$$(\alpha e_h + (-\alpha - 1)e_k)^2 + ((\alpha + 1)e_h - \alpha e_k)^2 = e_h^2 + e_k^2$$

to get $\alpha^2 + (\alpha + 1)^2 = 1$ with solutions $\alpha = 0, -1$, hence $x = v_h, v_k$ as desired. $\square$

Appendix C. Determinantal Varieties

In this section we think of a marking $\ell = \sum_{i=1}^{m} a_i v_i$ coming from the edges (for $q = 1$ we have $\pm v_i \pm v_j$) as a *map* from $V^{\oplus m}$ to V. Here V is a vector space where the v_i belong. Thus a list of k markings is thought of as a map $\rho : V^{\oplus m} = V \otimes \mathbb{C}^m \to V^{\oplus k} = V \otimes \mathbb{C}^k$. Such a map is given by a $k \times m$ matrix A and $\rho = 1_V \otimes A$ so that $Im(\rho) = V \otimes Im(A),\ \ker(\rho) = V \otimes \ker(A)$.

When $\dim(V) = n$ we shall be interested in particular in n–tuples of markings. In this case we have

Lemma 15. *An n–tuple of markings $m_i := \sum_j a_{ij} v_j$ is formally linearly independent – that is the $n \times m$ matrix of the a_{ij} has rank n– if and only if the associated map $\rho : V^{\oplus m} \to V^{\oplus n}$ is surjective.*

We may identify $V^{\oplus n}$ with $n \times n$ matrices and we have the determinantal variety D_n of $V^{\oplus n}$, defined by the vanishing of the determinant det (an irreducible polynomial), and formed by all the n–tuples of vectors $v_1, \dots, v_n$ which are linearly dependent. The variety D_n defines a similar irreducible determinantal variety $D_\rho := \rho^{-1}(D_n)$ in $V^{\oplus m}$ which depends on the map ρ. This is a proper hypersurface if and only if ρ is surjective. We have already remarked that, in this case, D_ρ is an irreducible hypersurface with equation the irreducible polynomial $\det \circ \rho$. We need to see when different lists of markings give rise to different determinantal varieties in $V^{\oplus m}$.

Lemma 16. *Given a surjective map $\rho : V^{\oplus m} \to V^{\oplus n}$, a vector $a \in V^{\oplus m}$ is such that $a + b \in D_\rho,\ \forall b \in D_\rho$ if and only if $\rho(a) = 0$.*

Proof. Clearly if $\rho(a) = 0$ then a satisfies the condition. Conversely if $\rho(a) \neq 0$, we think of $\rho(a)$ as a non zero matrix A and it is easily seen that there is a matrix $B = \rho(b) \in D_n$ such that $A + B = \rho(a + b) \notin D_n$. □

Let $\rho_1, \rho_2 : V^{\oplus m} \to V^{\oplus n}$ be two surjective maps, given by two $n \times m$ matrices $A = (a_{i,j}),\ B = (b_{i,j});\ a_{i,j}, b_{i,j} \in \mathbb{C}$.

Proposition 16. *$\rho_1^{-1}(D_n) = \rho_2^{-1}(D_n)$ if and only if the two matrices A, B have the same kernel.*

Proof. The two matrices A, B have the same kernel if and only if ρ_1, ρ_2 have the same kernel. By Lemma 16, if $\rho_1^{-1}(D_n) = \rho_2^{-1}(D_n)$ then the two matrices A, B have the same kernel. Conversely if the two matrices A, B have the same kernel we can write $B = CA$ with C invertible. Clearly $CD_n = D_n$ and the claim follows. □

We shall also need the following well known fact:

Lemma 17. *Consider the determinantal variety D, given by $d(X) = 0$, of $n \times n$ complex matrices of determinant zero. The real points of D are Zariski dense in D.*[14]

Proof. Consider in D the set of matrices of rank exactly $n - 1$. This set is dense in D and obtained from a fixed matrix (for instance the diagonal matrix I_{n-1} with all 1 except one 0) by multiplying $AI_{n-1}B$ with A, B invertible matrices. If a polynomial f vanishes on the real points of D then $F(A, B) := f(AI_{n-1}B)$ vanishes for all A, B

[14] This means that a polynomial vanishing on the real points of D vanishes also on the complex points.

invertible matrices and real. This set is the set of points in $\mathbb{R}^{2n^2}$, where a polynomial (the product of the two determinants) is non zero. But a polynomial which vanishes in all the points of any space $\mathbb{R}^s$ where another polynomial is non zero is necessarily the zero polynomial. So f vanishes also on complex points. This is the meaning of Zariski dense. □

References

1. Bambusi, D., Grébert, B.: Birkhoff normal form for partial differential equations with tame modulus. Duke Math. J. **135**(3), 507–567 (2006)
2. Berti, M., Bolle, Ph.: Quasi-periodic solutions Sobolev regularity of NLS on $\mathbb{T}^d$ with a multiplicative potential. to appear on Eur. Jour. Math., http://arxiv.org/abs/1012.1427v1 [math.Ap], 2010
3. Bombieri, E., Pila, J.: The number of integral points on arcs and ovals. Duke Math. J. **59**(2), 337–357 (1989)
4. Bourgain, J.: Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations. Ann. of Math. (2) **148**(2), 363–439 (1998)
5. Bourgain, J.: *Green's function estimates for lattice Schrödinger operators and applications*. Volume **158** of *Annals of Mathematics Studies*. Princeton, NJ: Princeton University Press, 2005
6. Chavaudret, C.: Reducibility of quasiperiodic cocycles in linear Lie groups. Erg. The. Dyn. Sys. **31**(3), 741–769 (2011)
7. Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T.: Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation. Invent. Math. **181**(1), 39–113 (2010)
8. Craig, W., Wayne, C.E.: Newton's method and periodic solutions of nonlinear wave equations. Comm. Pure Appl. Math. **46**(11), 1409–1498 (1993)
9. Eliasson, L.H.: Almost reducibility of linear quasi-periodic systems. Proc. Symp. Pure Math. **69**, 679–705 (2001)
10. Geng, J., Yi, Y.: Quasi-periodic solutions in a nonlinear Schrüdinger equation. J. Diff. Eqs. **233**, 512–542 (2007)
11. Geng, J., You, J., Xu, X.: An infinite dimensional KAM theorem and its application to the two dimensional cubic Schrödinger equation. Adv. Math. **226**(6), 5361–5402 (2011)
12. Gentile, G., Procesi, M.: Periodic solutions for the Schrödinger equation with nonlocal smoothing nonlinearities in higher dimension. J. Diff. Eqs. **245**(11), 3253–3326 (2008)
13. Gentile, G., Procesi, M.: Periodic solutions for a class of nonlinear partial differential equations in higher dimension. Commun. Math. Phys. **289**(3), 863–906 (2009)
14. Kuksin, S., Pöschel, J.: Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation. Ann. of Math. (2) **143**(1), 149–179 (1996)
15. Pöschel, J.: A KAM-theorem for some nonlinear partial differential equations. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **23**(1), 119–148 (1996)
16. Procesi, M.: A normal form for beam and non-local nonlinear Schrödinger equations. J. of Physics A: Math. Theor. **43**(43), 434028 (2010)
17. Wang, W.M.: Quasi-periodic solutions of the Schrödinger equation with arbitrary algebraic nonlinearities. Preprint, http://arxiv.org/abs/0907.3409v2 [math.Ap], 2009
18. Wang, W.M.: Supercritical nonlinear Schrödinger equations i: Quasi-periodic solutions. Preprint, http://arxiv.org/abs/1007.0156v1 [math.Ap], 2010
19. Wang, W.M.: Supercritical nonlinear Schrödinger equations ii: Almost global existence. Preprint, http://arxiv.org/abs/1007.0154v1 [math.Ap], 2010

Communicated by G. Gallavotti