The Norm map on the compactified Jacobian, the Prym stack

and

Spectral data for G-Higgs pairs

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Contents

Introduction and statement of results

Let C be a fixed smooth and projective curve over $\mathbb{C}$. A (L-twisted) *Higgs pair* on C is a pair (E, Φ) of a vector bundle E on C and a endomorphism $\Phi : E \to E \otimes L$ with values in a line bundle L on C. Higgs pairs were introduced first in the case $L = K_C$ under the name of Higgs bundles by N. Hitchin, in his study about dimensionally reduced self-duality equations of Yang-Mills gauge theory([**Hit87b**], [**Hit87a**]), and then by C. Simpson, in his study of nonabelian Hodge theory ([**Sim92**], [**Sim94a**], [**Sim94b**]). L-twisted Higgs pairs were introduced by N. Nitsure [**Nit92**].

If E has a G-structure, where G is some complex reductive group, and Φ satisfies some extra conditions depending on G, we speak of G-Higgs pairs and G-Higgs bundles. Hitchin [**Hit87a**], followed by R. Donagi [**Don93**], showed that the moduli space M_G of semistable G-Higgs bundles is endowed with a proper map H_G to a vector space, whose generic fiber is a complex Lagrangian torus and an abelian variety. Such map is called *G-Hitchin fibration* and makes M_G an algebraically completely integrable system; moreover, the smooth locus of M_G has the structure of a hyperkähler manifold. More generally, the Hitchin fibration H_G on the moduli space M_G^L of semistable G-Higgs pairs was introduced by N. Nitsure (loc. cit.); when $G = \mathrm{GL}$ and $L \otimes K_C^{-1}$ is effective, E. Markman [**Mar94**] and F. Bottacin [**Bot95**] proved that M_G^L is endowed with a Poisson structure (depending upon the choice of a section of $L \otimes K_C^{-1}$) with respect to which H_G becomes an algebraically completely integrable system.

T. Hausel and M. Thaddeus [**HT03**] related the G-Hitchin fibration to mirror symmetry, while A. Kapustin and E. Witten [**KW07**] pointed out on physical grounds that Hitchin's system for a complex reductive Lie group G is dual to Hitchin's system for the Langlands dual

group ^LG. This was proved in an algebro-geometric setting by R. Donagi and T. Pantev [**DP12**]; they also conjectured the *classical limit of the geometric Langlands correspondence* as a canonical equivalence between the derived categories of coherent sheaves over the moduli stacks of G-Higgs bundles and ^LG-Higgs bundles, which intertwines the action of the classical limit tensorization functors with the action of the classical limit Hecke functors. Recently, M. Groechenig, D. Wyss and P. Ziegler [**GWZ17**] proved, using an arithmetic perspective, a conjecture by Hausel-Thaddues stating that the moduli spaces of SL and PGL-Higgs bundles are mirror partners and that appropriately defined Hodge numbers of such spaces agree.

The connection between Hitchin system and geometric Langlands program led to the study of Lagrangian submanifolds of the moduli space of Higgs bundles supporting holomorphic sheaves (A-branes), and their dual objects (B-branes). This study was introduced by A. Kapustin and E. Witten in loc. cit. (see also E. Witten [**Wit15**]), followed by L. Schaposhnik and D. Baraglia ([**BS14**], [**BS16**]) and L. Branco [**Bra18**].

More recently, M. de Cataldo, T. Hausel and L. Migliorini [**dCHM12**] followed by J. Shen and Z. Zhang [**SZ18**] and M. de Cataldo, D. Maulik and J. Shen [**dCMS19**] studied the "P=W" conjecture, stating that the canonical isomorphism

$$H^*(\mathcal{M}_B, \mathbb{Q}) \simeq H^*(\mathcal{M}(r), \mathbb{Q})$$

induced by canonical diffeomorphism between the moduli space $\mathcal{M}(r)$ of rank r Higgs bundle and the corresponding character variety $\mathcal{M}_B$ of rank r stable local systems, identifies the weight filtration and the perverse filtration associated with the Hitchin fibration.

The generic fiber of the GL-Hitchin morphism is the Jacobian of a curve associated to the fiber, called the *spectral curve*; when $G = \mathrm{SL}, \mathrm{Sp}, \mathrm{SO}$, the generic fiber of the G-Hitchin morphisms are Prym varieties associated to certain morphisms from the spectral curve. This fact, also known as *spectral correspondence* or *abelianization process*, has been pointed out first in [**Hit87a**], followed by [**Hit07**] and [**Sch13**].

The duality of Hitchin systems manifests itself in the statement that the dual of the abelian variety for the generic fiber of $\mathcal{H}_G$ is the abelian variety for the generic fiber of $\mathcal{H}_{{}^L G}$. More recently, N. Hitchin and L. Schaposnik introduced in [**HS14**] a *non-abelianization* process in order to study Higgs bundles that correspond (by solving the gauge-theoretic Higgs bundle equations) to flat connections on C with holonomy in some real Lie groups.

The study of the Hitchin morphism restricted to the fibers whose associated spectral curve is integral played a crucial role in B. C. Ngô's proof of the fundamental lemma ([**Ngô06**] and [**Ngô10**]); more generally, the study of the Hitchin morphism restricted to the fibers whose associated spectral curve is reduced was a key ingredient in Chaudard-Laumon's proof of the weighted fundamental lemma ([**CL10**] and [**CL12**]).

For $G = \mathrm{GL}$ the spectral correspondence has been generalized to non-generic fibers with integral spectral curve by A. Beauville, M. Narasimhan and S. Ramanan in [**BNR89**], and to any fiber by D. Schaub [**Sch98**], followed by P.-H. Chaudouard and G. Laumon [**CL16**] and M. A. De Cataldo [**dC17**]. The spectral correspondence for non-generic fibers involves a wider moduli space than the Jacobian scheme, namely the *compactified Jacobian* parametrizing torsion-free rank-1 sheaves.

For G different from GL, the spectral correspondence for non-generic fibers has been studied by [**HP12**] for the case of SL, and by [**Bra18**] for the Hitchin map associated to some semisimple real Lie groups.

This thesis is divided in two parts, introduced by a preliminary Chapter about torsion-free rank-1 sheaves and Higgs pairs (Chapter 1). In the first part (Chapter 2 and 3), motivated by the spectral correspondence for $G = \mathrm{SL}$, we study the Norm map Nm_π on the compactified Jacobian associated to a finite, flat morphism $X \xrightarrow{\pi} Y$ between projective curves; the Norm map happens to be well-defined only if Y is smooth. In such case, we define the Prym stack of X over Y as the (stacky) fiber $\mathrm{Nm}^{-1}(\mathcal{O}_Y)$. In the case that X is reduced with locally planar singularities, we show that the usual Prym scheme is

contained in the Prym stack as an open and dense subset. In the second part (Chapter 4) we study the spectral correspondence for G-Higgs pairs, in the case of $G = \mathrm{SL}(r, \mathbb{C})$, $\mathrm{PGL}(r, \mathbb{C})$, $\mathrm{Sp}(2r, \mathbb{C})$, $\mathrm{GSp}(2r, \mathbb{C})$, $\mathrm{PSp}(2r, \mathbb{C})$, over any fiber.

In the future work, we aim to study the spectral data also for other classical groups such as $\mathrm{SO}(2r, \mathbb{C})$ and $\mathrm{SO}(2r + 1, \mathbb{C})$. Moreover, we aim to study the geometric properties of the moduli loci arising in the description of the spectral data. Finally, we are interested in characterizing, for various G, the spectral data corresponding to semistable pairs, and to consider also such data up to S-equivalence of the corresponding pairs; this would allow to describe spectral data for the scheme-theoretic version of the Hitchin morphism.

0.1. Preliminaries

In Chapter 1 we first introduce generalized line bundles and torsion-free rank-1 sheaves, giving the following general definitions.

DEFINITION 0.1.1. Let X be a Noetherian scheme of pure dimension 1. A coherent sheaf $\mathcal{F}$ on X is said:

(1) *torsion-free* if the support of $\mathcal{F}$ has dimension 1 and the maximal subsheaf $T(\mathcal{F}) \subset \mathcal{F}$ of dimension 0 is equal to 0;
(2) *rank-1* if for any generic point $\xi \in X$, the length of $\mathcal{F}_\xi$ as an $\mathcal{O}_{X,\xi}$-module is equal to the length of $\mathcal{O}_{X,\xi}$ as a module over itself.

DEFINITION 0.1.2. Let X be a Noetherian scheme of pure dimension 1. A *generalized line bundle* is a torsion-free sheaf $\mathcal{F}$ on X such that for any generic point $\xi \in X$, the stalk $\mathcal{F}_\xi$ is isomorphic to $\mathcal{O}_{X,\xi}$ as an $\mathcal{O}_{X,\xi}$-module.

In the case of X projective over a field, generalized line bundles and torsion-free rank-1 sheaves are both particular cases of torsion-free sheaves with polarized rank 1 with respect to any fixed polarization.

DEFINITION 0.1.3. Let X be a projective scheme of pure dimension 1 over a base field k and let H be a polarization of degree $\deg H = \delta$. Let $\mathcal{F}$ be a torsion-free sheaf on X. The *polarized rank* of $\mathcal{F}$ is the rational number $r_H(\mathcal{F})$ determined by the Hilbert polinomial of $\mathcal{F}$ with respect to H:

$$P(\mathcal{F}, n, H) := \chi(\mathcal{F} \otimes \mathcal{O}_X(nH)) = \delta r_H(\mathcal{F})n + \chi(X, \mathcal{F}).$$

The polarized rank of a torsion-free sheaf is related with its rank at the generic points.

THEOREM 0.1.4. *Let X be a projective scheme of pure dimension 1 over a field, with irreducible components $X_1, \ldots, X_s$ and let $\mathcal{F}$ be a torsion-free sheaf on X. For each i, let ξ_i be the generic point of X_i and let $\mathrm{rk}_{X_i}(\mathcal{F}) := \ell_{\mathcal{O}_{X,\xi_i}}(\mathcal{F}_{\xi_i})/\ell_{\mathcal{O}_{X,\xi_i}}(\mathcal{O}_{X,\xi_i})$ be the rank of $\mathcal{F}$ at X_i. Let H be a polarization on X. Then, the following formula for $r_H(\mathcal{F})$ holds:*

$$r_H(\mathcal{F}) = \frac{\sum_{i=1}^s \mathrm{rk}_{X_i}(\mathcal{F}) \deg H_{|X_i}}{\sum_{i=1}^s \deg H_{|X_i}}.$$

In particular, if $\mathcal{F}$ is a torsion-free rank-1 sheaf, then $r_H(\mathcal{F}) = 1$ for any polarization H on X.

We consider the following moduli spaces for torsion-free sheaves.

DEFINITION 0.1.5. Let X be a projective scheme of pure dimension 1 over a field k and let $d \in \mathbb{Z}$ be an integer number.

(1) The *Jacobian scheme* of degree d on X is the algebraic scheme $J^d(X)$ representing the sheafification of the functor that associates to any k-scheme T the set of isomorphism classes of line bundles of degree d on $X \times_k T$. The union of the Jacobians of all degrees is denoted as $J(X)$.

(2) The *generalized Jacobian stack* of degree d on X is the algebraic stack $G\mathbb{J}(X, d)$ such that, for any k-scheme T, $G\mathbb{J}(X, d)(T)$ is the groupoid of T-flat coherent sheaves on $X \times_k T$ whose fibers over T are generalized line bundles of degree d on $X \simeq X \times_k \{t\}$.

(3) The *compactified Jacobian stack* of degree d on X is the algebraic stack $\overline{\mathbb{J}}(X, d)$ such that for any k-scheme T, $\overline{\mathbb{J}}(X, d)(T)$ is the groupoid of T-flat coherent sheaves on $X \times_k T$ whose fibers over T are torsion-free sheaves of rank 1 and degree d on $X \simeq X \times_k \{t\}$.

(4) Let H be a polarization on X The *Simpson Jacobian stack* of degree d on X is the algebraic stack $\overline{\mathbb{J}}_{tf}(X, H, d)$ such that for any k-scheme T, $\overline{\mathbb{J}}_{tf}(X, H, d)(T)$ is the groupoid of T-flat coherent sheaves on $X \times_k T$ whose fibers over T are torsion-free sheaves of polarized rank 1 and polarized degree d on $X \simeq X \times_k \{t\}$ with respect to H.

The locus of semi-stable object can be defined in any of such moduli spaces, depending on the choiche of the polarization H. Then, we have the following good moduli spaces.

(5) The *generalized Jacobian scheme* $GJ(X, H, d)$ representing S-equivalence classes of H-semistable generalized line bundles of degree d on X.

(6) The *compactified Jacobian scheme* $\overline{J}(X, H, d)$ representing S-equivalence classes of H-semistable torsion-free sheaves of rank 1 and degree d on X.

(7) The *Simpson Jacobian* $\overline{J}_{tf}(X, H, d)$ representing S-equivalence classes of H-semistable torsion-free sheaves of polarized rank 1 and polarized degree d on X.

These moduli spaces satisfy the following chains of inclusion:

$$J^d(X) \overset{(1)}{\subseteq} G\mathbb{J}(X, d) \overset{(2)}{\subseteq} \overline{\mathbb{J}}(X, d) \overset{(3)}{\subseteq} \overline{\mathbb{J}}_{tf}(X, H, d),$$

and:

$$J^d(X) \overset{(1')}{\subseteq} GJ(X, H, d) \overset{(2')}{\subseteq} \overline{J}(X, H, d) \overset{(3')}{\subseteq} \overline{J}_{tf}(X, H, d).$$

All the inclusions above are open embeddings, non strict in general; inclusions (3) and (3′) are also closed. When X satisfies additional conditions, some of them are actually equalities.

- If X is irreducible, inclusion (3) (resp. (3)′) is an equality.
- If X is reduced, inclusion (2) (resp. (2′)) is an equality.

8

- If X is integral and smooth, inclusions (1), (2) and (3) (resp. (1'), (2') and (3')) are equalities.

The moduli space $\overline{J}_{tf}(X, H, d)$ of semistable torsion-free sheaves of polarized rank 1 and degree d is projective by the work of C. Simpson ([**Sim94a**] and [**Sim94b**]); hence, the compactified Jacobian $\overline{J}(X, H, d)$ is a projective subscheme and union of connected components.

The compactified Jacobian plays an important role in the spectral correspondence for Higgs pairs. Let C be a fixed smooth curve over the field of complex numbers and let L be a fixed line bundle on C with degree $\ell = \deg L$. The algebraic stack $\mathcal{M}(r, d)$ of all L-twisted Higgs pairs (E, Φ) on C of rank r and degree d is endowed with a morphism, called the *Hitchin morphism*, defined as:

$$\mathcal{H}_{r,d} : \mathcal{M}(r, d) \to \mathcal{A}(r) = \bigoplus_{i=1}^{r} H^0(C, L^i)$$

$$(E, \Phi) \mapsto (a_1(E, \Phi), \ldots, a_r(E, \Phi))$$

where $L^i = L^{\otimes i}$, $a_i(E, \Phi) := (-1)^i \operatorname{tr}(\wedge^i \Phi)$ and $\mathcal{A}(r)$ is called the *Hitchin base*. Similarly, the good moduli space $M(r, d)$ parametrizing S-equivalence classes of semistable L-twisted Higgs pairs of rank r and degree d is endowed with a flat projective morphism, called the *Hitchin fibration*:

$$H_{r,d} : M(r, d) \to \mathcal{A}(r) = \bigoplus_{i=1}^{r} H^0(C, L^i)$$

$$(E, \Phi) \mapsto (a_1(E, \Phi), \ldots, a_r(E, \Phi)).$$

Let $\underline{a} \in \mathcal{A}(r)$ be any characteristic. The spectral curve $X_{\underline{a}} \xrightarrow{\pi_{\underline{a}}} C$ is the projective scheme defined in the total space of L, $P = \mathbb{P}(\mathcal{O}_C \oplus L^{-1}) \xrightarrow{p} C$, by the homogeneous equation

$$x^r + p^*(a_1)x^{r-1}y + \cdots + p^*(a_r)y^r = 0$$

where x is the section of $\mathcal{O}_P(1) \otimes p^*(L)$ whose pushforward via p corresponds to the constant section $(1, 0)$ of $L \otimes \mathcal{O}_C$ and y is the section of $\mathcal{O}_P(1)$ whose pushforward via p corresponds to the constant section $(1, 0)$ of $\mathcal{O}_C \otimes L^{-1}$. The spectral curve $X_{\underline{a}}$ has pure dimension 1 and canonical sheaf $\omega_{X_{\underline{a}}} = \pi_{\underline{a}}^*(\omega_c \otimes L^{r-1})$. The following spectral correspondence is a classical result.

PROPOSITION 0.1.6. (Spectral correspondence) *Let $\underline{a} \in \mathcal{A}(r)$ be any characteristic and let $X_{\underline{a}} \xrightarrow{\pi_{\underline{a}}} C$ be the associated spectral curve. Let $\mathcal{O}_C(1)$ be an ample line bundle on C; let $\mathcal{O}_{X_{\underline{a}}}(1) := \pi_{\underline{a}}^*\big(\mathcal{O}_C(1)\big)$ be the ample line bundle on $X_{\underline{a}}$ obtained by pullback and denote with H the associated polarization on $X_{\underline{a}}$.*

(1) For any integer d, there is an isomorphism of stacks:

$$\mathcal{H}_{r,d}^{-1}(\underline{a}) \xrightarrow[\Pi]{\sim} \overline{\mathbb{J}}(X_{\underline{a}}, d'),$$

where $d' = d + r(1-g) - \chi(\mathcal{O}_{X_{\underline{a}}}) = d + \frac{r(r-1)}{2}\ell$ and $g = g(C)$ is the genus of C. If $\mathcal{M}$ is a torsion-free sheaf of rank 1 on $X_{\underline{a}}$, then $\Pi(\mathcal{M}) := (E, \Phi)$ where

$E = \pi_{\underline{a},*}(\mathcal{M})$

$\Phi = \pi_{\underline{a},*}(\cdot x) : \pi_{\underline{a},*}(\mathcal{M}) \to L \otimes \pi_{\underline{a},*}(\mathcal{M}) \simeq \pi_{\underline{a},*}(\pi_{\underline{a}}^*L \otimes \mathcal{M}).$

(2) A torsion-free sheaf of rank 1 $\mathcal{M}$ on $X_{\underline{a}}$ is H-semistable if and only if the associated Higgs pair $\Pi(\mathcal{M})$ is semistable on C. Hence, the above correspondence yields an isomorphism of schemes:

$$H_{r,d}^{-1}(\underline{a}) \simeq \overline{J}(X_{\underline{a}}, H, d').$$

The inverse direction in the spectral correspondence is resumed by the following proposition.

PROPOSITION 0.1.7. *Let $\underline{a} \in \mathcal{A}(r)$ be any characteristic and let $X_{\underline{a}} \xrightarrow{\pi_{\underline{a}}} C$ be the associated spectral curve. Let $\mathcal{M} \in \overline{\mathbb{J}}(X_{\underline{a}}, d')$ be a torsion-free rank-1 sheaf on $X_{\underline{a}}$ corresponding to the Higgs pair $(E, \Phi) \in \mathcal{H}_{r,d}^{-1}(\underline{a})$ on C. Then, the following exact sequence holds:*

$$0 \to \mathcal{M} \otimes \pi_{\underline{a}}^*(L^{1-r}) \to \pi_{\underline{a}}^*E \xrightarrow{\pi_{\underline{a}}^*(\Phi)-x} \pi_{\underline{a}}^*E \otimes \pi_{\underline{a}}^*L \xrightarrow{\text{ev}} \mathcal{M} \otimes \pi_{\underline{a}}^*L \to 0$$

where ev *is induced by the evaluation map $\pi_{\underline{a}}^*\pi_{\underline{a},*}(\mathcal{M}) \to \mathcal{M}$.*

0.2. Direct image of generalized divisors and Norm map

Let $X \xrightarrow{\pi} Y$ be a finite, flat morphism between projective schemes of pure dimension 1 over a field, such that Y is smooth. In Chapter 2 we introduce two pairs of important morphisms associated to π.

DEFINITION 0.2.1.   (1) The *direct and inverse image map* between the Hilbert schemes Hilb_X^d and Hilb_Y^d parametrizing 0-dimensional subschemes of X (resp. of Y) with Hilbert polynomial equal to $d \geq 0$ are defined on the T-valued points as:

$$\pi_*(T): \quad \text{Hilb}_X^d(T) \longrightarrow \text{Hilb}_Y^d(T)$$
$$D \subseteq X \times_k T \longmapsto \mathcal{Z}\left(\text{Fitt}_0(\pi_{T,*}(\mathcal{O}_D))\right)$$

$$\pi^*(T): \quad \text{Hilb}_Y^d(T) \longrightarrow \text{Hilb}_X^d(T)$$
$$D \subseteq Y \times_k T \longmapsto \mathcal{Z}\left(\pi_T^{-1}(\mathcal{I}_D) \cdot \mathcal{O}_{X \times_k T}\right)$$

where $\pi_T : X \times_k T \to Y \times_k T$ is the morphism induced by base change of π, Fitt_0 denotes the 0-th Fitting ideal of a sheaf of modules and $\mathcal{Z}$ denotes the closed subscheme defined by a sheaf of ideals.

(2) The *Norm and the inverse image map* between the compactified Jacobians of any degree d on X and Y are defined on the T-valued points as:

$$\text{Nm}_\pi(T): \quad \overline{\mathbb{J}}(X,d)(T) \longrightarrow J^d(Y)(T)$$
$$\mathcal{L} \longmapsto \det\left(\pi_{T,*}(\mathcal{L})\right) \otimes \det\left(\pi_{T,*}\mathcal{O}_{X \times_k T}\right)^{-1}$$

$$\pi^*(T): \quad J^d(Y)(T) \longrightarrow J^d(X) \subseteq \overline{\mathbb{J}}(X,d)(T)$$
$$\mathcal{N} \longmapsto \pi_T^*(\mathcal{N}).$$

Recall that, for any line bundle M of degree e on X, the M-twisted Abel map relates Hilb_X^d with the generalized Jacobian of degree $-d+e$:

$$\mathcal{A}_M: \quad \text{Hilb}_X^d \longrightarrow G\mathbb{J}(X, -d+e) \subseteq \overline{\mathbb{J}}(X, -d+e)$$
$$D \longmapsto \mathcal{I}_D \otimes M.$$

PROPOSITION 0.2.2. *Let $X \xrightarrow{\pi} Y$ be a finite, flat morphism between projective schemes of pure dimension 1 over a field, such that Y is smooth. The direct image between Hilbert schemes and the Norm map between compactified Jacobians are related for any $d \geq 0$ by the*

commutative diagram:

$$\begin{CD} \operatorname{Hilb}_X^d @>\pi_*>> {}^\ell\operatorname{Hilb}_Y^d \\ @V\mathcal{A}_M VV @VV\mathcal{A}_{\operatorname{Nm}_\pi(M)}V \\ G\mathbb{J}(X,-d+e) \subseteq \overline{\mathbb{J}}(X,-d+e) @>\operatorname{Nm}_\pi>> J^{-d+e}(Y). \end{CD}$$

Similarly, the inverse image between Hilbert schemes and the inverse image between compactified Jacobians are related for any $d \geq 0$ by the commutative diagram:

$$\begin{CD} {}^\ell\operatorname{Hilb}_Y^d @>\pi^*>> {}^\ell\operatorname{Hilb}_X^d \\ @V\mathcal{A}_N VV @VV\mathcal{A}_{\pi^*(N)}V \\ J(Y,-d+e) @>\pi^*>> J(X,-d+e). \end{CD}$$

The Norm and the inverse image map between compactified Jacobians satisfy some expected properties.

PROPOSITION 0.2.3. *Let $X \xrightarrow{\pi} Y$ be a finite, flat morphism between projective schemes of pure dimension 1 over a field, such that Y is smooth. Let Nm_π and π^* be the Norm and inverse image map between the compactified Jacobans of X and Y and let T be any k-scheme.*

(1) *Let $\mathcal{L}, \mathcal{M} \in \overline{\mathbb{J}}(X)(T)$ such that $\mathcal{M}$ is a T-flat family of line bundles. Then, $\operatorname{Nm}_\pi(\mathcal{L} \otimes \mathcal{M}) \simeq \operatorname{Nm}_\pi(\mathcal{L}) \otimes \operatorname{Nm}_\pi(\mathcal{M})$.*

(2) *Let $\mathcal{N}, \mathcal{N}' \in J(Y)(T)$. Then, $\pi^*(\mathcal{N} \otimes \mathcal{N}') \simeq \pi^*(\mathcal{N}) \otimes \pi^*(\mathcal{N}')$.*

(3) *Let $\mathcal{N} \in J(Y)(T)$. Then, $\pi^*(\mathcal{N})$ is a T-flat family of line bundles over X and $\operatorname{Nm}_\pi(\pi^*(\mathcal{N})) \simeq \mathcal{N}^{\otimes n}$.*

The fibers of the direct image and of the Norm map are studied in Chapter 3, in the case that X is reduced with locally planar singularities.

PROPOSITION 0.2.4. *Let $X \xrightarrow{\pi} Y$ be a finite, flat morphism between projective schemes of pure dimension 1 over a field, such that Y is smooth and X is reduced with locally planar singularities.*

(1) *Let $E \in \operatorname{Hilb}_Y$ be an effective divisor of degree d on Y and let $\pi_*^{-1}(E)$ be the corresponding fiber in Hilb_X. Then, the locus of Cartier divisors in $\pi_*^{-1}(E)$ is an open and non-empty dense subset of $\pi_*^{-1}(E)$.*

(2) Let $\mathcal{N} \in J(Y)$ be any line bundle on Y. Then, the fiber $\mathrm{Nm}_\pi^{-1}(\mathcal{N})$ is non-empty and contains $\mathrm{Nm}_\pi^{-1}(\mathcal{N}) \cap J(Y)$ as an open and dense subset.

In particular, the fiber of Nm_π over $\mathcal{O}_Y$ is called the *Prym stack* of X associated to π. When Y is smooth and X is reduced with locally planar singularities, the Prym variety contained in the Prym stack of X associated to π as an open and dense subset.

0.3. Spectral data for G-Higgs pairs

In Chapter 4 we study the spectral correspondence for G-Higgs pairs, where $G = \mathrm{SL}(r, \mathbb{C}), \mathrm{PGL}(r, \mathbb{C}), \mathrm{Sp}(2r, \mathbb{C}), \mathrm{GSp}(2r, \mathbb{C}), \mathrm{PSp}(2r, \mathbb{C})$. Throughout this chapter, C denotes a fixed smooth curve over the field of complex numbers and L a fixed line bundle on C with degree $\ell = \deg L$.

DEFINITION 0.3.1. Let G be a complex reductive Lie group. A *(L-twisted) G-Higgs pair* is a pair (P, ϕ) where P is a principal G-bundle over C and the G-Higgs field ϕ is a section of $H^0(\mathrm{ad}(P) \otimes L)$.

Let $\delta \in \pi_1(G)$ and let $p_1, \ldots, p_k$ be a homogeneous basis for the algebra of invariant polynomials of the Lie algebra $\mathfrak{g}$ of G, such that p_i has degree d_i. Then, the algebraic moduli stack $\mathcal{M}_G(\delta)$ of all L-twisted G-Higgs pairs (P, ϕ) such that P has topological type δ is endowed with a morphism, called the *G-Hitchin morphism*, defined as:

$$\mathcal{H}_{G,\delta}^{p_1,\ldots,p_k} = \mathcal{H}_{G,\delta} : \mathcal{M}_G(\delta) \to \mathcal{A}_G = \bigoplus_{i=1}^{k} H^0(C, L^{d_i})$$

$$(P, \phi) \mapsto (p_1(\phi), \ldots, p_k(\phi)).$$

If $G \overset{\rho}{\hookrightarrow} \mathrm{GL}(r, \mathbb{C})$, any G-Higgs pair (P, ϕ) gives rise via the associated bundle construction to a classical Higgs pair (E, Φ), where $E = P \times_\rho \mathbb{C}^r$ and Φ is the image ϕ with respect to the morphism $\mathrm{ad}(P) \to \mathrm{End}(E)$ induced by the embedding ρ; such E has rank r and degree d corresponding to δ under the map $\pi_1(G) \xrightarrow{\pi_1(\rho)} \pi_1(\mathrm{GL}(r, \mathbb{C}))$ induced by ρ. In this case, we denote as

$$\mathcal{M}_G(r, d) := \coprod_{\pi_1(\rho)(\delta)=d} \mathcal{M}_G(\delta)$$

13

the moduli space of G-Higgs pairs whose associated vector bundle is of rank r and degree d; the G-Hitchin morphism on $\mathcal{M}_G(r, d)$ is denoted as $\mathcal{H}_{G,r,d}$.

0.3.1. $G = \mathbf{SL}(r, \mathbb{C})$. In this case, the datum (P, ϕ) of a $SL(r, \mathbb{C})$-Higgs pair on C corresponds univocally, via the associated bundle construction, to the datum of (E, Φ, λ), where (E, Φ) is a Higgs pair of rank r on C with trace zero, and λ is a trivialization of $\det E$. The isomorphism $\det E \simeq \mathcal{O}_C$ implies in particular that E has degree equal to 0.

A basis for the invariant polynomials of $\mathfrak{sl}(r, \mathbb{C})$ is given by $\{p_2, \ldots, p_r\}$ where $p_i(P, \phi) := (-)^i \operatorname{tr}(\wedge^i \phi)$. The SL-Hitchin morphism of rank r can be defined as:

$$\mathcal{H}_{\mathrm{SL},r} : \mathcal{M}_{\mathrm{SL}}(r) := \mathcal{M}_{\mathrm{SL}}(r, 0) \longrightarrow \mathcal{A}_{\mathrm{SL}}(r) = \bigoplus_{i=2}^{r} H^0(C, L^i)$$

$$(P, \phi) \longmapsto (p_2(P, \phi), \ldots, p_r(P, \phi)).$$

PROPOSITION 0.3.2. *Let $\underline{a} \in \mathcal{A}_{\mathrm{SL}}(r)$ be any characteristic, let $X = X_{\underline{a}} \xrightarrow{\pi} C$ be the associated spectral curve, and denote $B := \det(\pi_* \mathcal{O}_X)^{-1}$. The fiber $\mathcal{H}_{\mathrm{SL},r}^{-1}(\underline{a})$ of the SL-Hitchin morphism is isomorphic, via the spectral correspondence, to the fiber $\operatorname{Nm}_\pi^{-1}(B)$ of the Norm map from $\overline{\mathbb{J}}(X, d')$ to $J(Y, d')$ induced by π for $d' = \frac{r(r-1)}{2}\ell$. If the spectral curve $X_{\underline{a}}$ is reduced, then*

$$\mathcal{H}_{\mathrm{SL},r}^{-1}(a) \simeq \overline{\operatorname{Pr}}(X, C).$$

0.3.2. $G = \mathbf{PGL}(r, \mathbb{C})$. Any $\mathrm{PGL}(r, \mathbb{C})$-Higgs pair admits a lifting $(\tilde{P}, \tilde{\phi})$ to a $\mathrm{GL}(r, \mathbb{C})$-Higgs pair, corresponding to a Higgs pair (E, Φ) via the associated bundle construction. Then, the datum of (P, ϕ) corresponds uniquely to the datum of the equivalence class $[(E, \Phi)]$ of Higgs pairs on C with trace zero, under the equivalence relation $\sim_{J(C)}$ defined by:

$$(E, \Phi) \sim_{J(C)} (E \otimes N, \Phi \otimes 1_N) \qquad \text{for any } N \in J(C).$$

Denote with $\mathcal{M}^{\mathrm{tr}=0}(r)$ the closed substack of $\mathcal{M}(r) = \coprod_{d \in \mathbb{Z}} \mathcal{M}(r, d)$ given by Higgs pairs of rank r with trace zero. Then, $[(E, \Phi)]$ is the

orbit of (E, Φ) under the action of $J(C)$ on $\mathcal{M}^{\mathrm{tr}=0}(r)$ defined by:

$$\mathcal{M}^{\mathrm{tr}=0}(r) \times J(C) \longrightarrow \mathcal{M}^{\mathrm{tr}=0}(r)$$

$$((E, \Phi), N) \longmapsto (E \otimes N, \Phi \otimes 1_N)$$

DEFINITION 0.3.3. Let (P, ϕ) be a PGL-Higgs pair and let (E, Φ) be a Higgs pair with trace zero and degree $d \in \mathbb{Z}$ corresponding to a lifting $(\tilde{P}, \tilde{\phi})$ of (P, ϕ) to a GL-Higgs pair. The *degree of* (P, ϕ) is the congruence class $\overline{d} \in \mathbb{Z}/r\mathbb{Z}$.

The first homotopy group $\pi_1(\mathrm{PGL}(r, \mathbb{C}))$ of $\mathrm{PGL}(r, \mathbb{C})$ is isomorphic to $\mathbb{Z}/r\mathbb{Z}$ and the degree of (P, ϕ) characterizes uniquely the topological type of P. Moreover, up to the action of multiples of $\mathcal{O}_C(1)$ on (E, Φ), the $J(C)$-orbit $[(E, \Phi)]$ corresponding to a PGL-Higgs pair with degree $\overline{d} \in \mathbb{Z}/r\mathbb{Z}$ is the orbit of a Higgs pair of degree d with $d \in \{0, \ldots, r-1\}$. Then, a PGL-Higgs pair of degree $\overline{d}$ is identified uniquely with the orbit in $\mathcal{M}^{\mathrm{tr}=0}(r, d)$ of a Higgs pair of trace zero *and degree d* with respect to the action of line bundles *of degree 0 on C*.

A basis for the invariant polynomials of $\mathfrak{pgl}(r, \mathbb{C}) = \mathfrak{sl}(r, \mathbb{C})$ is given by $\{p_2, \ldots, p_r\}$ where $p_i(P, \phi) := (-)^i \mathrm{tr}(\wedge^i \phi)$. Hence, we have the following PGL-Hitchin morphism of rank r and degree $\overline{d}$:

$$\mathcal{H}_{\mathrm{PGL}, r, \overline{d}} : \mathcal{M}_{\mathrm{PGL}}(r, \overline{d}) \longrightarrow \mathcal{A}_{\mathrm{PGL}}(r) = \bigoplus_{i=2}^{r} H^0(C, L^i)$$

$$(P, \phi) \longmapsto (p_2(P, \phi), \ldots, p_r(P, \phi)).$$

PROPOSITION 0.3.4. *Let $\underline{a} \in \mathcal{A}_{\mathrm{PGL}}(r)$ be any characteristic and let $X = X_{\underline{a}} \xrightarrow{\pi} C$ be the associated spectral curve. Let $\overline{d} \in \mathbb{Z}/r\mathbb{Z}$ with $d \in \{0, \ldots, r-1\}$ be any degree. The fiber $\mathcal{H}_{\mathrm{PGL}, r, \overline{d}}^{-1}(\underline{a})$ of the PGL-Hitchin morphism is isomorphic, via the spectral correspondence, to the quotient moduli space*

$$\overline{\mathbb{J}}(X, d') / \pi^* J^0(C)$$

of torsion-free sheaves of rank 1 and degree d' up to the action of line bundles of degree 0 on C by tensor product, with $d' = d + \frac{r(r-1)}{2} \ell$.

The case of $\mathrm{PGL}(r, \mathbb{C})$-Higgs pairs of degree 0 deserves special attention.

PROPOSITION 0.3.5. *Let $\underline{a} \in \mathcal{A}_{\mathrm{PGL}}(r)$ be any characteristic, let $X = X_{\underline{a}} \xrightarrow{\pi} C$ be the associated spectral curve and denote $B := \det(\pi_* \mathcal{O}_X)^{-1}$. Let $d' = \frac{r(r-1)}{2} \ell$ and let Nm_π be the Norm map induced by π on $\overline{\mathbb{J}}(X, d')$. Let $J^0(C)[r]$ be the group stack of line bundles with r-th torsion on C, acting on $\mathrm{Nm}_\pi^{-1}(B)$ as follows:*

$$\mathrm{Nm}_\pi^{-1}(B) \times \pi^* J^0(C)[r] \longrightarrow \mathrm{Nm}_\pi^{-1}(B)$$

$$\big((\mathcal{M}, \epsilon), (\pi^* N, \pi^*(\eta : N^r \xrightarrow{\sim} \mathcal{O}_C))\big) \longmapsto (\mathcal{M} \otimes \pi^* N, \epsilon_{N,\eta})$$

where $\epsilon_{N,\eta}$ is equal to the following composition:

$$\epsilon_{N,\eta} : \mathrm{Nm}(\mathcal{M} \otimes \pi^* N) \xrightarrow{\sim} \mathrm{Nm}(\mathcal{M}) \otimes \mathrm{Nm}(\pi^* N) \xrightarrow{\sim} \mathrm{Nm}(\mathcal{M}) \otimes N^r \xrightarrow{\epsilon \otimes \eta} B.$$

Then, the fiber $\mathcal{H}_{\mathrm{PGL},r,\bar{0}}^{-1}(\underline{a})$ of the PGL-Hitchin morphism is isomorphic, via the spectral correspondence, to the quotient moduli space

$$\mathrm{Nm}_\pi^{-1}(B)/\pi^*(J^0 C)[r].$$

0.3.3. $G = \mathbf{Sp(2r, \mathbb{C})}$. In this case, the datum of a $\mathrm{Sp}(2r, \mathbb{C})$-Higgs pair (P, ϕ) on C corresponds univocally to the datum of (E, Φ, ω) where (E, Φ) is a Higgs pair of rank $2r$ and degree 0 and $\omega : E \otimes E \to \mathcal{O}_C$ is a non-degenerate symplectic form on E satisfying the condition:

$$\omega(\Phi v, w) = -\omega(v, \Phi w).$$

A basis for the invariant polynomials of $\mathfrak{sp}(2r, \mathbb{C})$ is given by $\{p_{2i}\}_{i=1,\dots,r}$ where $p_{2i}(P, \phi) := \mathrm{tr}(\wedge^{2i}\phi)$. The corresponding Sp-Hitchin morphism takes the form:

$$\mathcal{H}_{\mathrm{Sp},2r} : \mathcal{M}_{\mathrm{Sp}}(2r) := \mathcal{M}_{\mathrm{Sp}}(2r, 0) \longrightarrow \mathcal{A}_{\mathrm{Sp}}(2r) = \bigoplus_{l=1}^{r} H^0(C, L^{2l})$$

$$(P, \phi) \longmapsto (p_2(P, \phi), p_4(P, \phi), \dots, p_{2r}(P, \phi)).$$

For any characteristic $\underline{a} \in \mathcal{A}_{\mathrm{Sp}}(2r)$, the spectral curve $\pi : X_{\underline{a}} \to C$ is defined in the total space $p : \mathbb{P}(\mathcal{O}_C \oplus L^{-1}) \to C$ by the equation

$$x^{2r} y + a_2 x^{2r-2} y^2 + \cdots + a_{2r-2} x^2 y^{2r-2} + a_{2r} y^{2r} = 0.$$

Hence, the curve $X_{\underline{a}}$ has an involution σ defined by $\sigma(x) = -x$.

PROPOSITION 0.3.6. *Let $\underline{a} \in \mathcal{A}_{\mathrm{Sp}}(r)$ be any characteristic and let $X = X_{\underline{a}} \xrightarrow{\pi} C$ be the associated spectral curve with involution $\sigma : X \to X$. The fiber $\mathcal{H}_{\mathrm{Sp},2r}^{-1}(\underline{a})$ of the Sp-Hitchin morphism is isomorphic, via*

the spectral correspondence, to the equalizer $\mathcal{E}_{\underline{a}}$ of the two maps

$$\_^* := \mathcal{H}om(\_, \mathcal{O}_X) : \overline{\mathbb{J}}(X, d') \to \overline{\mathbb{J}}(X, -d')$$

$$\sigma^* \_ \otimes \pi^* L^{1-2r} : \overline{\mathbb{J}}(X, d') \to \overline{\mathbb{J}}(X, -d'),$$

where $d' = r(2r - 1)\ell$.

0.3.4. $G = \mathbf{GSp}(2r, \mathbb{C})$. In this case, the datum of a $\mathrm{GSp}(2r, \mathbb{C})$-Higgs pair (P, ϕ) corresponds uniquely, via the associated bundle construction and the translation of the Higgs field, to the datum $(E, \Phi', \omega, M, \mu)$ of a Higgs pair (E, Φ') of rank $2r$ and degree rd, a non-degenerate symplectic form $\omega : E \otimes E \to M$ on E with values in a line bundle M of degree d, and a global global section $\mu \in H^0(C, L)$, such that $\Phi' \in H^0(C, \mathrm{End}(E) \otimes L)$ satisfies

$$\omega(\Phi' v, w) + \omega(v, \Phi' w) = 0.$$

The affine space $\mathcal{A}_{\mathrm{Sp}}(2r) \oplus H^0(C, L)$ can be taken as basis of a (translated) GSp-Hitchin morphism $\widetilde{\mathcal{H}}$, defined as:

$$\widetilde{\mathcal{H}}_{\mathrm{GSp}, 2r, rd} : \mathcal{M}_{\mathrm{GSp}}(2r, rd) \longrightarrow \mathcal{A}_{\mathrm{Sp}}(2r) \oplus H^0(C, L)$$

$$(P, \phi) \longmapsto (\underline{a}', \mu) = (a_2(E, \Phi'), a_4(E, \Phi'), \ldots, a_{2r}(E, \Phi'), \mu).$$

where (E, Φ) is the Higgs pair associated to (P, ϕ) via the associated bundle construction, μ is the global section $\frac{\mathrm{tr}\, \Phi}{2r} \in H^0(C, L)$ and $\Phi' = \Phi - \mu \, \mathrm{id}_E$ is the translated Higgs field.

PROPOSITION 0.3.7. *Let $\underline{a}' \in \mathcal{A}_{\mathrm{Sp}}(2r)$ be any characteristic and let $\mu \in H^0(C, L)$ be any section. Let $X = X_{\underline{a}'} \xrightarrow{\pi} C$ be the spectral curve associated to $\underline{a}'$, with involution $\sigma : X \to X$. Let $d' = rd + r(2r - 1)\ell$ and denote with $\mathcal{P}(d', n) = \overline{\mathbb{J}}(X, d') \times J(C, d)$ the Cartesian product of the Simpson Jacobian of degree d' on X and the Jacobian of degree d on C, endowed with the projection maps p_X and p_C on $\overline{\mathbb{J}}(X, d')$ and $J(C, d)$ respectively. Let $\mathcal{E}_{\underline{a}'}$ be the equalizer of the two maps*

$$(\mathcal{H}om_{\mathcal{O}_X}(\_, \mathcal{O}_X) \circ p_X) \otimes (\pi^* \circ p_C) : \mathcal{P}(d', d) \to \overline{\mathbb{J}}(X, rd - r(2r - 1)\ell)$$

$$(\sigma^* \circ p_X) \otimes \pi^* L^{1-2r} : \mathcal{P}(d', d) \to \overline{\mathbb{J}}(X, rd - r(2r - 1)\ell).$$

Then, the fiber $\widetilde{\mathcal{H}}_{\mathrm{GSp}, 2r, rd}^{-1}(\underline{a}', \mu)$ is isomorphic, via the spectral correspondence, to $\mathcal{E}_{\underline{a}'}$.

0.3.5. $G = \mathbf{PSp}(2r, \mathbb{C})$. In this case, any $\mathrm{PSp}(2r, \mathbb{C})$-Higgs pair (P, ϕ) has a lifting $(\tilde{P}, \tilde{\phi})$ to a $\mathrm{GSp}(r, \mathbb{C})$-Higgs pair, corresponding via the associated bundle construction to the datum (E, Φ, M, ω) of a Higgs pair (E, Φ) of rank $2r$ and degree rd, a non-degenerate symplectic form $\omega : E \otimes E \to M$ with values in a line bundle M of degree d on C, and a Higgs field Φ with trace zero satisfying the condition

$$\omega(\Phi v, w) + \omega(v, \Phi w) = 0.$$

Then, the datum of (P, ϕ) corresponds uniquely to the datum of the equivalence class $[(E, \Phi, M, \omega)]$ of Higgs pairs of rank $2r$ with a non-degenerate symplectic form, under the equivalence relation $\sim_{J(C)}$ defined by:

$$(E, \Phi, M, \omega) \sim_{J(C)} (E \otimes N, \Phi \otimes 1_N, M \otimes N^2, \omega_N) \qquad \text{for any } N \in J(C)$$

where

$$\omega_N : (E \otimes N) \otimes (E \otimes N) \to M \otimes N^2$$

is obtained by extension of scalars.

Let $\mathcal{M}_{\mathrm{GSp}}(2r) = \coprod_{d \in \mathbb{Z}} \mathcal{M}_{\mathrm{GSp}}(2r, rd)$ be the moduli stack of GSp-Higgs pairs of rank $2r$ in any degree and denote with $\mathcal{M}_{\mathrm{GSp}}^{\mathrm{tr}=0}(2r)$ the closed substack of $\mathcal{M}_{\mathrm{GSp}}(2r)$ given by GSp-Higgs pairs of rank $2r$ with trace zero. Then, $[(E, \Phi, M, \omega)]$ is the orbit of (E, Φ, M, ω) under the action of $J(C)$ on $\mathcal{M}_{\mathrm{GSp}}^{\mathrm{tr}=0}(2r)$ defined by:

$$\mathcal{M}_{\mathrm{GSp}}^{\mathrm{tr}=0}(2r) \times J(C) \longrightarrow \mathcal{M}_{\mathrm{GSp}}^{\mathrm{tr}=0}(2r)$$

$$((E, \Phi, M, \omega), N) \longmapsto (E \otimes N, \Phi \otimes 1_N, M \otimes N^2, \omega_N).$$

DEFINITION 0.3.8. Let (P, ϕ) be a PSp-Higgs pair and let (E, Φ, M, ω) be the datum of a Higgs pair with trace zero and degree rd endowed with a M-valued symplectic form ω, corresponding to a lifting $(\tilde{P}, \tilde{\phi})$ of (P, ϕ) to a GSp-Higgs pair. The *degree of* (P, ϕ) is the congruence class $\overline{rd} \in \mathbb{Z}/2r\mathbb{Z}$.

Up to the action on (E, Φ, M, ω) of a line bundle of degree 1 on C, it is straightforward to see that the $J(C)$-orbit $[(E, \Phi, M, \omega)]$ corresponding to a PGL-Higgs pair with degree $\overline{rd} \in \mathbb{Z}/2r\mathbb{Z}$ is always the orbit of a datum whose Higgs pair has degree 0 or rd. If we restrict the correspondence to Higgs pairs with fixed degree 0 or rd, we see that

a PGL-Higgs pair of degree $\overline{rd}$ is identified uniquely by the orbit of a Higgs pair with trace zero *and degree* 0 or rd with respect to the action of line bundles *of degree* 0 on C.

A basis for the invariant polynomials of $\mathfrak{psp}(2r, \mathbb{C}) = \mathfrak{sp}(2r, \mathbb{C})$ is given by $\{p_{2i}\}_{i=1,\ldots,r}$ where $p_{2i}(P, \phi) := \mathrm{tr}(\wedge^{2i}\phi)$. The corresponding PSp-Hitchin morphism takes the form:

$$\mathcal{H}_{\mathrm{PSp},2r,\overline{rd}} : \mathcal{M}_{\mathrm{PSp}}(2r, \overline{rd}) \longrightarrow \mathcal{A}_{\mathrm{PSp}}(2r) = \bigoplus_{i=1}^{r} H^0(C, L^{2l})$$

$$(P, \phi) \longmapsto (p_2(P, \phi), p_4(P, \phi), \ldots, p_{2r}(P, \phi)).$$

As in the case of $\mathrm{Sp}(2r, \mathbb{C})$-Higgs pairs, the curve $X_{\underline{a}}$ has an involution defined by $\sigma(x) = -x$.

PROPOSITION 0.3.9. *Let $\underline{a} \in \mathcal{A}_{\mathrm{PSp}}(2r)$ be any characteristic, let $X = X_{\underline{a}} \xrightarrow{\pi} C$ be the associated spectral curve with involution $\sigma : X \to X$. Let $d \in \{0, 1\}$, $d' = rd + r(2r - 1)\ell$ and denote with $\mathcal{P}(d', n) = \overline{\mathbb{J}}(X, d') \times J^d(C)$ the Cartesian product of the compactified Jacobian of degree d' on X and the Jacobian of degree d on C, endowed with the projection maps p_X and p_C on $\overline{\mathbb{J}}(X, d')$ and $J^d(C)$ respectively. Let $\mathcal{E}_{\underline{a}'}$ be the equalizer of the two maps*

$$(\mathcal{H}om_{\mathcal{O}_X}(\_, \mathcal{O}_X) \circ p_X) \otimes (\pi^* \circ p_C) : \mathcal{P}(d', d) \to \overline{\mathbb{J}}(X, rd - r(2r - 1)\ell)$$

$$(\sigma^* \circ p_X) \otimes \pi^* L^{1-2r} : \mathcal{P}(d', d) \to \overline{\mathbb{J}}(X, rd - r(2r - 1)\ell).$$

The group $J^0(C)$ of line bundles of degree 0 on C acts on $\mathcal{E}_{\underline{a}}$ as follows:

$$\mathcal{E}_{\underline{a}} \times J^0(C) \longrightarrow \mathcal{E}_{\underline{a}}$$

$$((\mathcal{M}, M, \lambda), N) \longmapsto (\mathcal{M} \otimes \pi^* N, M \otimes N^2, \lambda_N)$$

where λ_N is given by the composition of $\lambda \otimes \mathrm{id}_{\pi^* N}$ with the canonical isomorphisms:

$$(\mathcal{M} \otimes \pi^* N)^* \otimes \pi^*(M \otimes N^2) \xrightarrow{\sim} \mathcal{M}^* \otimes (\pi^* N)^{-1} \otimes \pi^* M \otimes (\pi^* N)^2 \xrightarrow{\sim}$$

$$\xrightarrow{\sim} \mathcal{M}^* \otimes \pi^* M \otimes \pi^* N \xrightarrow{\lambda \otimes \mathrm{id}_{\pi^* N}}$$

$$\xrightarrow{\sim} \sigma^* \mathcal{M} \otimes \pi^* L^{1-2r} \otimes \pi^* N \xrightarrow{\sim}$$

$$\xrightarrow{\sim} \sigma^*(\mathcal{M} \otimes \pi^* N) \otimes \pi^* L^{1-2r}.$$

Then, the fiber $\mathcal{H}^{-1}_{\mathrm{PSp},2r,\overline{rd}}(\underline{a})$ of the PSp-*Hitchin morphism is isomorphic, via the spectral correspondence, to the quotient $\mathcal{E}_{\underline{a}}/J^0(C)$.*

The case of $\mathrm{PSp}(2r,\mathbb{C})$-Higgs pairs of degree 0 deserves special attention.

PROPOSITION 0.3.10. *Let $\underline{a} \in \mathcal{A}_{\mathrm{PSp}}(2r)$ be any characteristic, let $X = X_{\underline{a}} \xrightarrow{\pi} C$ be the associated spectral curve with involution $\sigma : X \to X$. Let $d' = r(2r-1)\ell$ and let $\mathcal{E}_{\underline{a}}$ be the equalizer stack of the two maps:*

$$\_^* := \mathcal{H}om_{\mathcal{O}_X}(\_, \mathcal{O}_X) : \overline{\mathbb{J}}(X, d') \to \overline{\mathbb{J}}(X, -d')$$

$$\sigma^*\_ \otimes \pi^* L^{1-2r} : \overline{\mathbb{J}}(X, d') \to \overline{\mathbb{J}}(X, -d')$$

The group stack $J^0(C)[2]$ of 2-torsion line bundles on C acts on $\mathcal{E}_{\underline{a}}$ as follows:

$$\mathcal{E}_{\underline{a}} \times \pi^* J^0(C)[2] \longrightarrow \mathcal{E}_{\underline{a}}$$

$$((\mathcal{M}, \lambda), (\pi^*N, \pi^*\epsilon)) \longmapsto (\mathcal{M} \otimes \pi^*N, \lambda \otimes \pi^*\epsilon).$$

Then, the fiber $\mathcal{H}^{-1}_{\mathrm{PSp},2r,\overline{0}}(\underline{a})$ of the PSp-Hitchin morphism is isomorphic, via the spectral correspondence, to the quotient $\mathcal{E}_{\underline{a}}/\pi^* J^0(C)[2]$.

CHAPTER 1

Preliminaries

1.1. Rank and degree of coherent sheaves

Let X be any Noetherian scheme of pure dimension 1. In this section, we study classes of coherent sheaves on X for which the notions of rank on X are well defined. Moreover, we discuss the notions of degree for sheaves with well-defined rank.

The simplest class is clearly given by vector bundles or, equivalently, locally free sheaves of constant rank. If $\mathcal{F}$ is a locally free sheaf on X, we say that $\mathcal{F}$ has *(free) rank* r if $\mathcal{F}_x \simeq \mathcal{O}_{X,x}^{\oplus r}$ for any point x of X.

For any coherent sheaf $\mathcal{F}$ on X, we give now the following definitions.

DEFINITION 1.1.1. The *support* of $\mathcal{F}$ is the closed set

$$\mathrm{Supp}(\mathcal{F}) = \{x \in X | \mathcal{F}_x \neq 0\} \subseteq X.$$

The *dimension* of $\mathcal{F}$ is the dimension of its support and is denoted $\dim(\mathcal{F})$.

DEFINITION 1.1.2. The *torsion subsheaf* of $\mathcal{F}$ is the maximal subsheaf $T(\mathcal{F}) \subseteq \mathcal{F}$ of dimension 0. If $\mathcal{F} = T(\mathcal{F})$, we say that $\mathcal{F}$ is a *torsion sheaf*. If $T(\mathcal{F}) = 0$ we say that $\mathcal{F}$ is *torsion-free*.

Equivalently, $\mathcal{F}$ is torsion-free if $\dim(\mathcal{E}) = 1$ for any non-trivial coherent subsheaf $\mathcal{E} \subseteq \mathcal{F}$, i.e. $\mathcal{F}$ is pure of dimension 1. For any sheaf $\mathcal{F}$, the quotient sheaf $\mathcal{F}/T(\mathcal{F})$ is either zero or torsion-free.

The first class of torsion-free sheaves with well-defined rank is given by generalized vector bundles.

DEFINITION 1.1.3. $\mathcal{F}$ is a *generalized vector bundle* if it is torsion-free and there exists a positive integer r such that $\mathcal{F}_\xi \simeq \mathcal{O}_\xi^{\oplus r}$ for any generic point ξ of X, where $\mathcal{F}_\xi$ denotes the stalk of $\mathcal{F}$ at ξ. The integer

r is called the *rank* of the generalized vector bundle $\mathcal{F}$. A generalized vector bundle of rank 1 is also called a *generalized line bundle*.

Clearly, a vector bundle of rank r is also a generalized vector bundle of the same rank.

REMARK 1.1.4. Generalized line bundles were introduced in [**BE00**] in order to study limit linear series on a ribbon. As pointed out in [**EG95**], generalized line bundles are strictly related to generalized divisors introduced indipendently by Hartshone in [**Har86**] (followed by [**Har94**] and [**Har07**]). See also Chapter 2, 2.2 for details. The generalization to any rank r is due to [**Sav19**].

We introduce now the notion of rank for any coherent sheaf. Denote with $X_1, \ldots, X_s$ the irreducible components of X and with ξ_i the generic point of each irreducible component X_i. Denote with $C_i = X_{i,red}$ the reduced subscheme underlying X_i for each i.

DEFINITION 1.1.5. The *multiplicity* of C_i in X is defined as the positive integer:
$$\operatorname{mult}_X(C_i) := \ell_{\mathcal{O}_{X,\xi_i}}(\mathcal{O}_{X,\xi_i}).$$
where $\ell_{\mathcal{O}_{X,\xi_i}}$ denotes the length as $\mathcal{O}_{X,\xi_i}$-module.

DEFINITION 1.1.6. (Rank and multirank of a coherent sheaf) The *rank* of $\mathcal{F}$ on X_i is defined as the positive rational number
$$\operatorname{rk}_{X_i}(\mathcal{F}) = \frac{\ell_{\mathcal{O}_{X,\xi_i}}(\mathcal{F}_{\xi_i})}{\operatorname{mult}_X(C_i)}.$$

The *multirank* of $\mathcal{F}$ on X is the n-uple $\underline{r} = (r_1, \ldots, r_n)$ where r_i is the rank of $\mathcal{F}$ at X_i. The sheaf $\mathcal{F}$ has *rank* r on X if it has rank r at each X_i.

Note that, since length of modules is additive in short exact sequences, the rank and multirank of coherent sheaves are additive in short exact sequences too.

PROPOSITION 1.1.7. *A generalized vector bundle of rank r on X has rank r as a coherent sheaf. If X is reduced, any torsion-free sheaf of rank r is a generalized vector bundle of the same rank.*

PROOF. The first statement follows from 1.1.6 and the fact that isomorphic modules have the same length. For the second assertion, note that on an integral curve X there exists for any torsion-free sheaf $\mathcal{F}$ an open dense subset $U \subset X$ such that $\mathcal{F}_{|U}$ is locally free, hence the stalk of $\mathcal{F}$ at the generic point of X is a free $\mathcal{O}_{X,\xi}$-module of rank $\mathrm{rk}_X(\mathcal{F})$. If X is reducible, the assertion follows by considering any irreducible component. $\qquad\square$

REMARK 1.1.8. There are in literature other notions for the rank of a coherent sheaf on a irreducible component. A classical notion is the *reduced rank* of $\mathcal{F}$ on X_i:

$$\mathbf{rank}_{X_i}(\mathcal{F}) = \dim_{\kappa(\xi_i)}(\mathcal{F}_{\xi_i} \otimes_{\mathcal{O}_{X,\xi}} \kappa(\xi_i)).$$

This definition computes the rank of the sheaf restricted to its reduced support; it agrees with the other definitions for generalized vector bundles. However, there are cases when the reduced rank of a torsion-free sheaf differs from its rank, as it happens for quasi-locally free sheaves on a ribbon (see [**Sav19**, §1.4]). They also provide examples of torsion-free sheaves which are not generalized vector bundles, even if their rank is well defined.

The definition of $\mathrm{rk}_H(\mathcal{F})$ comes at least from [**Sch98**, Définition 1.2], where it is given in the context of projective k-schemes without embedded points. The notion of *generalized rank* introduced in [**Dre04**] and [**Sav19**] is essentially equivalent. Finally, a common definition in projective algebraic geometry is the polarized rank, that we will discuss later on in the present section.

In order to introduce the degree of coherent sheaves, from now on we **assume that X is projective curve over a base field** k. Recall that the Euler characteristic of a coherent sheaf $\mathcal{F}$ is

$$\chi(\mathcal{F}) := \sum (-)^i \dim_k H^i(X, \mathcal{F}).$$

DEFINITION 1.1.9. (Degree of a coherent sheaf) Suppose that $\mathcal{F}$ has rank r on X. Then, the *degree* of $\mathcal{F}$ on X is defined as the fractional number:

$$\deg \mathcal{F} = \chi(X, \mathcal{F}) - r\chi(X, \mathcal{O}_X).$$

REMARK 1.1.10. If the rank of $\mathcal{F}$ is integer, the degree is integer as well. If X is integral, the degree of a sheaf is an integer and coincides with the classical notion of degree for sheaves on integral curves.

The following technical lemma is very useful.

LEMMA 1.1.11. *Assume that X is irreducible. Let $\mathcal{F}$ be a coherent sheaf of rank r on X and let $\mathcal{E}$ be a locally free sheaf of rank n on X. Then*

$$\chi(X, \mathcal{F} \otimes \mathcal{E}) = r \deg(\mathcal{E}) + n \chi(X, \mathcal{F}).$$

PROOF. The proof is inspired by [**Sta19**, Tag 0AYV] and uses dévissage for coherent sheaves as stated in [**Sta19**, Tag 01YI]. Let $\mathcal{P}$ be the property of coherent sheaves $\mathcal{F}$ on X expressing that the formula of the Lemma holds. By additivity of rank and Euler characteristic in short exact sequences, $\mathcal{P}$ satisfy the two-out-of-three property. The integral subschemes Z of X are the reduced subscheme C with support equal to X and closed points. For $Z = C$, the formula of the Lemma for $\mathcal{O}_C$ is:

$$\chi(X, \mathcal{E} \otimes \mathcal{O}_C) = \frac{\deg(\mathcal{E})}{\mathrm{mult}_X C} + n \chi(X, \mathcal{O}_C).$$

This is true by definition of degree of $\mathcal{E} \otimes \mathcal{O}_C = \mathcal{E}_{|C}$ on C and the fact that $\deg(\mathcal{E}) = \mathrm{mult}_X(C) \deg(\mathcal{E}_{|C})$. Then $\mathcal{P}(\mathcal{O}_C)$ holds. If $i : Z \hookrightarrow X$ is a closed point, the formula of the Lemma is true for $i_*\mathcal{O}_Z$ since it is a torsion sheaf of rank 0 and $\chi(X, \mathcal{E} \otimes i_*\mathcal{O}_Z) = n \chi(X, i_*\mathcal{O}_Z)$. Then $\mathcal{P}(i_*\mathcal{O}_Z)$ holds. $\qquad\square$

Let H be any polarization of X; then, the notion of polarized rank and degree of sheaves can be defined. Denote with $H_{|C_i}$ the restriction of H to the closed subscheme $C_i \subseteq X$. Since H is locally free of rank 1, the degree of H on X is related to the degrees of each $H_{|C_i}$ on C_i by the formula:

$$\deg H = \sum_{i=1}^{s} \mathrm{mult}_X(C_i) \deg H_{|C_i}.$$

DEFINITION 1.1.12. (Polarized rank and degree of a torsion-free sheaf) Let H be any polarization of X with degree $\deg H = \delta$ and let $\mathcal{F}$ be a torsion-free sheaf on X. The *polarized rank and degree* of $\mathcal{F}$ are the rational numbers $r_H(\mathcal{F})$ and $d_H(\mathcal{F})$ determined by the Hilbert

polinomial of $\mathcal{F}$ with respect to H:

$$P(\mathcal{F}, n, H) := \chi(\mathcal{F} \otimes \mathcal{O}_X(nH)) = \delta r_H(\mathcal{F})n + d_H(\mathcal{F}) + r_H(\mathcal{F})\chi(\mathcal{O}_X).$$

The polarized rank and degree of a sheaf depend strictly on the degrees of the restrictions $H_{|C_i}$, as the following theorem shows.

THEOREM 1.1.13. *Let H be a polarization of X and let $\mathcal{F}$ be a torsion-free sheaf on X. The polarized rank of $\mathcal{F}$ is related to the multirank of $\mathcal{F}$ by the formula*

$$r_H(\mathcal{F}) = \frac{\sum_{i=1}^s \mathrm{rk}_{X_i}(\mathcal{F}) \, \mathrm{mult}_X(C_i) \deg H_{|C_i}}{\sum_{i=1}^s \mathrm{mult}_X(C_i) \deg H_{|C_i}}.$$

PROOF. Since the Hilbert polynomial of $\mathcal{F}$ with respect to H has degree 1, its leading term can be computed in terms of Euler characteristic as

$$\chi(X, \mathcal{F} \otimes \mathcal{O}_X(nH)) - \chi(X, \mathcal{F}).$$

Since $\delta = \sum_{i=1}^n \mathrm{mult}_X(C_i) \deg_{C_i} H$ is the degree of H on X, by Definition 1.1.12 we have to prove:

$$\chi(X, \mathcal{F} \otimes \mathcal{O}_X(nH)) - \chi(X, \mathcal{F}) = n \sum_{i=1}^s \mathrm{rk}_{X_i}(\mathcal{F}) \, \mathrm{mult}_X(C_i) \deg_{C_i} H.$$

First, we reduce to the case of X irreducible. Consider the exact sequence:

$$0 \to \mathcal{F} \to \bigoplus_i \mathcal{F}_{|X_i} \to T \to 0$$

where T is a torsion sheaf supported only at the intersections of the irreducible components. Tensoring by nH, we obtain another exact sequence:

$$0 \to \mathcal{F} \otimes \mathcal{O}_X(nH) \to \bigoplus_i (\mathcal{F} \otimes \mathcal{O}_X(nH))_{|X_i} \to T \otimes \mathcal{O}_X(nH) \to 0.$$

By additivity of the Euler characteristic with respect to exact sequences we have:

$$\sum_i \left(\chi(X_i, (\mathcal{F} \otimes \mathcal{O}_X(nH))_{|X_i}) - \chi(X_i, \mathcal{F}_{|X_i}) \right) =$$

$$= \chi(X, \mathcal{F} \otimes \mathcal{O}_X(nH)) - \chi(X, \mathcal{F}) + \chi(X, T \otimes \mathcal{O}_X(nH)) - \chi(X, T).$$

Since T has dimension 0, the characteristics $\chi(X, T \otimes \mathcal{O}_X(nH))$ and $\chi(X, T)$ are the same, hence:

$$\chi(X, \mathcal{F} \otimes \mathcal{O}_X(nH)) - \chi(X, \mathcal{F}) = \sum_i \left(\chi(X_i, (\mathcal{F} \otimes \mathcal{O}_X(nH))_{|X_i}) - \chi(X_i, \mathcal{F}_{|X_i}) \right).$$

We are then reduced to prove the statement on each irreducible component X_i. In other words we have to prove that, if X is a irreducible (possibly non-reduced) curve with reduced structure C, then:

$$\chi(X, \mathcal{F} \otimes \mathcal{O}_X(nH)) - \chi(X, \mathcal{F}) = n \deg_X(H) \operatorname{rk}_X(\mathcal{F}).$$

This is exactly the content of Lemma 1.1.11, with $\mathcal{E} = \mathcal{O}_X(nH)$. $\qquad \square$

COROLLARY 1.1.14. *Let $\mathcal{F}$ be any torsion-free sheaf with well-defined rank. Then, its polarized rank $r_H(\mathcal{F})$ and degree $d_H(\mathcal{F})$ are equal respectively to $\operatorname{rk}_X(\mathcal{F})$ and $\deg_X(\mathcal{F})$ for any polarization H on X. f X is irreducible, the rank and degree of a sheaf coincide with the polarized counterparts, for any polarization.*

PROOF. The statement about the rank follows from Theorem 1.1.13, where $r_i = r$ for each irreducible component X_i. To prove the statement about the degree, note that for any polarization H:

$$d_H(\mathcal{F}) = \chi(X, \mathcal{F}) - r_H(\mathcal{F}) \chi(X, \mathcal{O}_X).$$

If $r_H(\mathcal{F}) = \operatorname{rk}(\mathcal{F})$, then $d_H(\mathcal{F}) = \deg(\mathcal{F})$ by Definition 1.1.9. The last statement follows from the fact that the $\operatorname{rk}_X(\mathcal{F})$ is well-defined for any torsion-free sheaf when X has only one irreducible component. $\qquad \square$

REMARK 1.1.15. For reducible curves, the definitions of polarized rank and degree are the most general. As pointed out in [**Lop04**, §2], being of polarized rank 1 for a torsion-free sheaf does not ensure that the sheaf is supported on the whole curve. Moreover, there are cases of torsion-free sheaves of polarized rank 1 whose polarized rank on the restrictions to the irreducible components is different from 1 and not an integer, even for reduced curves.

1.2. Moduli spaces of torsion-free sheaves "of rank 1"

In this section, X is a (possibly reducible, non-reduced) projective curve over a base field k, with irreducible components $X_1, \ldots, X_s$. We introduce the definition of several moduli spaces parametrizing classes

of torsion-free sheaves "of rank 1" (with different meanings) that will play an important role in the present work. We start by recalling the definition of the moduli space parametrizing line bundles on X.

DEFINITION 1.2.1. Let $d \in \mathbb{Z}$ be an integer number. The *Jacobian scheme* of degree d on X is the algebraic scheme $J^d(X)$ representing the sheafification of the functor that associates to any k-scheme T the set of isomorphism classes of line bundles of degree d on $X \times_k T$. The union of the Jacobians of all degrees is denoted as $J(X)$.

REMARK 1.2.2. The existence of the representing scheme $J^d(X)$ is a classical result in algebraic geometry, see for example [**Kle05**, Corollary 4.18.3]. Following a number of authors (see for example [**MRV17a**] or [**Kas15**]) we prefer to use the term "Jacobian" instead of "Picard" even if we consider line bundles in any degree; this is motivated only by the simpler notation.

Since we are dealing with moduli spaces, we recall also the definition of slope and stability for torsion-free sheaves.

DEFINITION 1.2.3. Let H be a polarization on $\mathcal{F}$ and $\mathcal{F}$ be a torsion-free sheaf on X with polarized rank $r_H(\mathcal{F})$ and polarized degree $d_H(\mathcal{F})$.

The *slope* of $\mathcal{F}$ with respect to H is defined as the rational number $\mu_H(\mathcal{F}) = d_H(\mathcal{F})/r_H(\mathcal{F})$.

A coherent sheaf $\mathcal{F}$ on X is *H-stable* (respectively *H-semi-stable*) if it is torsion-free and for any proper subsheaf $\mathcal{E} \subset \mathcal{F}$ the equality $\mu_H(\mathcal{E}) < \mu_H(\mathcal{F})$ holds (respectively $\leq$).

We start from the more general moduli space, parametrizing for torsion-free sheaves of polarized rank 1.

DEFINITION/LEMMA 1.2.4. Let H be a polarization on X and let $d \in \mathbb{Z}$ be an integer number. The *Simpson Jacobian stack* of degree d on X is the algebraic stack $\overline{\mathbb{J}}_{tf}(X, H, d)$ such that for any k-scheme T, $\overline{\mathbb{J}}_{tf}(X, d)(T)$ is the groupoid of T-flat coherent sheaves on $X \times_k T$ whose fibers over T are torsion-free sheaves of polarized rank 1 and polarized degree d on $X \simeq X \times_k \{t\}$ with respect to H.

The *Simpson Jacobian* of degree d is the projective scheme $\overline{J}_{tf}(X, H, d)$ which is the good moduli space representing S-equivalence classes of H-semistable torsion-free sheaves of polarized rank 1 and polarized degree d on X.

The union of the Simpson Jacobians of all degrees is denoted $\overline{\mathbb{J}}_{tf}(X, H)$ (resp. $\overline{J}_{tf}(X, H)$).

PROOF. The algebraicity of $\overline{\mathbb{J}}_{tf}(X, H, d)$ follows from the algebraicity of the stack of coherent sheaves on X (see [**Lie06**, 2.1] and [**CW17**, Theorem 7.20]). The moduli space of semistable sheaves with fixed polarized rank and degree was constructed by Simpson in [**Sim94a**]. $\square$

In the following proposition, we show that the multirank of sheaves splits $\overline{\mathbb{J}}_{tf}(X, H, d)$ in unions of connected components.

LEMMA 1.2.5. *Let H be a fixed polarization on X. For any $\underline{r} = (r_1, \ldots, r_s) \in \mathbb{Q}_{\geq 0}^s$, let $W_{\underline{r}} \subseteq \overline{\mathbb{J}}_{tf}(X, H, d)$ be the substack parametrizing torsion-free sheaves with multirank equal to $\underline{r}$. Then $\{W_{\underline{r}} | W_{\underline{r}} \neq \emptyset\}_{\underline{r} \in \mathbb{Q}_{\geq 0}^s}$ is a finite collection of pairwise disjoint substacks of $\overline{\mathbb{J}}_{tf}(X, H, d)$ that covers the whole space. Moreover, each $W_{\underline{r}}$ is open and closed, hence union of connected components of $\overline{\mathbb{J}}_{tf}(X, H, d)$.*

PROOF. The multi-rank of a sheaf is well-defined, hence the substacks $W_{\underline{r}}$ are pairwise disjoint.

Let $\mathcal{F}$ be a sheaf with polarized rank r_H equal to 1. By Theorem 1.1.13 its multirank $\underline{r}$ must satisfy:

$$1 = r_H(\mathcal{F}) = \frac{\sum_{i=1}^s r_i(t) \operatorname{mult}_X(C_i) \deg_{C_i} H}{\sum_{i=1}^n \operatorname{mult}_X(C_i) \deg_{C_i} H}.$$

Since each r_i is a non-negative fraction with integer numerator and denominator equal to $\operatorname{mult}_X(C_i)$, the possible values for $\underline{r}$ form a finite subset of $\mathbb{Q}_{\geq 0}^s$, once that H and r_H are fixed. Then, only a finite number of $W_{\underline{r}}$ is non-empty, hence $\{W_{\underline{r}} | W_{\underline{r}} \neq \emptyset\}$ is a finite collection.

We claim that each $W_{\underline{r}}$ is closed; since the collection is finite and the $W_{\underline{r}}$'s are pairwise disjoint, this implies that each $W_{\underline{r}}$ is open and closed, and hence union of connected components.

To prove that $W_{\underline{r}}$ is closed, let T be any k-scheme and let $\mathcal{F}$ be any family of torsion-free sheaves of polarized rank 1 on $X \times_k T \xrightarrow{\pi} T$. Suppose that $\mathcal{F}$ has generically multirank $\underline{r}$ on X, meaning that

$\mathrm{rk}_{X_i}(\mathcal{F}_{|\pi^{-1}(\eta)}) = r_i$ for each irreducible component X_i of X and any generic point η of T. Let t be any point in the closure of $\{\eta\}$. Since the length is upper-semicontinuous, the rank of $\mathcal{F}_{|\pi^{-1}(t)}$ at each X_i is a rational number $r_i(t)$ greater or equal than r_i. On the other hand, $\mathcal{F}_{|\pi^{-1}(t)}$ have polarized rank 1; hence, thanks to Theorem 1.1.13, we can write:

$$1 = r_H(\mathcal{F}) = \frac{\sum_{i=1}^s r_i(t) \, \mathrm{mult}_X(C_i) \deg_{C_i} H}{\sum_{i=1}^n \mathrm{mult}_X(C_i) \deg_{C_i} H} \geq$$
$$\geq \frac{\sum_{i=1}^s r_i \, \mathrm{mult}_X(C_i) \deg_{C_i} H}{\sum_{i=1}^n \mathrm{mult}_X(C_i) \deg_{C_i} H} = 1.$$

By difference we obtain:

$$\frac{\sum_{i=1}^s (r_i(t) - r_i) \, \mathrm{mult}_X(C_i) \deg_{C_i} H}{\sum_{i=1}^n \mathrm{mult}_X(C_i) \deg_{C_i} H} = 0.$$

Since all the terms of the expression are non-negative, the only possibility is that $r_i(t) = r_i$ for each i. We conclude that the whole family $\mathcal{F}$ belongs to $W_{\underline{r}}$. $\qquad\square$

We come now to torsion-free sheaves with well-defined rank 1 on X.

DEFINITION/LEMMA 1.2.6. Let $d \in \mathbb{Z}$ be an integer number. The *compactified Jacobian stack* of degree d on X is the algebraic stack $\overline{\mathbb{J}}(X, d)$ such that for any k-scheme T, $\overline{\mathbb{J}}(X, d)(T)$ is the groupoid of T-flat coherent sheaves on $X \times_k T$ whose fibers over T are torsion-free sheaves of rank 1 and degree d on $X \simeq X \times_k \{t\}$.

Let H be a polarization on X. The *compactified Jacobian scheme* of degree d is the good moduli space $\overline{J}(X, H, d)$ representing S-equivalence classes of H-semistable torsion-free sheaves of rank 1 and degree d on X.

The union of the compactified Jacobians of all degrees is denoted $\overline{\mathbb{J}}(X)$ (resp. $\overline{J}(X, H)$).

PROOF. The property of having rank 1 on X implies that the polarized rank is equal to 1 for any polarization; moreover, the degree is well defined and equal to the polarized degree for any polarization. Fix any polarization H on X. Then, $\overline{\mathbb{J}}(X, d)$ can be seen as the open and closed substack $W_{(1,\ldots,1)}$ of $\overline{\mathbb{J}}_{tf}(X, H, d)$ (Lemma 1.2.5), and hence

is algebraic. The same argument holds for $\overline{J}(X, H, d)$, with the difference that the notion of semi-stability strictly depends on the choice of H. □

The following proposition shows that $\overline{J}(X, H, d)$ is actually a compactification of $J^d(X)$.

PROPOSITION 1.2.7. *For any polarization H on X, $\overline{J}(X, H, d)$ is a projective scheme, union of connected components of $\overline{J}_{tf}(X, H, d)$, and contains $J^d(X)$ as an open subscheme.*

PROOF. Since $\overline{\mathbb{J}}(X, d)$ is an open and closed substack of $\overline{\mathbb{J}}_{tf}(X, H, d)$, then $\overline{J}(X, H, d)$ is an open and closed subscheme of $\overline{J}_{tf}(X, H, d)$ and union of connected components. It is non-empty, as it contains the structure sheaf $\mathcal{O}_X$. Moreover $\overline{J}_{tf}(X, H, d)$ is projective, hence $\overline{J}(X, H, d)$ is projective too. Finally, the condition of being locally free is open and implies both stability and rank 1; hence, $J^d(X)$ is an open subscheme of $\overline{J}(X, H, d)$. □

We come now to the last moduli space, which parametrizes generalized line bundles.

DEFINITION/LEMMA 1.2.8. Let $d \in \mathbb{Z}$ be an integer number. The *generalized Jacobian stack* of degree d on X is the algebraic stack $G\mathbb{J}(X, d)$ such that, for any k-scheme T, $G\mathbb{J}(X, d)(T)$ is the groupoid of T-flat coherent sheaves on $X \times_k T$ whose fibers over T are generalized line bundles of degree d on $X \simeq X \times_k \{t\}$.
Let H be a polarization on X. The *generalized Jacobian scheme* of degree d is the good moduli space $GJ(X, H, d)$ representing S-equivalence classes of H-semistable generalized line bundles of degree d on X.
The union of the generalized Jacobians of all degrees is denoted $G\mathbb{J}(X)$ (resp. $GJ(X, H)$).

PROOF. Generalized line bundles are torsion-free sheaves of rank 1; moreover, the condition of being generically locally free is open, hence $G\mathbb{J}(X, d)$ can be seen as an open substack of $\overline{\mathbb{J}}(X, d)$. In particular, $G\mathbb{J}(X, d)$ is an algebraic stack. Similarly, $GJ(X, H, d)$ is an open subscheme of $\overline{J}(X, H, d)$. □

To sum up, we defined a number of moduli spaces satisfying the following chain of inclusions:

$$J^d(X) \overset{(1)}{\subseteq} G\mathbb{J}(X,d) \overset{(2)}{\subseteq} \overline{\mathbb{J}}(X,d) \overset{(3)}{\subseteq} \overline{\mathbb{J}}_{tf}(X,H,d),$$

and:

$$J^d(X) \overset{(1')}{\subseteq} GJ(X,H,d) \overset{(2')}{\subseteq} \overline{J}(X,H,d) \overset{(3')}{\subseteq} \overline{J}_{tf}(X,H,d).$$

REMARK 1.2.9. Inclusions (1) and (2) (resp. (1′) and (2′)) above are open embeddings; inclusion (3) (resp. (3′)) is an open and closed embedding. In general they are strict and not dense, in the sense that for any of them there exists a (possibly non-reduced or reducible) curve X such that the closure of the smaller space in the bigger one is contained strictly. Anyway, when X satisfies additional conditions, some of the inclusions above are actually equalities.

- If X is irreducible, inclusion (3) (resp. (3)′) is an equality.
- If X is reduced, inclusion (2) (resp. (2′)) is an equality.
- If X is integral and smooth, inclusions (1), (2) and (3) (resp. (1′), (2′) and (3′)) are equalities.

1.3. Moduli spaces of Higgs pairs

In this section, following [**Sch14**, Chapter 2] and [**MRV17b**, §10], we review the definitions of Higgs pairs and G-Higgs pairs and of the fibrations involving their moduli spaces. Throughout this section, C denotes a fixed smooth curve over the field of complex numbers and L a fixed line bundle on C with degree $\ell = \deg L$.

1.3.1. Classical Higgs pairs. We begin with the case of classical Higgs pairs.

DEFINITION 1.3.1. A *(L-twisted) Higgs pair* (or simply a *Higgs pair*) is a pair (E, Φ) consisting of a vector bundle E on C and a section $\Phi \in H^0(\Sigma, \mathrm{End}(E) \otimes L)$. The map Φ is called the *Higgs field*. The degree and the rank of the Higgs pair are the degree and the rank of the underlying bundle E.

DEFINITION 1.3.2. The algebraic stack $\mathcal{M}(r,d)$ of all L-twisted Higgs pairs (E, Φ) on C of rank r and degree d is endowed with a

morphism, called the *Hitchin morphism*, defined as:

$$\mathcal{H}_{r,d} = \mathcal{H} : \mathcal{M}(r,d) \to \mathcal{A}(r) = \bigoplus_{i=1}^{r} H^0(C, L^i)$$

$$(E, \Phi) \mapsto (a_1(E, \Phi), \dots, a_r(E, \Phi))$$

where $L^i = L^{\otimes i}$, $a_i(E, \Phi) := (-1)^i \operatorname{tr}(\wedge^i \Phi)$ and $\mathcal{A} = \mathcal{A}(r)$ is called the *Hitchin base*.

DEFINITION 1.3.3. A vector subbundle $F \subseteq E$ such that $\Phi(F) \subseteq F \otimes L$ is said to be a Φ-*invariant subbundle* of E.

A Higgs pair (E, Φ) is said *stable* (respectively *semi-stable*) if for each proper Φ-invariant subbundle $F \subset E$ one has $\mu(F) < \mu(E)$ (respectively $\mu(F) \leq \mu(E)$).

DEFINITION 1.3.4. The good moduli space $M(r,d)$ of S-equivalence classes of semistable L-twisted Higgs pairs of rank r and degree d is endowed with a flat projective morphism, called the *Hitchin fibration*:

$$H_{r,d} = H : M(r,d) = M \to \mathcal{A} = \mathcal{A}(r) = \bigoplus_{i=1}^{r} H^0(C, L^i)$$

$$(E, \Phi) \mapsto (a_1(E, \Phi), \dots, a_r(E, \Phi)).$$

1.3.2. G-Higgs pairs. Let G be a complex reductive Lie group.

DEFINITION 1.3.5. A *(L-twisted) G-Higgs pair* is a pair (P, ϕ) where P is a principal G-bundle over C and the G-Higgs field ϕ is a section of $H^0(\operatorname{ad}(P) \otimes L)$.

RECALL 1.3.6. When $G \overset{\rho}{\hookrightarrow} \operatorname{GL}(r, \mathbb{C})$, to any G-principal bundle P is associated a vector bundle of rank r, defined as:

$$E := P \times_\rho \mathbb{C}^r = (P \times \mathbb{C}^r)/ \sim$$

where $(s \cdot g, v) \sim (s, \rho(g) \cdot v)$ for any section s of P and any vector $v \in \mathbb{C}^r$. In terms of cocycles, if $\{U_\alpha\}_A$ is a trivializing cover for P on C and $\{g_{\alpha\beta}\}$ is a collection of transition elements, then E is the vector bundle defined by the $\rho(g_{\alpha\beta})$ as cocycles.

When, $G = \operatorname{GL}(r, \mathbb{C})$ the construction can be reversed; indeed, given any vector bundle E of rank r on C, the frame bundle of all ordered basis $P = \operatorname{Fr}(E)$ is a $\operatorname{GL}(r, \mathbb{C})$-principal bundle such that $E \simeq P \times_\rho \mathbb{C}^r$.

RECALL 1.3.7. Let P be any G-principal bundle and $E = P \times_\rho \mathbb{C}^r$ as above. The adjoint bundle $\mathrm{ad}(P)$ is defined as $P \times_{\mathrm{ad}_G} \mathfrak{g}$ where $\mathfrak{g} = \mathrm{Lie}(G)$ is the Lie algebra associated to the Lie group G and $\mathrm{ad}_G : G \to \mathrm{Aut}(\mathfrak{g})$ is the adjoint representation. If $G \overset{\rho}{\hookrightarrow} \mathrm{GL}(r, \mathbb{C})$, then the embedding ρ induces at the level of Lie algebras a G-equivariant map $\psi := \mathrm{ad}\,\rho : \mathfrak{g} \hookrightarrow \mathfrak{gl}(r, \mathbb{C}) = \mathrm{End}(\mathbb{C}^r)$; hence, we obtain a morphism

$$\mathrm{ad}\, P = P \times_{\mathrm{ad}_G} \mathfrak{g} \xrightarrow{1 \times \psi} P \times_{\psi(\mathrm{ad}_G)} \mathrm{End}(\mathbb{C}^r) = \mathrm{End}(E).$$

When $G = \mathrm{GL}(r, \mathbb{C})$, the adjoint bundle $\mathrm{ad}\, P$ is canonically isomorphic to $\mathrm{End}(E)$.

The two remarks above allows us to formulate the following:

PROPOSITION 1.3.8. *Let $G \overset{\rho}{\hookrightarrow} \mathrm{GL}(r, \mathbb{C})$ be a complex reductive Lie group emebedding in $\mathrm{GL}(r, \mathbb{C})$. Then, any G-Higgs pair (P, ϕ) gives rise via the associated bundle construction to a classical Higgs pair (E, Φ) of rank r, where $E = P \times_\rho \mathbb{C}^r$ and Φ is the image of ϕ with respect to the morphism $\mathrm{ad}(P) \to \mathrm{End}(E)$ induced by ρ.*

When $G = \mathrm{GL}(r, \mathbb{C})$, G-Higgs pairs are in one-to-one correspondence, via the associated bundle construction, with classical Higgs pairs of rank r.

Similarly to Higgs pairs, the moduli space of G-Higgs pairs admits a morphism to an affine space, which plays an important role in its study.

DEFINITION 1.3.9. Let G be a complex reductive Lie group, let $\delta \in \pi_1(G)$ and let $p_1, \ldots, p_k$ be a homogeneous basis for the algebra of invariant polynomials of the Lie algebra $\mathfrak{g}$ of G, such that p_i has degree d_i. Then, the algebraic moduli stack $\mathcal{M}_G(\delta)$ of all L-twisted G-Higgs pairs (P, ϕ) such that P has topological type δ is endowed with a morphism, called the *G-Hitchin morphism*, defined as:

$$\mathcal{H}_{G,\delta}^{p_1,\ldots,p_k} = \mathcal{H}_{G,\delta} : \mathcal{M}_G(\delta) \to \mathcal{A}_G = \bigoplus_{i=1}^k H^0(C, L^{d_i})$$

$$(P, \phi) \mapsto (p_1(\phi), \ldots, p_k(\phi)).$$

When $G \overset{\rho}{\hookrightarrow} \mathrm{GL}(r, \mathbb{C})$, the associated bundle construction induces a morphism of algebraic stack:

$$\mathcal{M}_G(\delta) \overset{\alpha}{\to} \mathcal{M}(r, d).$$

where d corresponds to δ by the canonical map $\pi_1(G) \xrightarrow{\pi_1(\rho)} \pi_1(\mathrm{GL}(r,\mathbb{C}))$.

DEFINITION 1.3.10. Let $G \xhookrightarrow{\rho} \mathrm{GL}(r,\mathbb{C})$. We denote as

$$\mathcal{M}_G(r,d) := \coprod_{\pi_1(\rho)(\delta)=d} \mathcal{M}_G(\delta)$$

the moduli space of G-Higgs pairs whose associated vector bundle is of rank r and degree d. The G-Hitchin morphism on $\mathcal{M}_G(r,d)$ is denoted as $\mathcal{H}_{G,r,d}$. When $G = \mathrm{GL}(r,\mathbb{C})$, the moduli space of GL-Higgs pairs $\mathcal{M}_{\mathrm{GL}}(r,d)$ is isomorphic via the associated bundle construction to the moduli space $\mathcal{M}(r,d)$ of classical Higgs pairs of rank r and degree d.

REMARK 1.3.11. For classical groups like SL, Sp, SO, the basis $p_1,\ldots,p_k$ can be chosen such that there is a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{M}_G(r,d) & \xrightarrow{\ \ \alpha\ \ } & \mathcal{M}(r,d) \\
\downarrow{\scriptstyle \mathcal{H}_{G,r,d}} & & \downarrow{\scriptstyle \mathcal{H}_{r,d}} \\
\mathcal{A}_G(r) & \subseteq & \mathcal{A}(r).
\end{array}
$$

Finally, the notion of stability can be defined for G-Higgs pairs, such that it is compatible with the notion of stability for the associated Higgs pairs when $G \subseteq \mathrm{GL}(r,\mathbb{C})$ (see [**Sch13**, Definition 2.4] for details). Then, we have the following:

DEFINITION 1.3.12. Let G be a complex reductive Lie group, let $\delta \in \pi_1(G)$ and denote with p_i for $i = 1,\ldots,k$ a homogeneous basis for the algebra of invariant polynomials of the Lie algebra $\mathfrak{g}$ of G, with degrees d_i. The good moduli space $M_G(\delta)$ of S-equivalence classes of semistable G-Higgs pairs (P,ϕ) such that P has topological type δ is endowed with a flat projective morphism, called the G-Hitchin fibration:

$$H_{G,\delta}^{p_1,\ldots,p_k} = H_{G,\delta} : M_G(\delta) \to \mathcal{A}_G = \bigoplus_{i=1}^k H^0(C, L^{d_i})$$

$$(P,\phi) \mapsto (p_1(\phi),\ldots,p_k(\phi)).$$

1.4. The spectral correspondence for Higgs pairs

In this section, following [**MRV17b**], [**CL16**] and [**dC17**], we resume known facts about the spectral correspondence, which describes

any fiber of the Hitchin morphism $\mathcal{H}$ in terms of torsion-free rank-1 sheaves on an associated spectral curve.

Let $\underline{a} \in \mathcal{A}(r)$ be any characteristic. The spectral curve $X_{\underline{a}} \xrightarrow{\pi_{\underline{a}}} C$ is the projective scheme defined in the total space of L, $P = \mathbb{P}(\mathcal{O}_C \oplus L^{-1}) \xrightarrow{p} C$, by the homogeneous equation

$$x^r + p^*(a_1)x^{r-1}y + \cdots + p^*(a_r)y^r = 0$$

where x is the section of $\mathcal{O}_P(1) \otimes p^*(L)$ whose pushforward via p corresponds to the constant section $(0,1)$ of $L \oplus \mathcal{O}_C$ and y is the section of $\mathcal{O}_P(1)$ whose pushforward via p corresponds to the constant section $(1,0)$ of $\mathcal{O}_C \oplus L^{-1}$. Note that the restriction of y to $X_{\underline{a}}$ is everywhere non-zero and hence the restriction of $\mathcal{O}_P(1)$ to $X_{\underline{a}}$ is trivial. Moreover, the restriction of x to $X_{\underline{a}}$ can be considered as a section of $\pi_{\underline{a}}^*(L)$.

We can now compute the canonical sheaf of $X_{\underline{a}}$.

LEMMA 1.4.1. (Canonical sheaf of the spectral curve) *The canonical sheaf of $X_{\underline{a}}$ is equal to*

$$\omega_{X_{\underline{a}}} = \pi_{\underline{a}}^*(\omega_c \otimes L^{r-1}).$$

PROOF. By [**Har77**, Chap. V, Lemma 2.10] the canonical sheaf of P is equal to

$$\omega_P = \mathcal{O}_P(-2) \otimes p^*(\omega_C \otimes L^{-1})$$

The canonical sheaf of $X_{\underline{a}}$ can be computed with the adjunction formula

$$\omega_{X_{\underline{a}}} = (\omega_P(X_{\underline{a}}))_{|X_{\underline{a}}} = \left(\mathcal{O}_P(r-2) + p^*(\omega_c \otimes L^{r-1})\right)_{|X_{\underline{a}}}.$$

Since $\mathcal{O}_P(1)_{|X_{\underline{a}}}$ is trivial, we conclude that

$$\omega_{X_{\underline{a}}} = \pi_{\underline{a}}^*(\omega_c \otimes L^{r-1}).$$

$\square$

COROLLARY 1.4.2. *Denote with g the genus of C and with ℓ the degree of L on C. Then*

$$\chi(X_{\underline{a}}) = r(1-g) - \frac{r(r-1)}{2}\ell.$$

PROOF. The pullback of a line bundle by a finite map has degree equal to the degree of the bundle times the degree of the morphism.

Then

$$\deg(\omega_{X_{\underline{a}}}) = r \deg(\omega_C) + r(r-1) \deg(L) = 2r(g-1) + r(r-1)\ell.$$

The statement follows then by the fact that $\chi(X_{\underline{a}}) = -\deg(\omega_{X_{\underline{a}}})/2.$ $\quad\square$

REMARK 1.4.3. The spectral curve $X_{\underline{a}}$ has pure dimension 1 and is embedded in the smooth surface P, hence it has only locally planar singularities. The Hitchin base $\mathcal{A}$ admits three notable open subsets $\mathcal{A}^{\mathrm{sm}} \subseteq \mathcal{A}^{\mathrm{ell}} \subseteq \mathcal{A}^{\mathrm{reg}} \subseteq \mathcal{A}$, called respectively the smooth locus, the elliptic locus and the regular locus, defined as:

$$\mathcal{A}^{\mathrm{sm}} := \{\underline{a} \in \mathcal{A} : X_{\underline{a}} \text{ is smooth and connected}\},$$
$$\mathcal{A}^{\mathrm{ell}} := \{\underline{a} \in \mathcal{A} : X_{\underline{a}} \text{ is integral}\},$$
$$\mathcal{A}^{\mathrm{reg}} := \{\underline{a} \in \mathcal{A} : X_{\underline{a}} \text{ is reduced and connected}\}.$$

We come now to the spectral correspondence for Higgs pairs. The following spectral correspondence is due to [**Sch98**], [**CL16**] and [**dC17**], as a generalization of the spectral correspondence stated in [**BNR89**] for smooth characteristics.

PROPOSITION 1.4.4. (Spectral correspondence) *Let $\underline{a} \in \mathcal{A}(r)$ be any characteristic and let $X_{\underline{a}} \xrightarrow{\pi_{\underline{a}}} C$ be the associated spectral curve. Let $\mathcal{O}_C(1)$ be an ample line bundle on C; let $\mathcal{O}_{X_{\underline{a}}}(1) := \pi_{\underline{a}}^*(\mathcal{O}_C(1))$ be the ample line bundle on $X_{\underline{a}}$ obtained by pullback and denote with H the associated polarization on $X_{\underline{a}}$.*

(1) For any integer d, there is an isomorphism of stacks:

$$\mathcal{H}_{r,d}^{-1}(\underline{a}) \xrightarrow[\Pi]{\sim} \overline{\mathbb{J}}(X_{\underline{a}}, d'),$$

where $d' = d + r(1-g) - \chi(\mathcal{O}_{X_{\underline{a}}}) = d + \frac{r(r-1)}{2}\ell$ and $g = g(C)$ is the genus of C. If $\mathcal{M}$ is a torsion-free sheaf of rank 1 on $X_{\underline{a}}$, then $\Pi(\mathcal{M}) := (E, \Phi)$ where

$$E = \pi_{\underline{a},*}(\mathcal{M})$$

$$\Phi = \pi_{\underline{a},*}(\cdot x) : \pi_{\underline{a},*}(\mathcal{M}) \to L \otimes \pi_{\underline{a},*}(\mathcal{M}) \simeq \pi_{\underline{a},*}(\pi_{\underline{a}}^* L \otimes \mathcal{M}).$$

(2) A torsion-free sheaf of rank 1 $\mathcal{M}$ on $X_{\underline{a}}$ is H-semistable if and only if the associated Higgs pair $\Pi(\mathcal{M})$ is semistable on C. Hence, the above correspondence yields an isomorphism of

schemes:

$$H_{r,d}^{-1}(\underline{a}) \simeq \overline{J}(X_{\underline{a}}, H, d').$$

Finally, in the following proposition, inspired by [**BNR89**, Remark 3.7], we show that the torsion-free sheaf $\mathcal{M}$ corresponding to an Higgs pair (E, Φ) can be always resumed, up to twisting, as the eigenspace of $\pi^*\Phi$ with eigenvalue x.

PROPOSITION 1.4.5. *Let $\underline{a} \in \mathcal{A}(r)$ be any characteristic and let $X_{\underline{a}} \xrightarrow{\pi_{\underline{a}}} C$ be the associated spectral curve. Let $\mathcal{M} \in \overline{\mathbb{J}}(X_{\underline{a}}, d')$ be a torsion-free rank-1 sheaf on $X_{\underline{a}}$ corresponding to the Higgs pair $(E, \Phi) \in \mathcal{H}_{r,d}^{-1}(\underline{a})$ on C. Then, the following exact sequence holds:*

$$0 \to \mathcal{M} \otimes \pi_{\underline{a}}^*(L^{1-r}) \to \pi_{\underline{a}}^* E \xrightarrow{\pi_{\underline{a}}^*(\Phi) - x} \pi_{\underline{a}}^* E \otimes \pi_{\underline{a}}^* L \xrightarrow{\text{ev}} \mathcal{M} \otimes \pi_{\underline{a}}^* L \to 0$$

where ev *is induced by the evaluation map $\pi_{\underline{a}}^* \pi_{\underline{a},*}(\mathcal{M}) \to \mathcal{M}$.*

PROOF. We first prove the exact sequence locally. Let $U = \operatorname{Spec} R \subseteq C$ be any open affine subset where L is trivial. The preimage $V = \pi^{-1}(U)$ is then equal to $\operatorname{Spec}(R[x]/P(x))$ where $P(x) = x^r + a_1 x^{r-1} + \cdots + a_{r-1} x + a_r$ and the a_i's denote (with a slight abuse of notation) the elements in R corresponding to the entries of $\underline{a}$ after choosing the trivialization of L. Denote $R[x]/P(x) = S$ and note that the inclusion $R \subset S$ makes S into an R-module with basis $1, x, \ldots, x^{r-1}$.

Recall that $E = \pi_{\underline{a},*}(\mathcal{M})$ and $\Phi = \pi_{\underline{a},*}(\cdot x)$. Denote with M the S-module corresponding to $\mathcal{M}$ on $V = \operatorname{Spec} S$. The pushforward $\pi_{\underline{a},*}(\mathcal{M})$ corresponds on V to the restriction of scalars of M from S to R. Then, $\pi_{\underline{a}}^* E$ correspond on V to the S-module $S \otimes_R M$, with the structure of S-module given by multiplication on the left-hand side of the tensor product. On the other hand, any element $p(x) \in S$ acts on $S \otimes_R M$ also on the right-hand side. We denote such action as

$$\overline{p(x)} : S \otimes_R M \longrightarrow S \otimes_R M$$
$$q(x) \otimes m \longmapsto q(x) \otimes p(x) \cdot m.$$

Regarding the evaluation map, it can be defined at the level of S-modules as

$$\text{ev} : S \otimes_R M \longrightarrow M$$
$$p(x) \otimes m \longmapsto p(x) \cdot m.$$

With this notation, we are left to prove that the following exact sequence of S-modules holds:

$$0 \to M \to S \otimes_R M \xrightarrow{\Psi = \overline{x} - x} S \otimes_R M \xrightarrow{\text{ev}} M \to 0.$$

The proof of the exactness on the right follows exactly the proof of [**Bou89**, III 10, Proposition 18], with the difference that $S \otimes_R M$ has a finite S-basis.

To prove the exactness on the left, first consider the following morphism of R-modules:

$$Q = \sum_{i=0}^{r-1} x^i \sum_{j=0}^{r-1-i} \overline{x}^{r-1-i-j} \overline{a}_j : M \longrightarrow S \otimes_R M$$

$$m \longmapsto \sum_{i=0}^{r-1} x^i \otimes \left(\sum_{j=0}^{r-1-i} (x^{r-1-i-j} a_j) \cdot m \right)$$

where we put $a_0 = 1$. We prove first that $\Psi \circ Q = 0$. Let $m \in M$ be any element. If we compute $x \cdot Q(m)$, recalling that $x^r = -\sum_{i=0}^{r-1} a_{r-i} x^i$, we obtain

$$x \cdot Q(m) = x \cdot \left(\sum_{i=0}^{r-1} x^i \otimes \left(\sum_{j=0}^{r-1-i} (x^{r-1-i-j} a_j) \cdot m \right) \right) =$$

$$= \sum_{i=0}^{r-2} x^{i+1} \otimes \left(\sum_{j=0}^{r-1-i} (x^{r-1-i-j} a_j) \cdot m \right) - \sum_{i=0}^{r-1} \left(a_{r-i} x^i \right) \otimes m.$$

Changing $i+1$ with i in the first sum and moving the a_{r-i} by linearity, we obtain

$$x \cdot Q(m) = \sum_{i=1}^{r-1} x^i \otimes \left(\sum_{j=0}^{r-i} (x^{r-i-j} a_j) \cdot m \right) - \sum_{i=0}^{r-1} x^i \otimes (a_{r-i} \cdot m) =$$

$$= \sum_{i=1}^{r-1} x^i \otimes \left(\sum_{j=0}^{r-i-1} x^{r-i-j} a_j + a_{r-i} \right) \cdot m - 1 \otimes (a_r \cdot m) =$$

$$= \sum_{i=1}^{r-1} x^i \otimes \left(\sum_{j=0}^{r-i-1} (x^{r-i-j} a_j) \cdot m \right) - 1 \otimes (a_r \cdot m).$$

Finally, recalling that $-a_r = \sum_{j=0}^{r-1} x^{r-j} a_j$, we can write

$$x \cdot Q(m) = \sum_{i=0}^{r-1} x^i \otimes \left(\sum_{j=0}^{r-i-1} (x^{r-i-j} a_j) \cdot m \right).$$

On the other hand, the computation of $\overline{x}(Q(m))$ gives

$$\overline{x}(Q(m)) = \overline{x} \left(\sum_{i=0}^{r-1} x^i \otimes \left(\sum_{j=0}^{r-1-i} (x^{r-1-i-j} a_j) \cdot m \right) \right) =$$

$$= \sum_{i=0}^{r-1} x^i \otimes \left(\sum_{j=0}^{r-1-i} (x^{r-i-j} a_j) \cdot m \right).$$

We conclude that $\Psi(Q(m)) = \overline{x}(Q(m)) - x \cdot Q(m) = 0$ for arbitrary $m \in M$; then $\Psi \circ Q = 0$.

We can now prove that Q is a morphism of S-modules. Since Q is R-linear by definition, it remains only to show that $Q(x \cdot m) = x \cdot Q(m)$; but it is clear by the definition of $\overline{x}$ that $Q(x \cdot m) = \overline{x}(Q(m))$, which is equal to $x \cdot Q(m)$ by the previous computation.

Note also that Q is injective. Indeed, after choosing the monomials $1, x, \ldots, x^{r-1}$ as a R-basis for S, we have a canonical isomorphism $S \simeq \bigoplus_{i=0}^{r-1} Rx^i$ that induces an isomorphism $S \otimes_R M \simeq \bigoplus_{i=0}^{r-1} (x^i \otimes M)$. To see that Q is injective, let $m \in M$ be any element such that $m \neq 0$. Then, the component of degree $r - 1$ of $Q(m)$ is $x^{r-1} \otimes m$, which is different from 0; we conclude that $Q(m) \neq 0$.

We are left to prove that $\ker(\Psi) \subseteq \mathrm{Im}(Q)$. Let $z = \sum_{i=0}^{r-1} x^i \otimes m_i$ be any element in $S \otimes_R M$ such that $\Psi(z) = 0$ or, equivalently,

$$\sum_{i=0}^{r-1} x^{i+1} \otimes m_i = \sum_{i=0}^{r-1} x^i \otimes x \cdot m_i.$$

Recalling that $x^r = -\sum_{i=0}^{r-1} a_{r-i} x^i$, by R-linearity we obtain

$$(1) \quad \sum_{i=1}^{r-1} x^i \otimes (m_{i-1} - a_{r-i} \cdot m_{r-1}) - a_r \otimes m_{r-1} = \sum_{i=0}^{r-1} x^i \otimes x \cdot m_i.$$

Let $m := m_{r-1}$. Looking at the components in degree $r - 1$ of Equation 1, we obtain

$$-a_1 \cdot m + m_{r-2} = x \cdot m,$$

or, explicitely,

$$m_{r-2} = (x + a_1) \cdot m = \sum_{j=0}^{1} (x^{1-j} a_j) \cdot m$$

Proceeding by induction on i decreasing from $r - 2$ to 1, suppose that the equality

$$m_i = \sum_{j=0}^{r-1-i} (x^{r-1-i-j} a_j) \cdot m$$

is proven. Then, looking at the components in degree i of Equation 1, we obtain:

$$m_{i-1} - a_{r-i} \cdot m = x \cdot m_i,$$

hence:

$$m_{i-1} = x \sum_{j=0}^{r-1-i} (x^{r-1-i-j} a_j) \cdot m + a_{r-i} \cdot m =$$

$$= \sum_{j=0}^{r-i} (x^{r-i-j} a_j) \cdot m.$$

Hence, we conclude that $m_i = \sum_{j=0}^{r-1-i} (x^{r-1-i-j} a_j) \cdot m$ for any i from 0 to $r - 1$; in other terms:

$$z = \sum_{i=0}^{r-1} x^i \otimes \left(\sum_{j=0}^{r-1-i} (x^{r-1-i-j} a_j) \cdot m \right) = Q(m) \in \operatorname{Im}(Q).$$

Finally, note that the map $Q = \sum_{i=0}^{r-1} x^i \sum_{j=0}^{r-1-i} \overline{x}^{r-1-i-j} \overline{a}_j$ has been defined at the level of S-modules, so it defines only a local map of sheaves on V. In order to define a global map of sheaves, note that any power of x and $\overline{x}$, considered as global maps, introduces a twist by $\pi_{\underline{a}}^* L$ up to the same tensor power, and any $\overline{a}_j$ introduces a twist by $\pi_{\underline{a}}^*(L^{\otimes j})$. The sum of these twistings in the definition of Q, for any i and j in the sum, is equal to $i + (r - 1 - i - j) + j = r - 1$. Hence, in order to extend the map Q globally, one has to consider a map

$$\mathcal{Q} : \mathcal{M} \otimes \pi_{\underline{a}}^*(L^{1-r}) \to \pi_{\underline{a}}^* E$$

whose local expression equals the one of Q on any affine open $V = \pi_{\underline{a}}^{-1}(U)$ such that L is trivial on U. Then, such map fits the global

exact sequence:

$$0 \to \mathcal{M} \otimes \pi_{\underline{a}}^*(L^{1-r}) \xrightarrow{\mathcal{Q}} \pi_{\underline{a}}^* E \xrightarrow{\pi_{\underline{a}}^*(\Phi)-x} \pi_{\underline{a}}^* E \otimes \pi_{\underline{a}}^* L \xrightarrow{\text{ev}} \mathcal{M} \otimes \pi_{\underline{a}}^* L \to 0.$$

$\square$

REMARK 1.4.6. Proposition 1.4.5 suggests that the spectral correspondence stated in [**HP12**, Proposition 6.1] is somehow misstated. Indeed, by definition of the spectral curve, the morphism $\pi_{\underline{a}}^* \Phi - x$ has non-trivial kernel at each point of $X_{\underline{a}}$. In particular, any spectral data $\mathcal{M}$ fitting in the exact sequence above needs to be supported on the whole $X_{\underline{a}}$. This is in general not true for torsion-free sheaves of polarized rank equal to 1, even for connected planar curves, as showed in [**Lop04**, §2].

CHAPTER 2

The Norm map on the compactified Jacobian

In this chapter we study how to generalize the Norm map for line bundles to the compactified Jacobian stack. We look first at the geometric counterpart of a smaller problem, regarding the direct image of Cartier divisors and generalized divisors, in order to understand how to proceed in the general situation of torsion-free sheaves with rank 1.

Let $X \xrightarrow{\pi} Y$ be any finite, flat morphism between embeddable noetherian schemes of pure dimension 1. In the first part of this chapter (Sections 2.1, 2.2), we review the classical Norm map and the theory of generalized divisors on curves, in relation with the objects introduced in Chapter 1. In the second part (Section 2.3) we propose the definitions of the direct and inverse image for generalized divisors and generalized line bundles, and we study their properties. In the third part of the chapter (Sections 2.4 and 2.5), restricting to the case when X and Y are projective curves over a field, we consider the same notions for families. On one hand, families of effective generalized divisors on a curve X are essentially families of subschemes of finite length, hence they are parametrized by the Hilbert scheme. On the other hand, families of generalized line bundles on X are parametrized by the generalized Jacobian $G\mathbb{J}(X)$, and in this case the direct image map is called *Norm map* in analogy with the Norm map defined between the Jacobians of the curves. In both cases, we show that giving the definitions of the direct image and the Norm map on such moduli spaces is possible only when the curve Y is smooth. Finally, we show that the Norm map can be extended under the same hypothesis to the compactified Jacobian $\overline{\mathbb{J}}(X)$.

Notation. In the rest of the chapter, in the absence of further specifications, by *curve* we refer to a noetherian scheme of pure dimension 1 which is embeddable (i.e. it can be embedded as a closed subscheme

of a regular scheme). This implies that the canonical (or dualizing) sheaf ω of the curve is well defined.

2.1. Review of the Norm map

We resume now the definition and properties of the direct and inverse image for Cartier divisors and the Norm map for line bundles, associated to a finite, flat morphism between curves. For a complete treatment, the standard reference is [**Gro67**, §21] together with [**Gro61**, §6.5]. We start with the definition of the norm at the level of sheaves of algebras.

Let $\pi : X \to Y$ be a finite, flat morphism between curves of degree n. Since Y is noetherian, this is equivalent to require that f is finite and locally free, i.e. that $\pi_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-algebra [**Sta19**, Tag 02K9].

DEFINITION 2.1.1. The sheaf $\pi_*\mathcal{O}_X$ is endowed with a homomorphism of $\mathcal{O}_Y$-modules, called the norm and defined on local sections by:

$$\mathcal{N}_{Y/X} : \quad \pi_*\mathcal{O}_X \longrightarrow \mathcal{O}_Y$$
$$s \longmapsto \det(\cdot s)$$

where $\cdot s : \pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_X$ is the multiplication map induced by s and $\det(\cdot s)$ is given locally by the determinant of the matrix with entries in $\mathcal{O}_Y$ associated to $\cdot s$.

By the standard properties of determinants, for local sections s, s' of $\pi_*\mathcal{O}_X$ and any section μ of $\mathcal{O}_Y$, we have:

(2)
$$\mathcal{N}_{Y/X}(s \cdot s') = \mathcal{N}_{Y/X}(s) \cdot \mathcal{N}_{Y/X}(s'), \qquad \mathcal{N}_{Y/X}(\mu s) = \mu^n \mathcal{N}_{Y/X}(s).$$

Before giving the definitions of the present section, we need a technical lemma.

LEMMA 2.1.2. *Let $\pi : X \to Y$ be a finite, flat morphism of degree n between curves and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then, $\pi_*\mathcal{L}$ is an invertible $\pi_*\mathcal{O}_X$-module and there exists an open affine cover $\{V_i\}_{i \in I}$*

of Y s.t. $\pi_*\mathcal{O}_X$ is trivial on each V_i and $\pi_*\mathcal{L}$ is trivial both as a $\pi_*\mathcal{O}_X$-module and as an $\mathcal{O}_Y$-module on each V_i.

PROOF. By [**Sta19**, Tag 02K9], $\pi_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module; denote with $\{W_\alpha\}_{\alpha\in A}$ an open affine cover such that $\pi_*\mathcal{O}_{X|W_\alpha} \simeq \mathcal{O}_{Y|W_\alpha}^n$ for each $\alpha \in A$. By [**Gro61**, Proposition 6.1.12], $\pi_*\mathcal{L}$ is an invertible $\pi_*\mathcal{O}_X$-module; denote with $\{W'_\beta\}_{\beta\in B}$ an open affine cover such that $\pi_*\mathcal{L}_{|W'_\beta} \simeq \pi_*\mathcal{O}_{X|W'_\beta}$ for each $\beta \in B$. Let $\{V_{\alpha,\beta} = W_\alpha \cap W'_\beta\}_{(\alpha,\beta)\in A\times B}$ be a common refinement. Then, for $I = A \times B$, $\{V_i\}_{i\in I}$ is an open affine cover of Y such that $\pi_*\mathcal{O}_X$ is trivial on each open of the cover and $\pi_*\mathcal{L}$ is trivial both as $\pi_*\mathcal{O}_X$-module and as $\mathcal{O}_Y$-module on each open. $\square$

2.1.1. Direct and inverse image of Cartier divisors. We recall now the definitions of direct and inverse image for Cartier divisors. For any curve X, denote with $\mathcal{K}_X$ the sheaf of total quotient rings of the curve. Recall that the set of *Cartier divisors* on X is the set of global sections of the quotient sheaf of multiplicative groups $\mathcal{K}_X^*/\mathcal{O}_X^*$:

$$\mathrm{CDiv}(X) = \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*).$$

Although the group operation on $\mathcal{K}_X^*/\mathcal{O}_X^*$ is multiplication, the group operation on $\mathrm{CDiv}(X)$ is denoted additively. The group of Cartier divisors of X contains the subgroup $\mathrm{Prin}(X)$ of *principal divisors* defined as the image of the canonical homomorphism

$$\Gamma(X, \mathcal{K}_X^*) \longrightarrow \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*).$$

It is well known [**Gro67**, §21.2] that the set of Cartier divisors is in one-to-one correspondence with the set of invertible fractional ideals, i.e. the set of subsheaves $\mathcal{I} \subseteq \mathcal{K}_X$ that are also invertible $\mathcal{O}_X$-modules [**Gro67**, Proposition 21.2.6]. If $D \in \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ is represented by an open cover $\{U_i\}_{i\in I}$ and a collection of sections $f_i \in \Gamma(U_i, \mathcal{K}_X^*)$ such that $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$, the corresponding fractional ideal $\mathcal{I}_D$ is the sub $\mathcal{O}_X$-module of $\mathcal{K}_X$ equal to $\mathcal{O}_{X|U_i} \cdot f_i$ on any U_i.[1] Under this correspondence, the sum of Cartier divisors corresponds to the multiplication of fractional ideals.

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[1]Differently from Grothendieck's EGA, we pick f_i instead of f_i^{-1}. This does not affect the other results.

DEFINITION/LEMMA 2.1.3. Let $\pi : X \to Y$ be a finite, flat morphism between curves and let $\pi^\sharp : \mathcal{O}_Y \to \pi_* \mathcal{O}_X$ be the associated canonical map of sheaves of modules.

Let $D \in \mathrm{CDiv}(X)$ be a Cartier divisor on X corresponding to the invertible fractional ideal $\mathcal{I}$ and let $\{V_i\}_{i \in I}$ be an affine cover as in Lemma 2.1.2. Then, on each V_i, $\pi_* \mathcal{I}_{|V_i}$ is equal to the subsheaf $h_i \cdot (\pi_* \mathcal{O}_X)_{|V_i}$ of $(\pi_* \mathcal{K}_X)_{|V_i}$ generated by a meromorphic regular section $h_i = f_i / g_i$, with $f_i, g_i \in \Gamma(V_i, \pi_* \mathcal{O}_X^*)$. The *direct image of D*, denoted $\pi_*(D)$, is the Cartier divisor on Y corresponding to the fractional ideal generated on any V_i by the meromorphic regular section $\mathcal{N}_{Y/X}(f_i)/\mathcal{N}_{Y/X}(g_i) \in \Gamma(V_i, \mathcal{K}_Y)$.

Let $M \in \mathrm{CDiv}(Y)$ be a Cartier divisor on Y corresponding to the invertible fractional ideal $\mathcal{J} \subseteq \mathcal{K}_Y$, and let $\{V_i\}_{i \in I}$ be an affine cover of Y such that, on each V_i, $\mathcal{J}_{|V_i}$ is equal to the fractional ideal of $\mathcal{O}_{Y|V_i}$-modules generated by a meromorphic regular section $u_i = s_i / t_i$ with $s_i, t_i \in \Gamma(V_i, \mathcal{O}_Y^*)$. The *inverse image* of M, denoted $\pi^*(M)$, is the Cartier divisor on X corresponding to the fractional ideal generated on any $U_i = \pi^{-1}(V_i)$ by the meromorphic regular section $\pi^\sharp(s_i)/\pi^\sharp(t_i) \in \Gamma(U_i, \mathcal{K}_X)$.

PROOF. The definition of direct image is a reformulation of the one given in [**Gro67**, §21.5.5]. The fact that $\mathcal{N}_{Y/X}(f_i)/\mathcal{N}_{Y/X}(g_i)$ is a regular meromorphic section follows from the discussion in [**Gro67**, §21.5.3]; the definition is independent of the choice of the h_i's since the norm of sheaves is multiplicative.

The definition of inverse image is a reformulation of [**Gro67**, Definition 21.4.2], obtained as a consequence of [**Gro67**, §21.4.3], together with the result of [**Gro67**, Proposition 21.4.5] that ensures that $\pi^\sharp(s_i)/\pi^\sharp(t_i)$ is regular. $\square$

PROPOSITION 2.1.4. *Let $\pi : X \to Y$ be a finite, flat morphism of curves of degree n. The direct and inverse image for Cartier divisors induce homomorphisms of groups:*

$$\pi_* : \mathrm{CDiv}(X) \longrightarrow \mathrm{CDiv}(Y)$$
$$\pi^* : \mathrm{CDiv}(Y) \longrightarrow \mathrm{CDiv}(X),$$

such that $\pi_* \circ \pi^*$ is the multiplication map by n.

PROOF. The direct image is a homomorphism thanks to [**Gro67**, §21.5.5.1], while the inverse image is a homomorphism as a consequence of [**Gro67**, Definition 21.5.5.1]. The result on $\pi_* \circ \pi^*$ is stated in [**Gro67**, Proposition 21.5.6] $\hfill\square$

2.1.2. Norm of line bundles. We come now to the definition of the Norm map for invertible $\mathcal{O}_X$-modules.

DEFINITION/LEMMA 2.1.5. Let $\pi : X \to Y$ be a finite, flat morphism between curves and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\{V_i\}_{i \in I}$ be an affine cover of Y as in Lemma 2.1.2. In particular, there is for any $i \in I$ an isomorphism $\lambda_i : (\pi_* \mathcal{L})_{|V_i} \to (\pi_* \mathcal{O}_X)_{|V_i}$. For any $i, j \in I$, the isomorphism $\omega_{ij} := \lambda_i \circ \lambda_j^{-1}$ can be interpreted as a section of $\pi_* \mathcal{O}_X$ over $V_i \cap V_j$. The collection of norms $\{\mathcal{N}_{Y/X}(\omega_{ij})\}_{i,j \in I}$ is a 1-cocycle with values in $\mathcal{O}_Y^*$.

The cocyle $\{\mathcal{N}_{Y/X}(\omega_{ij})\}_{i,j \in I}$ defines up to isomorphism an invertible $\mathcal{O}_Y$-module, which is called the *Norm of $\mathcal{L}$ relative to π* and is denoted as $\mathrm{Nm}_\pi(\mathcal{L})$ or $\mathrm{Nm}_{Y/X}(\mathcal{L})$.

PROOF. If $\mathcal{L}'$ is an invertible $\mathcal{O}_X$-module isomorphic to $\mathcal{L}$ through an isomorphism $h : \mathcal{L}' \to \mathcal{L}$, then a local trivialization for $\pi_* \mathcal{L}'$ over V_i is given by $\lambda_i \circ (\pi_* h)$. Running over all $i \in I$, the resulting 1-cocycle $\{\mathcal{N}_{Y/X}(\lambda_i \circ \pi_* h \circ \pi_* h^{-1} \circ \lambda_j^{-1})\}_{i,j \in I}$ is the same as for $\mathcal{L}$. $\hfill\square$

Recall that, for any curve X, the Picard group of X is the set $\mathrm{Pic}(X)$ of isomorphism classes of invertible $\mathcal{O}_X$-modules, endowed with the operation of tensor product.

PROPOSITION 2.1.6. *Let $\pi : X \to Y$ be a finite, flat morphism of curves of degree n. The Norm and the inverse image map for line bundles induce homomorphism of groups:*

$$\mathrm{Nm}_\pi : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y)$$
$$\pi^* : \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X),$$

such that $\mathrm{Nm}_\pi \circ \pi^*$ is the n-th tensor power.

PROOF. The inverse image for line bundles is a homomorphism since it commutes obviously with tensor products. The other results follow from (2), using the fact that tensor product of line bundles corresponds to multiplication at the level of defining cocycles. $\hfill\square$

PROPOSITION 2.1.7. *Let $\pi : X \to Y$ be a finite, flat morphism between curves. For any invertible $\mathcal{O}_X$-modules $\mathcal{L}$, we have*

(3)
$$\mathrm{Nm}_\pi(\mathcal{L}) \simeq \det(\pi_*\mathcal{L}) \otimes \det(\pi_*\mathcal{O}_X)^{-1}.$$

PROOF. See [**HP12**, Corollary 3.12].   □

As we show in the next proposition, the direct image for Cartier divisors and the Norm map for line bundles are related, as well as the inverse image maps. Recall that, on any curve X, the Picard group is canonically isomorphic to the group of Cartier divisors modulo the subgroup $\mathrm{Prin}(X)$ of principal divisors. This gives rise to a canonical quotient of groups:

$$q_X : \mathrm{CDiv}(X) \longrightarrow \mathrm{CDiv}(X)/\mathrm{Prin}(X) = \mathrm{Pic}(X)$$
$$D \longmapsto \mathcal{I}_D.$$

PROPOSITION 2.1.8. *Let $\pi : X \to Y$ be a finite, flat morphism between curves. The direct and inverse image for Cartier divisors are compatible via the quotient maps q_X and q_Y respectively with the Norm and the inverse image map for line bundles; i.e. the following diagrams of groups are commutative:*

$$
\begin{array}{ccc}
\mathrm{CDiv}(X) & \xrightarrow{\ \pi_*\ } & \mathrm{CDiv}(Y) \\
\downarrow{\scriptstyle q_X} & & \downarrow{\scriptstyle q_Y} \\
\mathrm{Pic}(X) & \xrightarrow{\ \mathrm{Nm}_\pi\ } & \mathrm{Pic}(Y)
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{CDiv}(Y) & \xrightarrow{\ \pi^*\ } & \mathrm{CDiv}(X) \\
\downarrow{\scriptstyle q_Y} & & \downarrow{\scriptstyle q_X} \\
\mathrm{Pic}(Y) & \xrightarrow{\ \pi^*\ } & \mathrm{Pic}(X).
\end{array}
$$

PROOF. The commutativity of the first diagram follows from correspondence between D and $\mathcal{I}_D$ and the definitions of π_* and Nm_π. The second diagram is commutative as consequence of [**Gro67**, Picture 21.4.2.1] together with [**Gro67**, Proposition 21.4.5].   □

We are finally interested in reviewing the Norm map for families of line bundles.[2] To study families of line bundles, we **assume that X and Y are projective curves over a base field k** and that $\pi : X \to Y$ is a finite, flat morphism of degree n defined over k. Families of line bundles are parametrized by the Jacobian schemes of X and Y (denoted $J(X)$ and $J(Y)$, see Chapter 1, Section 1.2).

---

[2]For the sake of completeness, we note that a complete discussion regarding families of Cartier divisors and their direct and inverse images is done in [**Gro67**, §21.15].

DEFINITION/LEMMA 2.1.9. Let $\pi : X \to Y$ be a finite, flat morphism of projective curves over a base field k. For any k-scheme T, the *Norm map for line bundles associated to π* is defined on the T-valued points of the Jacobian of X as:

$$\mathrm{Nm}_\pi(T) : \quad J(X)(T) \longrightarrow J(Y)(T)$$
$$\mathcal{L} \longmapsto \det\left(\pi_{T,*}(\mathcal{L})\right) \otimes \det\left(\pi_{T,*}\mathcal{O}_{X\times_k T}\right)^{-1}.$$

where $\pi_T : X \times_k T \to Y \times_k T$ is induced by pullback from π.

PROOF. We need to check that the map is well defined. Since $\mathcal{L}$ is a line bundle on $X \times_k T$, its pushforward $\pi_{T,*}(\mathcal{L})$ is a locally free sheaf of rank n on $Y \times_k T$, as for $\pi_{T,*}\mathcal{O}_{X\times_k T}$. Taking determinants produces line bundles, so $\det\left(\pi_{T,*}(\mathcal{L})\right) \otimes \det\left(\pi_{T,*}\mathcal{O}_{X\times_k T}\right)^{-1}$ is a well-defined line bundle on $Y \times_k T$. The whole construction is functorial, so the above definition gives a well-defined morphism of schemes

$$\mathrm{Nm}_\pi : \quad J(X) \longrightarrow J(Y).$$

$\square$

2.2. Review of generalized divisors

In this section, we review known facts about generalized divisors and generalized line bundles on curves. Moreover, we compare their moduli spaces with the compactified Jacobian.

REMARK 2.2.1. The theory of generalized divisors is developed by Hartshorne in his papers [**Har86**], [**Har94**] and [**Har07**], in order to generalize the notion of Cartier divisor on schemes satisfying condition S_2 of Serre. Since we are dealing with schemes of dimension 1, the condition S_2 of Serre coincides with the condition S_1, which in turn coincides with the fact of not having embedded components (i.e. embedded points). In particular, any scheme of pure dimension 1 is also S_1, and hence S_2. Similarly, a coherent sheaf on a curve satisfies condition S_2 if and only if it is torsion-free.

2.2.1. Generalized divisors.

DEFINITION 2.2.2. Let X be a curve and let $\mathcal{K}_X$ be the sheaf of total quotient rings on X. A *generalized divisor* on X is a nondegenerate fractional ideal of $\mathcal{O}_X$-modules, i.e. a subsheaf $\mathcal{I} \subseteq \mathcal{K}_X$ that is a

coherent sheaf of $\mathcal{O}_X$-modules and such that $\mathcal{I}_\eta = \mathcal{K}_{X,\eta}$ for any generic point $\eta \in X$. It is *effective* if $\mathcal{I} \subseteq \mathcal{O}_X$. It is *Cartier* if $\mathcal{I}$ is an invertible $\mathcal{O}_X$-module, or equivalently locally principal. It is *principal* if $\mathcal{I} = \mathcal{O}_X \cdot f$ (also denoted (f)) for some global section $f \in \Gamma(X, \mathcal{K}_X^*)$.

The set of generalized divisors on X is denoted with $\mathrm{GDiv}(X)$, the subset of Cartier divisors with $\mathrm{CDiv}(X)$ and the set of principal divisors with $\mathrm{Prin}(X)$.

Using the usual notion of subscheme associated to a sheaf of ideals, the set $\mathrm{GDiv}^+(X)$ of effective generalized divisors on X is in one-to-one correspondence with the set of closed subschemes $D \subset X$ of pure codimension one (i.e. of dimension zero). With a slight abuse of notation, also for non-effective divisors, we denote with D the generalized divisor and we refer to $\mathcal{I}$ (or $\mathcal{I}_D$) as the *fractional ideal* of D (also called *defining ideal* of D, or *ideal sheaf* of D if D is effective).

Let D_1, D_2 be generalized divisors on X, with fractional ideals $\mathcal{I}_1, \mathcal{I}_2$. The *sum* $D_1 + D_2$ is the generalized divisor associated to the fractional ideal $\mathcal{I}_1 \cdot \mathcal{I}_2 \subseteq \mathcal{K}_X$. The sum is commutative, associative, with neutral element 0 defined by the trivial ideal $\mathcal{O}_X$. The *inverse* of a generalized divisor D associated to $\mathcal{I}$ is the generalized divisor $-D$ associated to the fractional ideal $\mathcal{I}^{-1}$, i.e. the sheaf which on any open subset U of X is defined as $\{f \in \Gamma(U, \mathcal{K}_X) \mid \mathcal{I}(U) \cdot f \subseteq \mathcal{O}_X(U)\}$.

The inverse operation behaves well with the sum only for Cartier divisors. For any pair of generalized divisors D, E on X with E Cartier, $-(D + E) = (-D) + (-E)$, but $D + (-D) = 0$ if and only if D is a Cartier divisor. As a consequence, the set $\mathrm{GDiv}(X)$ of all generalized divisors over X endowed with the sum operation is not a group, but the subset $\mathrm{CDiv}(X)$ of Cartier divisors is. The set $\mathrm{Prin}(X)$ of principal divisors is a subgroup of $\mathrm{CDiv}(X)$ and both the groups act by addition on the set $\mathrm{GDiv}(X)$.

Tthe following Lemma, inspired by [**Har94**, Proposition 2.11], shows that any generalized divisor is equal to the sum of an effective generalized divisor and the inverse of an effective Cartier divisor, as

LEMMA 2.2.3. *Let X be a curve and let $D \in \mathrm{GDiv}(X)$ be any generalized divisor on X. Then, there exist an effective generalized*

divisor D' and an effective Cartier divisor E on X such that $D = D' + (-E)$.

PROOF. Cover X by open affines $U_i = \mathrm{Spec}(A_i)$, $i = 1, \ldots, r$. For each i, denote with I_i the fractional ideal of D restricted to U_i. This is a finitely generated A_i-module, so there exists a nonzero-divisor $f_i \in A_i$ such that $f \cdot I_i \subseteq A$. Let $Y_i \subset U_i$ be the closed subscheme defined by f_i, which is an effective Cartier divisor of U_i; after the composition with $U_i \hookrightarrow X$, Y_i becomes an effective Cartier divisor of X. Now, the sum of divisors $D + \sum_{i=1}^r Y_i$ is effective as it is sum of effective divisors on each U_i. By putting $E = \sum_{i=1}^r Y_i$ and $D' = D + E$ we get the result. $\square$

Two generalized divisors D_1, D_2 over X are *linearly equivalent* (written $D_1 \sim D_2$) if there is a divisor $(f) \in \mathrm{Prin}(X)$ such that $D_1 + (f) = D_2$. We define the *generalized Picard of X* as the set of divisors on X modulo linear equivalence:

$$\mathrm{GPic}(X) = \mathrm{GDiv}(X)/\mathrm{Prin}(X).$$

We have the following commutative diagram of sets, where the vertical maps are quotients by $\mathrm{Prin}(X)$:

$$
\begin{array}{ccc}
\mathrm{CDiv}(X) & \subseteq & \mathrm{GDiv}(X) \\
\downarrow & & \downarrow \\
\mathrm{Pic}(X) & \subseteq & \mathrm{GPic}(X)
\end{array}
$$

Taking inverse and sums preserve linear equivalence, so the two operations are well defined also on $\mathrm{GPic}(X)$ and the subset $\mathrm{Pic}(X) \subseteq \mathrm{GPic}(X)$ is a group that acts on $\mathrm{GPic}(X)$ by addition. For a curve X, the condition $\mathrm{GDiv}(X) = \mathrm{CDiv}(X)$ is also equivalent to $\mathrm{GPic}(X) = \mathrm{Pic}(X)$, which is also equivalent to the curve X being smooth.

The set $\mathrm{GPic}(X)$ has an alternative description in terms of generalized line bundles. Let D be a generalized divisor on X; then, its fractional ideal $\mathcal{I}$ is a generalized line bundle. If D' is another generalized line bundle, it is linearly equivalent to D if and only if its fractional ideal $\mathcal{I}'$ is a generalized line bundle isomorphic to X as an $\mathcal{O}_X$-module. Viceversa, any generalized line bundle $\mathcal{F}$ on X is isomorphic to the fractional ideal of some generalized divisor D [**Har07**,

Proposition 2.4]. Then, $\mathrm{GPic}(X)$ *can be also defined as the set of generalized line bundles of X, up to isomorphism.* Under this description, classes of Cartier divisors correspond to isomorphism classes of line bundles, and the operations of sum by a Cartier divisor and inverse of a divisor are replaced with tensor product and taking dual of the corresponding sheaves.

2.2.2. Degree of generalized divisors. We now **assume that X is a projective curve over a base field k.** In this case, the notion of *degree of a divisor* can be introduced.

DEFINITION/LEMMA 2.2.4. Let $D \in \mathrm{GDiv}^+(X)$ be an effective generalized divisor on X with ideal sheaf $\mathcal{I}_D \subseteq \mathcal{O}_X$, and let $x \in X$ be any point of X in codimension 1. We define the *degree of D at x* as the non-negative integer:

$$\deg_x(D) = \ell_{\mathcal{O}_{X,x}}\big(\mathcal{O}_{X,x}/\mathcal{I}_{D,x}\big)[\kappa(x) : k].$$

The degree of any generalized divisor $D \in \mathrm{GDiv}(X)$ at x is defined as $\deg_x(D) = \deg_x(E) - \deg_x(F)$, where $D = E - F$ with E, F effective generalized divisors and F Cartier by Lemma 2.2.3.

The degree of D on X (also denoted as $\deg(D)$ when there is no ambiguity) is equal to the sum of the degrees of D at all points of X in codimension 1:

$$\deg_X(D) = \sum_{x \text{ cod } 1} \deg_x(D).$$

PROOF. First, note that the local ring $\mathcal{O}_{X,x}/\mathcal{I}_{D,x}$ in nonzero if and only if x is in the support of D. In such case the local ring has Krull dimension 0, so it is Artinian and hence has finite length.

The definition of degree at a point of any generalized divisor is well-given thanks to [**Har94**, Lemma 2.17], and $\deg_x(-D) = -\deg_x(D)$.

Finally, the support of D contains only finitely many points, hence the degree of D on X is a well-defined integer. $\qquad\square$

The degree of a generalized divisor is strictly related to the degree of its fractional ideal, as the following proposition shows.

PROPOSITION 2.2.5. *Let D be a generalized divisor on X with fractional ideal $\mathcal{I}_D \subseteq \mathcal{K}_X$. Then, the degree of $\mathcal{I}_D$ as a torsion-free sheaf*

is well defined and equal to $-\deg_X(D)$. *Moreover, linearly equivalent divisors have the same degree on* X.

PROOF. The fractional ideal $\mathcal{I}_D$ is a generalized line bundle, hence it has rank 1 on X and well-defined degree as a torsion-free sheaf. In order to compute $\deg(\mathcal{I}_D)$ consider the short exact sequence

$$0 \to \mathcal{I}_D \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_D \to 0.$$

By definition of degree, $\deg(\mathcal{I}_D) = \chi(\mathcal{I}_D) - \chi(\mathcal{O}_X)$. Then, by additivity of Euler characteristic with respect to exact sequences, we deduce that $\deg(\mathcal{I}_D) = -\chi(\mathcal{O}_X/\mathcal{I}_D)$. Since $\mathcal{O}_X/\mathcal{I}_D$ has dimension 0, its Euler characteristic is equal to $h^0(\mathcal{O}_X/\mathcal{I}_D) := \dim_k H^0(X, \mathcal{O}_X/\mathcal{I}_D)$. The support of $\mathcal{O}_D = \mathcal{O}_X/\mathcal{I}_D$ is by definition the support of D and $H^0(X, \mathcal{O}_X/\mathcal{I}_D) = \oplus_{x \in \mathrm{Supp}\, D} \mathcal{O}_{D,x}$. Then we compute:

$$\deg(\mathcal{I}_D) = -\chi(\mathcal{O}_X/\mathcal{I}_D) = -\sum_{x \in \mathrm{Supp}\, D} \dim_k \mathcal{O}_{D,x} =$$

$$= -\sum_{x \in \mathrm{Supp}\, D} \dim_{\kappa(x)}\big(\mathcal{O}_{D,x}\big)[\kappa(x):k] =$$

$$= -\sum_{x \in \mathrm{Supp}\, D} \ell_{\mathcal{O}_X,x}\big(\mathcal{O}_{D,x}\big)[\kappa(x):k] =$$

$$= -\deg_X(D).$$

Consider now a generalized divisor D with fractional ideal $\mathcal{I}_D$ and let $D = E - F$ with E, F effective and F Cartier by Lemma 2.2.3. Since F is effective, consider first the short exact sequence:

(4) $$0 \to \mathcal{I}_F \to \mathcal{O}_X \to T \to 0$$

where T is a sheaf of dimension 0. Since F is Cartier, we can tensor by $\mathcal{I}_F^{-1}$ to obtain:

(5) $$0 \to \mathcal{O}_X \to \mathcal{I}_F^{-1} \to \mathcal{I}_F^{-1} \otimes T \to 0.$$

Using additivity of Euler characteristics in the exact sequences 4 and 5 and the fact that $\chi(\mathcal{I}_F^{-1} \otimes T) = \chi(T)$, we see that:

$$\chi(\mathcal{I}_F^{-1}) - \chi(\mathcal{O}_X) = \chi(\mathcal{O}_X) - \chi(\mathcal{I}_F).$$

Also E is effective, hence there is a short exact sequence:

(6) $$0 \to \mathcal{I}_E \to \mathcal{O}_X \to Q \to 0$$

where Q is a sheaf of dimension 0. Tensoring again by $\mathcal{I}_F^{-1}$ we obtain:

$$(7) \qquad\qquad 0 \to \mathcal{I}_D \to \mathcal{I}_F^{-1} \to \mathcal{I}_F^{-1} \otimes Q \to 0$$

where $\mathcal{I}_D \simeq \mathcal{I}_E \otimes \mathcal{I}_F^{-1}$ since $D = E - F$ and F is Cartier. Using addivity of Euler characteristics in the exact sequences 6 and 7 and the fact that $\chi(\mathcal{I}_F^{-1} \otimes Q) = \chi(Q)$, we can compute the degree of $\mathcal{I}_D$:

$$\begin{aligned}
\deg(\mathcal{I}_D) = \chi(\mathcal{I}_D) - \chi(\mathcal{O}_X) = \\
= \chi(\mathcal{I}_F^{-1}) - \chi(\mathcal{I}_F^{-1} \otimes Q) - \chi(\mathcal{O}_X) = \\
= \chi(\mathcal{I}_F^{-1}) - \chi(Q) - \chi(\mathcal{O}_X) = \\
= \chi(\mathcal{I}_F^{-1}) - 2\chi(\mathcal{O}_X) + \chi(\mathcal{I}_E) = \\
= \chi(\mathcal{O}_X) - \chi(\mathcal{I}_F) + \chi(\mathcal{I}_E) - \chi(\mathcal{O}_X).
\end{aligned}$$

Since E and F are effective, we conclude that:

$$\deg(\mathcal{I}_D) = -(\deg_X(E) - \deg_X(F)) = -\deg_X(D).$$

To prove the last statement, let D and D' be linearly equivalent divisors; then, their fractional ideal $\mathcal{I}_D$ and $\mathcal{I}_{D'}$ are isomorphic. Hence,

$$\deg_X(D) = -\deg(\mathcal{I}_D) = -\deg(\mathcal{I}_D') = \deg_X(D').$$

$\square$

2.2.3. Moduli of generalized divisors and the Abel Map. We start with considering the moduli space for generalized divisors. Since we are dealing with families of sheaves and related moduli problems, we **assume that X is a projective curve over a base field k.**

REMARK 2.2.6. In general $\mathrm{GDiv}(X)$ cannot be represented by a geometric object of finite type, even for fixed degrees. Also, it is not easy to give a correct definition for flat families of non-effective generalized divisors. Hence, only families of effective divisors are considered, using the Hilbert scheme.

DEFINITION 2.2.7. The *Hilbert scheme of effective generalized divisors of degree d on X* is the Hilbert scheme Hilb_X^d parametrizing 0-dimensional subschemes of X, with Hilbert polynomial equal to a costant integer d. Recall that, given a k-scheme T, a T-valued point of Hilb_X^d is a T-flat subscheme $D \subset X \times_k T$ such that D restricted to the

fiber over any $t \in T$ is a 0-dimensional subscheme of $X \times_k \{t\} \simeq X$, of degree d. In other words, the corresponding ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_{X \times_k T}$ restricted to any fiber over T defines a generalized divisor D_t on X.

The *Hilbert scheme of effective Cartier divisors of degree d* is the open subscheme ${}^\ell \mathrm{Hilb}_X^d \subseteq \mathrm{Hilb}_X^d$ parametrizing subschemes of X of degree d whose ideal sheaf is locally principal.

Note that ${}^\ell \mathrm{Hilb}_X^d = \mathrm{Hilb}_X^d$ when X is smooth.

Families of effective generalized divisors can be added with families of Cartier divisors, giving rise to a morphism:

$$\mathrm{Hilb}_X^d \times {}^\ell \mathrm{Hilb}_X^e \longrightarrow \mathrm{Hilb}_X^{d+e}$$
$$(D, E) \longmapsto \mathcal{Z}(\mathcal{I}_D \cdot \mathcal{I}_E).$$

Effective generalized divisors on X up to linear equivalence are generalized line bundles, hence they are parametrized by the generalized Jacobian $G\mathbb{J}(X)$.

By Remark 1.2.9, the Jacobian scheme $J(X)$ of X is contained in $G\mathbb{J}(X)$ as an open subscheme. The operations of tensor products and taking inverse are well defined on $J(X)$ and make $J(X)$ into an algebraic group, acting on $G\mathbb{J}(X)$ via tensor product:

$$G\mathbb{J}(X) \times J(X) \longrightarrow G\mathbb{J}(X)$$
$$(\mathcal{L}, \mathcal{E}) \longmapsto \mathcal{L} \otimes \mathcal{E}.$$

Finally, we recall the definition of the geometric Abel map. Let $\mathrm{Hilb}_X = \bigsqcup_{d \geq 0} \mathrm{Hilb}_X^d$ be the Hilbert scheme of effective generalized divisor of any degree on X.

DEFINITION 2.2.8. Let M be a line bundle of degree e on X. The *(M-twisted) Abel map* of degree d is defined as:

$$\mathcal{A}_M^d : \quad \mathrm{Hilb}_X^d \longrightarrow G\mathbb{J}(X, -d+e)$$
$$D \longmapsto \mathcal{I}_D \otimes M.$$

The M-twisted Abel map in any degree is defined similary as a map:

$$\mathcal{A}_M : \quad \mathrm{Hilb}_X \longrightarrow G\mathbb{J}(X)$$

The restriction of $\mathcal{A}_M^d$ to ${}^\ell \mathrm{Hilb}_X^d$ takes values in $J(X, -d+e)$. Taking care of the twisting, the Abel map is equivariant with respect to the

sum of effective Cartier divisors: for any pair of line bundles M and N on X and for any $D \in \mathrm{Hilb}_X$, $E \in {}^\ell\mathrm{Hilb}_X$,

$$\mathcal{A}_{M \otimes N}(D + E) \simeq \mathcal{A}_M(D) \otimes \mathcal{A}_N(E).$$

2.3. The direct and inverse image for generalized divisors and generalized line bundles

Let $\pi : X \to Y$ be a finite, flat morphism of degree n between curves. In the present section, we extend the notion of direct and inverse image from Cartier divisors to generalized divisors and generalized line bundles.

2.3.1. The Fitting ideal.

A fundamental tool for the definition of the direct image for generalized divisors is the *Fitting ideal of a module* (and *of a sheaf of modules*), of which we recall here the definition. See [**Eis95**, Chapter 20], [**Vas04**, Chapter 2.4] and [**EH06**, Chapter V.1.3] for a detailed treatment.

DEFINITION/LEMMA 2.3.1. Let X be a scheme and let $\mathcal{F}$ be a coherent sheaf on X. Let

$$\mathcal{E}_1 \xrightarrow{\ \psi\ } \mathcal{E}_0 \longrightarrow \mathcal{F} \to 0$$

be any finite presentation of $\mathcal{F}$, with $\mathcal{E}_0$ locally free of rank r. The 0-*th Fitting ideal* of $\mathcal{F}$, denoted $\mathrm{Fitt}_0(\mathcal{F})$, is defined as the image of the map:

$$\textstyle\bigwedge^r \mathcal{E}_1 \otimes (\bigwedge^r \mathcal{E}_0)^{-1} \xrightarrow{\ \Psi\ } \mathcal{O}_X$$

induced by $\bigwedge^r \psi : \bigwedge^r \mathcal{E}_1 \to \bigwedge^r \mathcal{E}_0$ and is independent of the choiche of the presentation for $\mathcal{F}$. If $\mathcal{E}_1$ is locally free, then ψ can be locally represented by a matrix and the 0-th Fitting ideal is generated locally by the minors of size r of such matrix, with the convention that the determinant of the 0×0 matrix is 1.

PROOF. The definition is the sheaf-theoretic reformulation of [**Eis95**, Corollary-Definition 20.4]. □

As a consequence of the definition, the formation of Fitting ideals commutes with restrictions and with base change [**Eis95**, Corollary 20.5].

DEFINITION/LEMMA 2.3.2. Let X be a scheme and $\mathcal{F}$ be a coherent sheaf on X. The 0-*th Fitting scheme* of $\mathcal{F}$ is the subscheme of X defined as the zero locus of $\mathrm{Fitt}_0(\mathcal{F}) \subseteq \mathcal{O}_X$ in X. The 0-th Fitting scheme contains the support of $\mathcal{F}$, as a closed subscheme with the same underlying topological space.

PROOF. The result is stated in [**EH06**, Definition V-10]. $\square$

2.3.2. The direct image. Here, we define the notion of direct image for generalized divisors. First, we start with the case of an effective generalized divisor. Let $D \in \mathrm{GDiv}^+(X)$ be an effective generalized divisor on X, with ideal sheaf $\mathcal{I}$. Since π is finite and flat, the pushforward $\pi_*(\mathcal{O}_X/\mathcal{I})$ is a coherent $\mathcal{O}_Y$-module.

DEFINITION/LEMMA 2.3.3. (Direct image of an effective generalized divisor) Let $D \in \mathrm{GDiv}^+(X)$ be an effective generalized divisor on X, with ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$. The *direct image of D with respect to π*, denoted with $\pi_*(D)$, is the effective generalized divisor on Y defined by the 0-th Fitting ideal of $\pi_*(\mathcal{O}_X/\mathcal{I})$.

PROOF. Let us prove that the 0-th Fitting ideal of $\pi_*(\mathcal{O}_X/\mathcal{I})$ defines an effective generalized divisor on Y. First note that $\mathrm{Fitt}_0\,\pi_*(\mathcal{O}_X/\mathcal{I})$ is a subsheaf of $\mathcal{O}_Y$, and is a coherent $\mathcal{O}_Y$-module; hence, it is an effective fractional ideal. To prove that $\mathrm{Fitt}_0\,\pi_*(\mathcal{O}_X/\mathcal{I})$ is nondegenerate, consider a generic point $\eta \in Y$. Since the map π is dominant, the preimage $\pi^{-1}(\eta) = \{\eta_i\}$ is a finite set of generic points of X. Then,

$$\mathrm{Fitt}_0(\pi_*(\mathcal{O}_X/\mathcal{I}))_\eta = \mathrm{Fitt}_0(\pi_*(\mathcal{O}_X/\mathcal{I})_\eta) =$$
$$= \mathrm{Fitt}_0((0)_\eta) = \mathcal{O}_{X,\eta} = \mathcal{K}_{X,\eta}.$$

$\square$

LEMMA 2.3.4. (Linearity w.r.t effective Cartier divisors) *Let $D \in \mathrm{GDiv}^+(X)$ be an effective generalized divisor and $E \in \mathrm{CDiv}^+(X)$ be an effective Cartier divisor on X. Then, $\pi_*(E)$ is a Cartier divisor on Y and $\pi_*(D + E) = \pi_*(D) + \pi_*(E)$.*

PROOF. Let $\mathcal{J}$ be the ideal sheaf of E and let $\{V_i\}$ be an open affine cover of Y such that $\pi_*\mathcal{O}_X$ and $\pi_*\mathcal{J}$ are trivial on each V_i as in Lemma 2.1.2. The sheaf $\pi_*\mathcal{O}_X/\mathcal{J}$ is locally presented by the exact

sequence:

$$(\pi_*\mathcal{O}_X)_{|V_i} \xrightarrow{\cdot h_i} (\pi_*\mathcal{O}_X)_{|V_i} \to (\pi_*\mathcal{O}_X/\mathcal{J})_{|V_i} \to 0$$

where $h_i \in \Gamma(\pi^{-1}(V_i), \mathcal{O}_X)$ is a local generator for $\mathcal{J} \subset \mathcal{O}_X$ on $\pi^{-1}(V_i)$. Since $(\pi_*\mathcal{O}_X)_{|V_i}$ is a free $(\mathcal{O}_Y)_{|V_i}$-module of rank n, there is a $n \times n$ matrix H_i with entries in $(\mathcal{O}_Y)_{|V_i}$ representing the multiplication by h_i. Then, by Definition 2.3.1, the 0-th Fitting ideal of $\pi_*\mathcal{O}_X/\mathcal{J}$ is the principal ideal generated locally by $\det(\cdot h_i)$ on V_i. In particular, $\pi_*(E)$ is a Cartier divisor.

Let $\mathcal{I}$ be the ideal sheaf of D, so that $\mathcal{I} \cdot \mathcal{J}$ is the ideal sheaf of $D + E$. To prove the remaining part of the thesis, we show that the equality

$$\mathrm{Fitt}_0(\pi_*\mathcal{O}_X/\mathcal{I}\mathcal{J}) = \mathrm{Fitt}_0(\pi_*\mathcal{O}_X/\mathcal{I}) \cdot \mathrm{Fitt}_0(\pi_*\mathcal{O}_X/\mathcal{J})$$

holds locally around any point $y \in Y$. Let V be an open neighborhood of y such that $(\pi_*\mathcal{O}_X)_{|V}$ is a free $\mathcal{O}_Y$-module, $\pi_*\mathcal{I}_{|V}$ is generated by sections $s_1, \ldots, s_r$ of $\Gamma(V, \pi_*\mathcal{O}_X)$ and $\pi_*\mathcal{J}_{|V}$ is a principal ideal generated by a section h of $\Gamma(V, \pi_*\mathcal{O}_X)$. In terms of exact sequences:

$$(\pi_*\mathcal{O}_X)_{|V}^{\oplus r} \xrightarrow{(\cdot s_1, \ldots, \cdot s_r)} (\pi_*\mathcal{O}_X)_{|V} \to (\pi_*\mathcal{O}_X/\mathcal{I})_{|V} \to 0$$

$$(\pi_*\mathcal{O}_X)_{|V} \xrightarrow{\cdot h} (\pi_*\mathcal{O}_X)_{|V} \to (\pi_*\mathcal{O}_X/\mathcal{J})_{|V} \to 0$$

The ideal sheaf $\mathcal{I} \cdot \mathcal{J}$ is equal to $\mathcal{J} \cdot \mathcal{I}$, which is generated on V by the sections $hs_1, \ldots, hs_n$. In terms of exact sequences:

$$(\pi_*\mathcal{O}_X)_{|V}^{\oplus r} \xrightarrow{(\cdot hs_1, \ldots, \cdot hs_r)} (\pi_*\mathcal{O}_X)_{|V} \to (\pi_*\mathcal{O}_X/\mathcal{I}\mathcal{J})_{|V} \to 0$$

Denote with S_i and H the $(\mathcal{O}_Y)_{|V}$-matrices representing the multiplication by s_i and h respectively. The map $(\cdot hs_1, \ldots, \cdot hs_r)$ in the previous exact sequence is represented by the $n \times nr$ matrix

$$M = \left[\ HS_1\ \middle|\ \ldots\ \middle|\ HS_r\ \right] = H \left[\ S_1\ \middle|\ \ldots\ \middle|\ S_r\ \right]$$

The 0-th Fitting ideal of $\pi_*\mathcal{O}_X/\mathcal{I}\mathcal{J}$, restricted to V, is the ideal of $(\pi_*\mathcal{O}_X)_{|V}$ generated by the $n \times n$ minors of the matrix M. Any such minor is equal to a $n \times n$ minor of the matrix $\left[\ S_1\ \middle|\ \ldots\ \middle|\ S_r\ \right]$ multiplied by $\det H$. Then, by Definition 2.3.1,

$$\mathrm{Fitt}_0(\pi_*\mathcal{O}_X/\mathcal{I}\mathcal{J})_{|V} = \mathrm{Fitt}_0(\pi_*\mathcal{O}_X/\mathcal{I})_{|V} \cdot \mathrm{Fitt}_0(\pi_*\mathcal{O}_X/\mathcal{J})_{|V}$$

$\square$

DEFINITION/LEMMA 2.3.5. (Direct image of a generalized divisor) Let $D \in \mathrm{GDiv}(X)$ be a generalized divisor on X, such that $D = D' - E$ with $D' \in \mathrm{GDiv}^+(X)$ and $E \in \mathrm{CDiv}^+(X)$ by Lemma 2.2.3. The *direct image of D with respect to π*, denoted with $\pi_*(D)$, is the generalized divisor $\pi_*(D') - \pi_*(E)$.

PROOF. To prove that it is well defined, let $D = D' - E = \widetilde{D}' - \widetilde{E}$, with $D', \widetilde{D}'$ effective and $E, \widetilde{E}$ effective Cartier. Since $E, \widetilde{E}$ are Cartier, $D' + \widetilde{E} = \widetilde{D}' + E$; then, by Lemma 2.3.4 we have:

$$\pi_*(D') + \pi_*(\widetilde{E}) = \pi_*(\widetilde{D}') + \pi_*(E).$$

Since $\pi_*(E)$ and $\pi_*(\widetilde{E})$ are also Cartier by Lemma 2.3.4, they can be subtracted from each side in order to obtain:

$$\pi_*(D') - \pi_*(E) = \pi_*(\widetilde{D}') - \pi_*(\widetilde{E}).$$

This shows that $\pi_*(D)$ does not depend on the choice of D' and E. $\square$

We study now some properties of the direct image for generalized divisors.

PROPOSITION 2.3.6. (Properties of direct image)

(1) Let $D \in \mathrm{CDiv}(X)$ be a Cartier divisor. Then, $\pi_*(D)$ is a Cartier divisor and $\pi_*(-D) = -\pi_*(D)$. Moreover, $\pi_*(D)$ coincides with $\pi_*(D)$ of Definition 2.1.3.

(2) Let $D, E \in \mathrm{GDiv}(X)$ be generalized divisors, such that E is Cartier. Then, $\pi_*(D + E) = \pi_*(D) + \pi_*(E)$.

(3) Let $V \subset Y$ be an open subset, and denote with π_U the restriction of π to $U = \pi^{-1}(V) \subset X$. Let $D \in \mathrm{GDiv}(X)$ be a generalized divisor on X. Then,

$$(\pi_U)_*(D_{|U}) = \pi_*(D)_{|V}$$

(4) Let $D, D' \in \mathrm{GDiv}(X)$ be generalized divisors such that $D \sim D'$. Then, $\pi_*(D) \sim \pi_*(D')$.

PROOF. To prove (1), consider $D = E - F$ with E, F effective and F Cartier by Lemma 2.2.3. Since D is Cartier and $E = D + F$, then also E is Cartier. By Definition 2.3.5,

$$\pi_*(D) = \pi_*(E) - \pi_*(F).$$

By Lemma 2.3.4, it is a difference of Cartier divisors and hence it is Cartier. To compute $\pi_*(-D)$, note that since F is Cartier then $-D = F - E$, and it is a difference of effective Cartier divisors; then, apply Definition 2.3.5 to obtain

$$\pi_*(-D) = \pi_*(F) - \pi_*(E) =$$
$$= -\pi_*(D).$$

To compare $\pi_*(D)$ with Definition 2.1.3, let $\mathcal{I}$ and $\mathcal{J}$ be the ideal sheaves of E and F and let $\{V_i\}_{i \in I}$ be an open cover of Y such that $\pi_*\mathcal{I}_{|V_i}$ and $\pi_*\mathcal{J}_{|V_i}$ are non-degenerate principal ideals of $(\pi_*\mathcal{O}_X)_{|V_i}$-modules generated by the regular sections f_i and g_i of $\Gamma(V, \pi_*\mathcal{O}_X)$, respectively on each $i \in I$. The fractional ideal of D is then generated on each V_i by the meromorphic regular section $h_i = f_i/g_i$ of $\Gamma(V_i, \pi_*\mathcal{K}_X^*)$. By the proof of Lemma 2.3.4, the ideal sheaves of $\pi_*(E)$ and $\pi_*(F)$ are generated on each V_i by $\det(\cdot f_i)$ and $\det(\cdot g_i)$, so the fractional ideal of $\pi_*(D)$ is generated on each V_i by the meromorphic regular section $\det(\cdot f_i)/\det(\cdot g_i)$ of $\Gamma(V, \mathcal{K}_Y^*)$. Using the same cover in Definition 2.1.3 and applying Definition 2.1.1, the sheaf $\pi_*(D)$ defined above and the sheaf $\pi_*(D)$ defined in Section 2.1 have the same local generators, hence they are equal.

To prove (2), consider $D = D_1 - D_2$ and $E = E_1 - E_2$, with D_1, D_2, E_1, E_2 effective and D_2, E_2 Cartier. Note that $D + E = (D_1 + E_1) - (D_2 + E_2)$, and it is a difference of effective divisors with $D_2 + E_2$ Cartier. Then, applying Definition 2.3.5 and Lemma 2.3.4, we obtain:

$$\pi_*(D + E) = \pi_*((D_1 + E_1) - (D_2 + E_2)) =$$
$$= \pi_*(D_1 + E_1) - \pi_*(D_2 + E_2) =$$
$$= \pi_*(D_1) + \pi_*(E_1) - \pi_*(D_2) - \pi_*(E_2) =$$
$$= (\pi_*(D_1) - \pi_*(D_2)) - (\pi_*(E_1) - \pi_*(E_2)) =$$
$$= \pi_*(D) + \pi_*(E).$$

To prove (3), if D is effective, the result follows from Definition 2.3.1. Then, observe that the operations of product and inverse of fractional ideals commute with restrictions.

To prove (4), let $f \in \Gamma(X, \mathcal{K}_X)$ be a global section that generates a principal divisor $E = (f) \in \mathrm{Prin}(X)$. By linearity, it is sufficient to

prove that $\pi_*(E)$ is again principal. Let $\{V_i\}$ be an affine open cover of Y, such that locally

$$f_{|\pi^{-1}(V_i)} = g_i/h_i, \qquad g_i, h_i \in \Gamma(\pi^{-1}(V_i), \mathcal{O}_X)$$

In terms of divisors, this means that $(f)_{|\pi^{-1}(V_i)} = (g_i) - (h_i)$, and it is a difference of effective Cartier divisors on $\pi^{-1}(V_i)$. Then, applying Part (3) and reasoning similarly to Part (1), we obtain:

$$\pi_*((f))_{|V} = (\pi_i)_*((g_i)) - (\pi_i)_*((h_i)) =$$
$$= (\det[\cdot g_i]) - (\det[\cdot h_i]) =$$
$$= (\det[\cdot g_i]/\det[\cdot h_i]).$$

Since taking determinants is multiplicative, the local sections $\det[\cdot g_i]/\det[\cdot h_i]$ glue together to give a global section $\widetilde{f}$ of $\mathcal{K}_Y$, such that $\pi_*(E) = (\widetilde{f})$. □

We are now ready to define the direct image for generalized line bundles.

DEFINITION/LEMMA 2.3.7. (Direct image for generalized line bundles) The direct image for generalized divisors induce a direct image map between the sets of generalized line bundles, defined as:

$$[\pi_*]: \qquad \mathrm{GPic}(X) \longrightarrow \mathrm{GPic}(Y)$$
$$[D] \mapsto [\pi_*(D)].$$

PROOF. Recall that for any curve X, the set $\mathrm{GPic}(X)$ can be seen equivalently as the set of generalized line bundles or as the set of generalized divisors modulo linear equivalence. Here, $[\pi_*]$ is defined in terms of generalized divisors modulo linear equivalence. By Proposition 2.3.6, the direct images of linearly equivalent divisors are linearly equivalent, hence $[\pi_*]$ is well defined. □

In the remaining part of this subsection, we study an alternative formula for $[\pi_*]$ in terms of generalized line bundles. First, we need a technical lemma.

LEMMA 2.3.8. *Let $\mathcal{F}$ be a coherent sheaf on a curve X which is locally free of rank 1 at any generic point and let $\omega = \omega_X$ be the canonical, or dualizing sheaf of X. Let $T(\mathcal{F})$ be the torsion subsheaf of $\mathcal{F}$*

and $\mathcal{F}^{\omega\omega} = \mathcal{H}om(\mathcal{H}om(\mathcal{F},\omega),\omega)$ be the double ω-dual. There, there is a canonical isomorphism

$$\mathcal{F}^{tf} = \mathcal{F}/T(\mathcal{F}) \xrightarrow{\ \sim\ } \mathcal{F}^{\omega\omega}.$$

PROOF. The sheaf $\mathcal{F}^{tf}$ is endowed with a quotient map $q : \mathcal{F} \twoheadrightarrow \mathcal{F}^{tf}$, which is universal among all arrows from $\mathcal{F}$ to torsion-free sheaves (i.e. pure of dimension 1). Note that, since $\mathcal{F}$ is locally free of rank 1 at any generic point of X, then q is generically an isomorphism. Let $\alpha(\mathcal{F}) : \mathcal{F} \to \mathcal{F}^{\omega\omega}$ be the canonical map from $\mathcal{F}$ to its double ω-dual. Since taking double ω-duals is functorial, there is a commutative diagram of homomorphism of sheaves:

$$\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\ q\ } & \mathcal{F}^{tf} \\
\Big\downarrow{\scriptstyle \alpha(\mathcal{F})} & & \Big\downarrow{\scriptstyle \alpha(\mathcal{F}^{tf})} \\
\mathcal{F}^{\omega\omega} & \xrightarrow{\ q^{\omega\omega}\ } & (\mathcal{F}^{tf})^{\omega\omega}.
\end{array}$$

Using [**Har07**, Proposition 1.5] and recalling that $S_1 = S_2$ for sheaves on curves, we note that a sheaf on X is ω-reflexive if and only if it is S_1. In particular, $\mathcal{F}^{\omega\omega}$ is ω-reflexive by [**Har07**, Proposition 1.6] and hence it is S_1. Then, by the universal property of q, there is a unique map $\psi : \mathcal{F}^{tf} \to \mathcal{F}^{\omega\omega}$ such that $\psi \circ q = \alpha(\mathcal{F})$. Note that

$$q^{\omega\omega} \circ \psi \circ q = q^{\omega\omega} \circ \alpha(\mathcal{F}) = \alpha(\mathcal{F}^{tf}) \circ q,$$

so by surjectivity of q we conclude that $q^{\omega\omega} \circ \psi = \alpha(\mathcal{F}^{tf})$.

We show moreover that ψ is an isomorphism. By construction $\mathcal{F}^{tf}$ is pure, hence it is ω-reflexive and $\alpha(\mathcal{F}^{tf})$ is an isomorphism. Moreover, $q^{\omega\omega}$ is surjective and generically an isomorphism since q is surjective and generically an isomorphism. Then, the kernel K of $q^{\omega\omega}$ is a subsheaf of $\mathcal{F}^{\omega\omega}$ which is generically zero. Since $\mathcal{F}^{\omega\omega}$ is pure, we conclude that K is everywhere zero and then $q^{\omega\omega}$ is an isomorphism. This shows that $\psi = (q^{\omega\omega})^{-1} \circ \alpha(\mathcal{F}^{tf})$ is an isomorphism. $\qquad\square$

In order to study the formula for $[\pi_*]$, we study a preliminary formula for π_* in the case of effective generalized divisors.

LEMMA 2.3.9. *Let $D \in \mathrm{GDiv}^+(X)$ be an effective generalized divisor with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Then, there is an injection:*

$$\left(\textstyle\bigwedge^n(\pi_*\mathcal{I})\right)^{\omega\omega} \otimes \det(\pi_*\mathcal{O}_X)^{-1} \hookrightarrow \det(\pi_*\mathcal{O}_X) \otimes \det(\pi_*\mathcal{O}_X)^{-1} \xrightarrow{\sim} \mathcal{O}_Y$$

whose image in $\mathcal{O}_Y$ is the 0-th Fitting ideal of $\pi_*\mathcal{O}_X/\mathcal{I}$.

PROOF. Consider the short exact sequence:

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

Since π is finite and flat, the pushforward induces a a short exact sequence:

$$0 \to \pi_*\mathcal{I} \xrightarrow{\varphi} \pi_*\mathcal{O}_X \to \pi_*(\mathcal{O}_X/\mathcal{I}) \to 0.$$

In particular, the last exact sequence is a finite presentation of $\pi_*(\mathcal{O}_X/\mathcal{I})$ whose middle term is locally free. Hence, by Definition 2.3.1, the 0-th Fitting ideal of $\pi_*(\mathcal{O}_X/\mathcal{I})$ is equal to the image of the morphism:

$$\bigwedge\nolimits^n\big(\pi_*\mathcal{I}\big) \otimes \det\big(\pi_*\mathcal{O}_X\big)^{-1} \xrightarrow{\det\varphi\otimes 1} \det(\pi_*\mathcal{O}_X) \otimes \det(\pi_*\mathcal{O}_X)^{-1} \xrightarrow{\sim} \mathcal{O}_Y.$$

Consider now the determinant map $\det\varphi$. Since $\det(\pi_*\mathcal{O}_X)$ is also a pure sheaf, applying the universal property of the torsion-free quotient together with Lemma 2.3.8 we obtain the following commutative diagram:

$$
\begin{array}{ccc}
\bigwedge^n\big(\pi_*\mathcal{I}\big) & \xrightarrow{\det\varphi} & \det(\pi_*\mathcal{O}_X) \\
\Big\downarrow{\scriptstyle\alpha} & {\scriptstyle\beta}\nearrow & \\
\big(\bigwedge^n\big(\pi_*\mathcal{I}\big)\big)^{\omega\omega} & &
\end{array}
$$

The canonical map α is surjective by Lemma 2.3.8. Since $\mathcal{I}$ is locally free of rank 1 at the generic points of X, both $\det\phi$ and α are generically isomorphisms; hence, the map β is generically an isomorphism. Since its domain $\big(\bigwedge^n\big(\pi_*\mathcal{I}\big)\big)^{\omega\omega}$ is pure, we conclude that its kernel is zero, hence β is injective. We have then factorized $\det\varphi$ as the composition of a surjective map α followed by an injective map β.

Tensoring by $\det(\pi_*\mathcal{O}_X)^{-1}$, we obtain then the following commutative diagram:

$$
\begin{array}{ccccc}
\bigwedge^n\big(\pi_*\mathcal{I}\big) \otimes \det(\pi_*\mathcal{O}_X)^{-1} & \xrightarrow{\det\varphi\otimes 1} & \det(\pi_*\mathcal{O}_X) \otimes \det(\pi_*\mathcal{O}_X)^{-1} & \xrightarrow[\eta]{\sim} & \mathcal{O}_Y \\
\Big\downarrow{\scriptstyle\alpha\otimes 1} & {\scriptstyle\beta\otimes 1}\nearrow & & & \\
\big(\bigwedge^n\big(\pi_*\mathcal{I}\big)\big)^{\omega\omega} \otimes \det(\pi_*\mathcal{O}_X)^{-1} & & & &
\end{array}
$$

The map $\alpha \otimes 1$ is surjective, so the composition $\eta \circ (\beta \otimes 1)$ is an injective map whose image in $\mathcal{O}_Y$ is equal to the image of the map $\eta \circ (\det\varphi \otimes 1)$. By the previous remark, such image coincides with the 0-th Fitting ideal of $\pi_*(\mathcal{O}_X/\mathcal{I})$, proving the lemma. $\qquad\square$

We are now ready to give a sheaf-theoretic formula for $[\pi_*]$. Recall that, for any curve X, the set $\mathrm{GPic}(X)$ can be seen as the set of isomorphism classes of generalized line bundles on on X.

PROPOSITION 2.3.10. (Formula for the direct image of generalized line bundles) *Let $\mathcal{L}$ be a generalized line bundle on X. Then,*

$$[\pi_*](\mathcal{L}) \simeq \left(\textstyle\bigwedge^n(\pi_*\mathcal{L})\right)^{\omega\omega} \otimes \det(\pi_*\mathcal{O}_X)^{-1}.$$

PROOF. By the surjectivity of $\mathrm{GDiv}(X) \twoheadrightarrow \mathrm{GPic}(X)$, we can pick a generalized divisor $D \in \mathrm{GDiv}(X)$ with fractional ideal $\mathcal{I}$ isomorphic to $\mathcal{L}$; then,

$$[\pi_*](\mathcal{L}) = [\pi_*]([D]) = [\pi_*(D)]$$

by Definition 2.3.7. By Lemma 2.2.3, there are effective generalized divisors E, F on X such that $D = E - F$ and F Cartier. Denote with $\mathcal{I}'$ the ideal sheaf of E and with $\mathcal{J}$ the ideal sheaf of F. Since F is Cartier, the condition $D = E - F$ can be rewritten as $E = D + F$, or in terms of sheaves:

$$\mathcal{I}' = \mathcal{I} \cdot \mathcal{J}.$$

Consider the direct images of E and F. By Lemma 2.3.9, the ideal sheaf of $\pi_*(E)$ is isomorphic to

$$\left(\textstyle\bigwedge^n(\pi_*\mathcal{I}')\right)^{\omega\omega} \otimes \det(\pi_*\mathcal{O}_X)^{-1},$$

while the ideal sheaf of $\pi_*(F)$ is isomorphic to

$$\left(\textstyle\bigwedge^n(\pi_*\mathcal{J})\right)^{\omega\omega} \otimes \det(\pi_*\mathcal{O}_X)^{-1} \simeq \textstyle\bigwedge^n(\pi_*\mathcal{J}) \otimes \det(\pi_*\mathcal{O}_X)^{-1}.$$

By Definition 2.3.5, $\pi_*(D) = \pi_*(E) - \pi_*(F)$. Then, the fractional ideal of $\pi_*(D)$ is isomorphic to:

$$\left(\textstyle\bigwedge^n(\pi_*\mathcal{I}')\right)^{\omega\omega} \otimes \det(\pi_*\mathcal{O}_X)^{-1} \otimes \left(\textstyle\bigwedge^n(\pi_*\mathcal{J})\right)^{-1} \otimes \det(\pi_*\mathcal{O}_X),$$

which in turn is isomorphic to:

$$\left(\textstyle\bigwedge^n(\pi_*\mathcal{I}')\right)^{\omega\omega} \otimes \left(\textstyle\bigwedge^n(\pi_*\mathcal{J})\right)^{-1}.$$

Then, we are left to prove that:

$$\left(\textstyle\bigwedge^n(\pi_*\mathcal{I}')\right)^{\omega\omega} \otimes \left(\textstyle\bigwedge^n(\pi_*\mathcal{J})\right)^{-1} \simeq \left(\textstyle\bigwedge^n(\pi_*\mathcal{I})\right)^{\omega\omega} \otimes \det(\pi_*\mathcal{O}_X)^{-1},$$

or, equivalently, that:

$$\left(\textstyle\bigwedge^n(\pi_*\mathcal{I}')\right)^{\omega\omega} \otimes \det(\pi_*\mathcal{O}_X) \simeq \left(\textstyle\bigwedge^n(\pi_*\mathcal{I})\right)^{\omega\omega} \otimes \left(\textstyle\bigwedge^n(\pi_*\mathcal{J})\right),$$

under the assumptions $\mathcal{I}' = \mathcal{I} \cdot \mathcal{J}$ and $\mathcal{J}$ locally principal. Consider an open cover $\{V_i\}_{i \in I}$ of Y such that $\mathcal{J}$ is trivial on each $U_i = \pi^{-1}(V_i)$, i.e. there is an isomorphism $\lambda_i : \mathcal{J}_{|U_i} = (f_i) \xrightarrow{\sim} \mathcal{O}_{X|U_i}$ for each $i \in I$. On the intersections $U_i \cap U_j$, the collection $\{\lambda_i \circ \lambda_j^{-1}\}$ of automorphisms of $\mathcal{O}_{X|U_i \cap U_j}$ is a cochain of elements of $\mathcal{O}_{X|U_i \cap U_j}^*$ that measures the obstruction for the λ_i's to glue to a global isomorphism. We define now an isomorphism

$$\alpha : \left(\textstyle\bigwedge^n (\pi_* \mathcal{I}')\right)^{\omega\omega} \otimes \det(\pi_* \mathcal{O}_X) \longrightarrow \left(\textstyle\bigwedge^n (\pi_* \mathcal{I})\right)^{\omega\omega} \otimes \left(\textstyle\bigwedge^n (\pi_* \mathcal{J})\right)$$

by glueing a collection of isomorphisms α_i defined on each V_i. To do so, we define each α_i as the following composition of arrows:

$$
\begin{array}{ccc}
\left(\left(\bigwedge^n (\pi_* \mathcal{I}')\right)^{\omega\omega} \otimes \det(\pi_* \mathcal{O}_X)\right)_{|V_i} & \xrightarrow{\ \alpha_i\ } & \left(\left(\bigwedge^n (\pi_* \mathcal{I})\right)^{\omega\omega} \otimes \bigwedge^n (\pi_* \mathcal{J})\right)_{|V_i} \\[2mm]
\| \Big\downarrow & & \Big\uparrow \wr \\[2mm]
\left(\bigwedge^n \left(\pi_* \mathcal{I}'_{|U_i}\right)\right)^{\omega\omega} \otimes \det(\pi_* \mathcal{O}_{X|U_i}) & & \left(\bigwedge^n \left(\pi_* \mathcal{I}_{|U_i}\right)\right)^{\omega\omega} \otimes \left(\bigwedge^n \left(\pi_* \mathcal{J}_{|U_i}\right)\right)^{\omega\omega} \\[2mm]
\scriptstyle(\bigwedge^n(\pi_*\lambda_i))^{\omega\omega}\otimes\mathrm{id}\Big\downarrow & & \Big\uparrow\scriptstyle\mathrm{id}\otimes\left(\bigwedge^n(\pi_*\lambda_i^{-1})\right)^{\omega\omega} \\[2mm]
\left(\bigwedge^n \left(\pi_* \mathcal{I}_{|U_i}\right)\right)^{\omega\omega} \otimes \det(\pi_* \mathcal{O}_{X|U_i}) & \simeq & \left(\bigwedge^n \left(\pi_* \mathcal{I}_{|U_i}\right)\right)^{\omega\omega} \otimes \left(\bigwedge^n \left(\pi_* \mathcal{O}_{X|U_i}\right)\right)^{\omega\omega}
\end{array}
$$

i.e. $\alpha_i := \left(\bigwedge^n (\pi_* \lambda_i)\right)^{\omega\omega} \otimes \left(\bigwedge^n (\pi_* \lambda_i^{-1})\right)^{\omega\omega}$. Since $\bigwedge^n (\pi_{*\text{-}})$ and $(\text{-})^{\omega\omega}$ are functorial, the obstruction $\alpha_i \circ \alpha_j^{-1}$ is trivial on any $V_i \cap V_j$, whence the α_i's glue together to a global isomorphism α. $\qquad\square$

COROLLARY 2.3.11. *Let $\mathcal{L}$ be a line bundle on X. Then,*

$$\mathrm{Nm}_\pi(\mathcal{L}) = [\pi_*](\mathcal{L}).$$

PROOF. By Lemma 2.1.7,

$$\mathrm{Nm}_\pi(\mathcal{L}) \simeq \det(\pi_* \mathcal{L}) \otimes \det(\pi_* \mathcal{O}_X)^{-1}.$$

On the other side, by Proposition 2.3.10,

$$[\pi_*](\mathcal{L}) \simeq (\det(\pi_* \mathcal{L}))^{\omega\omega} \otimes \det(\pi_* \mathcal{O}_X)^{-1}.$$

Since $\mathcal{L}$ is locally free, $\det(\pi_* \mathcal{L})$ is a line bundle and in particular is a pure coherent sheaf. Then, $\det(\pi_* \mathcal{L}) \simeq (\det(\pi_* \mathcal{L}))^{\omega\omega}$, proving the thesis. $\qquad\square$

COROLLARY 2.3.12. *Let $\mathcal{L}$ be a generalized line bundle on X. Suppose that Y is smooth. Then:*

$$[\pi_*](\mathcal{L}) \simeq \bigwedge^n (\pi_*\mathcal{L}) \otimes \det(\pi_*\mathcal{O}_X)^{-1}.$$

PROOF. First note that, by Proposition 2.3.10,

$$[\pi_*](\mathcal{L}) \simeq \left(\bigwedge^n (\pi_*\mathcal{L})\right)^{\omega\omega} \otimes \det(\pi_*\mathcal{O}_X)^{-1}.$$

Second, observe that the pure sheaf $\pi_*\mathcal{L}$ is locally free of rank n since Y is smooth (see [**HL10**, Example 1.1.16]), and $\bigwedge^n (\pi_*\mathcal{L})$ is a line bundle. Then,

$$\left(\bigwedge^n (\pi_*\mathcal{L})\right)^{\omega\omega} \simeq \bigwedge^n (\pi_*\mathcal{L}).$$

Combining these two facts, we have proved the thesis. $\qquad\square$

2.3.3. The inverse image. In this subsection, we define the inverse image for generalized divisors and generalized line bundles and we study the relation of the inverse image with the direct image. We start from the case of effective divisors.

DEFINITION/LEMMA 2.3.13. (Inverse image of an effective generalized divisor) *Let $D \in \mathrm{GDiv}^+(Y)$ be an effective generalized divisor with ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_Y$. The inverse image of D relative to π, denoted $\pi^*(D)$, is the effective generalized divisor with ideal sheaf $\pi^{-1}(\mathcal{I}) \cdot \mathcal{O}_X$.*

PROOF. The inverse image ideal $\pi^{-1}(\mathcal{I})$ is an ideal sheaf of $\pi^{-1}(\mathcal{O}_Y)$-modules and can be extended to $\mathcal{O}_X$-modules via the canonical map of sheaves of rings $\pi^\sharp : \pi^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$. It is coherent since $\mathcal{I}$ is coherent. If η is a generic point of X, then $\pi(\eta)$ is a generic point of Y, hence

$$(\pi^{-1}(\mathcal{I}) \cdot \mathcal{O}_X)_\eta = \mathcal{I}_{\pi(\eta)} \cdot \mathcal{O}_{X,\eta} = \mathcal{O}_{X,\eta}$$

since $\mathcal{I}_{\pi(\eta)} = \mathcal{O}_{Y,\pi(\eta)}$. $\qquad\square$

REMARK 2.3.14. In the setting of Definition 2.3.13, consider the short exact sequence:

$$0 \to \mathcal{I} \to \mathcal{O}_Y \to \mathcal{O}_D \to 0.$$

Since π is flat and surjective, the pullback functor $_-\otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{O}_X$ is exact as well as the inverse image functor π^{-1}. Then, the previous exact sequence induces the following exact sequence:

$$0 \to \pi^*\mathcal{I} \to \pi^*\mathcal{O}_Y \to \pi^*\mathcal{O}_D \to 0.$$

Since $\pi^*\mathcal{O}_Y = \mathcal{O}_X$, the pullback sheaf $\pi^*(\mathcal{I})$ has a canonical injection in $\mathcal{O}_X$, whose image is exactly the inverse image ideal $\pi^{-1}(\mathcal{I}) \cdot \mathcal{O}_X$

LEMMA 2.3.15. *Let $D \in \mathrm{GDiv}^+(Y)$ be a generalized effective divisor and $E \in \mathrm{CDiv}^+(Y)$ be an effective Cartier divisor on Y. Then, the inverse image divisor $\pi^*(E)$ is Cartier and $\pi^*(D+E) = \pi^*(D) + \pi^*(E)$.*

PROOF. Let $\mathcal{I}$ and $\mathcal{J}$ be the ideal sheaves of D and E respectively. The ideal sheaf $\mathcal{J}$ is locally principal, hence its inverse image $\pi^{-1}(\mathcal{J}) \cdot \mathcal{O}_X$ is again locally principal; then, $\pi^*(E)$ is Cartier. The generalized divisor $D + E$ is defined by the ideal sheaf $\mathcal{I} \cdot \mathcal{J}$, whose inverse image is:

$$\pi^{-1}(\mathcal{I} \cdot \mathcal{J}) \cdot \mathcal{O}_X = (\pi^{-1}(\mathcal{I}) \cdot \pi^{-1}(\mathcal{J})) \cdot \mathcal{O}_X =$$
$$= (\pi^{-1}(\mathcal{I}) \cdot \mathcal{O}_X) \cdot (\pi^{-1}(\mathcal{J}) \cdot \mathcal{O}_X),$$

which is the defining ideal of $\pi^*(D) + \pi^*(E)$. $\qquad\square$

Now, we can extend the definition of inverse image to any generalized divisor and study its properties.

DEFINITION/LEMMA 2.3.16. (Inverse image of a generalized divisor) Let $D \in \mathrm{GDiv}(Y)$ be any generalized divisor and let $D = E - F$ with E, F effective generalized divisors and F Cartier by Lemma 2.2.3. The *inverse image of D relative to π*, denoted $\pi^*(D)$, is the generalized divisor $\pi^*(E) - \pi^*(F)$.

PROOF. To prove that it is well defined, let $D = D' - E = \widetilde{D}' - \widetilde{E}$, with $D', \widetilde{D}'$ effective and $E, \widetilde{E}$ effective Cartier. Since $E, \widetilde{E}$ are Cartier, $D' + \widetilde{E} = \widetilde{D}' + E$; then, by Lemma 2.3.15 we have:

$$\pi^*(D') + \pi^*(\widetilde{E}) = \pi^*(\widetilde{D}') + \pi^*(E).$$

Again by Lemma 2.3.15 $\pi^*(E)$ and $\pi^*(\widetilde{E})$ are Cartier, so they can be subtracted from each side in order to obtain:

$$\pi^*(D') - \pi^*(E) = \pi^*(\widetilde{D}') - \pi^*(\widetilde{E}),$$

so $\pi^*(D)$ does not depend on the choice of D' and E. $\qquad\square$

PROPOSITION 2.3.17. (Properties of inverse image)

(1) Let $D \in \mathrm{CDiv}(Y)$ be a Cartier divisor on Y. Then, $\pi^*(D)$ is a Cartier divisor and $\pi^*(-D) = -\pi^*(D)$. Moreover, $\pi^*(D)$ coincides with $\pi^*(D)$ of Definition 2.1.3.

(2) Let $D, E \in \mathrm{GDiv}(Y)$ be generalized divisors, such that E is Cartier. Then, $\pi^*(D + E) = \pi^*(D) + \pi^*(E)$.

(3) Let $V \subset Y$ be an open subset, and denote with π_U the restriction of π to $U = \pi^{-1}(V) \subset X$. Let $D \in \mathrm{GDiv}(Y)$ be a generalized divisor on Y. Then,

$$(\pi_U)^*(D_{|V}) = \pi^*(D)_{|U}$$

(4) Let $D, D' \in \mathrm{GDiv}(Y)$ be generalized divisors such that $D \sim D'$. Then, $\pi^*(D) \sim \pi^*(D')$.

PROOF. To prove (1), consider $D = E - F$ with E, F effective and F Cartier by Lemma 2.2.3 on Y. Since D is Cartier and $E = D + F$, then also E is Cartier. By Definition 2.3.16,

$$\pi^*(D) = \pi^*(E) - \pi^*(F).$$

By Lemma 2.3.15, it is a difference of Cartier divisors and hence it is Cartier. To compute $\pi(-D)$, note that since F is Cartier then $-D = F - E$, and it is a difference of effective Cartier divisors; then, apply Definition 2.3.16 to obtain

$$\pi^*(-D) = \pi^*(F) - \pi^*(E) =$$
$$= -\pi^*(D).$$

To compare $\pi^*(D)$ with Definition 2.1.3, let $\mathcal{I}$ and $\mathcal{J}$ be the ideal sheaves of E and F and let $\{V_i\}_{i \in I}$ be an open cover of Y such that $\mathcal{I}_{|V_i}$ and $\mathcal{J}_{|V_i}$ are principal ideals of $\mathcal{O}_{Y|V_i}$-modules generated by regular sections s_i and t_i of $\Gamma(V_i, \mathcal{O}_Y)$, respectively on each $i \in I$. The fractional ideal of D is generated on each V_i by the meromorphic regular section $u_i = s_i/t_i$ of $\Gamma(V_i, \mathcal{K}_Y)$. By Definition 2.3.13, the ideal sheaves of $\pi^*(E)$ and $\pi^*(F)$ are generated on each $U_i = \pi^{-1}(V_i)$ by $\pi^\sharp_{U_i}(s_i)$ and $\pi^\sharp_{U_i}(t_i)$ respectively. Then, by Definition 2.3.16, the fractional ideal of $\pi^*(D)$ is generated on each U_i by the meromorphic regular section $\pi^\sharp_{U_i}(s_i)/\pi^\sharp_{U_i}(t_i)$ of $\Gamma(U_i, \mathcal{K}_X)$. These are exactly the local generators for $\pi^*(D)$ as defined in Definition 2.1.3.

To prove (2), consider $D = D_1 - D_2$ and $E = E_1 - E_2$, with D_1, D_2, E_1, E_2 effective and D_2, E_2 Cartier. Note that $D + E = (D_1 + E_1) - (D_2 + E_2)$, and it is a difference of effective divisors with $D_2 + E_2$ Cartier. Then, applying Definition 2.3.16 and Lemma 2.3.15, we obtain:

$$\pi^*(D + E) = \pi^*((D_1 + E_1) - (D_2 + E_2)) =$$
$$= \pi^*(D_1 + E_1) - \pi^*(D_2 + E_2) =$$
$$= \pi^*(D_1) + \pi^*(E_1) - \pi^*(D_2) - \pi^*(E_2) =$$
$$= (\pi^*(D_1) - \pi^*(D_2)) - (\pi^*(E_1) - \pi^*(E_2)) =$$
$$= \pi^*(D) + \pi^*(E).$$

To prove (3), if D is effective, the result follows the fact that the inverse image functor π^{-1} commutes with restrictions. Then, observe that the operations of product and inverse of fractional ideals also commute with restrictions.

To prove (4), let $f \in \Gamma(Y, \mathcal{K}_Y)$ be a global section that generates a principal divisor $E = (f) \in \mathrm{Prin}(Y)$ such that $D = D' + E$. The inverse image $\pi^*(E)$ is the principal divisor $(\pi^\sharp(f)) \in \mathrm{Prin}(X)$. Then, by Lemma 2.3.15, $\pi^*(D) = \pi^*(D') + (\pi^\sharp(f))$; so $\pi^*(D)$ and $\pi^*(D')$ are linearly equivalent. $\qquad\square$

We are now ready to define the inverse image for generalized line bundles.

DEFINITION/LEMMA 2.3.18. (Inverse image for generalized line bundles) The inverse image for generalized divisors induce a inverse image map between the sets of generalized line bundles, defined as:

$$[\pi^*]: \qquad \mathrm{GPic}(Y) \longrightarrow \mathrm{GPic}(X)$$
$$[D] \mapsto [\pi^*(D)].$$

PROOF. Recall that for any curve X, the set $\mathrm{GPic}(X)$ can be seen equivalently as the set of generalized line bundles or as the set of generalized divisors modulo linear equivalence. Here, $[\pi^*]$ is defined in terms of generalized divisors modulo linear equivalence. By Proposition 2.3.17, the direct images of linearly equivalent divisors are linearly equivalent, hence $[\pi^*]$ is well defined. $\qquad\square$

REMARK 2.3.19. If $\mathcal{L}$ is a generalized line bundle on Y and D is a generalized divisor with fractional ideal $\mathcal{I}$ isomorphic to $\mathcal{L}$, then the inverse image divisor $\pi^*(D)$ has fractional ideal isomorphic to the pullback sheaf $\pi^*(\mathcal{I})$ by Remark 2.3.14. Since $\pi^*(\mathcal{I})$ is isomorphic to $\pi^*(\mathcal{L})$ as abstract $\mathcal{O}_X$-modules, we conclude that the inverse image $[\pi^*](\mathcal{L})$ of the generalized line bundle $\mathcal{L}$ is equal to the pullback sheaf $\pi^*(\mathcal{L})$.

We prove now a result on the composition of the direct image with the inverse image of generalized divisors.

PROPOSITION 2.3.20. (Composition of the direct image with the inverse image) *Let $D \in \mathrm{GDiv}(Y)$ be a generalized divisor on Y. Then,*

$$\pi_*(\pi^*(D)) = n \cdot D.$$

PROOF. Since both of the terms are linear with respect to the sum of Cartier divisor, we can suppose that D is effective with ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_Y$. First note that, from the exact sequence:

$$0 \to \pi^*\mathcal{I} \to \pi^*\mathcal{O}_Y \to \pi^*\mathcal{O}_D \to 0$$

together with Remark 2.3.14, we obtain $\mathcal{O}_X/(\pi^{-1}\mathcal{I}\cdot\mathcal{O}_X) = \mathcal{O}_X/\pi^*\mathcal{I} = \pi^*(\mathcal{O}_Y/\mathcal{I})$. To prove the thesis, we show that the equality

$$\mathrm{Fitt}_0\left(\pi_*(\mathcal{O}_X/\pi^*\mathcal{I})\right) = \mathcal{I}^n$$

holds locally around any point $y \in Y$. Let $V \subseteq Y$ be an open neighborhood of y such that $(\pi_*\mathcal{O}_X)_{|V} \simeq (\mathcal{O}_{Y|V})^{\oplus n}$ and $\mathcal{I}_{|V}$ is generated by sections $s_1, \ldots, s_r$ of $\Gamma(V, \mathcal{I})$. Then, consider the following presentation:

$$\mathcal{O}_{Y|V}^{\oplus r} \xrightarrow{(\cdot s_1, \ldots, \cdot s_r)} \mathcal{O}_{Y|V} \to (\mathcal{O}_Y/\mathcal{I})_{|V} \to 0.$$

Pulling back with π^*, we obtain the following exact sequence on $U = \pi^{-1}(V)$:

$$\mathcal{O}_{X|U}^{\oplus r} \xrightarrow{(\cdot s_1, \ldots, \cdot s_r)} \mathcal{O}_{X|U} \to (\mathcal{O}_X/\pi^*\mathcal{I})_{|U} \to 0.$$

In order to compute $\mathrm{Fitt}_0\left(\pi_*(\mathcal{O}_X/\pi^*\mathcal{I})\right)_{|V}$, we consider then the pushforward sequence:

$$\left(\pi_*\mathcal{O}_X^{\oplus r}\right)_{|V} \xrightarrow{(\cdot s_1, \ldots, \cdot s_r)} \left(\pi_*\mathcal{O}_X\right)_{|V} \to \pi_*(\mathcal{O}_X/\pi^*\mathcal{I})_{|V} \to 0.$$

Since $\pi_*(\mathcal{O}_X/\pi^*\mathcal{I})_{|V} \simeq (\mathcal{O}_Y^{\oplus n})_{|V}$, the map on the left is represented by the following $n \times nr$ matrix with entries in $\Gamma(V, \mathcal{O}_Y)$:

$$M = \begin{bmatrix} s_1 & & & & s_n & & \\ & \ddots & & \cdots & & \ddots & \\ & & s_1 & & & & s_n \end{bmatrix}.$$

Now, $\mathrm{Fitt}_0\left(\pi_*(\mathcal{O}_X/\pi^*\mathcal{I})\right)_{|V}$ is generated by the $n \times n$ minors of M, i.e. all the possible products of n generators of $\mathcal{I}$ on V, with ripetitions. This shows that $\mathrm{Fitt}_0\left(\pi_*(\mathcal{O}_X/\pi^*\mathcal{I})\right)_{|V} = \mathcal{I}_{|V}^n$. $\qquad \square$

2.3.4. Relation with the degree. In this subsection we show that π_* preserves the degree of divisors under the condition that Y is smooth over k. In general, however, the direct image of a generalized divisor D may not have the same degree of D, as the following example shows. Since we are dealing with degrees, we **assume that X and Y are projective curves over a base field k**.

EXAMPLE 2.3.21. Fix k an algebraically closed field. Let $A = k[x, y]/(y^2 - x^4)$ be the affine coordinate ring of a curve $X = \mathrm{Spec}\, A$ with a tacnode at the point P corresponding to the maximal ideal $\mathfrak{p} = (x, y)$.

The involution σ on A defined by $x \mapsto -x, y \mapsto y$ induces a involution σ_X on the curve X. The geometric quotient $Y = X/\sigma_X$ is an affine curve with coordinate ring equal to the ring of invariants $A^\sigma = k[x^2, y]/(y^2 - x^4)$, that is isomorphic to $B = k[s, t]/(t^2 - s^2)$ putting $s \mapsto x^2$ and $t \mapsto y$. The quotient curve Y has a simple node at the point Q corresponding to the maximal ideal $\mathfrak{q} = (s, t)$.

The inclusion map $A^\sigma \subset A$ gives to A the structure of free B-module, with basis $\{1, x\}$; so, the corresponding morphism of curve $\pi : X \to Y = X/\sigma_X$ is a finite, locally free map of degree 2 sending P to Q.

Let D be the generalized divisor on X defined by the ideal $I = (x^2, y) \subset A$; note that D is supported only on the tacnode P. Since we want to compare $\deg(D) = \deg_P(D)$ with $\deg(\pi_*(D)) = \deg_Q(\pi_*(D))$, we can restrict to work locally around P and Q. Let $B_{\mathfrak{q}}$ be the local ring of Y at Q. The induced map $B_{\mathfrak{q}} \to A_{\mathfrak{p}}$ makes $A_{\mathfrak{p}}$ a free $B_{\mathfrak{q}}$-module of rank 2.

Let $E = A_{\mathfrak{p}}/I_{\mathfrak{p}} \simeq k[x]/(x^2)$ be the local ring of D at the P. We have:

$$\deg_P(D) = \ell(E) = 2.$$

Observe that E has the following free presentation as $A_{\mathfrak{p}}$-module:

$$A_{\mathfrak{p}}^{\oplus 2} \xrightarrow{(\cdot x^2, \cdot y)} A_{\mathfrak{p}} \to E \to 0.$$

Since $A_{\mathfrak{p}}$ is a free $B_{\mathfrak{q}}$-module of rank 2, E has also a presentation as $B_{\mathfrak{q}}$-module:

$$B_{\mathfrak{q}}^{\oplus 4} \xrightarrow{\varphi} B_{\mathfrak{q}}^{\oplus 2} \to E \to 0$$

where

$$\varphi = \left[\begin{array}{cc|cc} s & 0 & t & 0 \\ 0 & s & 0 & t \end{array} \right].$$

Then, the 0-th Fitting ideal of E as $B_{\mathfrak{q}}$-module is $F_0(E) = (s^2, t^2, st) \subset B_{\mathfrak{q}}$. We have:

$$\deg_Q(\pi_*(D)) = \ell(B_{\mathfrak{q}}/F_0(E)) = 3.$$

Note that there are divisors of degree 2 on X, supported at P, whose direct image has degree 2 on Y. For example, take $D' = (x)$.

REMARK 2.3.22. The previous example shows, in particular, that Proposition 2.33 in [**Vas04**] is false.

We now prove that the degree is preserved if the direct image is Cartier. We show first a proposition that computes the degree at any point where the direct image is locally principal.

PROPOSITION 2.3.23. (Degree of the direct image of a generalized divisor) *Let $D \in \mathrm{GDiv}(X)$ be a generalized divisor on X and let $y \in Y$ be a point in codimension 1 of the support of $\pi_*(D)$. Suppose that $\pi_*(D)$ is locally principal at y. Then,*

$$\deg_y(\pi_*(D)) = \sum_{\pi(x)=y} \deg_x(D).$$

PROOF. First, suppose that D is effective. Let $V = \mathrm{Spec}(B)$ be an affine open neighborhood of y with affine pre-image $U = \pi^{-1}(V) = \mathrm{Spec}(A)$, and let $I \subset A$ denote the ideal of D restricted to U. The coordinate ring of D on U is the Artin ring A/I whose spectrum is

equal to $\mathrm{Spec}(A) \cap \mathrm{Supp}(D) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$; hence we have:

$$A/I = \prod_{i=1}^{s} (A/I)_{\mathfrak{p}_i}.$$

Let $\mathfrak{q} \subset B$ denote the maximal ideal corresponding to y in $\mathrm{Spec}(B)$ and let $B_\mathfrak{q}$ be the associated local ring of dimension 1. Then, the localization of A/I at $\mathfrak{q}$, denoted E, is the coordinate ring of D restricted to the fiber of y:

$$E = (A/I)_\mathfrak{q} = \prod_{\pi^{-1}(\mathfrak{p}_i)=\mathfrak{q}} (A/I)_{\mathfrak{p}_i}.$$

Since $\pi_*(D)$ is effective, the degree of $\pi_*(D)$ at y is:

$$\deg_y(\pi_*(D)) = \ell\big(\mathcal{O}_{Y,y}/F_0(\pi_*\mathcal{O}_D)_y\big)[\kappa(y) : k] =$$
$$= \ell\big(B_\mathfrak{q}/F_0(E)\big)[\kappa(y) : k].$$

By hypothesis $F_0(E)$ is an invertible module, so by [**Vas04**, Proposition 2.32], we have $\ell\big(B_\mathfrak{q}/F_0(E)\big) = \ell(E)$. On the other hand, thanks to [**Sta19**, Tag 02M0], note that:

$$\ell(E) = \ell \left(\prod_{\pi^{-1}(\mathfrak{p}_i)=\mathfrak{q}} (A/I)_{\mathfrak{p}_i} \right) =$$
$$= \sum_{\pi^{-1}(\mathfrak{p}_i)=\mathfrak{q}} \ell(A_{\mathfrak{p}_i}/I_{\mathfrak{p}_i})][k(\mathfrak{p}_i) : k(\mathfrak{q})] =$$
$$= \sum_{\pi(x)=y} \ell\big(\mathcal{O}_{X,x}/\mathcal{I}_x\big)[\kappa(x) : \kappa(y)] =$$
$$= \sum_{\pi(x)=y} \deg_x(D)[\kappa(x) : \kappa(y)].$$

Putting everything together we have:

$$\deg_y(\pi_*(D)) = \ell(E)[\kappa(y) : k] =$$

$$= \left(\sum_{\pi(x)=y} \deg_x(D)[\kappa(x) : \kappa(y)] \right) [\kappa(y) : k] =$$

$$= \sum_{\pi(x)=y} \deg_x(D)[\kappa(x) : k] =$$

$$= \sum_{\pi(x)=y} \deg_x(D)$$

for D effective generalized divisor on X.

Let now $D = E - F$ be a generalized divisor, written as a difference of effective generalized divisors with F Cartier by Lemma 2.2.3. Since $\pi_*(F)$ is Cartier, using the result for effective divisors establibed before, we have:

$$\deg_y(\pi_*(D)) = \deg_y(\pi_*(E) - \pi_*(F)) =$$

$$= \deg_y(\pi_*(E)) - \deg_y(\pi_*(F)) =$$

$$= \sum_{\pi(x)=y} [\deg_x(E) - \deg_x(F)] =$$

$$= \sum_{\pi(x)=y} \deg_x(D).$$

$\square$

COROLLARY 2.3.24. *Let $D \in \mathrm{GDiv}(X)$ be a generalized divisor on X such that $\pi_*(D)$ is Cartier. Then,*

$$\deg_Y(\pi_*(D)) = \deg_X(D).$$

PROOF. Applying Definition 2.2.4 and Proposition 2.3.23, we get:

$$\deg_Y(\pi_*(D)) = \sum_{y \text{ cod } 1} \deg_y(\pi_*(D)) =$$

$$= \sum_{y \text{ cod } 1} \sum_{\pi(x)=y} \deg_x(D).$$

By the properties of the Fitting image, $\pi_*(D)$ is supported on the set-theoretic image of the support of D; hence, the x appearing in the last sum are all the x for which $\deg_x(D)$ is not zero. Then, the previous

sum gives:

$$\deg_Y(\pi_*(D)) = \sum_{x \text{ cod } 1} \deg_x(D) = \deg_X(D).$$

$\square$

COROLLARY 2.3.25. *Suppose that Y is smooth. Then, for any generalized divisor D on X,*

$$\deg_Y(\pi_*(D)) = \deg_X(D).$$

PROOF. Let D be a generalized divisor on X. The direct image $\pi_*(D)$ is a generalized divisor on Y smooth, hence Cartier. Then, apply Corollary 2.3.24. $\square$

Proposition 2.3.23 yields another useful corollary about the surjectivity of the direct image morphism.

COROLLARY 2.3.26. (Surjectivity of the direct image for effective divisors) *Suppose that Y is smooth and k is algebraically closed. Then, the direct image for effective divisors:*

$$\pi_*^+ : \mathrm{GDiv}^+(X) \to \mathrm{CDiv}^+(Y)$$

is surjective.

PROOF. Let $E \in \mathrm{CDiv}^+(Y)$ be an effective Cartier divisor on Y and let $V = \mathrm{Spec}(R) \subseteq Y$ be an affine open subset of Y such that E is supported on V. Let $\mathrm{Supp}(E) = \{y_1, \ldots, y_r\}$ be the support of E and for each $i = 1, \ldots, r$ let

$$d_i = \deg_{y_i}(E)/[\kappa(y_i) : k] = \ell_{\mathcal{O}_{Y,y_i}}(\mathcal{O}_{E,y_i}).$$

Let $U = \mathrm{Spec}(S) = \pi^{-1}(V)$ be the preimage of V. For each i, pick one element $x_i \in \pi^{-1}(y_i)$ in the finite fiber of y_i and let $\mathfrak{M}_i$ be the maximal ideal of S corresponding to the point x_i. Then, the ideal $I = \mathfrak{M}_1^{d_1} \cdot \cdots \cdot \mathfrak{M}_r^{d_r}$ in S defines a divisor D on U (and hence of X) such that $\deg_{y_i}(E) = \deg_{x_i}(D)$. Looking at its direct image $\pi_*(D)$, it is an effective divisor on Y with the same support of E and the same degree at any point by Propostion 2.3.23. Since Y is smooth, we conclude that $\pi_*(D) = E$. $\square$

2.4. The direct and inverse image

for families of generalized divisors

Let $\pi : X \to Y$ be a **finite, flat map of degree** n **between projective curves over a field** k. In the present section, we discuss the definition of direct and inverse image for families of effective generalized divisors. Under suitable conditions, recalling Definition 2.2.7, we aim to define a pair of geometric morphisms:

$$\pi_* : \mathrm{Hilb}_X \to \mathrm{Hilb}_Y$$

$$\pi^* : \mathrm{Hilb}_Y \to \mathrm{Hilb}_X$$

that, on k-valued points, coincide with the direct and inverse image between $\mathrm{GDiv}^+(X)$ and $\mathrm{GDiv}^+(Y)$.

REMARK 2.4.1. Giving a definition of the direct image for families of effective generalized divisors is not possible in general when the curve Y is not smooth over k. Consider, for example, the setting of Example 2.3.21; since X and Y are reduced curves with planar singularities, their Hilbert schemes of generalized divisors of given degree are connected (see [**AIK77**], [**BGS81**]). The effective divisors D_1 and D_2 on X defined by (x^2, y) and (x) on X have both degree 2, but their direct images on the quotient node Y have degree respectively equal to 3 and 2. Then, their k-points D_1, D_2 in the connected component Hilb_X^2 of Hilb_X are sent to different connected components of Hilb_Y. This shows that *the direct image of divisors in general is not defined as a geometric map.*

In the rest of the section, consider $\pi : X \to Y$ a finite, flat map of degree n between projective curves over k, and **suppose that** Y **is smooth over** k. Recall that, in such case, $\mathrm{Hilb}_Y = {}^\ell\mathrm{Hilb}_Y$.

DEFINITION/LEMMA 2.4.2. (Direct image for families of effective generalized divisors) Let T be any k-scheme. The *direct image map for the Hilbert scheme of effective generalized divisors* is defined on the T-valued points as:

$$\pi_*(T) : \quad \mathrm{Hilb}_X(T) \longrightarrow {}^\ell\mathrm{Hilb}_Y(T)$$

$$D \subseteq X \times_k T \longmapsto \mathcal{Z}\left(\mathrm{Fitt}_0(\pi_{T,*}(\mathcal{O}_D))\right)$$

where $\pi_T : X \times_k T \to Y \times_k T$ is the morphism induced by base change of π.

PROOF. Let $D \subseteq X \times_k T$ be a T-flat family of effective divisors of X, defined by an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{X \times_k T}$ such that $\mathcal{O}_D = \mathcal{O}_{X \times_k T}/\mathcal{I}$ is flat over S. From the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X \times_k T} \to \mathcal{O}_D \to 0$$

we deduce that also $\mathcal{I}$ is flat over T. Since π is finite and flat, $\pi_{T,*}(\mathcal{I})$ is also flat over T, fiberwise locally free since Y is smooth and hence locally free on $Y \times_k T$ by [**HL10**, Lemma 2.1.7]. Moreover, it fits the exact sequence

$$0 \to \pi_{T,*}(\mathcal{I}) \xrightarrow{\varphi} \pi_{T,*}(\mathcal{O}_{X \times_k T}) \to \pi_{T,*}(\mathcal{O}_D) \to 0.$$

Then, by Definition 2.3.1, the 0-th Fitting ideal of $\pi_{T,*}(\mathcal{O}_D)$ is the image of the canonical injection

$$\det\left(\pi_{T,*}(\mathcal{I})\right) \otimes \det\left(\pi_{T,*}\mathcal{O}_{X \times_k T}\right)^{-1} \xrightarrow{\det(\varphi)} \mathcal{O}_{Y \times_k T}$$

and this is locally free over $Y \times_k T$, hence flat over T. Then, it defines a T-flat family of effective divisors of Y. $\qquad\square$

REMARK 2.4.3. For any T-family of effective generalized divisors $D \subseteq X \times_k T$ and for any point $t \in T$, the fiber $\pi_*(T)(D)_t$ is equal to the direct image $\pi_*(D_t)$ defined for the effective divisor D_t on X. Moreover, since Y is smooth, by Corollary 2.3.25 we have:

$$\deg_Y(\pi_*(D_t)) = \deg_X(D_t).$$

Then, for any $d \geq 0$, π_* restricts to a map:

$$\pi_*^d : \mathrm{Hilb}_X^d \longrightarrow {}^\ell\mathrm{Hilb}_Y^d.$$

DEFINITION/LEMMA 2.4.4. (Inverse image for families of effective generalized divisors) Let T be any k-scheme. The *inverse image map for the Hilbert scheme of effective Cartier divisors* is defined on the T-valued points as:

$$\pi^*(T): \quad {}^\ell\mathrm{Hilb}_Y(T) \longrightarrow {}^\ell\mathrm{Hilb}_X(T)$$
$$D \subseteq Y \times_k T \longmapsto \mathcal{Z}\left(\pi_T^{-1}(\mathcal{I}_D) \cdot \mathcal{O}_{X \times_k T}\right)$$

where $\pi_T : X \times_k T \to Y \times_k T$ is the morphism induced by base change of π and $\mathcal{I}_D \subseteq \mathcal{O}_{Y \times_k T}$ is the ideal sheaf of D.

PROOF. Since $\mathcal{I}$ is locally principal, $\pi_T^{-1}(\mathcal{I}) \cdot \mathcal{O}_{X \times_k T} \subseteq \mathcal{O}_{X \times_k T}$ is locally principal. The restriction of $\pi^*(T)(D)$ to the fiber over any $t \in T$ has ideal sheaf $\pi^{-1}(\mathcal{I}_t) \cdot \mathcal{O}_{X \times_k t}$, which is equal to the defining ideal of $\pi^*(D_t)$ by Definition 2.3.16. Hence, $\pi^*(T)(D)$ is a locally principal subscheme of $X \times_k T$, such that all fibers over T are effective Cartier divisors by Lemma 2.3.15. Then, $\pi^*(T)(D)$ is T-flat by [**Sta19**, Tag 062Y], hence it is a T-family of Cartier divisors. $\square$

REMARK 2.4.5. For any integer $d \geq 0$, π^* restricts to a map:

$$\pi_d^* : {}^\ell\mathrm{Hilb}_Y^d \longrightarrow {}^\ell\mathrm{Hilb}_X^{nd} .$$

We study now some properties of the direct and inverse image for families of effective generalized divisors. With a slight abuse of notation, we will write π_* and π^* instead of $\pi_*(T)$ and $\pi^*(T)$, when it is clear that we are working on T-points.

PROPOSITION 2.4.6. (Properties of the direct and inverse image for families of effective generalized divisors) *Let T be any k-scheme. Then, the following fact holds.*

(1) *Let D, E be T-families of effective divisors over X such that E is a family of Cartier divisors. Then, $\pi_*(D + E) = \pi_*(D) + \pi_*(E)$.*

(2) *Let F, G be T-families of effective divisors over Y. Then, $\pi^*(F + G) = \pi^*(F) + \pi^*(G)$.*

(3) *If F is a T-family of effective divisors over Y, then $\pi_*(\pi^*(F)) = nF$.*

PROOF. The proof of parts (1) and (2) follows the proofs of the second part of Lemma 2.3.4 and 2.3.15 respectively. The proof of part (3) follows the proof of Proposition 2.3.20. $\square$

2.5. The Norm and the inverse image
for families of torsion-free rank-1 sheaves

In the present section, we provide the definition of the Norm map for families torsion-free sheaves of rank 1 and the related inverse image

map. Since we are dealing with families of sheaves, we **assume that X and Y are projective curves over a field k.**

In order to be compatible with the direct image for generalized line bundles, our definition of the Norm map on $\overline{\mathbb{J}}(X)$ will be inspired by the sheaf-theoretic formula of Proposition 2.1.7. By Proposition 2.3.10 and its corollaries, the generalization of such formula to generalized divisors and torsion-free sheaves involves taking the double ω-dual of the exterior power of the pushforward of generalized line bundles. In general, this operation does not behave well in families if Y is not smooth.

Then, in accordance with the previous section, we **suppose that Y is smooth over k;** in such case, the moduli space $\overline{\mathbb{J}}(Y)$ is actually equal to the Jacobian $J(Y)$. Then, the aim of this chapter is to provide a pair of geometric morphism:

$$\mathrm{Nm}_\pi : \overline{\mathbb{J}}(X) \to J(Y)$$
$$\pi^{-1} : J(Y) \to J(X) \subseteq \overline{\mathbb{J}}(X)$$

that, on k-valued points, coincide with the direct and inverse image maps between $\mathrm{GPic}(X)$ and $\mathrm{GPic}(Y) = \mathrm{Pic}(Y)$.

Recall also (Definition 2.2.8) that, for any line bundle M on X, the compactified Jacobian of X is related to the Hilbert scheme of effective generalized divisors via the twisted Abel map $\mathcal{A}_M$. We will show that the direct image map and the Norm map are compatible as well as the inverse image maps, meaning that for any $M \in J(X)$ and $N \in J(Y)$ there are commutative diagrams of k-schemes:

$$
\begin{array}{ccc}
\mathrm{Hilb}_X & \xrightarrow{\quad \pi_* \quad} & {}^\ell\mathrm{Hilb}_Y \\
\downarrow{\scriptstyle \mathcal{A}_M} & & \downarrow{\scriptstyle \mathcal{A}_{\mathrm{Nm}_\pi(M)}} \\
G\mathbb{J}(X) \;\subseteq\; \overline{\mathbb{J}}(X) & \xrightarrow{\mathrm{Nm}_\pi} & J(Y)
\end{array}
\qquad
\begin{array}{ccc}
{}^\ell\mathrm{Hilb}_Y & \xrightarrow{\;\pi^*\;} & {}^\ell\mathrm{Hilb}_X \\
\downarrow{\scriptstyle \mathcal{A}_N} & & \downarrow{\scriptstyle \mathcal{A}_{\pi^*(N)}} \\
J(Y) & \xrightarrow{\;\pi^*\;} & J(X).
\end{array}
$$

We give first the definition for the Norm map on $\overline{\mathbb{J}}(X)$.

DEFINITION/LEMMA 2.5.1. (Norm map for torsion-free rank-1 sheaves) Let T be any k-scheme. The *Norm map between compactified Jacobians*

associated to π is defined on the T-valued points as:

$$\mathrm{Nm}_\pi(T): \qquad \overline{\mathbb{J}}(X)(T) \longrightarrow J(Y)(T)$$

$$\mathcal{L} \longmapsto \det\left(\pi_{T,*}(\mathcal{L})\right) \otimes \det\left(\pi_{T,*}\mathcal{O}_{X\times_k T}\right)^{-1}.$$

PROOF. Let $\mathcal{L}$ be a T-family of torsion-free sheaves of rank 1 on X, i.e. a T-flat coherent sheaf on $X \times_k T$, whose fibers over T are torsion-free sheaves of rank 1. The push-forward $\pi_{T,*}(\mathcal{L})$ is a T-flat coherent sheaf on $Y \times_k T$ such that, for any $t \in T$, the fiber $(\pi_{T,*}\mathcal{L})_t$ equals $\pi_*(\mathcal{L}_t)$ on $Y \simeq Y \times_k t$, . Since Y is smooth, $\pi_*(\mathcal{L}_t)$ is a locally free sheaf of rank n for any $t \in T$. Then, by [**HL10**, Lemma 2.1.7], $\pi_{T,*}(\mathcal{L})$ is a locally sheaf of rank n on $Y \times_k T$. Its determinant bundle is a line bundle on $Y \times_k T$, hence flat over T. $\qquad\square$

The definition of the inverse image is standard but we give it for the sake of completeness.

DEFINITION/LEMMA 2.5.2. (Inverse image map for line bundles) Let T be any k-scheme. The *inverse image map between Jacobians* associated to π is defined on the T-valued points as:

$$\pi^*(T): \qquad J(Y)(T) \longrightarrow J(X) \subseteq \overline{\mathbb{J}}(X)(T)$$

$$\mathcal{N} \longmapsto \pi_T^*(\mathcal{N}).$$

PROOF. By hypothesis, $\mathcal{N}$ is a T-flat coherent sheaf on $Y \times_k T$ that is a line bundle on any fiber over T; then, by [**HL10**, Lemma 2.1.7], it is a line bundle on $Y \times_k T$. We conclude that $\pi^*\mathcal{N}$ is a line bundle on $X \times_k T$ and hence a T-flat family of line bundles over T. $\qquad\square$

REMARK 2.5.3. When there is no ambiguity, we will write Nm_π and π^* instead of $\mathrm{Nm}_\pi(T)$ and $\pi^*(T)$. The Norm and the inverse image for generalized line bundles define morphisms of algebraic stacks

$$\mathrm{Nm}_\pi: \qquad \overline{\mathbb{J}}(X) \longrightarrow J(Y)$$

$$\pi^*: \qquad J(Y) \longrightarrow J(X).$$

For any integer d, they restricts to:

$$\mathrm{Nm}_\pi^d: \qquad \overline{\mathbb{J}}(X,d) \longrightarrow J^d(Y)$$

$$\pi_d^*: \qquad J^d(Y) \longrightarrow J^{nd}(X).$$

Moreover, the Norm for torsion-free rank-1 sheaves, restricted to the locus of line bundles $J(X) \subseteq \overline{\mathbb{J}}(X)$, coincides with the classical Norm map from $J(X)$ to $J(Y)$ of Definition 2.1.9.

We study now some properties of the Norm map on $\overline{\mathbb{J}}(X)$ and the inverse image map. First, we need a technical lemma.

LEMMA 2.5.4. *Let T be a fixed k-scheme. For any T-flat family $\mathcal{L}$ of torsion-free rank-1 sheaves on $X \times_k T$ and any line bundle $\mathcal{M}$ on $X \times_k T$, there is an isomorphism:*

$$\det\left(\pi_{T,*}(\mathcal{L} \otimes \mathcal{M})\right) \otimes \det\left(\pi_{T,*}\mathcal{O}_{X \times_k T}\right) \simeq$$
$$\simeq \det\left(\pi_{T,*}(\mathcal{L})\right) \otimes \det\left(\pi_{T,*}(\mathcal{M})\right).$$

PROOF. The proof is similar to the second part of the proof of Proposition 2.3.10.

Since $\pi_T : X \times_k T \to Y \times_k T$ is finite and flat, $\pi_{T,*}(\mathcal{L})$ is a T-flat coherent sheaf on $Y \times_k T$, that is locally free on any fiber over T since Y is smooth; then, $\pi_{T,*}(\mathcal{L})$ is locally free of rank n by [**HL10**, Lemma 2.1.7]. The same holds for $\pi_{T,*}(\mathcal{L} \otimes \mathcal{M})$. On the other hand, by [**Gro61**, Proposition 6.1.12], $\pi_{T,*}\mathcal{M}$ is a locally free $\pi_{T,*}\mathcal{O}_X$-module of rank 1. In particular, there is open cover $\{V_i\}_I$ of Y such that $\mathcal{M}$ is a trivial $\mathcal{O}_{X|U_i}$-module on each $U_i = \pi_T^{-1}(V_i)$, i.e. there are isomorphisms $\lambda_i : \mathcal{M}_{|U_i} \xrightarrow{\sim} \mathcal{O}_{X|U_i}$ for each $i \in I$. On the intersections $U_i \cap U_j$, the collection $\{\lambda_i \circ \lambda_j^{-1}\}$ of automorphisms of $\mathcal{O}_{X \times_k T|U_i \cap U_j}$ is a cochain that measures the obstruction for the λ_i's to glue to a global isomorphism. We define now an isomorphism

$$\alpha : \det\left(\pi_{T,*}(\mathcal{L} \otimes \mathcal{M})\right) \otimes \det\left(\pi_{T,*}\mathcal{O}_{X \times_k T}\right) \longrightarrow \det\left(\pi_{T,*}(\mathcal{L})\right) \otimes \det\left(\pi_{T,*}(\mathcal{M})\right)$$

by glueing a collection of isomorphisms α_i defined on each V_i. To do so, we define each α_i as the following composition of arrows:

$$
\begin{array}{ccc}
\left(\det\left(\pi_{T,*}(\mathcal{L} \otimes \mathcal{M})\right) \otimes \det(\pi_{T,*}\mathcal{O}_X)\right)_{|V_i} & \xrightarrow{\alpha_i} & \left(\det\left(\pi_{T,*}\mathcal{L}\right) \otimes \det\left(\pi_{T,*}\mathcal{M}\right)\right)_{|V_i} \\
\| \downarrow & & \uparrow \| \\
\det\left(\pi_{T,*}(\mathcal{L} \otimes \mathcal{M})_{|U_i}\right) \otimes \det(\pi_{T,*}\mathcal{O}_{X|U_i}) & & \det\left(\pi_{T,*}\mathcal{L}_{|U_i}\right) \otimes \det\left(\pi_{T,*}\mathcal{M}_{|U_i}\right) \\
\det\left(\pi_{T,*}\lambda_i\right) \otimes \mathrm{id} \downarrow & & \uparrow \mathrm{id} \otimes \det\left(\pi_{T,*}\lambda_i^{-1}\right) \\
\det\left(\pi_{T,*}\mathcal{L}_{|U_i}\right) \otimes \det(\pi_{T,*}\mathcal{O}_{X|U_i}) & = & \det\left(\pi_{T,*}\mathcal{L}_{|U_i}\right) \otimes \det\left(\pi_{T,*}\mathcal{O}_{X|U_i}\right)
\end{array}
$$

i.e. $\alpha_i := \det\left(\pi_{T,*}\lambda_i\right) \otimes \det\left(\pi_{T,*}\lambda_i^{-1}\right)$. Since $\det\left(\pi_{T,*-}\right)$ is functorial, the obstruction $\alpha_i \circ \alpha_j^{-1}$ is trivial on any $V_i \cap V_j$, whence the α_i's glue together to a global isomorphism α. □

PROPOSITION 2.5.5. (Properties of the Norm and the inverse image map) *Let T be any k-scheme. Then, the following facts hold.*

(1) *Let $\mathcal{L}, \mathcal{M} \in \overline{\mathbb{J}}(X)(T)$ such that $\mathcal{M}$ is a T-flat family of line bundles. Then, $\mathrm{Nm}_\pi(\mathcal{L} \otimes \mathcal{M}) \simeq \mathrm{Nm}_\pi(\mathcal{L}) \otimes \mathrm{Nm}_\pi(\mathcal{M})$.*

(2) *Let $\mathcal{N}, \mathcal{N}' \in J(Y)(T)$. Then, $\pi^*(\mathcal{N} \otimes \mathcal{N}') \simeq \pi^*(\mathcal{N}) \otimes \pi^*(\mathcal{N}')$.*

(3) *Let $\mathcal{N} \in J(Y)(T)$. Then, $\pi^*(\mathcal{N})$ is a T-flat family of line bundles over X and $\mathrm{Nm}_\pi(\pi^*(\mathcal{N})) \simeq \mathcal{N}^{\otimes n}$.*

PROOF. To prove (1), note that $\mathcal{M}$ is a T-flat coherent sheaf on $X \times_k T$ that is a line bundle on any fiber over T; hence it is a line bundle on $X \times_k T$ by [**HL10**, Lemma 2.1.7]. Then, applying Definition 2.5.1 and Lemma 2.5.4, we have:

$$\mathrm{Nm}_\pi(\mathcal{L} \otimes \mathcal{M}) = \det\left(\pi_{T,*}(\mathcal{L} \otimes \mathcal{M})\right) \otimes \det\left(\pi_{T,*}\mathcal{O}_{X \times_k T}\right)^{-1} \simeq$$
$$\simeq \det\left(\pi_{T,*}(\mathcal{L})\right) \otimes \det\left(\pi_{T,*}(\mathcal{M})\right) \otimes \det\left(\pi_{T,*}\mathcal{O}_{X \times_k T}\right)^{-2} \simeq$$
$$\simeq \mathrm{Nm}_\pi(\mathcal{L}) \otimes \mathrm{Nm}_\pi(\mathcal{M}).$$

Part (2) follows from the associative properties of the tensor product. To prove (3), compute by Definitions 2.5.1 and 2.5.2:

$$\mathrm{Nm}_\pi(\pi^*(\mathcal{N})) = \det\left(\pi_{T,*}(\pi_T^*(\mathcal{N}))\right) \otimes \det\left(\pi_{T,*}\mathcal{O}_{X \times_k T}\right)^{-1}.$$

By the projection formula [**Sta19**, Tag 01E8] and the standard properties of determinants, we have:

$$\mathrm{Nm}_\pi(\pi^*(\mathcal{N})) = \det\left(\mathcal{N} \otimes \pi_{T,*}\mathcal{O}_{X \times_k T}\right) \otimes \det\left(\pi_{T,*}\mathcal{O}_{X \times_k T}\right)^{-1} \simeq$$
$$\simeq \mathcal{N}^n \otimes \det\left(\pi_{T,*}\mathcal{O}_{X \times_k T}\right) \otimes \det\left(\pi_{T,*}\mathcal{O}_{X \times_k T}\right)^{-1} \simeq$$
$$\simeq \mathcal{N}^n.$$

□

We compare now the Norm and the inverse image maps between the compactified Jacobians with the direct and inverse image maps between the Hilbert schemes of divisors, respectively.

PROPOSITION 2.5.6. (Comparison of the direct image and the Norm map via the Abel map) *For any line bundle M of degree e on X and*

for any $d \geq 0$, there is a commutative diagram of algebraic stacks over k:

$$
\begin{array}{ccc}
\mathrm{Hilb}_X^d & \xrightarrow{\quad \pi_*^d \quad} & {}^\ell\mathrm{Hilb}_Y^d \\[2mm]
\Big\downarrow{\scriptstyle \mathcal{A}_M^d} & & \Big\downarrow{\scriptstyle \mathcal{A}_{\mathrm{Nm}_\pi(M)}^d} \\[2mm]
G\mathbb{J}(X, -d+e) \;\; \subseteq \;\; \overline{\mathbb{J}}(X, -d+e) & \xrightarrow{\mathrm{Nm}_\pi^{-d+e}} & J^{-d+e}(Y)
\end{array}
$$

PROOF. Let T be any k-scheme, let D be a T-flat family of effective divisors of degree d on X with ideal sheaf $\mathcal{I}$, and denote with $\mathcal{M}$ the pullback of M to $X \times_k T$. Following the bottom-left side of the square, combining Definitions 2.2.8 and 2.5.1 we get:

$$
\mathrm{Nm}_\pi^{-d+e}(\mathcal{A}_M^d(D)) = \det\left(\pi_{T,*}(\mathcal{I} \otimes \mathcal{M})\right) \otimes \det\left(\pi_{T,*}\mathcal{O}_{X \times_k T}\right)^{-1}.
$$

Following the top-right side of the square, combining definitions 2.4.2, 2.5.1 and 2.2.8 we get:

$$
\mathcal{A}_{\mathrm{Nm}_\pi(M)}^d(\pi_*^d(D)) = \det\left(\pi_{T,*}(\mathcal{I})\right) \otimes \det\left(\pi_{T,*}(\mathcal{M})\right) \otimes \det\left(\pi_{T,*}\mathcal{O}_{X \times_k T}\right)^{-2}.
$$

We are left to prove that:

$$
\det\left(\pi_{T,*}(\mathcal{I} \otimes \mathcal{M})\right) \otimes \det\left(\pi_{T,*}\mathcal{O}_{X \times_k T}\right) \simeq
$$
$$
\simeq \det\left(\pi_{T,*}(\mathcal{I})\right) \otimes \det\left(\pi_{T,*}(\mathcal{M})\right).
$$

Now, $\mathcal{I}$ is a T-flat family of generalized line bundles by hypothesis and $\mathcal{M}$ is a line bundle on $X \times_k T$ since it is the pull-back of a line bundle on X. Then, the assertion is true by Lemma 2.5.4. $\qquad\square$

PROPOSITION 2.5.7. (Comparison of the inverse image maps via the Abel map) *For any line bundle N of degree e on Y and any $d \geq 0$, there is a commutative diagram of algebraic stacks over k:*

$$
\begin{array}{ccc}
{}^\ell\mathrm{Hilb}_Y^d & \xrightarrow{\quad \pi_d^* \quad} & {}^\ell\mathrm{Hilb}_X^d \\[2mm]
\Big\downarrow{\scriptstyle \mathcal{A}_N^d} & & \Big\downarrow{\scriptstyle \mathcal{A}_{\pi^*(N)}^d} \\[2mm]
J(Y, -d+e) & \xrightarrow{\pi_{-d+e}^*} & J(X, -d+e)
\end{array}
$$

PROOF. Let T be any k-scheme, let D be a T-flat family of divisors of degree d on Y with ideal sheaf $\mathcal{I} \subseteq Y \times_k T$, and denote with $\mathcal{N}$ the pullback of N to $Y \times_k T$. By Definition 2.2.8, Proposition 2.5.5(2) and

Remark 2.3.19, we have:

$$\pi_d^*(\mathcal{A}_N^d(D)) = \pi_T^*(\mathcal{I} \otimes \mathcal{N}) \simeq \pi_T^*(\mathcal{I}) \otimes \pi_T^*(\mathcal{N}) \simeq$$
$$\simeq \mathcal{A}_{\pi^*(N)}^d\big(\pi_{-d+e}^*(\mathcal{N})\big).$$

$\square$

CHAPTER 3

The fibers of the Norm map and the Prym stack

In the present chapter we study the fibers of the Norm map defined in Chapter 2. The purpose is to generalize the Prym scheme associated to a finite, flat morphism between projective curves to the context of torsion-free sheaves of rank 1. Fix an algebraically closed field k of characteristic 0. Let X, Y be projective curves over k, such that Y is smooth, and let $\pi : X \to Y$ be a finite, flat morphism of degree n. We start by recalling the following definition.

DEFINITION 3.0.1. The *Prym scheme* of X associated to π is the locus $\mathrm{Pr}(X, Y) \subseteq J(X)$ of line bundles whose Norm with respect to π is trivial. In other words:

$$\mathrm{Pr}(X, Y) = \{\mathcal{L} \in J(X) : \mathrm{Nm}_\pi(\mathcal{L}) \simeq \mathcal{O}_Y\}.$$

We can now extend the definition of the Prym stack to the context of torsion-free rank 1 sheaves using the Norm map defined in the previous chapter.

DEFINITION 3.0.2. The *Prym stack* of X associated to π is the locus $\overline{\mathrm{Pr}}(X, Y) \subseteq \overline{\mathbb{J}}(X)$ of torsion-free rank-1 sheaves whose Norm with respect to π is trivial. In other words:

$$\overline{\mathrm{Pr}}(X, Y) := \{(\mathcal{L}, \lambda) : \mathcal{L} \in \overline{\mathbb{J}}(X) \text{ and } \lambda : \mathrm{Nm}_\pi(\mathcal{L}) \xrightarrow{\sim} \mathcal{O}_Y\}.$$

REMARK 3.0.3. It follows from the definition that:

$$\mathrm{Pr}(X, Y) = \overline{\mathrm{Pr}}(X, Y) \cap J(X).$$

Since being locally-free is an open condition, $\mathrm{Pr}(X, Y)$ is open in $\overline{\mathrm{Pr}}(X, Y)$.

The Prym stack is a fiber of the Norm map associated to π. In the present chapter, we study the fibers of the Norm map assuming that the curve X is reduced with locally planar singularities. The study of such fibers involves the study of the fibers of the direct image map for

effective divisors defined in Chapter 2, and the study of the fibers of the Hilbert-Chow morphism. We start from the last.

3.1. The fibers of the Hilbert-Chow morphism

In this section, we refer to [**Ber12**] for notations and known results.

DEFINITION 3.1.1. Let X be a projective curve over k and let d be a positive integer. The *d-symmetric power* $X^{(d)}$ of X is the quotient X^d/Σ_d of the Cartesian product X^n by the action of the symmetric group Σ_d in d-letters permuting the factors. Note that, for $d = 1$, $X^{(1)} = X$. For $d = 0$, we put $X^{(0)} = \{0\}$. The *scheme of effective 0-cycles* (or of *effective Weil divisors*) of X is the algebraic scheme:

$$\mathrm{WDiv}^+(X) = \bigsqcup_{d \geq 0} X^{(d)} = \bigsqcup_{d \geq 0} \mathrm{WDiv}^d(X).$$

The *Hilbert-Chow morphism* of degree d associated to X is the map of schemes

$$\rho_X^d : \mathrm{Hilb}_X^d \to X^{(d)}$$
$$D \mapsto \sum_{x \text{ cod } 1} \deg_x(D) \cdot [x]$$

Denote with $^\ell\rho_X^d$ the restriction of ρ_X^d to the subscheme $^\ell\mathrm{Hilb}_X^d \subseteq \mathrm{Hilb}_X^d$ parametrizing Cartier divisors. The collection of Hilbert-Chow morphisms in non-negative degrees gives rise to the Hilbert-Chow morphism in any degree:

$$\rho_X : \mathrm{Hilb}_X \to \mathrm{WDiv}^+(X).$$

Finally, denote with $^\ell\rho_X$ the restriction of ρ_X to $^\ell\mathrm{Hilb}_X \subseteq \mathrm{Hilb}_X$.

REMARK 3.1.2. If X is a smooth projective curve, then ρ_X is an isomorphism and coincides with $^\ell\rho_X$.

We study now the fibers of the Hilbert-Chow morphism in the case when X is reduced with locally planar singularities.

PROPOSITION 3.1.3. *Suppose that X is reduced with locally planar singularities and that Y is smooth. Let $W \in \mathrm{WDiv}^d(X)$ be an effective Weil divisor of degree d on X and let $\rho_X^{-1}(W)$ be the corresponding fiber in Hilb_X. The locus $^\ell\rho_X^{-1}(W)$ of Cartier divisors in $\rho_X^{-1}(W)$ is an open and dense subset of $\rho_X^{-1}(W)$.*

PROOF. First, note that ${}^{\ell}\rho_X^{-1}(W) = \rho_X^{-1}(W) \cap {}^{\ell}\mathrm{Hilb}_X$. Since ${}^{\ell}\mathrm{Hilb}_X \subseteq \mathrm{Hilb}_X$ is open (see for example [**MRV17a**, Fact 2.4]), ${}^{\ell}\rho_X^{-1}(W)$ is open in $\rho_X^{-1}(W)$.

To prove that it is dense, let $W = \sum_{i=1}^{s} n_i \cdot [x_i]$ with the x_i's are s distinct points. For each i, the fiber of ρ_X over the cycle $n_i \cdot [x_i]$ is equal by definition to the punctual Hilbert scheme $\mathrm{Hilb}_{X,x_i}^{n_i}$ parametrizing 0-dimensional subschemes of X supported at x_i having length n_i over k. Since the x_i are distinct, we have:

$$\rho_X^{-1}(W) = \rho_X^{-1}\left(\sum_{i=1}^{s} n_i \cdot [x_i]\right) =$$

$$= \prod_{i=1}^{s} \rho_X^{-1}(n_i \cdot [x_i]) = \prod_{i=1}^{s} \mathrm{Hilb}_{X,x_i}^{n_i}.$$

For each i, let ${}^{\ell}\mathrm{Hilb}_{X,x_i}^{n_i} = \mathrm{Hilb}_{X,x_i}^{n_i} \cap {}^{\ell}\mathrm{Hilb}_X$. An effective divisor is Cartier if and only if it is Cartier at each point of its support, hence:

$$ {}^{\ell}\rho_X^{-1}(W) = \prod_{i=1}^{s} \left(\mathrm{Hilb}_{X,x_i}^{n_i} \cap {}^{\ell}\mathrm{Hilb}_X\right) = \prod_{i=1}^{s} {}^{\ell}\mathrm{Hilb}_{X,x_i}^{n_i}.$$

For each i, ${}^{\ell}\mathrm{Hilb}_{X,x_i}^{n_i}$ is the smooth locus of $\mathrm{Hilb}_{X,x_i}^{n_i}$ by [**BGS81**, Proposition 2.3], so it is dense; hence ${}^{\ell}\rho_X^{-1}(W)$ is dense in $\rho_X^{-1}(W)$. $\square$

3.2. The fibers of the direct image between Hilbert schemes

In the present section, we study the fibers of the direct image map π_* defined between the Hilbert schemes of generalized divisors of X and Y.

First, we introduce a similar notion for Weil divisors and we see how it relates to the direct image for generalized divisors.

DEFINITION 3.2.1. The *direct image for Weil divisors* associated to π is the morphism of schemes:

$$\pi_* : \mathrm{WDiv}^+(X) \to \mathrm{WDiv}^+(Y)$$

given on the level of points by

$$\sum_{i=1}^{d} n_i \cdot [x_i] \longmapsto \sum_{i=1}^{d} n_i \cdot [\pi(x_i)].$$

PROPOSITION 3.2.2. *Assume that Y is smooth. There is a commutative square of schemes over k:*

$$
\begin{array}{ccc}
\mathrm{Hilb}_X & \xrightarrow{\ \pi_*\ } & {}^{\ell}\mathrm{Hilb}_Y \\
\downarrow{\scriptstyle\rho_X} & & \downarrow{\scriptstyle\rho_Y} \\
\mathrm{WDiv}(X) & \xrightarrow{\ \pi_*\ } & \mathrm{WDiv}(Y).
\end{array}
$$

PROOF. Let D be an effective generalized divisor on X. Following the bottom-left side of the square, we obtain:

$$
\pi_*(\rho_X(D)) = \sum_{x \text{ cod } 1} \deg_x(D) \cdot [\pi(x)],
$$

while on the top-right side of the square we have

$$
\rho_Y(\pi_*(D)) = \sum_{y \text{ cod } 1} \deg_y(\pi_*(D)) \cdot [y].
$$

Since Y is smooth, $\pi_*(D)$ is Cartier locally at each point of Y. By Proposition 2.3.23, for any point y in codimension 1 of Y

$$
\deg_y(\pi_*(D)) = \sum_{\pi(x)=y} \deg_x(D).
$$

Then we compute:

$$
\rho_Y(\pi_*(D)) = \sum_{y \text{ cod } 1} \deg_y(\pi_*(D)) \cdot [y] =
$$

$$
= \sum_{y \text{ cod } 1} \left(\sum_{\pi(x)=y} \deg_x(D) \right) \cdot [y] =
$$

$$
= \sum_{y \text{ cod } 1} \sum_{\pi(x)=y} \deg_x(D) \cdot [\pi_*(x)] =
$$

$$
= \sum_{x \text{ cod } 1} \deg_x(D) \cdot [y] = \pi_*(\rho_X(D)).
$$

$\square$

We are now ready to study the fibers of the direct image for generalized divisors.

PROPOSITION 3.2.3. *Assume that Y is smooth and X is reduced with locally planar singularities. Let $E \in \mathrm{Hilb}_Y$ be an effective divisor of degree d on Y and let $\pi_*^{-1}(E)$ be the corresponding fiber in*

Hilb_X. *Then,* $\pi_*^{-1}(E) \cap {}^{\ell}\text{Hilb}_X$ *is an open and non-empty dense subset of* $\pi_*^{-1}(E)$.

Proof. By Proposition 3.2.2, there is a commutative diagram of schemes over k:

$$
\begin{array}{ccc}
\text{Hilb}_X & \xrightarrow{\ \pi_*\ } & {}^{\ell}\text{Hilb}_Y \\
\downarrow{\scriptstyle \rho_X} & & \downarrow{\scriptstyle \rho_Y} \\
\text{WDiv}(X) & \xrightarrow{\ \pi_*\ } & \text{WDiv}(Y).
\end{array}
$$

First note that, since Y is smooth, the map π_* is surjective by Corollary 2.3.26, hence $\pi_*^{-1}(E)$ is non-empty. Moreover, the Hilbert-Chow morphism ρ_Y is an isomorphism. Then,

$$
\pi_*^{-1}(E) = \rho_X^{-1}\pi_*^{-1}(\rho_Y(E)).
$$

Let $S = \{y_1, \dots, y_r\}$ be the support of E and let $d_i = \deg_{y_i} E$ for each i; then

$$
\rho_Y(E) = \sum_{i=1}^{r} d_i \cdot [y_i].
$$

For each i, denote with $\{x_i^1, \dots, x_i^{n_i}\}$ the discrete fiber of π over y_i. Since points can occur with multeplicity in the geometric fibers, $n_i \leq n$ holds for each i. The fiber of $\sum_{i=1}^{r} d_i \cdot [y_i]$ is then the discrete subset of $\text{WDiv}(X)$ given by:

$$
\pi_*^{-1}\left(\sum_{i=1}^{r} d_i \cdot [y_i] \right) = \left\{ \sum_{i=1}^{r} \sum_{j=1}^{n_i} c_{ij} \cdot [x_i^j] \ \middle|\ c_{ij} \in \mathbb{Z}_{\geq 0}, \ \sum_{j=1}^{n_i} c_{ij} = d_i \text{ for any } i \right\}.
$$

Denote with Z the set of all tuples (c_{ij}) of positive integers with the conditions that $\sum_{j=1}^{n_i} c_{ij} = d_i$ for any i. Then, we can write the previous set as

$$
\pi_*^{-1}\left(\sum_{i=1}^{r} d_i \cdot [y_i] \right) = \bigsqcup_{(c_{ij}) \in Z} \left\{ \sum_{i=1}^{r} \sum_{j=1}^{n_i} c_{ij} \cdot [x_i^j] \right\}.
$$

Then, applying ρ_*^{-1}, we otbain:

$$\rho_X^{-1}\pi_*^{-1}(\rho_Y(E)) = \rho_X^{-1}\pi_*^{-1}\left(\sum_{i=1}^{r} d_i \cdot [y_i]\right) =$$

$$= \rho_X^{-1}\left(\bigsqcup_{(c_{ij})\in Z}\left\{\sum_{i=1}^{r}\sum_{j=1}^{n_i} c_{ij} \cdot [x_i^j]\right\}\right) =$$

$$= \bigsqcup_{(c_{ij})\in Z} \rho_X^{-1}\left(\sum_{i=1}^{r}\sum_{j=1}^{n_i} c_{ij} \cdot [x_i^j]\right),$$

where the last disjoint union is a disjoint union of topological subspaces of Hilb_X in the same connected component. Then, $\pi_*^{-1}(E)$ is a non-emtpy disjoint uniont of a finite number of fibers of the Hilbert-Chow morphism. By Proposition 3.1.3, the intersection of any such fiber with the Cartier locus $^\ell\mathrm{Hilb}_X$ is open and dense in the same fiber. Taking the disjoint union, $\pi_*^{-1}(E) \cap {}^\ell\mathrm{Hilb}_X$ is a non-empty open and dense subset of $\pi_*^{-1}(E)$. $\qquad\square$

3.3. The fibers of the Norm map and $\overline{\mathrm{Pr}}(X, Y)$

In this section, assuming that Y is smooth and X is reduced with locally planar singularities, we prove that the generalized Prym of X with respect to Y is non-empty, open and dense in the Prym stack of X with respect to Y. The theorem is based upon the following two auxiliaries proposition.

PROPOSITION 3.3.1. *Assume that Y is smooth and let $\mathcal{N} \in J(Y)$ be any line bundle on Y. Then, there is a line bundle $\mathcal{L}$ on X such that $\mathrm{Nm}_\pi(\mathcal{L}) \simeq \mathcal{N}$.*

PROOF. Let D be a Cartier divisor on Y such that $\mathcal{N} \in [D]$ and let $D = E - F$ where E, F are effective Cartier divisors by Lemma 2.2.3. By Proposition 3.2.3 there are effective Cartier divisors H, K on X such that $\pi_*(H) = E$ and $\pi_*(K) = F$. Then, by Lemma 2.3.4 $\pi_*(H - K) = D$. Set $H - K = C$ and let $\mathcal{L}$ be the defining ideal of C, seen as a line bundle. By Lemma 2.3.11 we conclude that $\mathrm{Nm}_\pi(\mathcal{L}) \simeq \mathcal{N}$. $\qquad\square$

PROPOSITION 3.3.2. *Assume that Y is smooth and X is reduced with locally planar singularities and let $\mathcal{N} \in J(Y)$ be any line bundle*

on Y. Then, the fiber $\mathrm{Nm}_\pi^{-1}(\mathcal{N})$ is non-empty and contains $\mathrm{Nm}_\pi^{-1}(\mathcal{N}) \cap J(Y)$ as an open and dense substack.

PROOF. First, note that $\overline{\mathbb{J}}(X) = G\mathbb{J}(X)$ since X is reduced. The fiber $\mathrm{Nm}_\pi^{-1}(\mathcal{N})$ is non-empty by Proposition 3.3.1 and the substack $\mathrm{Nm}_\pi^{-1}(\mathcal{N}) \cap J(Y)$ is open in $\mathrm{Nm}_\pi^{-1}(\mathcal{N})$ since being locally free is an open condition. To prove that it is dense, recall that for any fixed line bundle $M \in J(X)$ there is a commutative diagram of stacks over k:

$$\begin{array}{ccc} \mathrm{Hilb}_X & \xrightarrow{\ \pi_* \ } & {}^\ell\mathrm{Hilb}_Y \\ \downarrow{\scriptstyle \mathcal{A}_M} & & \downarrow{\scriptstyle \mathcal{A}_{\mathrm{Nm}_\pi(M)}} \\ \overline{\mathbb{J}}(X) & \xrightarrow{\ \mathrm{Nm}_\pi \ } & J(Y) \end{array}$$

Moreover, by [**MRV17a**, Proposition 2.5] there is a cover of $\overline{\mathbb{J}}(X)$ by k-finite type open subsets $\{U_\beta\}$ such that, for each β, there is a line bundle $M_\beta \in J(X)$ with the property that $\mathcal{A}_{M_\beta | V_\beta} : \mathcal{A}_{M_\beta}^{-1}(U_\beta) = V_\beta \to U_\beta$ is smooth and surjective.

Since density can be checked locally, fix $M = M_\beta$, $U = U_\beta$ and $V = V_\beta$. Then, we have:

$$\mathrm{Nm}_\pi^{-1}(\mathcal{N}) \cap U = \mathcal{A}_M \left(V \cap \pi_*^{-1} \left(\mathcal{A}_{\mathrm{Nm}_\pi(M)}^{-1}(\mathcal{O}_Y) \right) \right)$$

and

$$\mathrm{Nm}_\pi^{-1}(\mathcal{N}) \cap J(X) \cap U = \mathcal{A}_M \left(V \cap \pi_*^{-1} \left(\mathcal{A}_{\mathrm{Nm}_\pi(M)}^{-1}(\mathcal{O}_Y) \right) \cap {}^\ell\mathrm{Hilb}_X \right).$$

Put $\mathbf{K} = \mathcal{A}_{\mathrm{Nm}_\pi(M)}^{-1}(\mathcal{O}_Y)$. Then, the topological space underlying $\pi_*^{-1}(\mathbf{K})$ contains the topological union of fibers of the points in $\mathbf{K}$, so that:

$$\pi_*^{-1}(\mathbf{K}) \supseteq \bigcup_{E \in \mathbf{K}} \pi_*^{-1}(E).$$

By Proposition 3.2.3, $\pi_*^{-1}(E) \cap {}^\ell\mathrm{Hilb}_X$ is a non-empty open and dense subscheme of $\pi_*^{-1}(E)$ for any closed point $E \in \mathbf{K}$. Hence, $\pi_*^{-1}(\mathbf{K}) \cap {}^\ell\mathrm{Hilb}_X$ is non-empty, open and dense in $\pi_*^{-1}(\mathbf{K})$. Intersecting with V and composing with $\mathcal{A}_M$, we get the thesis. $\qquad \square$

We finally come to the Prym stack of X with respect to Y. Recall that by definition:

$$\mathrm{Pr}(X, Y) = \mathrm{Nm}_\pi^{-1}(\mathcal{O}_Y) \cap J(X),$$
$$\overline{\mathrm{Pr}}(X, Y) = \mathrm{Nm}_\pi^{-1}(\mathcal{O}_Y).$$

COROLLARY 3.3.3. $\Pr(X, Y)$ *is non-empty, open and dense in* $\overline{\Pr}(X, Y)$.

PROOF. Set $\mathcal{N} = \mathcal{O}_Y$ and apply Proposition 3.3.2. $\qquad\square$

CHAPTER 4

Spectral data for G-Higgs pairs

In this chapter we study how the spectral correspondence (see Chapter 1, Section 1.4) specializes for G-Higgs pairs, where $G = \mathrm{SL}(r, \mathbb{C})$, $\mathrm{PGL}(r, \mathbb{C})$, $\mathrm{Sp}(2r, \mathbb{C})$, $\mathrm{GSp}(2r, \mathbb{C})$, $\mathrm{PSp}(2r, \mathbb{C})$. Throughout this chapter, C denotes a fixed smooth curve over the field of complex numbers and L a fixed line bundle on C with degree $\ell = \deg L$.

4.1. $\mathrm{SL}(r, \mathbb{C})$-Higgs pairs

The Special Linear Group $\mathrm{SL}(r, \mathbb{C})$ is defined as:

$$\mathrm{SL}(r, \mathbb{C}) = \{M \in \mathrm{GL}(r, \mathbb{C}) : \det M = 1\}.$$

The Lie algebra associated to $\mathrm{SL}(r, \mathbb{C})$ is

$$\mathfrak{sl}(r, \mathbb{C}) = \{X \in \mathfrak{gl}(r, \mathbb{C}) : \mathrm{tr}\, X = 0\}.$$

If (P, ϕ) is a $\mathrm{SL}(r, \mathbb{C})$-Higgs pair on C, the associated vector bundle E is endowed with a volume form $\lambda : \det E \xrightarrow{\sim} \mathcal{O}_C$ and the associated Higgs field Φ is a L-valued endomorphism with $\mathrm{tr}\, \Phi = 0$; we say that (E, Φ) is a Higgs pair *with trace zero*. Viceversa, let (E, Φ) be a Higgs pair of rank r such that $\mathrm{tr}\, \Phi = 0$ and let $\lambda : \det E \xrightarrow{\sim} \mathcal{O}_C$ be a volume form on E; then, the frame bundle $P = \mathrm{Fr}_{\mathrm{SL}}(E, \lambda)$ of all ordered basis whose λ-volume equals 1 is a principal $\mathrm{SL}(r, \mathbb{C})$-bundle, and the Higgs field Φ is the image of a unique global section ϕ of $\mathrm{ad}\, P \otimes L$ with respect to the morphism $\mathrm{ad}\, P \otimes L \to \mathrm{End}\, E \otimes L$ induced by the canonical embedding $\rho : \mathrm{SL}(r, \mathbb{C}) \hookrightarrow \mathrm{GL}(r, \mathbb{C})$.

To sum up, the datum (P, ϕ) of a $\mathrm{SL}(r, \mathbb{C})$-Higgs pair on C corresponds univocally, via the associated bundle construction, to the datum of (E, Φ, λ), where (E, Φ) is a Higgs pair of rank r with trace zero, and λ is a trivialization of $\det E$. Note that the isomorphism $\det E \simeq \mathcal{O}_C$ implies in particular that the Higgs pairs associated to SL-Higgs pairs have always degree equal to 0.

A basis for the invariant polynomials of $\mathfrak{sl}(r, \mathbb{C})$ is given by $\{p_2, \ldots, p_r\}$ where $p_i(P, \phi) := (-)^i \operatorname{tr}(\wedge^i \phi)$; in particular, if (E, Φ) is the associated Higgs pair, then $p_i(P, \phi) = a_i(E, \phi)$. Then, the SL-Hitchin morphism of rank r can be defined as:

$$\mathcal{H}_{\mathrm{SL},r} : \mathcal{M}_{\mathrm{SL}}(r) := \mathcal{M}_{\mathrm{SL}}(r, 0) \longrightarrow \mathcal{A}_{\mathrm{SL}}(r) = \bigoplus_{i=2}^{r} H^0(C, L^i)$$

$$(P, \phi) \longmapsto (p_2(P, \phi), \ldots, p_r(P, \phi)) =$$

$$= (a_2(E, \Phi), \ldots, a_r(E, \Phi))$$

$$\text{for } (E, \Phi) \text{ associated to } (P, \phi).$$

PROPOSITION 4.1.1. (Spectral correspondence for $\mathrm{SL}(r, \mathbb{C})$) *Let $\underline{a} \in \mathcal{A}_{\mathrm{SL}}(r)$ be any characteristic, let $X = X_{\underline{a}} \xrightarrow{\pi} C$ be the associated spectral curve, and denote $B := \det(\pi_* \mathcal{O}_X)^{-1}$. The fiber $\mathcal{H}_{\mathrm{SL},r}^{-1}(\underline{a})$ of the SL-Hitchin morphism is isomorphic, via the spectral correspondence, to the fiber $\mathrm{Nm}_\pi^{-1}(B)$ of the Norm map from $\overline{\mathbb{J}}(X, d')$ to $J(C, d')$ induced by π for $d' = \frac{r(r-1)}{2}\ell$.*

PROOF. The datum of a $\mathrm{SL}(r, \mathbb{C})$-Higgs pair (P, ϕ) with characteristic $\underline{a}$ corresponds uniquely, via the associated bundle construction, to the datum of (E, Φ, λ), where (E, Φ) is a Higgs pair of rank r and degree 0 with $\mathcal{H}_{r,0}(E, \Phi) = \underline{a}$ and λ is a trivialization of $\det E$.

The datum of the Higgs pair $(E, \Phi) \in \mathcal{H}_{r,0}^{-1}(\underline{a})$ corresponds uniquely, via the spectral correspondence (Proposition 1.4.4), to the datum of a torsion-free rank-1 sheaf $\mathcal{M}$ on X of degree d'. By Definition 2.5.1, $\mathrm{Nm}_\pi(\mathcal{M}) := \det(\pi_* \mathcal{M}) \otimes B$, so giving a isomorphism λ between $\det \pi_* \mathcal{M}$ and $\mathcal{O}_C$ is the same as giving a isomorphism ϵ between $\mathrm{Nm}_\pi(\mathcal{M})$ and B.

To sum up, the datum of (P, ϕ) corresponds to the datum $(\mathcal{M}, \epsilon)$ of an element $\mathcal{M} \in \overline{\mathbb{J}}(X, d')$ and an isomorphism $\epsilon : \mathrm{Nm}(\mathcal{M}) \xrightarrow{\sim} B$. This is an element of the fiber product stack:

$$\begin{array}{ccc} \mathrm{Nm}^{-1}(B) & \longrightarrow & \overline{\mathbb{J}}(X, d') \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{Nm}_\pi} \\ B & \longrightarrow & J(C, d') \end{array}$$

We conclude that $\mathcal{H}_{\mathrm{SL},r}^{-1}(\underline{a}) \simeq \mathrm{Nm}^{-1}(B)$. $\qquad\qquad \square$

PROPOSITION 4.1.2. *Let $\underline{a} \in \mathcal{A}_{\mathrm{SL}}^{reg}(r)$ be a characteristic such that the spectral curve X is reduced. Then,*

$$\mathcal{H}_{\mathrm{SL},r}^{-1}(a) \simeq \overline{\mathrm{Pr}}(X,C).$$

PROOF. Let $B = \det(\pi_* \mathcal{O}_X)^{-1} \in J(C)$. Since X is reduced with locally planar singularities, $\overline{\mathbb{J}}(X) = G\mathbb{J}(X)$ by Remark 1.2.9 and $\mathrm{Nm}_\pi^{-1}(B)$ contains at least a line bundle by Proposition 3.3.2; denote it with N. Tensoring by N in $\overline{\mathbb{J}}(X)$ induces then by Proposition 2.5.5 a well-defined isomorphism:

$$\overline{\mathrm{Pr}}(X,C) \longrightarrow \mathrm{Nm}_\pi^{-1}(B)$$
$$\mathcal{L} \longmapsto \mathcal{L} \otimes N.$$

$\square$

4.2. $\mathrm{PGL}(r,\mathbb{C})$-**Higgs pairs**

We come now to the case of $\mathrm{PGL}(r,\mathbb{C})$-Higgs pairs. Recall first that the Projective Linear Group $\mathrm{PGL}(r,\mathbb{C})$ is defined by the exact sequence:

$$(8) \qquad 0 \to \mathbb{C}^* \xrightarrow{\lambda \mapsto \lambda I_r} \mathrm{GL}(r,\mathbb{C}) \to \mathrm{PGL}(r,\mathbb{C}) \to 0.$$

From the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_C \to \mathcal{O}_C^* \to 0$ we deduce that $H^2(C, \mathcal{O}_C^*) = 0$; hence, the sequence 8 applied to the structure sheaves:

$$0 \to \mathcal{O}_C^* \to \mathrm{GL}(r, \mathcal{O}_C) \to \mathrm{PGL}(r, \mathcal{O}_C) \to 0$$

yields the cohomology exact sequence

$$(9) \quad H^1(C, \mathcal{O}_C^*) \to H^1(C, \mathrm{GL}(r, \mathcal{O}_C)) \xrightarrow{q} H^1(C, \mathrm{PGL}(r, \mathcal{O}_C)) \to 0.$$

The sequence 9, read in terms of cocycles, means that $\mathrm{PGL}(r,\mathbb{C})$-principal bundles are in one-to-one correspondence with equivalence classes of $\mathrm{GL}(r,\mathbb{C})$-principal bundles, with respect to the action on associated bundles given by tensor product of line bundles on C. If P is a $\mathrm{PGL}(r,\mathbb{C})$-principal bundle and $\tilde{P}$ is a $\mathrm{GL}(r,\mathbb{C})$-principal bundle such that $q(\tilde{P}) = P$, we say that $\tilde{P}$ is a *lifting* of P to a $\mathrm{GL}(r,\mathbb{C})$-principal bundle.

Moreover, the Lie algebra $\mathfrak{pgl}(r,\mathbb{C})$ associated to $\mathrm{PGL}(r,\mathbb{C})$ is equal to the Lie algebra $\mathfrak{sl}(r,\mathbb{C})$ of $\mathrm{SL}(r,\mathbb{C})$. If P is any $\mathrm{PGL}(r,\mathbb{C})$-principal

bundle and $\tilde{P}$ is a lifting of P to a $\mathrm{GL}(r,\mathbb{C})$-principal bundle, then a section ϕ of $H^0(C, \mathrm{ad}\, P \otimes L)$ determines uniquely a section $\tilde{\phi}$ of $H^0(C, \mathrm{ad}\, \tilde{P} \otimes L)$ with trace equal to 0, and viceversa. We say that $(\tilde{P}, \tilde{\phi})$ is a lifting of (P, ϕ) to a GL-Higgs pair and with a slight abuse of notation we write $q(\tilde{P}, \tilde{\phi}) = (P, \phi)$.

Tu sum up, any $\mathrm{PGL}(r,\mathbb{C})$-Higgs pair (P, ϕ) has a lifting $(\tilde{P}, \tilde{\phi})$ to a to a $\mathrm{GL}(r,\mathbb{C})$-Higgs pair, corresponding to a Higgs pair (E, Φ) with trace zero via the associated bundle construction. Then, the datum of (P, ϕ) corresponds uniquely to the datum of the equivalence class $[(E, \Phi)]$ of Higgs pairs on C with trace zero, under the equivalence relation $\sim_{J(C)}$ defined by:

$$(E, \Phi) \sim_{J(C)} (E \otimes N, \Phi \otimes 1_N) \qquad \text{for any } N \in J(C).$$

Let $\mathcal{M}(r) = \coprod_{d \in \mathbb{Z}} \mathcal{M}(r, d)$ be the moduli stack of Higgs pairs of rank r in any degree and denote with $\mathcal{M}^{\mathrm{tr}=0}(r)$ the closed substack of $\mathcal{M}(r)$ given by Higgs pairs of rank r with trace zero. Then, $[(E, \Phi)]$ is the orbit of (E, Φ) under the action of $J(C)$ on $\mathcal{M}^{\mathrm{tr}=0}(r)$ defined by:

$$\mathcal{M}^{\mathrm{tr}=0}(r) \times J(C) \longrightarrow \mathcal{M}^{\mathrm{tr}=0}(r)$$
$$((E, \Phi), N) \longmapsto (E \otimes N, \Phi \otimes 1_N).$$

The degree of Higgs pairs associated to a PGL-Higgs pair is defined only modulo r. Indeed, if N is any line bundle on C, then

$$\deg(E \otimes N) = \deg(\det(E \otimes N)) =$$
$$= \deg(\det(E) \otimes N^r) = \deg(E) + r \deg(N)$$

so the degree of $J(C)$-equivalent Higgs pairs may differ by multiples of r. Hence, we can give the following definition.

DEFINITION 4.2.1. Let (P, ϕ) be a PGL-Higgs pair and let (E, Φ) be a Higgs pair with trace zero and degree $d \in \mathbb{Z}$ corresponding to a lifting $(\tilde{P}, \tilde{\phi})$ of (P, ϕ) to a GL-Higgs pair. The *degree of* (P, ϕ) is the congruence class $\bar{d} \in \mathbb{Z}/r\mathbb{Z}$.

REMARK 4.2.2. Obviously, the same definition can be given just for PGL-principal bundles. Let P be a PGL-principal bundle ald let E be the vector bundle associated to any lifting of P to a GL-principal

bundle. The *degree* of a PGL-principal bundles P is the congruence class $\overline{d} \in \mathbb{Z}/r\mathbb{Z}$ of the degree $d = \deg E$.

As a matter of fact, the first homotopy group $\pi_1(\mathrm{PGL}(r,\mathbb{C}))$ of $\mathrm{PGL}(r,\mathbb{C})$ is isomorphic to $\mathbb{Z}/r\mathbb{Z}$ and the degree of (P,ϕ) characterizes uniquely the topological type of P.

Up to the action on (E,Φ) of a line bundle of degree 1 on C, it is straightforward to see that the $J(C)$-orbit $[(E,\Phi)]$ corresponding to a PGL-Higgs pair with degree $\overline{d} \in \mathbb{Z}/r\mathbb{Z}$ is always the orbit of a Higgs pair of degree d with $d \in \{0,\ldots,r-1\}$. If we restrict the correspondence to Higgs pairs with fixed degree $d \in \{0,\ldots,r-1\}$, then a PGL-Higgs pair of degree $\overline{d}$ is identified uniquely with the orbit of a Higgs pair of trace zero *and degree* d with respect to the action of line bundles *of degree* 0 on C. We have then the following proposition.

PROPOSITION 4.2.3. *Let $\overline{d} \in \mathbb{Z}/r\mathbb{Z}$ with $d \in \{0,\ldots,r-1\}$. Denote with $\mathcal{M}^{\mathrm{tr}=0}(r,d)$ the closed substack of $\mathcal{M}(r,d)$ given by Higgs pairs with trace zero. The moduli stack $\mathcal{M}_{\mathrm{PGL}}(r,\overline{d})$ of PGL-Higgs pairs of rank r and degree $\overline{d}$ is isomorphic to the quotient stack*

$$q : \mathcal{M}^{\mathrm{tr}=0}(r,d) \longrightarrow \mathcal{M}^{\mathrm{tr}=0}(r,d)/J^0(C) \xrightarrow{\sim} \mathcal{M}_{\mathrm{PGL}}(r,\overline{d})$$

$$(E,\Phi) \longmapsto [(E,\Phi)] \longmapsto (P,\phi) := q(\tilde{P},\tilde{\phi})$$

where (E,Φ) is associated to the GL-principal bundle $(\tilde{P},\tilde{\phi})$.

A basis for the invariant polynomials of $\mathfrak{pgl}(r,\mathbb{C}) = \mathfrak{sl}(r,\mathbb{C})$ is given by $p_2,\ldots,p_r$ where $p_i(P,\phi) := (-)^i \mathrm{tr}(\wedge^i \phi)$; moreover, if $[(E,\Phi)]$ is the associated $J^0(C)$-equivalence class of Higgs pairs with zero trace, then $p_i(P,\phi) = a_i(E,\Phi)$ for any (E,Φ) associated to a lifting of (P,ϕ). Hence, we have the following PGL-Hitchin morphism of rank r and degree $\overline{d}$:

$$\mathcal{H}_{\mathrm{PGL},r,\overline{d}} : \mathcal{M}_{\mathrm{PGL}}(r,\overline{d}) \longrightarrow \mathcal{A}_{\mathrm{PGL}}(r) = \bigoplus_{i=2}^{r} H^0(C,L^i)$$

$$(P,\phi) \longmapsto (p_2(P,\phi),\ldots,p_r(P,\phi)).$$

The PGL-Hitchin morphism fits the following commutative diagram:

$$
\begin{array}{ccccc}
\mathcal{M}(r,d) & \lhook\joinrel\relbar\joinrel\rightarrow & \mathcal{M}^{\mathrm{tr}=0}(r,d) & \relbar\joinrel\relbar\joinrel\twoheadrightarrow & \mathcal{M}_{\mathrm{PGL}}(r,\overline{d}) \\
\Big\downarrow{\scriptstyle\mathcal{H}_{r,d}} & & \Big\downarrow{\scriptstyle\mathcal{H}^{\mathrm{tr}=0}_{r,d}} & & \Big\downarrow{\scriptstyle\mathcal{H}_{\mathrm{PGL},r,\overline{d}}} \\
\mathcal{A}(r) & \lhook\joinrel\relbar\joinrel\rightarrow & \bigoplus_{i=2}^{r} H^0(C, L^i) & = & \mathcal{A}_{\mathrm{PGL}}(r)
\end{array}
$$

with $d \in \{0, \ldots, r-1\}$ and $\overline{d} \in \mathbb{Z}/r\mathbb{Z}$.

We are now ready to state the spectral correspondence for $\mathrm{PGL}(r,\mathbb{C})$-Higgs pairs.

PROPOSITION 4.2.4. (Spectral correspondence for $\mathrm{PGL}(r,\mathbb{C})$) *Let* $\underline{a} \in \mathcal{A}_{\mathrm{PGL}}(r)$ *be any characteristic and let* $X = X_{\underline{a}} \xrightarrow{\pi} C$ *be the associated spectral curve. Let* $\overline{d} \in \mathbb{Z}/r\mathbb{Z}$ *with* $d \in \{0, \ldots, r-1\}$ *be any degree. The fiber* $\mathcal{H}^{-1}_{\mathrm{PGL},r,\overline{d}}(\underline{a})$ *of the* PGL-*Hitchin morphism is isomorphic, via the spectral correspondence, to the quotient moduli space*

$$
\overline{\mathbb{J}}(X, d') / \pi^* J^0(C)
$$

of torsion-free sheaves of rank 1 and degree d' *up to the action of line bundles of degree* 0 *on* C *by tensor product, with* $d' = d + \frac{r(r-1)}{2}\ell$.

PROOF. The datum of a $\mathrm{PGL}(r,\mathbb{C})$-Higgs pair (P, ϕ) with degree $\overline{d}$ and characteristic $\underline{a}$ corresponds uniquely to the datum of a $J^0(C)$-equivalence class $[(E, \Phi)]$ of Higgs pairs of rank r and degree d with characteristic $\underline{a}$, where (E, Φ) is the Higgs pair associated, via the vector bundle construction, to a lifting of (P, ϕ) to a GL-Higgs pair. By the spectral correspondence, the datum of (E, Φ) corresponds uniquely to the datum of a pure sheaf $\mathcal{M} \in \overline{\mathbb{J}}(X, d')$ such that $\pi_* \mathcal{M} = E$ and $d' = d + \frac{r(r-1)}{2}\ell$. Let $N \in J^0(C)$ be any line bundle of degree 0 on C. By the projection formula,

$$
\pi_*(\mathcal{M} \otimes \pi^* N) \simeq \pi_* \mathcal{M} \otimes N,
$$

and the following square commutes:

$$
\begin{array}{ccc}
\pi_*(\mathcal{M} \otimes \pi^* N) & \xrightarrow{\ \sim\ } & \pi_* \mathcal{M} \otimes N \\
\Big\downarrow{\scriptstyle \pi_*(\cdot x)} & & \Big\downarrow{\scriptstyle \Phi_N} \\
\pi_*(\mathcal{M} \otimes \pi^* N) \otimes L & \xrightarrow{\ \sim\ } & (\pi^* \mathcal{M} \otimes N) \otimes L.
\end{array}
$$

Hence, the spectral correspondence is equivariant with respect to the action of $\pi^* J^0(C)$ by tensor product on $\overline{\mathbb{J}}(X, d')$ and the action of $J^0(C)$

on $\mathcal{M}^{\mathrm{tr}=0}(r,d)$. Then, passing to the quotient on both sides, the datum of the $J^0(C)$-orbit $[(E,\Phi)]$ corresponds uniquely to the datum of a $\pi^* J^0(C)$-orbit $[\mathcal{M}] \in \overline{\mathbb{J}}(X,d')/\pi^* J^0(C)$. $\qquad\qquad\qquad\qquad\qquad\square$

We study now the special case of $\mathrm{PGL}(r,\mathbb{C})$-Higgs pairs of degree 0. Since $\mathbb{C}$ is algebraically closed, the Projective Linear group is equal to the Projective Special Linear group $\mathrm{PSL}(r,\mathbb{C})$ defined by the exact sequence:

(10) $$0 \to \mu_r \xrightarrow{\lambda \mapsto \lambda I_r} \mathrm{SL}(r,\mathbb{C}) \to \mathrm{PSL}(r,\mathbb{C}) \to 0;$$

here we denote with μ_r the group of r-th roots of unity, as defined by the exact sequence:

$$0 \to \mu_r \to \mathbb{C}^* \xrightarrow{\lambda \mapsto \lambda^r} \mathbb{C}^* \to 0.$$

In particular, $H^2(C,\mu_r) = \mathbb{Z}/r\mathbb{Z}$, hence the exact sequence 10, applied to structure sheaves, induces the cohomology exact sequence:

(11)
$$H^1(C,\mu_r) \to H^1(C,\mathrm{SL}(r,\mathcal{O}_C)) \xrightarrow{q} H^1(C,\mathrm{PSL}(r,\mathcal{O}_C)) \xrightarrow{\deg} \mathbb{Z}/r\mathbb{Z} \to 0.$$

The sequence 11, read in terms of cocycles, means that a $\mathrm{PGL}(r,\mathbb{C})$-principal bundle P can be lifted to a principal $\mathrm{SL}(r,\mathbb{C})$-principal bundle P_0 if and only if P has degree $0 \in \mathbb{Z}/r\mathbb{Z}$; any other lifting P_0' of P differs from P_0 by the action on associated bundles of a r-th torsion line bundle by tensor product. Moreover, a Higgs field ϕ on P determines uniquely a Higgs field ϕ_0 on P_0, and viceversa.

To sum up, any $\mathrm{PGL}(r,\mathbb{C})$-principal bundle (P,ϕ) of degree zero has a lifting (P_0,ϕ_0) to a $\mathrm{SL}(r,\mathbb{C})$-principal bundle, corresponding to the datum (E,Φ,λ) of a Higgs pair (E,Φ) with trace zero and a volume form $\lambda : \det E \xrightarrow{\sim} \mathcal{O}_C$ via the associated bundle construction. Then, the datum of (P,ϕ) corresponds uniquely to the datum of the equivalence class $[(E,\phi,\lambda)]$ of Higgs pairs with trace zero, under the equivalence relation $\sim_{J^0(C)[r]}$ defined by:

$$(E,\Phi,\lambda) \sim_{J^0(C)[r]} (E \otimes N, \Phi \otimes 1_N, \lambda_{N,\epsilon}) \qquad \text{for any } (N,\epsilon) \in J^0(C)[r].$$

Here, $J^0(C)[r]$ is the group stack of r-torsion line bundles (N, η) on C, where η is the isomorphism $N^r \xrightarrow{\sim} \mathcal{O}_C$, and $\lambda_{N,\eta}$ denote the isomorphism:

$$\lambda_{N,\eta} : \det(E \otimes N) \xrightarrow{\sim} \det(E) \otimes N^r \xrightarrow[\lambda \otimes \eta]{\sim} \mathcal{O}_C.$$

We can prove now the following proposition.

PROPOSITION 4.2.5. (Spectral correspondence for $\mathrm{PGL}(r, \mathbb{C})$ of degree 0) *Let $\underline{a} \in \mathcal{A}_{\mathrm{PGL}}(r)$ be any characteristic, let $X = X_{\underline{a}} \xrightarrow{\pi} C$ be the associated spectral curve and denote $B := \det(\pi_* \mathcal{O}_X)^{-1}$. Let $d' = \frac{r(r-1)}{2}\ell$ and let Nm_π be the Norm map induced by π on $\overline{\mathbb{J}}(X, d')$. Let $J^0(C)[r]$ be the group stack of line bundles with r-th torsion on C, acting on $\mathrm{Nm}_\pi^{-1}(B)$ as follows:*

$$\mathrm{Nm}_\pi^{-1}(B) \times \pi^* J^0(C)[r] \longrightarrow \mathrm{Nm}_\pi^{-1}(B)$$
$$((\mathcal{M}, \epsilon), (\pi^* N, \pi^* \eta)) \longmapsto (\mathcal{M} \otimes \pi^* N, \epsilon_{N,\eta})$$

where $\epsilon_{N,\eta}$ is equal to the following composition:

$$\epsilon_{N,\eta} : \mathrm{Nm}(\mathcal{M} \otimes \pi^* N) \xrightarrow{\sim} \mathrm{Nm}(\mathcal{M}) \otimes \mathrm{Nm}(\pi^* N) \xrightarrow{\sim} \mathrm{Nm}(\mathcal{M}) \otimes N^r \xrightarrow{\epsilon \otimes \eta} B.$$

Then, the fiber $\mathcal{H}_{\mathrm{PGL},r,\overline{0}}^{-1}(\underline{a})$ of the PGL-Hitchin morphism is isomorphic, via the spectral correspondence, to the quotient moduli space

$$\mathrm{Nm}_\pi^{-1}(B) / \pi^*(J^0 C)[r].$$

PROOF. The datum of a $\mathrm{PGL}(r, \mathbb{C})$-Higgs pair (P, ϕ) with degree 0 and characteristic $\underline{a}$ corresponds uniquely to the datum of a $J^0(C)[r]$-equivalence class $[(E, \Phi, \lambda)]$ of Higgs pairs of rank r and degree 0 with characteristic $\underline{a}$, where (E, Φ, λ) is the datum of a Higgs pair with a volume form λ associated, via the vector bundle construction, to a lifting of (P, ϕ) to a SL-Higgs pair.

By Proposition 4.1.1, the datum of (E, Φ, λ) corresponds to the datum (M, ϵ) of a torsion-free rank-1 sheaf $\mathcal{M}$ of degree d' on X and an isomorphism $\epsilon : \mathrm{Nm}(\mathcal{M}) \xrightarrow{\sim} B$, i.e. an element of $\mathrm{Nm}^{-1}(B)$. If (N, η) is a line bundle on C with r-th torsion, the datum of $(E \otimes N, \Phi_N, \lambda_{N,\eta})$ corresponds by projection formula to the datum of $(M \otimes \pi^* N, \epsilon_{N,\eta})$. Then, the datum of the $J^0(C)[r]$-equivalence class $[(E, \Phi, \lambda)]$ corresponds uniquely to the datum of the $\pi^* J^0(C)[r]$-equivalence class $[(\mathcal{M}, \epsilon)]$.

$\square$

4.3. $\mathrm{Sp}(2r, \mathbb{C})$-**Higgs pairs**

The symplectic group $\mathrm{Sp}(2r, \mathbb{C})$ is defined as:

$$\mathrm{Sp}(2r, \mathbb{C}) = \{M \in \mathrm{GL}(2r, \mathbb{C}) : M\Omega M^t = \Omega\}$$

where $\Omega = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$.

The Lie algebra associated to $\mathrm{Sp}(2r, \mathbb{C})$ is

$$\mathfrak{sp}(2r, \mathbb{C}) = \left\{ X \in \mathfrak{gl}(2r, \mathbb{C}) : X\Omega + \Omega X^t = 0 \right\}.$$

If (P, ϕ) is a $\mathrm{Sp}(2r, \mathbb{C})$-Higgs pair on C, the associated vector bundle E is endowed with a non-degenerate symplectic form $\omega : E \otimes E \to \mathcal{O}_C$ and the associated Higgs field Φ is satisfies the condition $\omega(\Phi v, w) = -\omega(v, \Phi w)$ for all sections v, w of E. The existence of a non-degenerate symplectic form implies that E has trivial determinant and, in particular, that E has degree 0. Viceversa, let (E, Φ) be a Higgs pair of rank $2r$ and let ω be a non-degenerate symplectic form on E satisfying the condition $\omega(\Phi v, w) = -\omega(v, \Phi w)$; then, the frame bundle $P = \mathrm{Fr}_{\mathrm{Sp}}(E, \omega)$ of all ordered symplectic basis of (E, ω) is a principal $\mathrm{Sp}(2r, \mathbb{C})$-bundle, and the Higgs field Φ is the image of a unique global section ϕ of $\mathrm{ad}\, P \otimes L$ with respect to the morphism $\mathrm{ad}\, P \otimes L \to \mathrm{End}\, E \otimes L$ induced by the canonical embedding $\rho : \mathrm{Sp}(2r, \mathbb{C}) \hookrightarrow \mathrm{GL}(2r, \mathbb{C})$.

To sum up, the datum of a $\mathrm{Sp}(2r, \mathbb{C})$-Higgs pair (P, ϕ) on C corresponds univocally to the datum of (E, Φ, ω) where (E, Φ) is a Higgs pair of rank $2r$ and degree 0 and $\omega : E \otimes E \to \mathcal{O}_C$ is a non-degenerate symplectic form on E satisfying the condition:

$$\omega(\Phi v, w) = -\omega(v, \Phi w).$$

A basis for the invariant polynomials of $\mathfrak{sp}(2r, \mathbb{C})$ is given by $\{p_{2i}\}_{i=1,\dots,r}$ where $p_{2i}(P, \phi) := \mathrm{tr}(\wedge^{2i}\phi)$; if (E, Φ) is the associated Higgs pair, then $p_{2i}(P, \phi) = a_{2i}(E, \phi)$. The corresponding Sp-Hitchin morphism takes the form:

$$\mathcal{H}_{\mathrm{Sp},2r} : \mathcal{M}_{\mathrm{Sp}}(2r) := \mathcal{M}_{\mathrm{Sp}}(2r, 0) \longrightarrow \mathcal{A}_{\mathrm{Sp}}(2r) = \bigoplus_{l=1}^{r} H^0(C, L^{2l})$$

$$(P, \phi) \longmapsto (p_2(P, \phi), p_4(P, \phi), \dots, p_{2r}(P, \phi)).$$

For any characteristic $\underline{a} \in \mathcal{A}_{\mathrm{Sp}}(2r)$, the spectral curve $\pi : X_{\underline{a}} \to C$ is defined in the total space $p : \mathbb{P}(\mathcal{O}_C \oplus L^{-1}) \to C$ by the equation

$$x^{2r}y + a_2 x^{2r-2}y^2 + \cdots + a_{2r-2}x^2 y^{2r-2} + a_{2r}y^{2r} = 0.$$

Hence, the curve $X = X_{\underline{a}}$ has an involution σ defined by $\sigma(x) = -x$ and a quotient curve:

$$X \xrightarrow{\quad q \quad} X/\sigma = Y$$
$$\pi \searrow \qquad \swarrow \bar{\pi}$$
$$C$$

The involution σ induces by pullback an involution on the compactified Jacobian of torsion-free rank-1 sheaves with any degree d':

$$\sigma^* : \overline{\mathbb{J}}(X, d') \longrightarrow \overline{\mathbb{J}}(X, d')$$
$$\mathcal{L} \longmapsto \sigma^* \mathcal{L}.$$

PROPOSITION 4.3.1. (Spectral correspondence for $\mathrm{Sp}(2r, \mathbb{C})$) *Let $\underline{a} \in \mathcal{A}_{\mathrm{Sp}}(2r)$ be any characteristic and let $X = X_{\underline{a}} \xrightarrow{\pi} C$ be the associated spectral curve with involution $\sigma : X \to X$. The fiber $\mathcal{H}^{-1}_{\mathrm{Sp},2r}(\underline{a})$ of the Sp-Hitchin morphism is isomorphic, via the spectral correspondence, to the equalizer $\mathcal{E}_{\underline{a}}$ of the two maps*

$$\_^* := \mathcal{H}om_{\mathcal{O}_X}(\_, \mathcal{O}_X) : \overline{\mathbb{J}}(X, d') \to \overline{\mathbb{J}}(X, -d')$$
$$\sigma^*\_ \otimes \pi^* L^{1-2r} : \overline{\mathbb{J}}(X, d') \to \overline{\mathbb{J}}(X, -d'),$$

where $d' = r(2r-1)\ell$.

PROOF. The dual of a torsion-free sheaf on a Gorenstein curve is torsion-free sheaf with the same rank, hence the map $\mathcal{H}om_{\mathcal{O}_X}(\_, \mathcal{O}_X)$ is well-defined. Let (P, ϕ) be any Sp-Higgs pair with characteristic $\underline{a}$ and let (E, Φ, ω) be the associated datum of a Higgs pair of rank $2r$ and degree 0 with characteristic $\underline{a}$ and symplectic form ω. The torsion-free sheaf $\mathcal{M}$ on X associated to (E, Φ) by the spectral correspondence fits into the exact sequence:

$$0 \to \mathcal{M} \otimes \pi^* L^{1-2r} \to \pi^* E \xrightarrow{\pi^* \Phi - x} \pi^*(E \otimes L) \to \mathcal{M} \otimes \pi^* L \to 0.$$

Taking the dualized sequence and tensoring by $\pi^* L$ gives a left exact sequence:

$$0 \to \mathcal{M}^* \to \pi^* E^* \xrightarrow{\pi^* \Phi^t - x} \pi^* E^* \otimes \pi^* L.$$

On the other hand, applying σ^* to the first exact sequence gives:

$$0 \to \sigma^* \mathcal{M} \otimes \pi^* L^{1-2r} \to \pi^* E \xrightarrow{\pi^* \Phi + x} \pi^*(E \otimes L) \to \sigma^* \mathcal{M} \otimes \pi^* L \to 0.$$

Now, the symplectic form ω induce an isomorphism $\omega_E : E \simeq E^*$ and hence a commutative diagram with vertical isomorphisms:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{M}^* & \longrightarrow & \pi^* E^* & \xrightarrow{\pi^* \Phi^t - x} & \pi^* E^* \otimes \pi^* L \\
& & & & \omega_E \big\uparrow \sim & & \omega_E \big\uparrow \sim \\
0 & \longrightarrow & \sigma^* \mathcal{M} \otimes \pi^* L^{1-2r} & \longrightarrow & \pi^* E & \xrightarrow{\pi^* \Phi + x} & \pi^*(E \otimes L)
\end{array}
$$

Hence, we conclude that the sheaf $\mathcal{M}$ comes with a sheaf isomorphism

$$\lambda : \mathcal{M}^* \xrightarrow{\sim} \sigma^* \mathcal{M} \otimes \pi^* L^{1-2r}.$$

Viceversa, suppose that $\mathcal{M}$ is a torsion-free sheaf of rank 1 and degree d' on X with an isomorphism $\lambda : \mathcal{M}^* \xrightarrow{\sim} \sigma^* \mathcal{M} \otimes \pi^* L^{1-2r}$. Let (E, Φ) be the GL-Higgs pair with characteristic $\underline{a}$ corresponding to $\mathcal{M}$. By [**HS93**], the Relative duality formula holds for X/C with dualizing sheaf equal to the sheaf of relative differentials $\omega_{X/C}$. By Lemma 1.4.1 and Riemann-Hurwitz we obtain that $\omega_{X/C} \simeq \pi^* L^{2r-1}$. Applying Relative duality in dimension 0 and the projection formula, we obtain an isomorphism:

$$E^* = \pi_*(\mathcal{M})^* \simeq \pi_*(\mathcal{M}^* \otimes \pi^* L^{2r-1}) \simeq$$

$$\simeq \pi_*(\sigma^*(\mathcal{M})) = \pi_*(\sigma_* \sigma^* \mathcal{M}) \simeq$$

$$\simeq \pi_*(\sigma_* \mathcal{O}_X \otimes \mathcal{M}) = \pi_*(\mathcal{O}_X \otimes \mathcal{M}) \simeq$$

$$\simeq \pi_*(\mathcal{M}) = E;$$

that induces the symplectic structure $\omega : E \otimes E \to \mathcal{O}_C$.

To sum up, the datum of (P, ϕ) in the fiber $\mathcal{H}_{\mathrm{Sp}, 2r}^{-1}(\underline{a})$ corresponds uniquely to the datum $(\mathcal{M}, \lambda)$ of a torsion-free rank-1 sheaf $\mathcal{M} \in \bar{\mathbb{J}}(X, d')$ and an isomorphism $\lambda : \mathcal{M}^* \xrightarrow{\sim} \sigma^* \mathcal{M} \otimes \pi^* L^{1-2r}$, i.e. an element of the equalizer stack $\mathcal{E}_{\underline{a}}$. $\qquad \square$

4.4. GSp$(2r, \mathbb{C})$-Higgs pairs

The General Symplectic group GSp$(2r, \mathbb{C})$ is defined as:

$$\mathrm{GSp}(2r, \mathbb{C}) = \{M \in \mathrm{GL}(2r, \mathbb{C}) : M \Omega M^t = \lambda \Omega \text{ for some } \lambda \in \mathbb{C}^*\}$$

where $\Omega = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$.

The Lie algebra associated to $\mathrm{GSp}(2r, \mathbb{C})$ is

$$\mathfrak{gsp}(2r, \mathbb{C}) = \left\{ A \in \mathfrak{gl}(2r, \mathbb{C}) : A\Omega + \Omega A^t = \frac{\operatorname{tr} A}{r} \Omega \right\} \simeq$$

$$\simeq \mathfrak{sp}(2r, \mathbb{C}) \oplus \mathbb{C}$$

where the isomorphism is given by the decomposition $A = X + \frac{\operatorname{tr} A}{2r} I_{2r}$, with $X \in \mathfrak{sp}(2r, \mathbb{C})$ and $\operatorname{tr}(A) \in \mathbb{C}$.

If (P, ϕ) is a $\mathrm{GSp}(2r, \mathbb{C})$-Higgs pair on C, the associated vector bundle E is endowed with a non-degenerate symplectic form $\omega : E \otimes E \to M$ with values in a line bundle M on C and the associated Higgs field Φ satisfies the condition

$$\omega(\Phi v, w) + \omega(v, \Phi w) = \frac{\operatorname{tr} \Phi}{r} \omega(v, w)$$

for all sections v, w of E. The existence of a non-degenerate symplectic M-valued form implies that E has determinant isomorphic to M^r and, in particular, that E has degree equal to $r \deg M$. Viceversa, let (E, Φ) be a Higgs pair of rank $2r$ and let ω be a non-degenerate symplectic M-valued form on E satisfying the condition

$$\omega(\Phi v, w) + \omega(v, \Phi w) = \frac{\operatorname{tr} \Phi}{r} \omega(v, w);$$

then, the frame bundle $P = \mathrm{Fr}_{\mathrm{GSp}}(E, \omega)$ of all ordered symplectic basis of (E, ω) is a principal $\mathrm{GSp}(2r, \mathbb{C})$-bundle, and the Higgs field Φ is the image of a unique global section ϕ of $\operatorname{ad} P \otimes L$ with respect to the morphism $\operatorname{ad} P \otimes L \to \operatorname{End} E \otimes L$ induced by the canonical embedding $\rho : \mathrm{Sp}(2r, \mathbb{C}) \hookrightarrow \mathrm{GL}(2r, \mathbb{C})$.

Tu sum up, the datum of a $\mathrm{GSp}(2r, \mathbb{C})$-Higgs pair (P, ϕ) corresponds uniquely, via the associated bundle construction, to the datum (E, Φ, M, ω) of a Higgs pair (E, Φ) of rank $2r$ and degree rd and a non-degenerate symplectic form $\omega : E \otimes E \to M$ with values in a line bundle M of degree d, and $\Phi \in H^0(C, \operatorname{End}(E) \otimes L)$ satisfies

(12) $$\omega(\Phi v, w) + \omega(v, \Phi w) = \frac{\operatorname{tr} \Phi}{r} \omega(v, w).$$

Since $\mathfrak{gsp}(2r, \mathbb{C}) \subset \mathfrak{gl}(2r, \mathbb{C})$, the GSp-Hitchin morphism can be defined with the help of the GL-Hitchin morphism, as follows:

$$\mathcal{H}_{\mathrm{GSp}, 2r, rd} : \mathcal{M}_{\mathrm{GSp}}(2r, rd) \longrightarrow \mathcal{A}_{\mathrm{GSp}}(2r) \subseteq \bigoplus_{i=1}^{2r} H^0(C, L^i)$$

$$(P, \phi) \longmapsto (p_1(P, \phi), \ldots, p_{2r}(P, \phi)).$$

where $p_i(P, \phi) = (-)^i \operatorname{tr}(\wedge^i \phi)$ and $\mathcal{A}_{\mathrm{GSp}}(2r)$ is defined as the locus of characteristics in $\mathcal{A}_{\mathrm{GL}}(2r)$ resulting as characteristics of $\mathrm{GSp}(2r, \mathbb{C})$-Higgs pairs.

In order to study the space of characteristics $\mathcal{A}_{\mathrm{GSp}}(2r)$, let (E, Φ) be the Higgs pair associated to any GSp-Higgs pair (P, ϕ) and consider then the Higgs field

$$\Phi' = \Phi - \frac{\operatorname{tr} \Phi}{2r} \operatorname{id}_E \in H^0(C, \operatorname{End}(E) \otimes L).$$

Reformulating Equation 12, the following condition on Φ' holds:

(13) $$\omega(\Phi' v, w) + \omega(v, \Phi' w) = 0.$$

As in the case of Sp-Higgs pairs, this condition implies that $a_i(E, \Phi') = 0$ when $i = 2l+1$. In particular, the vector $\underline{a}' = (a_2(E, \Phi'), a_4(E, \Phi'), \ldots, a_{2r}(E, \Phi'))$ may assume any value in the affine space

$$\bigoplus_{l=1}^{r} H^0(C, L^{2l}) = \mathcal{A}_{\mathrm{Sp}}(2r) \subset \mathcal{A}_{\mathrm{GL}}(2r) = \mathcal{A}(2r).$$

Let $\underline{a} = (a_1(E, \Phi), a_2(E, \Phi), \ldots, a_{2r}(E, \Phi)) \in \mathcal{A}(2r)$ be the characteristic of (E, Φ). Since $a_i(E, \Phi) = p_i(P, \phi)$ for any $i = 0, \ldots, 2r$, we have that $\underline{a}$ is equal to the characteristic of (P, ϕ) in $\mathcal{A}_{\mathrm{GSp}}(2r)$. What is the relation between $\underline{a}$ and $\underline{a}'$? Denote with χ_Φ and $\chi_{\Phi'}$ respectively the characteristic polynomials of Φ and Φ', and set $\frac{\operatorname{tr} \Phi}{2r} = \mu \in H^0(C, L)$. By definition of Φ', the characteristic polynomials χ_Φ and $\chi_{\Phi'}$ are related by the following equality:

(14) $$\chi_\Phi(t + \mu) = \det(\Phi - (t + \mu) \operatorname{id}_E) = \det(\Phi' - t \operatorname{id}_E) = \chi_{\Phi'}(t).$$

Comparing the coefficients in Equation 14 and recalling that $\operatorname{tr} \Phi = a_1(E, \Phi)$, it follows that the vector $(\underline{a}', \mu)$ can be determined by a polynomial combination $\underline{P}(\underline{a})$ of the entries of $\underline{a}$. Viceversa, by the equality $\chi_\Phi(t) = \chi_{\Phi'}(t - \mu)$, it follows that the vector $\underline{a}$ can be determined back by a polynomial combination of the entries of the vector $\underline{a}'$ and

the scalar μ, denoted by $\underline{Q}(\underline{a}', \mu)$. In other words, $\underline{P}$ defines an isomorphism between $\mathcal{A}_{\mathrm{GSp}}(2r)$ and the affine space $\mathcal{A}_{\mathrm{Sp}}(2r) \oplus H^0(C, L)$, with inverse $\underline{Q}$.

By means of the previous discussion, the datum of a $\mathrm{GSp}(2r, \mathbb{C})$-Higgs pair (P, ϕ) corresponds uniquely, via the associated bundle construction and the translation of the Higgs field, to the datum $(E, \Phi', \omega, M, \mu)$ of a Higgs pair (E, Φ') of rank $2r$ and degree rd, a non-degenerate symplectic form $\omega : E \otimes E \to M$ with values in a line bundle M of degree d, and a global global section $\mu \in H^0(C, L)$, such that $\Phi' \in H^0(C, \mathrm{End}(E) \otimes L)$ satisfies

(15) $$\omega(\Phi' v, w) + \omega(v, \Phi' w) = 0.$$

Moreover, the affine space $\mathcal{A}_{\mathrm{Sp}}(2r) \oplus H^0(C, L)$ can be taken as basis of another GSp-Hitchin morphism $\widetilde{\mathcal{H}} = \underline{P} \circ \mathcal{H}$:

$$\widetilde{\mathcal{H}}_{\mathrm{GSp}, 2r, rd} : \mathcal{M}_{\mathrm{GSp}}(2r, rd) \longrightarrow \mathcal{A}_{\mathrm{Sp}}(2r) \oplus H^0(C, L)$$

$$(P, \phi) \longmapsto (\underline{a}', \mu) = \underline{P}(p_1(P, \phi), \ldots, p_{2r}(P, \phi)) =$$

$$= (a_2(E, \Phi'), a_4(E, \Phi'), \ldots, a_{2r}(E, \Phi'), \frac{\mathrm{tr}\, \Phi}{2r}).$$

PROPOSITION 4.4.1. (Spectral correspondence for $\mathrm{GSp}(2r, \mathbb{C})$) *Let $\underline{a}' \in \mathcal{A}_{\mathrm{Sp}}(2r)$ be any characteristic and let $\mu \in H^0(C, L)$ be any section. Let $X = X_{\underline{a}'} \xrightarrow{\pi} C$ be the spectral curve associated to $\underline{a}'$ and let σ be the involution defined on X as in Section 4.3. Let $d' = rd + r(2r - 1)\ell$ and denote with $\mathcal{P}(d', n) = \overline{\mathbb{J}}(X, d') \times J^d(C)$ the Cartesian product of the compactified Jacobian of degree d' on X and the Jacobian of degree d on C, endowed with the projection maps p_X and p_C on $\overline{\mathbb{J}}(X, d')$ and $J^d(C)$ respectively. Let $\mathcal{E}_{\underline{a}'}$ be the equalizer of the two maps*

$$(\mathcal{H}om_{\mathcal{O}_X}(\_, \mathcal{O}_X) \circ p_X) \otimes (\pi^* \circ p_C) : \mathcal{P}(d', d) \to \overline{\mathbb{J}}(X, rd - r(2r - 1)\ell)$$

$$(\sigma^* \circ p_X) \otimes \pi^* L^{1-2r} : \mathcal{P}(d', d) \to \overline{\mathbb{J}}(X, rd - r(2r - 1)\ell).$$

Then, the fiber $\widetilde{\mathcal{H}}^{-1}_{\mathrm{GSp}, 2r, rd}(\underline{a}', \mu)$ is isomorphic, via the spectral correspondence, to $\mathcal{E}_{\underline{a}'}$.

PROOF. The datum of a GSp-Higgs pair (P, ϕ) in the fiber $\widetilde{\mathcal{H}}^{-1}_{\mathrm{GSp}, 2r, rd}(\underline{a}', \mu)$ corresponds uniquely to the datum $(E, \Phi', \omega, M, \mu)$ of a Higgs pair (E, Φ') with rank $2r$ and characteristic $\underline{a}'$, a non-degenerate symplectic form ω on E with values in a line bundle M of degree d on C, and

$\mu \in H^0(C, L)$ fixed. The torsion-free sheaf $\mathcal{M}$ on X corresponding to (E, Φ') by the spectral correspondence fits into the exact sequence:

$$0 \to \mathcal{M} \otimes \pi^* L^{1-2r} \to \pi^* E \xrightarrow{\pi^* \Phi - x} \pi^*(E \otimes L) \to \mathcal{M} \otimes \pi^* L \to 0.$$

Taking the dualized sequence and tensoring by $\pi^*(L \otimes M)$ gives a left exact sequence:

$$0 \to \mathcal{M}^* \otimes M \to \pi^*(E^* \otimes M) \xrightarrow{\pi^* \Phi^t - x} \pi^*(E^* \otimes M) \otimes \pi^* L.$$

On the other hand, applying σ^* to the first exact sequence gives:

$$0 \to \sigma^* \mathcal{M} \otimes \pi^* L^{1-2r} \to \pi^* E \xrightarrow{\pi^* \Phi + x} \pi^*(E \otimes L) \to \sigma^* \mathcal{M} \otimes \pi^* L \to 0.$$

Now, the non-degenerate symplectic form ω induces an isomorphism $\omega_E : E \to E^* \otimes M$ and hence a commutative diagram with vertical isomorphisms:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{M}^* \otimes M & \longrightarrow & \pi^*(E^* \otimes M) & \xrightarrow{\pi^* \Phi^t - x} & \pi^*(E^* \otimes M) \otimes \pi^* L \\
& & & & \omega_E \uparrow \sim & & \omega_E \uparrow \sim \\
0 & \longrightarrow & \sigma^* \mathcal{M} \otimes \pi^* L^{1-2r} & \longrightarrow & \pi^* E & \xrightarrow{\pi^* \Phi + x} & \pi^*(E \otimes L)
\end{array}
$$

Hence, we conclude that the sheaf $\mathcal{M}$ comes with sheaf isomorphism

$$\lambda : \mathcal{M}^* \otimes \pi^* M \xrightarrow{\sim} \sigma^* \mathcal{M} \otimes \pi^* L^{1-2r}.$$

Viceversa, suppose that $\mathcal{M}$ is a torsion-free sheaf of rank 1 and degree d' on X with a line bundle M of degree d on C and an isomorphism $\lambda : \mathcal{M}^* \otimes \pi^* M \xrightarrow{\sim} \sigma^* \mathcal{M} \otimes \pi^* L^{1-2r}$. Let (E, Φ') be the Higgs pairs of rank $2r$ and degree rd corresponding to $\mathcal{M}$ by the spectral correspondence. Applying Relative duality in dimension 0 and the projection formula as in Section 4.3, we have an isomorphism:

$$
\begin{aligned}
E^* = \pi_*(\mathcal{M})^* &\simeq \pi_*(\mathcal{M}^* \otimes \pi^* L^{2r-1}) \simeq \\
&\simeq \pi_*(\sigma^*(\mathcal{M}) \otimes \pi^* M^{-1}) \simeq \\
&\simeq \pi_*(\sigma^*(\mathcal{M})) \otimes M^{-1} = \pi_*(\sigma_* \sigma^* \mathcal{M}) \otimes M^{-1} \simeq \\
&\simeq \pi_*(\sigma_* \mathcal{O}_X \otimes \mathcal{M}) \otimes M^{-1} = \pi_*(\mathcal{O}_X \otimes \mathcal{M}) \otimes M^{-1} \simeq \\
&\simeq \pi_*(\mathcal{M}) \otimes M^{-1} = E \otimes M^{-1};
\end{aligned}
$$

that induces the non-degenerate M-valued symplectic structure $\omega : E \otimes E \to M$.

To sum up, the datum of (P, ϕ) in the fiber $\mathcal{H}^{-1}_{\mathrm{GSp},2r,rd}(\underline{a})$ corresponds uniquely to the datum $(\mathcal{M}, M, \lambda)$ of a torsion-free rank-1 sheaf $\mathcal{M} \in \overline{\overline{\mathbb{J}}}(X, d')$, a line bundle $M \in J(C, d)$ and an isomorphism $\lambda : \mathcal{M}^* \otimes \pi^* M \xrightarrow{\sim} \sigma^* \mathcal{M} \otimes \pi^* L^{1-2r}$, i.e. an element of the equalizer stack $\mathcal{E}_{\underline{a}}$. $\quad\square$

4.5. $\mathrm{PSp}(2r, \mathbb{C})$-**Higgs pairs**

The Projective Symplectic group $\mathrm{PSp}(2r, \mathbb{C})$ is defined by the exact sequence:

$$(16) \qquad 0 \to \mathbb{C}^* \xrightarrow{\lambda \mapsto \lambda I_{2r}} \mathrm{GSp}(2r, \mathbb{C}) \to \mathrm{PSp}(2r, \mathbb{C}) \to 0$$

or by the exact sequence:

$$(17) \qquad 0 \to \{\pm 1\} \xrightarrow{1 \mapsto I_{2r}} \mathrm{Sp}(2r, \mathbb{C}) \to \mathrm{PSp}(2r, \mathbb{C}) \to 0.$$

In particular, the sheaf-theoretic version of sequence 16 yields the cohomology exact sequence:

$$(18) \quad H^1(C, \mathcal{O}_C^*) \to H^1(C, \mathrm{GSp}(2r, \mathcal{O}_C)) \xrightarrow{q} H^1(C, \mathrm{PSp}(2r, \mathcal{O}_C)) \to 0.$$

This surjection, read in terms of cocycles, means that $\mathrm{PSp}(2r, \mathbb{C})$-principal bundles are in one-to-one correspondence with equivalence classes of $\mathrm{GSp}(2r, \mathbb{C})$-principal bundles, with respect to the action on associated bundles given by tensor product of line bundles. If P is a $\mathrm{PSp}(2r, \mathbb{C})$-principal bundle and $\tilde{P}$ is a $\mathrm{GSp}(2r, \mathbb{C})$-principal bundle such that $q(\tilde{P}) = P$, we say that $\tilde{P}$ is a lifting of P to a $\mathrm{GSp}(2r, \mathbb{C})$-principal bundle.

Moreover, by sequence 17, $\mathrm{PSp}(2r, \mathbb{C})$ is the quotient of $\mathrm{Sp}(2r, \mathbb{C})$ by the action of a finite group, hence the associated Lie algebras $\mathfrak{psp}(2r, \mathbb{C})$ and $\mathfrak{sp}(2r, \mathbb{C})$ are equal. If P is any $\mathrm{PSp}(r2, \mathbb{C})$-principal bundle and $\tilde{P}$ is a lifting of P to a $\mathrm{GSp}(2r, \mathbb{C})$-principal bundle, then a section ϕ of $H^0(C, \mathrm{ad}\, P \otimes L)$ determines uniquely a section $\tilde{\phi}$ of $H^0(C, \mathrm{ad}\, \tilde{P} \otimes L)$ with trace equal to 0, and viceversa. We say that $(\tilde{P}, \tilde{\phi})$ is a lifting of (P, ϕ) to a $\mathrm{GSp}(2r, \mathbb{C})$-Higgs pair and with a slight abuse of notation we write $q(\tilde{P}, \tilde{\phi}) = (P, \phi)$.

To sum up, any $\mathrm{PSp}(2r, \mathbb{C})$-Higgs pair (P, ϕ) has a lifting $(\tilde{P}, \tilde{\phi})$ to a $\mathrm{GSp}(2r, \mathbb{C})$-Higgs pair with trace zero, corresponding via the associated bundle construction to the datum (E, Φ, M, ω) of a Higgs pair (E, Φ) of rank $2r$, a non-degenerate symplectic form $\omega : E \otimes E \to M$ with values

in a line bundle M of degree d on C, and a Higgs field Φ satisfying the condition

$$\omega(\Phi v, w) + \omega(v, \Phi w) = 0.$$

Then, the datum of (P, ϕ) corresponds uniquely to the datum of the equivalence class $[(E, \Phi, M, \omega)]$ of Higgs pairs of rank $2r$ with a non-degenerate symplectic form, under the equivalence relation $\sim_{J(C)}$ defined by:

$$(E, \Phi, M\omega) \sim_{J(C)} (E \otimes N, \Phi \otimes 1_N, M \otimes N^2, \omega_N) \qquad \text{for any } N \in J(C)$$

where

$$\omega_N : (E \otimes N) \otimes (E \otimes N) \to M \otimes N^2$$

is obtained by extension of scalars.

More precisely, let $\mathcal{M}_{\mathrm{GSp}}(2r) = \coprod_{d \in \mathbb{Z}} \mathcal{M}_{\mathrm{GSp}}(2r, rd)$ be the moduli stack of GSp-Higgs pairs of rank $2r$ in any degree and denote with $\mathcal{M}_{\mathrm{GSp}}^{\mathrm{tr}=0}(2r)$ the closed substack of $\mathcal{M}_{\mathrm{GSp}}(2r)$ given by GSp-Higgs pairs of rank $2r$ with trace zero. Then, $[(E, \Phi, M, \omega)]$ is the orbit of (E, Φ, M, ω) under the action of $J(C)$ on $\mathcal{M}_{\mathrm{GSp}}^{\mathrm{tr}=0}(2r)$ defined by:

$$\mathcal{M}_{\mathrm{GSp}}^{\mathrm{tr}=0}(2r) \times J(C) \longrightarrow \mathcal{M}_{\mathrm{GSp}}^{\mathrm{tr}=0}(2r)$$

$$((E, \Phi, M, \omega), N) \longmapsto (E \otimes N, \Phi \otimes 1_N, M \otimes N^2, \omega_N).$$

Recall that the isomorphism $\det E \simeq M^r$ implies that $\deg(E) = r \deg(M)$; in other words, the degree of the Higgs pair in the datum (E, Φ, M, ω) is determined by the degree of the line bundle M in the same datum. The action of $J(C)$ on such datum modifies the degree of M by multiples of 2, since $\deg(M \otimes N^2) = \deg(M) + 2\deg(N)$. This his fact implies that the degree of GSp-Higgs pairs associated to a PSp-Higgs pair is defined only modulo $2r$. Then, we can give the following definition.

DEFINITION 4.5.1. Let (P, ϕ) be a PSp$(2r, \mathbb{C})$-Higgs pair and let (E, Φ, M, ω) be the datum of a Higgs pair with trace zero and degree rd endowed with a M-valued symplectic form ω, corresponding to a lifting $(\tilde{P}, \tilde{\phi})$ of (P, ϕ) to a GSp-Higgs pair. The *degree of* (P, ϕ) is the congruence class $\overline{rd} \in \mathbb{Z}/2r\mathbb{Z}$.

REMARK 4.5.2. The same definition works for $\mathrm{PSp}(2r, \mathbb{C})$-principal bundles, without Higgs pair. The degree of a PSp-principal bundle P is the congruence class (modulo $2r$) of the degree of the vector bundles associated to any lifting $\tilde{P}$ of P to a GSp-principal bundle.

Clearly, only two cases are possible: the degree of a $\mathrm{PSp}(2r, \mathbb{C})$-Higgs pair (P, ϕ) is congruent either to 0 or to r modulo $2r$. This reflects the fact that the topological type of P is parametrized by $\pi_1(\mathrm{PSp}(2r, \mathbb{C})) \simeq \mathbb{Z}/2\mathbb{Z}$.

Up to the action on (E, Φ, M, ω) of a line bundle of degree 1 on C, it is straightforward to see that the $J(C)$-orbit $[(E, \Phi, M, \omega)]$ corresponding to a PGL-Higgs pair with degree $\overline{rd} \in \mathbb{Z}/2r\mathbb{Z}$ is always the orbit of a datum whose Higgs pair has degree 0 or rd. If we restrict the correspondence to Higgs pairs with fixed degree 0 or rd, we see that a PGL-Higgs pair of degree $\overline{rd}$ is identified uniquely by the orbit of a Higgs pair with trace zero *and degree* 0 or rd with respect to the action of line bundles *of degree* 0 on C.

A basis for the invariant polynomials of $\mathfrak{psp}(2r, \mathbb{C}) = \mathfrak{sp}(2r, \mathbb{C})$ is given by $\{p_{2i}\}_{i=1,\ldots,r}$ where $p_{2i}(P, \phi) := \mathrm{tr}(\wedge^{2i}\phi)$. The corresponding PSp-Hitchin morphism takes the form:

$$\mathcal{H}_{\mathrm{PSp},2r,\overline{rd}} : \mathcal{M}_{\mathrm{PSp}}(2r, \overline{rd}) \longrightarrow \mathcal{A}_{\mathrm{PSp}}(2r) = \bigoplus_{l=1}^{r} H^0(C, L^{2l})$$

$$(P, \phi) \longmapsto (p_2(P, \phi), p_4(P, \phi), \ldots, p_{2r}(P, \phi)).$$

If $(\tilde{P}, \tilde{\phi})$ is any lifting of (P, ϕ) to a $\mathrm{GSp}(2r, \mathbb{C})$-Higgs pair, then $\mathcal{H}_{\mathrm{GSp},2r,rd}(\tilde{P}, \tilde{\phi}) = \mathcal{H}_{\mathrm{PSp},2r,\overline{rd}}(P, \phi)$. Moreover, for any characteristic $\underline{a} \in \mathcal{A}_{\mathrm{PSp}}(r)$, the spectral curve $\pi : X_{\underline{a}} \to C$ is defined in the total space $p : \mathbb{P}(\mathcal{O}_C \oplus L^{-1}) \to C$ by the equation

$$x^{2r}y + a_2 x^{2r-2}y^2 + \cdots + a_{2r-2}x^2 y^{2r-2} + a_{2r}y^{2r} = 0.$$

Hence, as in the case of $\mathrm{Sp}(2r, \mathbb{C})$-Higgs pairs, the curve $X_{\underline{a}}$ has an involution defined by $\sigma(x) = -x$.

We can now state the following proposition.

PROPOSITION 4.5.3. (Spectral correspondence for $\mathrm{PSp}(2r, \mathbb{C})$) *Let* $\underline{a} \in \mathcal{A}_{\mathrm{PSp}}(2r)$ *be any characteristic, let* $X = X_{\underline{a}} \xrightarrow{\pi} C$ *be the associated spectral curve with involution* $\sigma : X \to X$. *Let* $d \in \{0, 1\}$, $d' = rd +$

$r(2r-1)\ell$ and denote with $\mathcal{P}(d',n) = \overline{\mathbb{J}}(X,d') \times J^d(C)$ the Cartesian product of the compactified Jacobian of degree d' on X and the Jacobian of degree d on C, endowed with the projection maps p_X and p_C on $\overline{\mathbb{J}}(X,d')$ and $J^d(C)$ respectively. Let $\mathcal{E}_{\underline{a}'}$ be the equalizer of the two maps

$$(\mathcal{H}om_{\mathcal{O}_X}(\_, \mathcal{O}_X) \circ p_X) \otimes (\pi^* \circ p_C) : \mathcal{P}(d',d) \to \overline{\mathbb{J}}(X, rd - r(2r-1)\ell)$$

$$(\sigma^* \circ p_X) \otimes \pi^* L^{1-2r} : \mathcal{P}(d',d) \to \overline{\mathbb{J}}(X, rd - r(2r-1)\ell).$$

The group $J^0(C)$ of line bundles of degree 0 on C acts on $\mathcal{E}_{\underline{a}}$ as follows:

$$\mathcal{E}_{\underline{a}} \times J^0(C) \longrightarrow \mathcal{E}_{\underline{a}}$$

$$((\mathcal{M}, M, \lambda), N) \longmapsto (\mathcal{M} \otimes \pi^*N, M \otimes N^2, \lambda_N)$$

where λ_N is given by the composition of $\lambda \otimes \mathrm{id}_{\pi^*N}$ with the canonical isomorphisms:

$$(\mathcal{M} \otimes \pi^*N)^* \otimes \pi^*(M \otimes N^2) \xrightarrow{\sim} \mathcal{M}^* \otimes (\pi^*N)^{-1} \otimes \pi^*M \otimes (\pi^*N)^2 \xrightarrow{\sim}$$

$$\xrightarrow{\sim} \mathcal{M}^* \otimes \pi^*M \otimes \pi^*N \xrightarrow{\lambda \otimes \mathrm{id}_{\pi^*N}}$$

$$\xrightarrow{\sim} \sigma^*\mathcal{M} \otimes \pi^*L^{1-2r} \otimes \pi^*N \xrightarrow{\sim}$$

$$\xrightarrow{\sim} \sigma^*(\mathcal{M} \otimes \pi^*N) \otimes \pi^*L^{1-2r}.$$

Then, the fiber $\mathcal{H}^{-1}_{\mathrm{PSp},2r,\overline{rd}}(\underline{a})$ of the PSp-*Hitchin morphism is isomorphic, via the spectral correspondence, to the quotient $\mathcal{E}_{\underline{a}}/J^0(C)$.

PROOF. Let (P, ϕ) be any $\mathrm{PSp}(2r, \mathbb{C})$-Higgs pair with characteristic $\underline{a}$ and let (E, Φ, M, ω) be the datum of a Higgs pair with a non-degenerate M-valued symplectic form such that $(P\phi)$ corresponds uniquely to the datum of the $J^0(C)$-equivalence class $[(E, \Phi, M, \omega)]$.

By Proposition 4.4.1, the datum of (E, Φ, M, ω) corresponds, via the spectral correspondence, to the datum $(\mathcal{M}, M, \lambda)$ of a torsion-free rank-1 sheaf $\mathcal{M} \in \overline{\mathbb{J}}(X, d')$, the line bundle $M \in J^d(C)$ and an isomorphism $\lambda : \mathcal{M}^* \otimes \pi^*M \xrightarrow{\sim} \sigma^*\mathcal{M} \otimes \pi^*L^{1-2r}$, i.e. an element of $\mathcal{E}_{\underline{a}}$. If N is any line bundle of degree 0 on C, then by the projection formula the datum of $(E \otimes N, \Phi \otimes 1_N, M \otimes N^2, \omega_N)$ corresponds to the datum of $(\mathcal{M} \otimes \pi^*N, M \otimes N^2, \lambda_N)$.

We conclude that the datum of the $J^0(C)$-orbit $[(E, \Phi, M, \omega)]$ corresponds, via the spectral correspondence, to the datum of the $J^0(C)$-orbit of $(\mathcal{M}, M, \lambda)$. $\square$

A particular case occurs for $d = 0$, i.e. for $\mathrm{PGL}(2r,\mathbb{C})$-Higgs pairs of degree 0. Indeed, recall the sequence 16:

$$0 \to \{\pm 1\} \xrightarrow{1 \mapsto I_{2r}} \mathrm{Sp}(2r,\mathbb{C}) \to \mathrm{PSp}(2r,\mathbb{C}) \to 0.$$

Recalla also that $\{\pm 1\} = \mu_2$ and $H^2(C, \mu_2) = \mathbb{Z}/2\mathbb{Z}$. Hence this sequence, applied to structure sheaves, induces the cohomology exact sequence:

$$H^1(C, \mu_2) \to H^1(C, \mathrm{Sp}(2r, \mathcal{O}_C)) \xrightarrow{q} H^1(C, \mathrm{PSp}(2r, \mathcal{O}_C)) \xrightarrow{\deg/r} \mathbb{Z}/2\mathbb{Z} \to 0$$

where the last map sends a $\mathrm{PSp}(2r,\mathbb{C})$-principal bundle of degree $\overline{rd}$ to the congruence class $\overline{d}$. This sequence, read in terms of cocycles, means that a $\mathrm{PSp}(2r,\mathbb{C})$-principal bundle P can be lifted to $\mathrm{Sp}(2r,\mathbb{C})$-principal bundle P_0 if and only P has degree equal to $0 \in \mathbb{Z}/2r\mathbb{Z}$; any other lifting P_0' of P differs from P_0 by the action on the associated bundles of a 2-torsion line bundle by tensor product. Moreover, a Higgs field ϕ on P determines uniquely a a Higgs field ϕ_0 on P and viceversa.

To sum up, any $\mathrm{PSp}(2r,\mathbb{C})$-Higgs pair (P, ϕ) of degree 0 has a lifting $(\tilde{P}, \tilde{\phi})$ to a $\mathrm{Sp}(r,\mathbb{C})$-Higgs pair, corresponding to the datum (E, Φ, ω) of a Higgs pair of rank $2r$ and degree 0 and a non-degenerate symplectic form ω via the associated bundle construction. Then, the datum of (P, ϕ) corresponds uniquely to the datum of the equivalence class $[(E, \Phi, \omega)]$ of Higgs pairs of rank $2r$ (and degree 0) with a non-degenerate symplectic form, under the equivalence relation $\sim_{J^0(C)[2]}$ defined by:

$$(E, \Phi, \omega) \sim_{J^0(C)[2]} (E \otimes N, \Phi \otimes 1_N, \omega_{N,\epsilon}) \qquad \text{for any } (N, \epsilon) \in J^0(C)[2].$$

Here, $J^0(C)[2]$ denotes the group stack parametrizing line bundles with 2-torsion (N, ϵ) on C, where ϵ is the isomorphism $N^{-1} \xrightarrow{\sim} N$, and $\omega_{N,\epsilon}$ is obtained by extension of scalars, as follows:

$$\omega_{N,\epsilon} : (E \otimes N) \otimes (E \otimes N) \to N^2 \xrightarrow[\epsilon^{-1} \otimes 1]{\sim} N^{-1} \otimes N \xrightarrow[\mathrm{ev}]{\sim} \mathcal{O}_C.$$

We can finally state the following proposition.

PROPOSITION 4.5.4. (Spectral correspondence for $\mathrm{PSp}(2r,\mathbb{C})$ of degree 0) *Let $\underline{a} \in \mathcal{A}_{\mathrm{PSp}}(2r)$ be any characteristic and let $X = X_{\underline{a}} \xrightarrow{\pi} C$ be the associated spectral curve with involution $\sigma : X \to X$. Let*

$d' = r(2r - 1)\ell$ and let $\mathcal{E}_{\underline{a}}$ be the equalizer stack of the two maps:

$$\_^* := \mathcal{H}om_{\mathcal{O}_X}(\_, \mathcal{O}_X) : \overline{\mathbb{J}}(X, d') \to \overline{\mathbb{J}}(X, -d')$$
$$\sigma^* \_ \otimes \pi^* L^{1-2r} : \overline{\mathbb{J}}(X, d') \to \overline{\mathbb{J}}(X, -d')$$

The group stack $J^0(C)[2]$ of 2-torsion line bundles on C acts on $\mathcal{E}_{\underline{a}}$ as follows:

$$\mathcal{E}_{\underline{a}} \times \pi^* J^0(C)[2] \longrightarrow \mathcal{E}_{\underline{a}}$$
$$((\mathcal{M}, \lambda), (\pi^* N, \pi^* \epsilon)) \longmapsto (\mathcal{M} \otimes \pi^* N, \lambda \otimes \pi^* \epsilon).$$

Then, the fiber $\mathcal{H}_{\mathrm{PSp}, 2r, \overline{0}}^{-1}(\underline{a})$ of the PSp-Hitchin morphism is isomorphic, via the spectral correspondence, to the quotient $\mathcal{E}_{\underline{a}}/\pi^* J^0(C)[2]$.

PROOF. Let (P, ϕ) be any $\mathrm{PSp}(2r, \mathbb{C})$-Higgs pair with degree 0 and characteristic $\underline{a}$ and let $(\tilde{P}, \tilde{\phi})$ be any lifting to a $\mathrm{Sp}(2r, \mathbb{C})$-Higgs pair. By construction, $(\tilde{P}, \tilde{\phi})$ is a $\mathrm{Sp}(2r, \mathbb{C})$-Higgs pair with characteristic $\underline{a}$ and corresponds uniquely, via the associated bundle construction, to the datum (E, Φ, ω) of a Higgs pair of rank $2r$ and degree 0 with characteristic $\underline{a}$ and a non-degenerate symplectic form. Then, the datum of (P, ϕ) corresponds uniquely to the datum of the $J^0(C)[2]$-equivalence class $[(E, \Phi, \omega)]$.

By Proposition 4.3.1, the datum of (E, Φ, ω) corresponds to the datum $(\mathcal{M}, \lambda)$ of a torsion-free rank-1 sheaf $\mathcal{M} \in \overline{\mathbb{J}}(X, d')$ and an isomorphism $\lambda : \mathcal{M}^* \xrightarrow{\sim} \sigma^* \mathcal{M} \otimes \pi^* L^{1-2r}$, i.e. an element of $\mathcal{E}_{\underline{a}}$. If N is any 2-torsion line bundle on C with isomorphism $\epsilon : N^{-1} \xrightarrow{\sim} N$, then by the projection formula the datum of $(E \otimes N, \Phi \otimes 1_N, \omega_N)$ corresponds to the datum of $(\mathcal{M} \otimes \pi^* N, \lambda \otimes \pi^* \epsilon)$.

We conclude that the datum of the $J^0(C)[2]$-orbit $[(E, \Phi, \omega)]$ corresponds to the datum of the $\pi^* J^0(C)[2]$-orbit of $(\mathcal{M}, \lambda)$. $\square$

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