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A topological version of Hilbert's Nullstellensatz ☆

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We prove that the space of radical ideals of a ring R, endowed with the hull-kernel topology, is a spectral space, and that it is canonically homeomorphic to the space of the non-empty Zariski closed subspaces of $\mathtt{Spec}(R)$, endowed with a Zariski-like topology.

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1. Introduction and preliminaries

Hilbert's Nullstellensatz establishes a fundamental relationship between geometry and algebra, relating algebraic sets in affine spaces to radical ideals in polynomial rings over algebraically closed fields. On the other hand, for any ring R, the set of radical ideals of R can be thought of as a set of representatives of the closed sets of $X := \mathtt{Spec}(R)$, in the sense that the map $\mathscr{J}$, sending a closed set C of X to the radical ideal $\mathscr{J}(C) := \bigcap\{P \mid P \in C\}$, is a natural order-reversing bijection, having as inverse the map $\mathtt{V}$ defined by sending a radical ideal H of R to the Zariski-closed subspace $\mathtt{V}(H) := \{P \in \mathtt{Spec}(R) \mid H \subseteq P\}$ of X.

In the present paper, we will put into a topological perspective the relationship between the geometry of $\mathtt{Spec}(R)$ and ideal theory of R, shedding new light onto the Nullstellensatz-type correspondence established by the maps $\mathscr{J}$ and $\mathtt{V}$.

Precisely, we consider $\mathtt{Rd}(R) := \{H \text{ ideal of } R \mid H = \mathrm{rad}(H) \subsetneq R\}$ endowed with the so-called *hull-kernel topology*, that is the topology defined by taking, as a subbasis of open sets, the collection of all the subsets of the form $\{H \in \mathtt{Rd}(R) \mid x_1, \ldots, x_n \notin H\}$, for $x_1, \ldots, x_n$ varying in the ring R. In this situation, we show that $\mathtt{Rd}(R)^{\mathtt{hk}}$ (i.e., $\mathtt{Rd}(R)$ with hull-kernel topology) is a spectral space (after Hochster [12]), using a general approach described below. On the other hand, we introduce a natural topology, called the *Zariski topology*, on the space $\boldsymbol{\mathcal{X}'}(R)$ of all the nonempty closed subspaces of the spectral space $\mathtt{Spec}(R)$, by declaring as a basis of open sets the collection of the sets of the form

$$\boldsymbol{\mathcal{U}'}(\Omega) := \{C \in \boldsymbol{\mathcal{X}'}(R) \mid C \cap \Omega = \emptyset\},$$

where Ω runs in the family of all quasi-compact open subspaces of $\mathtt{Spec}(R)$.

In such a way, $\boldsymbol{\mathcal{X}'}(R)$ becomes a T_0 topological space which can be considered as a natural order-reversing topological extension of $\mathtt{Spec}(R)$. More precisely, if $\mathtt{Spec}(R)$ is endowed with the inverse topology (as defined by Hochster; the definition will be recalled later), then the natural map $\varphi' : \mathtt{Spec}(R) \to \boldsymbol{\mathcal{X}'}(R)$, $P \mapsto \mathtt{Cl}(\{P\})$, turns out to be a topological embedding, where $\mathtt{Cl}(\{P\})$ denotes the Zariski-closure in $\mathtt{Spec}(R)$ of the singleton $\{P\}$, i.e., $\mathtt{Cl}(\{P\}) = \mathtt{V}(P)$.

Among the main results of the present paper, we show that the topological space $\boldsymbol{\mathcal{X}'}(R)$, endowed with the Zariski topology (denoted by $\boldsymbol{\mathcal{X}'}(R)^{\mathtt{zar}}$), is a spectral space. By linking algebraic and topological properties, we show that $\mathscr{J}$ establishes a homeomorphism between $\boldsymbol{\mathcal{X}'}(R)^{\mathtt{inv}}$ (that is, $\boldsymbol{\mathcal{X}'}(R)$ endowed with the inverse topology) and $\mathtt{Rd}(R)^{\mathtt{hk}}$ (Theorem 4.1).

The topological properties that we prove concerning the space $\mathtt{Rd}(R)$ are obtained as particular cases of a more general construction. Indeed, given a R-module M, we define in a standard way the hull-kernel topology on the set $\mathtt{SMod}(M|R)$ of all R-submodules of M, and we prove that this topological space is a spectral space, by using a characterization based on ultrafilters. Then, we focus on the subspace $\mathtt{Spec}_R(M)$ of $\mathtt{SMod}(M|R)$

given by the prime R-submodules of M (definition recalled later), and we show that $\texttt{Spec}_R(M)$ is spectral if and only if it is quasi-compact; this happens, for example, when M is finitely generated. Among other facts, we investigate whether some distinguished subspaces of $\texttt{SMod}(M|R)$ are closed, with respect to the constructible topology. We show that this happens to the space $\texttt{SMod}^c(M|R) := \{N \in \texttt{SMod}(M|R) \mid N = N^c\}$, where $c : \texttt{SMod}(R|M) \to \texttt{SMod}(R|M)$, $N \mapsto N^c$, is a closure operation of finite type; in particular, it is a spectral space, with the hull-kernel topology. Thus, keeping in mind that the set of all ideals of R, denoted by $\texttt{Id}(R)$, coincides with the spectral space $\texttt{SMod}(R|R)$ and that the mapping rad : $\texttt{Id}(R) \to \texttt{Id}(R)$ (sending an ideal I of R to its radical) is a closure operation of finite type, we deduce that $\texttt{Rd}(R)$ (with the hull-kernel topology) is a spectral space. Furthermore, we show that the Krull dimension of this spectral space can be evaluated by the formula $\dim(\texttt{Rd}(R)) = |\texttt{Spec}(R)| - 1 \geq \dim(\texttt{Spec}(R))$.

In the following, we will freely use some well known facts on spectral spaces [12]. However, for convenience of the reader we recall now briefly some basic definitions and background material.

1.1. *Spectral spaces*

Let X be a topological space. According to [12], X is called a *spectral space* if there exists a ring R such that $\texttt{Spec}(R)$, with the Zariski topology, is homeomorphic to X. Spectral spaces can be characterized in a purely topological way: a topological space X is spectral if and only if X is T_0 (this means that for every pair of distinct points of X, at least one of them has an open neighborhood not containing the other), quasi-compact (i.e., any open cover of X admits a finite subcover), admits a basis of quasi-compact open subspaces that is closed under finite intersections, and every irreducible closed subspace C of X has a unique generic point (i.e., there exists a unique point $x_C \in C$ such that C coincides with the closure of this point) [12, Proposition 4].

1.2. *The inverse topology on a spectral space*

Let X be a topological space and let Y be any subset of X. We denote by $\texttt{Cl}(Y)$ the closure of Y in the topological space X. Recall that the topology on X induces a natural preorder $\leq$ on X, defined by setting $x \leq y$ if $y \in \texttt{Cl}(\{x\})$. It is straightforward to see that $\leq$ is a partial order if and only if X is a T_0 space (e.g., this holds when X is spectral). The set $Y^{\texttt{gen}} := \{x \in X \mid y \in \texttt{Cl}(\{x\}),\text{ for some } y \in Y\}$ is called *closure under generizations of* Y. Similarly, using the opposite order, the set $Y^{\texttt{sp}} := \{x \in X \mid x \in \texttt{Cl}(\{y\}),\text{ for some } y \in Y\}$ is called *closure under specializations of* Y. We say that Y is *closed under generizations* (respectively, *closed under specializations*) if $Y = Y^{\texttt{gen}}$ (respectively, $Y = Y^{\texttt{sp}}$). For any two elements x, y in a spectral space X, we have:

$$x \leq y \quad \Leftrightarrow \quad \{x\}^{\texttt{gen}} \subseteq \{y\}^{\texttt{gen}} \quad \Leftrightarrow \quad \{x\}^{\texttt{sp}} \supseteq \{y\}^{\texttt{sp}}.$$

Suppose that X is a spectral space, then X can be endowed with another topology, introduced by Hochster [12, Proposition 8], whose basis of closed sets is the collection of all the quasi-compact open subspaces of X. This topology is called *the inverse topology on* X (called also *the* O*-topology in* [21]; see also [11]). For a subset Y of X, let $\texttt{Cl}^{\texttt{inv}}(Y)$ be the closure of Y, in the inverse topology of X; we denote by $X^{\texttt{inv}}$ the set X, equipped with the inverse topology. The name given to this new topology is due to the fact that, given $x, y \in X$, $x \in \texttt{Cl}^{\texttt{inv}}(\{y\})$ if and only if $y \in \texttt{Cl}(\{x\})$, i.e., the partial order induced by the inverse topology is the opposite order of the partial order induced by the given spectral topology [12, Proposition 8].

By definition, for any subset Y of X, we have

$$\texttt{Cl}^{\texttt{inv}}(Y) := \bigcap \{U \mid U \text{ open and quasi-compact in } X,\ U \supseteq Y\}.$$

In particular, keeping in mind that the inverse topology reverses the order of the given spectral topology, it follows that the closure under generizations $\{x\}^{\texttt{gen}}$ of a singleton is closed in the inverse topology of X, since

$$\{x\}^{\texttt{gen}} = \texttt{Cl}^{\texttt{inv}}(\{x\}) = \bigcap \{U \mid U \subseteq X \text{ quasi-compact and open, } x \in U\}$$

[12, Proposition 8]. On the other hand, it is trivial, by the definition, that the closure under specializations of a singleton $\{x\}^{\texttt{sp}}$ is closed in the given topology of X, since $\{x\}^{\texttt{sp}} = \texttt{Cl}(\{x\})$.

For recent developments in the use of the inverse topology in Commutative Algebra and spaces of valuation domains see, for example, [20].

1.3. The constructible topology on a spectral space

Let X be a spectral space. As it is well known, the topology of X is Hausdorff if and only if X is zero-dimensional. Following [9], there is a natural way to refine the topology of X in order to make X an Hausdorff space without losing compactness. Precisely, define *the constructible topology on* X to be the coarsest topology for which the quasi-compact open subspaces of X form a collection of clopen sets. In this way, X becomes a totally disconnected Hausdorff spectral space. Let $X^{\texttt{cons}}$ denote the set X endowed with the constructible topology. By [12, Proposition 9], any closed subset of $X^{\texttt{cons}}$ is a spectral subspace of X (with respect to the original spectral topology). Thus, in particular, any quasi-compact open subspace Ω of X is spectral, since Ω is clopen in the constructible topology, by definition. It is, in general, not so easy to describe the closed sets of $X^{\texttt{cons}}$. The following results provides both a criterion to characterize when a topological space X is spectral and to characterize the closed sets of $X^{\texttt{cons}}$. This result is based on the use of ultrafilters. For background material on this topic and application of ultrafilters to Commutative Ring Theory see, for example, [16] and [22].

Theorem 1.1. *[5, Corollary 3.3] Let X be a topological space.*

(1) *The following conditions are equivalent.*
  (i) *X is a spectral space.*
  (ii) *There exists a subbasis $\mathcal{S}$ of X such that, for any ultrafilter $\mathscr{U}$ on X, the set*

$$X(\mathscr{U}) := \{x \in X \mid [\forall S \in \mathcal{S}, \text{ the following holds: } x \in S \Leftrightarrow S \in \mathscr{U}]\}$$

  is nonempty.

(2) *If the previous equivalent conditions hold and $\mathcal{S}$ is as in* (ii)*, then a subset Y of X is closed, with respect to the constructible topology, if and only if for any ultrafilter $\mathscr{V}$ on Y we have*

$$Y(\mathscr{V}) := \{x \in X \mid [\forall S \in \mathcal{S} \text{ the following holds: } x \in S \Leftrightarrow S \cap Y \in \mathscr{V}]\} \subseteq Y.$$

Corollary 1.2. *Let X be a topological space satisfying the equivalent conditions of Theorem 1.1(1), and let $\mathcal{S}$ be as in Theorem 1.1(1,ii). Then $\mathcal{S}$ is a subbasis of quasi-compact open subspaces of X.*

Proof. By [5, Corollary 2.9, Propositions 2.11 and 3.2], $\mathcal{S}$ is a collection of clopen sets with respect to the constructible topology on the spectral space X. In the constructible topology, every clopen set is quasi-compact with respect to the given spectral topology. The claim follows. □

2. Spectral spaces of ideals and modules

The main purpose of the present section is to apply the general construction of the space of inverse-closed subspaces of the prime spectrum of a ring, considered in the previous section, to obtain a topological version of Hilbert's Nullstellensatz.

Let R be a ring and M be an R-module. On the set $\texttt{SMod}(M|R)$ of R-submodules of M we can define an *hull-kernel topology* having, as a subbasis for the closed sets, the subsets of the form

$$\boldsymbol{V}(x_1, x_2, \ldots, x_m) := \{N \in \texttt{SMod}(M|R) \mid x_1, x_2, \ldots, x_m \in N\},$$

where $x_1, x_2, \ldots, x_m$ varies among all finite subsets of M. Moreover, let

$$\boldsymbol{D}(x_1, x_2, \ldots, x_m) := \texttt{SMod}(M|R) \setminus \boldsymbol{V}(x_1, , x_2 \ldots, x_m).$$

Note that the hull-kernel topology is clearly T_0 and, by definition, the order induced by this topology on $\texttt{SMod}(M|R)$ coincides with the order provided by the set-theoretic inclusion $\subseteq$.

Proposition 2.1. *For any ring R and for any R-module M, $\texttt{SMod}(M|R)$ is a spectral space. Moreover, the collection of sets $\mathcal{S} := \{\boldsymbol{D}(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in M\}$ is a subbasis of quasi-compact open subspaces of $\texttt{SMod}(M|R)$.*

Proof. Let $\mathscr{U}$ be an ultrafilter on $\texttt{SMod}(M|R)$, and set $N_{\mathscr{U}} := \{y \in M \mid \boldsymbol{V}(y) \in \mathscr{U}\}$.

If $y_1, y_2, y \in N_{\mathscr{U}}$ and $r \in R$, then $\boldsymbol{V}(y_1)$, $\boldsymbol{V}(y_2)$ and $\boldsymbol{V}(y)$ are in $\mathscr{U}$. Since $\boldsymbol{V}(y_1 - y_2) \supseteq \boldsymbol{V}(y_1) \cap \boldsymbol{V}(y_2)$ and $\boldsymbol{V}(xr) \supseteq \boldsymbol{V}(y)$, by definition of ultrafilter we have $\boldsymbol{V}(y_1 - y_2) \in N_{\mathscr{U}}$ and $\boldsymbol{V}(ry) \in N_{\mathscr{U}}$, i.e., $y_1 - y_2, ry \in N_{\mathscr{U}}$. Therefore, $N_{\mathscr{U}}$ is a R-submodule of M.

From the definition, it follows easily that:

$$N_{\mathscr{U}} \in \texttt{SMod}(M|R)(\mathscr{U}) := \{N \in \texttt{SMod}(M|R) \mid [\forall \Omega \in \mathcal{S}, N \in \Omega \iff \Omega \in \mathscr{U}]\}.$$

Hence, by [5, Corollary 3.3], $\texttt{SMod}(M|R)$ is a spectral space. The last statement follows from Corollary 1.2. □

As particular cases of the spectral space of the submodules of a given module, we can consider the following distinguished cases.

(a) Given any ring R, let

$$\texttt{Id}(R) := \texttt{SMod}(R|R),$$
$$\texttt{Id}_\bullet(R) := \texttt{Id}(R) \setminus \{R\},$$

where $\texttt{Id}(R)$ (respectively, $\texttt{Id}_\bullet(R)$) is the set of all ideals (respectively, the set of all proper ideals).

(b) Given any integral domain D with quotient field K, let

$$\overline{\mathscr{F}}(D) := \texttt{SMod}(K|D) = \{E \mid E \text{ is a } D\text{-submodule of } K\}.$$

Corollary 2.2. *Let R be a ring and let D be an integral domain with quotient field K, $D \neq K$.*

(1) *The set $\texttt{Id}(R)$ (respectively, $\texttt{Id}_\bullet(R)$), endowed with the hull-kernel topology, is a spectral space.*

(2) *Let $\texttt{Rd}(R)$ be the set of proper radical ideals of R and consider the following topological embeddings with respect to the hull-kernel topology, induced from $\texttt{Id}(R)$,*

$$\texttt{Spec}(R) \subseteq \texttt{Rd}(R) \subseteq \texttt{Id}_\bullet(R) \subseteq \texttt{Id}(R).$$

Then, the hull-kernel topology induced on $\texttt{Spec}(R)$ coincides with the Zariski topology.

(3) *The space $\overline{\mathscr{F}}(D)$ endowed with the hull-kernel topology, is a spectral space.*

(4) *The space $\mathscr{F}(D)$ of all fractional ideals of D, endowed with the hull-kernel topology, is* not *a spectral space.*

Proof. (1) and (3). The statements for $\mathtt{Id}(R)$ for $\overline{\mathscr{F}}(D)$ are direct consequences of Proposition 2.1. The claim for $\mathtt{Id}_\bullet(R)$ follows if we show that $N_{\mathscr{U}} \neq R$, when $\mathscr{U}$ is an ultrafilter of $\mathtt{Id}_\bullet(R)$. If $N_{\mathscr{U}} = R$ then $1 \in N_{\mathscr{U}}$, i.e., $\boldsymbol{D}(1) \cap \mathtt{Id}_\bullet(R) \in \mathscr{U}$. Since $\boldsymbol{D}(1) \cap \mathtt{Id}_\bullet(R) = \emptyset$, we reach a contradiction. Hence, $N_{\mathscr{U}} \neq R$.

(2) is straightforward.

(4) If $\mathscr{F}(D)$ were a spectral space, then it would have proper maximal elements. If E is one of these, then there is an element $x \in K \setminus E$ (since K is not a fractional ideal of D if $D \neq K$) and so $E + xD$ is a fractional ideal properly containing E, against the hypothesized maximality. □

Remark 2.3. Since we have proved that $\mathtt{Id}_\bullet(R)$ is a spectral space (Corollary 2.2(1)), it is then natural to ask in general if similar cases might occur:

(Q.1) Is $\mathtt{SMod}^\bullet(M|R) := \mathtt{SMod}(M|R) \setminus \{(0)\}$ (with the hull-kernel topology) a spectral space?

(Q.2) Is $\mathtt{SMod}_\bullet(M|R) := \mathtt{SMod}(M|R) \setminus \{M\}$ (with the hull-kernel topology) a spectral space?

The answer to both question is negative: we shall see in Remark 3.7 a counterexample to question **(Q.1)**, while the problem of question **(Q.2)** will be completely settled in the following Proposition 2.4.

Proposition 2.4. *Let M be a R-module. Then, $\mathtt{SMod}_\bullet(M|R) := \mathtt{SMod}(M|R) \setminus \{M\}$ is a spectral space, endowed with the hull-kernel subspace topology, if and only if M is finitely generated.*

Proof. Consider the subbasis of open sets $\mathcal{S} := \{\boldsymbol{D}(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in M\}$ of $X := \mathtt{SMod}_\bullet(M|R)$ and assume first that M is finitely generated. If $\mathscr{U}$ is an ultrafilter on X, recall that the subset $N_{\mathscr{U}} := \{y \in M \mid \boldsymbol{V}(y) \cap X \in \mathscr{U}\}$ is a R-submodule of M, by the proof of Proposition 2.1. In the notation of Theorem 1.1, if we show that $N_{\mathscr{U}}$ is a proper submodule of M, it will follow immediately that $N_{\mathscr{U}} \in X(\mathscr{U})$, thus X will be spectral. Let F be a finite set of generators for M. If $N_{\mathscr{U}} = M$ then, by definition, $\boldsymbol{V}(F) \cap X \in \mathscr{U}$ and, since the empty set is not a member of any ultrafilter, we can pick a submodule $N \in \boldsymbol{V}(F) \cap X$. But $N \in \boldsymbol{V}(F)$ implies $M = \langle F \rangle = N$, a contradiction. Then $N_{\mathscr{U}} \neq M$ and thus the first part of the proof is complete.

Conversely, assume that M is not finitely generated, and note that the family of subsets $\{\boldsymbol{D}(x) \mid x \in M\}$ is obviously an open cover of X. Of course, for any finite subset F of M, the collection of open sets $\{\boldsymbol{D}(x) \mid x \in F\}$ is not a subcover of X, since the finitely generated submodule $N := \langle F \rangle$ of M is proper, by assumption, and thus

$N \in X \setminus \bigcup\{\boldsymbol{D}(x) \mid x \in F\}$. This shows that, if M is not finitely generated, then X is not quasi-compact and, a fortiori, is not spectral. □

Remark 2.5. In Corollary 2.2, we considered the space of ideals of a ring R as a special case of the space of R-submodules of a R-module M. It is possible, however, to reverse this relation, in the following way.

With the same proof of Proposition 2.1, we can first show that, given two ideals I and J with $J \subseteq I$, the set $\mathtt{Id}((I,J)|R) := \{H \in \mathtt{Id}(R) \mid J \subseteq H \subseteq I\}$ is a spectral space, with $\mathtt{Id}(R)$ being the special case with $J = (0)$ and $I = R$. Consider now an R-module M: then, M is an ideal of the idealization ring $\mathscr{R} := R \ltimes M$ [13, Section 25]. In this case, we have that $\mathtt{Id}((M,(0))|\mathscr{R})$ coincides with $\mathtt{SMod}(M)$ and so, from this fact, we can deduce that $\mathtt{SMod}(M)$ is a spectral space.

In the next proposition we show that the construction of the spectral space $\mathtt{SMod}(M|R)$ is functorial. Recall that a map $f : X \to Y$ of spectral spaces is called a *spectral map* provided that, for any open and quasi-compact subspace Ω of Y, the set $f^{-1}(\Omega)$ is open and quasi-compact. In particular, any spectral map of spectral spaces is continuous.

Proposition 2.6. *Let R be a ring. For every R-module homomorphism $f : M \to N$, set $\mathtt{SMod}(f) : \mathtt{SMod}(N|R) \to \mathtt{SMod}(M|R)$, defined by $\mathtt{SMod}(f)(L) := f^{-1}(L)$, for each $L \in \mathtt{SMod}(N|R)$. The assignment $M \mapsto \mathtt{SMod}(M|R)$, $f \mapsto \mathtt{SMod}(f)$ gives rise to a contravariant functor $\mathtt{SMod}$ from the category of R-modules and R-linear maps to the category of spectral spaces and spectral maps.*

Proof. By Proposition 2.1, $\mathtt{SMod}(M|R)$ and $\mathtt{SMod}(N|R)$ are spectral spaces. In order to show that $\mathtt{SMod}(f)$ is continuous and spectral, it is enough to note that, for each finite subset $\{x_1, x_2, \ldots, x_m\}$ of K,

$$\mathtt{SMod}(f)^{-1}(\boldsymbol{V}(x_1, x_2, \ldots, x_m)) = \boldsymbol{V}(f(x_1), f(x_2), \ldots, f(x_m))\,.$$

Moreover, it is clear that $\mathtt{SMod}(g \circ f) = \mathtt{SMod}(f) \circ \mathtt{SMod}(g)$, so that $\mathtt{SMod}$ is a (contravariant) functor. □

For example, let D be an integral domain with quotient field K and let $j : D \hookrightarrow K$ be the natural embedding. Then, the map $\mathtt{SMod}(j) : \mathtt{SMod}(K|D) = \overline{\mathscr{F}}(D) \to \mathtt{SMod}(D|D) = \mathtt{Id}(D)$, defined by $E \mapsto E \cap D$, is a spectral retraction (between spectral spaces endowed with the hull-kernel topology). In fact, if $i : \mathtt{Id}(D) \hookrightarrow \overline{\mathscr{F}}(D)$ is the natural (spectral) embedding, then $\mathtt{SMod}(j) \circ i$ is the identity of $\mathtt{Id}(D)$.

3. The prime spectrum of a module

Recall that a *prime submodule* of a R-module M is a submodule $P \neq M$ such that, whenever $am \in P$ for some $a \in R$, $m \in M$, we have $m \in P$ or $aM \subseteq P$ (see, for example, [17]). Denote by $\mathtt{Spec}_R(M)$ the set of prime submodules of M. Note that $\mathtt{Spec}_R(M)$ may be empty (e.g., if R is a domain, K its quotient field and $M = K/R$) and that when $M = R$ it coincides with the prime spectrum of R.

Proposition 3.1. *Let M be a R-module and endow $\mathtt{SMod}(M|R)$ with the hull-kernel topology.*

(1) $\mathtt{Spec}_R(M) \cup \{M\}$ *is a spectral subspace of* $\mathtt{SMod}(M|R)$.
(2) $\mathtt{Spec}_R(M)$ *is a spectral space if and only if it is quasi-compact.*
(3) *If M is finitely generated, then* $\mathtt{Spec}_R(M)$ *is a spectral space.*

Proof. (1) Let $\mathscr{U}$ be an ultrafilter on $\mathtt{Spec}_R(M)$; like in the proof of Proposition 2.1, it is enough to show that the set $N_{\mathscr{U}} := \{x \in M \mid \boldsymbol{V}(x) \cap \mathtt{Spec}_R(M) \in \mathscr{U}\}$ is a prime submodule of M, if it is different from M. To shorten the notation, set $\bar{\mathtt{S}} := \mathtt{Spec}_R(M) \cup \{M\}$, $\mathtt{S} := \mathtt{Spec}_R(M)$, $\boldsymbol{V}_{\mathtt{S}}(x) := \boldsymbol{V}(x) \cap \mathtt{Spec}_R(M)$ and $\boldsymbol{D}_{\mathtt{S}}(x) := \mathtt{Spec}_R(M) \setminus \boldsymbol{V}_{\mathtt{S}}(x)$.

The proof of Proposition 2.1 shows that $N_{\mathscr{U}}$ is a submodule of M. Suppose now that $a \in R$, $m \in M$, $am \in N_{\mathscr{U}}$, and that $m \notin N_{\mathscr{U}}$, so $N_{\mathscr{U}} \neq M$. By definition of a prime submodule, it follows easily that $T := \boldsymbol{V}_{\mathtt{S}}(am) \cap \boldsymbol{D}_{\mathtt{S}}(m) \subseteq \boldsymbol{V}_{\mathtt{S}}(ax)$, for any $x \in M$. Now, keeping in mind that $m \notin N_{\mathscr{U}}$, $am \in N_{\mathscr{U}}$ and that $\mathscr{U}$ is an ultrafilter on $\mathtt{Spec}_R(M)$, it follows that $T \in \mathscr{U}$ and, a fortiori, $\boldsymbol{V}_{\mathtt{S}}(ax) \in \mathscr{U}$, for any $x \in M$, that is, $xM \subseteq N_{\mathscr{U}}$. In other words, $N_{\mathscr{U}}$ is a prime submodule of M.

(2) If $\mathtt{S} = \mathtt{Spec}_R(M)$ is a spectral space then it is clearly quasi-compact. Conversely, keeping in mind that $\{M\}$ is the unique closed point in $\bar{\mathtt{S}}$, we have that $\mathtt{S}$ is open and quasi-compact in the spectral space $\bar{\mathtt{S}}$, and hence it is spectral.

(3) Let $\mathscr{U}$ and $N_{\mathscr{U}}$ be as in part (1). We need to prove that, if M is finitely generated, then $N_{\mathscr{U}} \neq M$. In fact, let $M = \langle x_1, x_2, \ldots, x_n \rangle$, if $N_{\mathscr{U}} = M$, then, by definition, the set $\bigcap_{i=1}^n \boldsymbol{V}(x_i) \in \mathscr{U}$. Thus, we can pick a prime submodule $P \in \bigcap_{i=1}^n \boldsymbol{V}(x_i)$, that is, $M = \langle x_1, \ldots, x_n \rangle \subseteq P$, reaching a contradiction. This proves that, if M is finitely generated, then $\mathtt{Spec}_R(M)$ is a closed set of $\bar{\mathtt{S}}$, with respect to the constructible topology, by Theorem 1.1(2). In particular, $\mathtt{Spec}_R(M)$ is quasi-compact, when endowed with the hull-kernel topology. The conclusion is then a consequence of part (2). □

Remark 3.2.

(1) The condition that M is finitely generated is not necessary for $\mathtt{Spec}_R(M)$ to be spectral. For example, if $R = D$ is an integral domain and $M = K$ is its quotient field, then $\mathtt{Spec}_D(K) = \{(0)\}$, which is compact and spectral. However, K is not finitely generated over D if $D \neq K$.
(2) $\mathtt{Spec}_R(M)$ might not be indeed quasi-compact: let R be any ring, $P \in \mathtt{Spec}(R)$, and let $M = \bigoplus_{\alpha\in\mathcal{A}} e_\alpha R$ be a non-finitely generated free module over R. We always have $\mathtt{Spec}_R(M) \subseteq \bigcup_{\alpha\in\mathcal{A}} \boldsymbol{D}(e_\alpha)$. If $\mathtt{Spec}_R(M)$ were quasi-compact, there would be $\alpha_1, \alpha_2 \ldots, \alpha_n \in \mathcal{A}$ such that $\mathtt{Spec}_R(M) \subseteq \boldsymbol{D}(e_{\alpha_1}) \cup \boldsymbol{D}(e_{\alpha_2}) \cup \cdots \cup \boldsymbol{D}(e_{\alpha_n})$, and so there would be no prime submodule containing all $e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_n}$. Since $\mathcal{A}$ is

infinite, there is an element $\beta \in \mathcal{A}$ such that $\beta \neq \alpha_i$ for every i, $1 \leq i \leq n$. Define a submodule N of M as follows:

$$N := \bigoplus_{\alpha \in \mathcal{A}} e_\alpha N_\alpha, \text{ where } \quad N_\alpha = R \quad \text{if } \alpha \neq \beta \quad \text{and} \quad N_\beta = P.$$

We have $M/N \simeq R/P$, so that N is a prime submodule of M. However, N contains $e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_n}$, against our hypothesis. Therefore, $\mathtt{Spec}_R(M)$ is not quasi-compact.

(3) In [17], the set $\mathtt{Spec}_R(M)$ (indicated with $\mathtt{Spec}(M)$) was endowed with a topology τ (which the author calls *Zariski topology*) whose closed sets are those in the form $V(N) := \{P \in \mathtt{Spec}_R(M) \mid (P : M) \subseteq (N : M)\}$, as N ranges among the submodules of M. This topology is in general weaker than the topology introduced in the present paper, and it is T_0 if and only if the map $\psi : \mathtt{Spec}_R(M) \to \mathtt{Spec}(R)$, defined by $P \mapsto (P : M)$, is injective. In [17], it was also shown that, if ψ is injective and its image is the closed subspace $\mathtt{V}(\mathrm{ann}(M))$, then it is an homeomorphism on its image (so that, in particular, $\mathtt{Spec}_R(M)$ endowed with the topology τ is spectral). Even when τ is T_0, however, this topology does not always coincide with the hull-kernel topology. Indeed, let $R := \mathbb{Z}$, $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ and let $M := \mathbb{Z}_2 \oplus \mathbb{Q}$. We have $\mathtt{Spec}_R(M) = \{P, Q\}$, where $P := \mathbb{Z}_2 \oplus (0)$ and $Q := (0) \oplus \mathbb{Q}$; hence both P and Q are closed points in the hull-kernel topology of $\mathtt{Spec}_R(M)$. On the other hand, both $V(P)$ and $V(Q)$ are irreducible closed subsets in the topology τ [17, Corollary 5.3]. However, $(P : M) = 2\mathbb{Z}$ and $(Q : M) = (0)$, so $V(P) = \{P\}$ and $V(Q) = \{P, Q\}$. It follows that $\mathtt{Spec}_R(M)$ is T_0 in the Zariski topology, but Q is not a closed point.

Denote by $\mathtt{Overr}(D)$ the set of all overrings of the integral domain D. We observe that $\mathtt{Overr}(D)$ is a subset of $\overline{\mathscr{F}}(D)$ (in fact, it is a subset of $\overline{\boldsymbol{F}}(D) := \overline{\mathscr{F}}(D) \setminus \{(0)\}$, the set of all nonzero D-submodules of K). On the other hand, the set $\mathtt{Overr}(D)$ can be endowed with a topology, called the *Zariski topology*, having as basic open sets the subsets of the type $\mathtt{B}(F) := \mathtt{Overr}(D[F]) = \{T \in \mathtt{Overr}(D) \mid F \subseteq T\}$, where F is varying among the finite subsets of K. If we denote by $\mathtt{Overr}(D)^{\mathtt{zar}}$ the topological space $\mathtt{Overr}(D)$ with the Zariski topology and $\overline{\mathscr{F}}(D)^{\mathtt{hk}}$ (respectively, $\overline{\boldsymbol{F}}(D)^{\mathtt{hk}}$) the space $\overline{\mathscr{F}}(D)$ (respectively, $\overline{\boldsymbol{F}}(D)$) with the hull-kernel topology (respectively, topology induced from the hull-kernel topology of $\overline{\mathscr{F}}(D)$) then the inclusion maps $\mathtt{Overr}(D) \subseteq \overline{\boldsymbol{F}}(D)$ and $\mathtt{Overr}(D) \subseteq \overline{\mathscr{F}}(D)$ are not continuous. In fact, the quotient field K is the generic point of $\mathtt{Overr}(D)^{\mathtt{zar}}$ but it is a closed point for $\overline{\boldsymbol{F}}(D)^{\mathtt{hk}}$ (and for $\overline{\mathscr{F}}(D)^{\mathtt{hk}}$).

Recall that $\mathtt{Overr}(D)^{\mathtt{zar}}$ is a spectral space [5, Proposition 3.5(2)] and denote by $\mathtt{Overr}(D)^{\mathtt{inv}}$ (respectively, $\mathtt{Overr}(D)^{\mathtt{hk}}$) the set $\mathtt{Overr}(D)$ with the inverse topology (respectively, with the hull-kernel topology, induced from $\overline{\mathscr{F}}(D)^{\mathtt{hk}}$).

Proposition 3.3. *For any domain D, $\mathtt{Overr}(D)^{\mathtt{hk}}$ coincides with $\mathtt{Overr}(D)^{\mathtt{inv}}$.*

Proof. By definition of the inverse topology, a basis for the closed sets of $\mathtt{Overr}(D)^{\mathtt{inv}}$ is given by the quasi-compact open subspaces of $\mathtt{Overr}(D)^{\mathtt{zar}}$, i.e., by the finite unions of the subsets $\mathtt{B}(F)$, where F is varying among the finite subsets of K. On the other hand, by definition, $\mathtt{Overr}(D[F]) = \boldsymbol{V}(F)$. Moreover, if G is any subset of K, then $\boldsymbol{V}(G) = \bigcap\{\boldsymbol{V}(F) \mid F \subseteq G \text{ and } F \text{ is finite}\}$, so that $\{\boldsymbol{V}(F) \mid F \text{ is finite subset of } K\}$ is a basis for the closed sets of the topological space $\mathtt{Overr}(D)^{\mathtt{hk}}$. Therefore, we conclude that $\mathtt{Overr}(D)^{\mathtt{hk}} = \mathtt{Overr}(D)^{\mathtt{inv}}$. □

Given a ring R, on any R-module M, a *closure operation* on $\mathtt{SMod}(M|R)$ is a map $(-)^c : \mathtt{SMod}(M|R) \to \mathtt{SMod}(M|R)$ that is extensive (i.e., $N \subseteq N^c$), order-preserving (i.e., $N_1 \subseteq N_2$ implies $N_1^c \subseteq N_2^c$) and idempotent (i.e., $(N^c)^c = N^c$). We also say that c is *of finite type* if, for any $N \in \mathtt{SMod}(M|R)$, $N^c = \bigcup\{L^c \mid L \subseteq N, L \in \mathtt{SMod}(M|R), L \text{ is finitely generated}\}$. For a deeper insight on this topic see, for example, [1,3,4,10], and [23].

Proposition 3.4. *Let M be an R-module and c be a closure operation of finite type on $\mathtt{SMod}(M|R)$. The set $\mathtt{SMod}^c(M|R) := \{N \in \mathtt{SMod}(M|R) \mid N = N^c\}$ is a spectral space. Moreover, $\mathtt{SMod}^c(M|R)$ is closed in $\mathtt{SMod}(M|R)$, endowed with the constructible topology.*

Proof. With the same notation of the proof of Proposition 2.1, to prove the first statement we only need to show that, if $\mathscr{U}$ is an ultrafilter on $\mathtt{SMod}^c(M|R)$, $N_{\mathscr{U}}$ is also in $\mathtt{SMod}^c(M|R)$.

Let $x \in (N_{\mathscr{U}})^c$. Since c is of finite type, there is a finitely generated R-module $L \subseteq N_{\mathscr{U}}$ such that $x \in L^c$. In particular, $x \in H^c$ for all $H \supseteq L$, i.e., for all $H \in \boldsymbol{V}(L)$; therefore, $\boldsymbol{V}(L) \cap \mathtt{SMod}^c(M|R) \subseteq \boldsymbol{V}(x) \cap \mathtt{SMod}^c(M|R)$. If $L = \ell_1 R + \ell_2 R + \cdots + \ell_n R$, then $\boldsymbol{V}(L) = \boldsymbol{V}(\ell_1) \cap \boldsymbol{V}(\ell_2) \cap \cdots \cap \boldsymbol{V}(\ell_n)$. Since each $\boldsymbol{V}(\ell_i) \cap \mathtt{SMod}^c(M|R)$ is in $\mathscr{U}$ (by definition of $N_{\mathscr{U}}$), then $\boldsymbol{V}(L) \cap \mathtt{SMod}^c(M|R) \in \mathscr{U}$. Hence, $\boldsymbol{V}(x) \cap \mathtt{SMod}^c(M|R) \in \mathscr{U}$, i.e., $x \in N_{\mathscr{U}}$. Thus, $N_{\mathscr{U}} = (N_{\mathscr{U}})^c$ and $\mathtt{SMod}^c(M|R)$ is a spectral space.

Finally, from Theorem 1.1(2) we deduce that $\mathtt{SMod}^c(M|R)$ is a closed subspace of $\mathtt{SMod}(M|R)$, endowed with the constructible topology. □

Corollary 3.5. *Let D be an integral domain and $\star$ be a semistar operation of finite type on D (for background material on semistar operations see, for instance, [3,8,19]). Then, the subspaces*

$$\overline{\mathscr{F}}(D)^{\star} := \{E \in \overline{\mathscr{F}}(D) \mid E^{\star} = E\} \quad \textit{and} \quad \mathtt{Overr}^{\star}(D) := \{T \in \mathtt{Overr}(D) \mid T = T^{\star}\}$$

of $\overline{\mathscr{F}}(D)^{\mathtt{hk}}$ are spectral spaces.

Proof. By applying Proposition 3.4 and the proof of [5, Proposition 3.5] we note that $\overline{\mathscr{F}}(D)^{\star}$ and $\mathtt{Overr}^{\star}(D)$ are closed in $\overline{\mathscr{F}}(D)$, endowed with the constructible topology. Then, the conclusion follows by [12, Proposition 9]. □

Corollary 3.6. *Let c be a closure operation of finite type on a ring R. Then,* $\mathtt{Id}^c(R) := \mathtt{SMod}^c(R|R)$ *(respectively,* $\mathtt{Id}^c_\bullet(R) := \mathtt{SMod}^c(R|R) \setminus \{R\}$*), endowed with the hull-kernel topology, is a spectral space.*

Proof. The statements follow from Proposition 3.4 and its proof, using the same argument of the proof of Corollary 2.2(1). □

Remark 3.7. If c is a closure operation of finite type on an R-module M, we can always consider a canonical surjective map $\psi_c : \mathtt{SMod}(M|R) \longrightarrow \mathtt{SMod}^c(M|R)$, by setting $\psi_c(N) := N^c$, for each $N \in \mathtt{SMod}(M|R)$. However, ψ_c is only rarely continuous (with respect to the hull-kernel topology). For example, let $M = R$ be any infinite ring such that the intersection of all nonzero ideals is (0) (such a ring is, for example, an integral domain that is not a field). Set $(0)^c := (0)$, and set I^c to be equal to R if $I \neq (0)$. Therefore, $\mathtt{SMod}^c(R|R) = \mathtt{Id}^c(R) = \{(0), R\}$. Note that $\psi_c^{-1}(R) = \{I \mid I \neq (0)\} = \mathtt{Id}(R) \setminus \{(0)\}$. Since R is a closed point in $\mathtt{SMod}(R|R) = \mathtt{Id}(R)$ (endowed with the hull-kernel topology) and $R = R^c$, then R is a closed point in $\mathtt{SMod}^c(R|R) = \mathtt{Id}^c(R)$ (endowed with the hull-kernel topology). If ψ_c were continuous, $\psi_c^{-1}(R) = \mathtt{Id}(R) \setminus \{(0)\}$ would be closed and thus (being a closed subset of a spectral space) it would be a spectral space itself. However, $\mathtt{Id}(R) \setminus \{(0)\}$ cannot be a spectral space, when endowed with the hull-kernel topology induced from $\mathtt{Id}(R)$, since $\mathtt{Id}(R) \setminus \{(0)\}$ is not quasi-compact. Indeed, by assumption, the intersection of all nonzero ideals of R is (0), and thus the collection of sets $\{\boldsymbol{D}(x) \setminus \{(0)\} \mid x \neq 0\}$ provides an infinite open cover of $\mathtt{Id}(R) \setminus \{(0)\}$ without finite subcovers.

As a particular case of the Proposition 3.4 and Corollary 3.6, we have the following.

Corollary 3.8. *Let R be a ring. The sets* $\mathtt{Rd}(R)$ *and* $\mathtt{Rd}(R) \cup \{R\}$*, endowed with the hull-kernel topology, are spectral spaces.*

Proof. As usual, let $\mathrm{rad}(I)$ denote the radical of an ideal I of R. If $x \in \mathrm{rad}(I)$, then $x \in \mathrm{rad}(x^n)$ for some $x^n \in I$, so rad is a closure operation of finite type in $\mathtt{Id}(R)$, i.e., $\mathtt{Rd}(R) \cup \{R\} = \mathtt{Id}^c(R)$, where $c = \mathrm{rad}$. The conclusion is now a consequence of Corollary 3.6. □

4. A topological version of Hilbert's Nullstellensatz

Let now X be a spectral space and let $\mathtt{Cl}\,(Y)$ denote the closure of a subspace Y in the given topology of X. Let $\boldsymbol{\mathcal{X}'}(X)$ be the space of nonempty closed sets of X, and endow it with a topology whose subbasic open sets are the family of sets

$$\boldsymbol{\mathcal{U}'}(\Omega) := \{Y \in \boldsymbol{\mathcal{X}'}(X) \mid Y \cap \Omega = \emptyset\},$$

as Ω ranges among the quasi-compact open subspaces of X. Note that the family of sets of the type $\boldsymbol{\mathcal{U}'}(\Omega)$ forms a basis, since $\boldsymbol{\mathcal{U}'}(\Omega_1) \cap \boldsymbol{\mathcal{U}'}(\Omega_2) = \boldsymbol{\mathcal{U}'}(\Omega_1 \cup \Omega_2)$. We call this topology the *Zariski topology* of the space $\boldsymbol{\mathcal{X}'}(X)$. The notation used here is chosen in analogy and for coherence with the construction of the space $\boldsymbol{\mathcal{X}}(X)$, which is sketched in [6] and elaborated upon in [7].

Note that there is a canonical injective map $\varphi' : X^{\texttt{inv}} \to \boldsymbol{\mathcal{X}'}(X)^{\texttt{zar}}$, defined by $\varphi'(x) := \{x\}^{\texttt{sp}}$, which is a topological embedding. Indeed, φ' is continuous since

$$\varphi'^{\,-1}(\boldsymbol{\mathcal{U}'}(\Omega)) = \{x \in X^{\texttt{sp}} \mid \{x\}^{\texttt{sp}} \cap \Omega = \emptyset\} = X \setminus \Omega\,,$$

which is, by definition, a subbasic open set of $X^{\texttt{inv}}$. Moreover, since the family of the sets of the type $X \setminus \Omega$, for Ω ranging among the quasi-compact open subspaces of X, forms a subbasis of $X^{\texttt{inv}}$, the calculation above shows that $\varphi'(X \setminus \Omega) = \boldsymbol{\mathcal{U}'}(\Omega) \cap \varphi'(X)$, and thus the map φ' is a topological embedding.

Now, we are in condition to state a “topological version” of the Hilbert Nullstellensatz.

Theorem 4.1. *Let R be a ring and let $\boldsymbol{\mathcal{X}'}(R) := \boldsymbol{\mathcal{X}'}(\texttt{Spec}(R))$ be the topological space of the non-empty Zariski closed subspaces of $\texttt{Spec}(R)$, endowed with the Zariski topology. Let $\texttt{Rd}(R)$ be the spectral space of all proper radical ideals of R with the inverse topology. Then, the map*

$$\begin{aligned}\mathscr{J} : \boldsymbol{\mathcal{X}'}(R)^{\texttt{zar}} &\to \texttt{Rd}(R)^{\texttt{inv}}\\ C &\mapsto \bigcap\{P \in \texttt{Spec}(R) \mid P \in C\}\end{aligned}$$

is a homeomorphism. In particular, $\boldsymbol{\mathcal{X}'}(R)$ is a spectral space. Moreover, the same map $\mathscr{J}$ defines a homeomorphism between $\boldsymbol{\mathcal{X}'}(R)^{\texttt{inv}}$ and $\texttt{Rd}(R)^{\texttt{hk}}$.

Proof. For each $x_1, \ldots, x_n \in R$, let $\boldsymbol{\Delta}(x_1, \ldots, x_n) := \{H \in \texttt{Rd}(R) \mid (x_1, \ldots, x_n) \nsubseteq H\} = \boldsymbol{D}(x_1, \ldots, x_n) \cap \texttt{Rd}(R)$ be a subbasic open set of $\texttt{Rd}(R)$ and let $\texttt{D}(x_1, \ldots, x_n) := \{P \in \texttt{Spec}(R) \mid x \notin P\}$ be a subbasic open set of $\texttt{Spec}(R)$. By the definition of the hull-kernel topology, by Corollaries 1.2, 3.8 and Proposition 2.1, it follows that $\mathcal{B} := \{\boldsymbol{\Delta}(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in R\}$ is a collection of quasi-compact open subspaces of $\texttt{Rd}(R)^{\texttt{hk}}$, that is, it is a subbasis of closed sets of $\texttt{Rd}(R)^{\texttt{inv}}$. Set $\boldsymbol{\mathcal{X}'} := \boldsymbol{\mathcal{X}'}(R)$. Then,

$$\begin{aligned}\mathscr{J}^{-1}(\boldsymbol{\Delta}(x_1, \ldots, x_n)) &= \{C \in \boldsymbol{\mathcal{X}'} \mid \mathscr{J}(C) \in \boldsymbol{\Delta}(x_1, \ldots, x_n)\} =\\ &= \{C \in \boldsymbol{\mathcal{X}'} \mid (x_1, \ldots, x_n) \nsubseteq \mathscr{J}(C)\} =\\ &= \{C \in \boldsymbol{\mathcal{X}'} \mid (x_1, \ldots, x_n) \nsubseteq \bigcap\{P \in \texttt{Spec}(R) \mid P \in C\}\} =\\ &= \big\{C \in \boldsymbol{\mathcal{X}'} \mid x_i \notin P \text{ for some } P \in C \text{ and some } i\big\} =\\ &= \{C \in \boldsymbol{\mathcal{X}'} \mid C \cap \texttt{D}(x_1, \ldots, x_n) \neq \emptyset\} =\\ &= \boldsymbol{\mathcal{X}'} \setminus \boldsymbol{\mathcal{U}'}(\texttt{D}(x_1, \ldots, x_n))\end{aligned}$$

which is, by definition, a closed set of $\boldsymbol{\mathcal{X}'}$. Hence, $\mathscr{J}$ is continuous (when $\mathtt{Rd}(R)$ is equipped with the inverse topology). In order to show that it is a closed map, it is enough to note that $\{\boldsymbol{\mathcal{X}'} \setminus \boldsymbol{\mathcal{U}'}(\mathtt{D}(x_1,\ldots,x_n)) \mid x_1,\ldots,x_n \in R\}$ is a basis of closed sets of $\boldsymbol{\mathcal{X}'}$ and that, by Hilbert Nullstellensatz, $\mathscr{J}$ is bijective; hence $\mathscr{J}(\boldsymbol{\mathcal{X}'} \setminus \boldsymbol{\mathcal{U}'}(\mathtt{D}(x_1,\ldots,x_n))) = \boldsymbol{\Delta}(x_1,\ldots,x_n)$ is closed in $\mathtt{Rd}(R)^{\mathtt{inv}}$. Thus, $\mathscr{J}$ is a homeomorphism.

The last claim follows directly from Hochster's duality, that is, more explicitly, from the fact that $(\mathtt{Rd}(R)^{\mathtt{inv}})^{\mathtt{inv}}$ coincides with $\mathtt{Rd}(R)^{\mathtt{hk}}$. □

In the following, if X is a topological space, we will denote by $\dim(X)$ (respectively, $|X|$) the dimension (respectively, the cardinality) of X.

Proposition 4.2. *Let R be a ring and let $\mathtt{Rd}(R)$ be the space of all proper radical ideals of R, endowed with the hull-kernel topology. Then*

$$\dim(\mathtt{Rd}(R)) = |\mathtt{Spec}(R)| - 1 \geq \dim(\mathtt{Spec}(R))\,.$$

Moreover, if $\mathtt{Spec}(R)$ is linearly ordered, then $\dim(\mathtt{Rd}(R)) = \dim(\mathtt{Spec}(R))$.

Proof. Let X be a nonempty finite subset of $\mathtt{Spec}(R)$, with $|X| = n$. Let P_n be a minimal element of X and, by induction, let P_i be a minimal element of $X \setminus \{P_n, \ldots, P_{i+1}\}$, for $1 \leq i \leq n-1$. Consider the radical ideals $H_i := \bigcap_{\ell=1}^{i} P_\ell$, for $i = 1, 2, \ldots, n$. By construction, we have $P_i \not\supseteq P_1, \ldots, P_{i-1}$, for $i = 2, \ldots n$, that is $P_i \not\supseteq H_{i-1}$. Thus, we get a strictly increasing chain of radical ideals of R

$$H_n \subsetneq H_{n-1} \subsetneq \ldots \subsetneq H_1 := P_1\,.$$

Since the order induced by the hull-kernel topology is the set-theoretic inclusion, this chain corresponds to a chain of length $n-1$ of irreducible closed subspaces of $\mathtt{Rd}(R)$. Thus, when $\mathtt{Spec}(R)$ is infinite, we can get, by applying the previous argument, chains of irreducible closed subsets of $\mathtt{Rd}(R)$ of arbitrary length. Thus, in this case, the equality $\dim(\mathtt{Rd}(R)) = |\mathtt{Spec}(R)| - 1$ is proved. Assume now that $\mathtt{Spec}(R)$ is finite. By applying the first part of the proof to $X := \mathtt{Spec}(R)$ we deduce immediately that $|\mathtt{Spec}(R)| - 1 \leq \dim(\mathtt{Rd}(R))$. Conversely, a chain of length t of irreducible closed subspaces of $\mathtt{Rd}(R)$ corresponds to a chain of radical ideals

$$L_0 \subsetneq L_1 \subsetneq \ldots \subsetneq L_t$$

and it provides the following chain of closed sets

$$\mathtt{V}(L_t) \subsetneq \mathtt{V}(L_{t-1}) \subsetneq \ldots \subsetneq \mathtt{V}(L_0)$$

of $\mathtt{Spec}(R)$. Since $\mathtt{Spec}(R)$ is finite, $\mathtt{V}(L_0)$ has at most $|\mathtt{Spec}(R)|$ elements. Since all inclusions are proper, it follows that $t \leq |\mathtt{Spec}(R)| - 1$. The first part of the proof is now complete. The last statement follows immediately by noting that $\mathtt{Rd}(R) = \mathtt{Spec}(R)$ if and only if $\mathtt{Spec}(R)$ is linearly ordered. $\square$

Topologies on the family of the closed subsets of a topological space were introduced and intensively studied since the beginning of 20th century, with applications to uniform spaces, Functional Analysis, Game Theory, etc. [2,14,15,18]. In this circle of ideas, one of the first contributions was made by L. Vietoris in [24]. We briefly recall his construction. Let X be any topological space and let, as before, $\boldsymbol{\mathcal{X}'}(X)$ denote the collection of all the nonempty closed subspaces of X (called also the *hyperspace of* X). For any open subspace U of X set

$$U^+ := \{C \in \boldsymbol{\mathcal{X}'}(X) \mid C \subseteq U\} \qquad U^- := \{C \in \boldsymbol{\mathcal{X}'}(X) \mid C \cap U \neq \emptyset\}.$$

The *upper Vietoris topology on* $\boldsymbol{\mathcal{X}'}(X)$ (respectively, *lower Vietoris topology*) is the topology on $\boldsymbol{\mathcal{X}'}(X)$ having as a basis (respectively, subbasis) of open sets the collection $\mathcal{V}^+ := \{U^+ \mid U \text{ open in } X\}$ (respectively, $\mathcal{V}^- := \{U^- \mid U \text{ open in } X\}$).

We now unveil a relation between the lower Vietoris topology and the Zariski topology $\boldsymbol{\mathcal{X}'}(X)$, considered at the beginning of the present section.

Proposition 4.3. *Let X be a spectral space. Then, the inverse topology of the spectral space $\boldsymbol{\mathcal{X}'}(X)^{\mathtt{zar}}$ and the lower Vietoris topology on $\boldsymbol{\mathcal{X}'}(X)$ are the same.*

Proof. Note first that, for any spectral space $\mathscr{X}$, if $\mathcal{B}$ is a basis of quasi-compact open subspaces of $\mathscr{X}$ (such a $\mathcal{B}$ exists, by definition of a spectral space), then $\mathcal{B}^{\mathtt{inv}} := \{\mathscr{X} \setminus B \mid B \in \mathcal{B}\}$ is a basis of open sets for $\mathscr{X}^{\mathtt{inv}}$.

Starting from the given spectral space X, with the notation introduced at the beginning of the present section, for any open and quasi-compact subspace Ω of X, we observe that the set $\boldsymbol{\mathcal{U}'}(\Omega)$ is quasi-compact, as a subspace of $\boldsymbol{\mathcal{X}'}(X)^{\mathtt{zar}}$. Indeed, note that $X \setminus \Omega \in \boldsymbol{\mathcal{U}'}(\Omega)$ and that, if $\boldsymbol{\mathcal{U}'}(\Omega) \subseteq \bigcup_{i\in I} \boldsymbol{\mathcal{U}'}(\Omega_i)$, with $\Omega_i \subseteq X$ open and quasi-compact, then $X \setminus \Omega \in \boldsymbol{\mathcal{U}'}(\Omega_i)$, for some i, that is $\Omega_i \subseteq \Omega$. Thus, a fortiori, $\boldsymbol{\mathcal{U}'}(\Omega) \subseteq \boldsymbol{\mathcal{U}'}(\Omega_i)$. This shows that the basis

$$\mathcal{B} := \{\boldsymbol{\mathcal{U}'}(\Omega) \mid \Omega \text{ quasi-compact open in } X\}$$

consists of quasi-compact open subspaces of $\boldsymbol{\mathcal{X}'}(X)^{\mathtt{zar}}$, and thus $\mathcal{B}^{\mathtt{inv}}$ is a basis of open sets for $\boldsymbol{\mathcal{X}'}(X)^{\mathtt{inv}}$. Since, by definition, the typical element in $\mathcal{B}^{\mathtt{inv}}$ is a set of closed subspaces hitting a fixed quasi-compact open subspace of X, it follows immediately that the inverse topology of $\boldsymbol{\mathcal{X}'}(X)^{\mathtt{zar}}$ is coarser than (or equal to) the lower Vietoris topology.

Conversely, let U be any open set of X and take a point $C \in U^- := \{F \in \boldsymbol{\mathcal{X}'}(X) \mid F \cap U \neq \emptyset\}$. If $x \in C \cap U$, there is a quasi-compact open subspace V of X such that $V \subseteq U$ and $x \in C \cap V$, since the collection of all the quasi-compact open subspaces of a spectral space forms a basis. Thus, $C \in V^- = \boldsymbol{\mathcal{X}'}(X) \setminus \boldsymbol{\mathcal{U}'}(V) \subseteq U^-$. This shows that U^- is open, in the inverse topology of $\boldsymbol{\mathcal{X}'}(X)^{\mathtt{zar}}$. The proof is now complete. $\square$

Remark 4.4. **(a)** The previous proposition shows that, given a spectral space X, the lower Vietoris topology on $\boldsymbol{\mathcal{X}}'(X)$ is always spectral. However, the same property can fail to hold for the upper Vietoris topology. To see this, let D be any integral domain with Jacobson radical $\mathfrak{J} \neq (0)$, let $X := \texttt{Spec}(D)$, let $Y := \texttt{V}(\mathfrak{J}) \in \boldsymbol{\mathcal{X}}'(X)$, and let $\Omega \subseteq \boldsymbol{\mathcal{X}}'(X)$ be any open neighborhood of Y, with respect to the upper Vietoris topology. Without loss of generality, we can assume that $\Omega = D(I)^+$, for some ideal I of D. Since each maximal ideal M of D belongs to Y, we have $I \nsubseteq M$, for each $M \in \texttt{Max}(D)$ and thus $I = D$, that is, $\Omega = \boldsymbol{\mathcal{X}}'(X)$. This proves that the unique open neighborhood of Y is $\boldsymbol{\mathcal{X}}'(X)$ and trivially the same holds for the point $X \in \boldsymbol{\mathcal{X}}'(X)$, with $Y \neq X$ since $\mathfrak{J} \neq (0)$. This shows that $\boldsymbol{\mathcal{X}}'(X)$, equipped with the upper Vietoris topology, does not satisfy the T_0 axiom and, a fortiori, it is not spectral.

Note also that the previous example shows that the inverse topology of the spectral space $\boldsymbol{\mathcal{X}}'(X)$, endowed with the lower Vietoris topology, is not the upper Vietoris topology on $\boldsymbol{\mathcal{X}}'(X)$.

(b) Following the idea of intertwining algebra and topology, it is possible to give an alternate proof of Proposition 4.3 based on Theorem 4.1.

Let $X = \texttt{Spec}(R)$, and let $\mathscr{J}_0$ be the map $\mathscr{J}$ defined in the statement of Theorem 4.1, but considered as a map from $\boldsymbol{\mathcal{X}}'(R)^{\texttt{loV}}$ (i.e., the space $\boldsymbol{\mathcal{X}}'(R)$ equipped with the lower Vietoris topology) to $\texttt{Rd}(R)^{\texttt{hk}}$ (i.e., the space $\texttt{Rd}(R)$ equipped with the hull-kernel topology). Obviously, $\mathscr{J}_0$ is bijective.

A subbasis of the space $\texttt{Rd}(R)^{\texttt{hk}}$ is composed by the sets of the form $\boldsymbol{D}(I) = \{H \in \texttt{Rd}(R) \mid I \nsubseteq H\}$, as I ranges among the ideals of R, while a subbasis of $\boldsymbol{\mathcal{X}}'(R)^{\texttt{loV}}$ is composed of the sets of the form $\boldsymbol{D}(I)^- = \{F \in \boldsymbol{\mathcal{X}}'(R) \mid F \cap \boldsymbol{D}(I) \neq \emptyset\}$, since the open sets of $\texttt{Spec}(R)$ are of the form $\boldsymbol{D}(I)$. However,

$$\begin{aligned} \mathscr{J}_0^{-1}(\boldsymbol{D}(I)) &= \{F \in \boldsymbol{\mathcal{X}}'(R)^{\texttt{loV}} \mid I \nsubseteq P \text{ for some prime ideal } P \in F\} = \\ &= \{F \mid F \nsubseteq \boldsymbol{V}(I)\} = \\ &= \{F \mid F \cap \boldsymbol{D}(I) \neq \emptyset\} = \boldsymbol{D}(I)^-, \end{aligned}$$

and thus $\mathscr{J}_0$ is a homeomorphism.

We thus have a chain of maps

$$\boldsymbol{\mathcal{X}}'(R)^{\texttt{loV}} \xrightarrow{\mathscr{J}_0} \texttt{Rd}(R)^{\texttt{hk}} \xrightarrow{\texttt{id}} ((\texttt{Rd}(R)^{\texttt{hk}})^{\texttt{inv}})^{\texttt{inv}} \xrightarrow{(\mathscr{J}^{-1})^{\texttt{inv}}} \boldsymbol{\mathcal{X}}'(R)^{\texttt{inv}},$$

where $\texttt{id}$ is the identity on the set $\texttt{Rd}(R)$ and $(\mathscr{J}^{-1})^{\texttt{inv}}$ indicates the map $\mathscr{J}^{-1}$ in the inverse topology. By Hochster's duality, $\texttt{id}$ is a homeomorphism, while $(\mathscr{J}^{-1})^{\texttt{inv}}$ and $\mathscr{J}_0$ are homeomorphism, respectively, by Theorem 4.1 and the above reasoning. Since the composition $(\mathscr{J}^{-1})^{\texttt{inv}} \circ \mathrm{id} \circ \mathscr{J}_0$ is clearly the identity on the set $\boldsymbol{\mathcal{X}}'(R)$, we conclude that the lower Vietoris topology and the inverse topology on $\boldsymbol{\mathcal{X}}'(R)$ are identical, as claimed.

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