Illinois Journal of Mathematics
Volume 60, Numbers 3–4, Fall and Winter 2016, Pages 791–809
S 0019-2082

NON-COMPACT SUBSETS OF THE ZARISKI SPACE OF AN INTEGRAL DOMAIN

DARIO SPIRITO

ABSTRACT. Let V be a minimal valuation overring of an integral domain D and let $\mathrm{Zar}(D)$ be the Zariski space of the valuation overrings of D. Starting from a result in the theory of semistar operations, we prove a criterion under which the set $\mathrm{Zar}(D)\setminus\{V\}$ is not compact. We then use it to prove that, in many cases, $\mathrm{Zar}(D)$ is not a Noetherian space, and apply it to the study of the spaces of Kronecker function rings and of Noetherian overrings.

1. Introduction

The *Zariski space* $\mathrm{Zar}(K|D)$ of the valuation rings of a field K containing a domain D was introduced (under the name *abstract Riemann surface*) by O. Zariski, who used it to show that resolution of singularities holds for varieties of dimension 2 or 3 over fields of characteristic 0 ([31], [32]). In particular, Zariski showed that $\mathrm{Zar}(K|D)$, endowed with a natural topology, is always a compact space [33, Chapter VI, Theorem 40]; this result has been subsequently improved by showing that $\mathrm{Zar}(K|D)$ is a spectral space (in the sense of Hochster [17]), first in the case where K is the quotient field of D ([4], [5]), and then in the general case [8, Corollary 3.6(3)]. The topological aspects of the Zariski space has subsequently been used, for example, in real and rigid algebraic geometry ([18], [34]) and in the study of representation of integral domains as intersections of valuation overrings ([26], [27], [28]). In the latter context, that is, when K is the quotient field of D, two important properties for subspaces of $\mathrm{Zar}(K|D)$ to investigate are the properties of compactness and of Noetherianess.

Received September 19, 2015; received in final form May 3, 2017.
This work was partially supported by *GNSAGA* of *Istituto Nazionale di Alta Matematica*.
2010 *Mathematics Subject Classification.* Primary 13F30. Secondary 13A15, 13A18, 13B22, 54D30.

In this paper, we concentrate on the case where K is the quotient field of D, studying subspaces of $\mathrm{Zar}(K|D)=\mathrm{Zar}(D)$ that are *not* compact. The starting point is a criterion based on semistar operations, proved in [8, Theorems 4.9 and 4.13] (see also [11, Proposition 4.5] for a slightly stronger version) and integrated, as in [9, Example 3.7], with the use of the two-faced definition of the integral closure/b-operation, either through valuation overrings or through equations of integral dependence (see, e.g., [19, Chapter 6]). In particular, we analyze sets of the form $\mathrm{Zar}(D)\setminus\{V\}$, where V is a minimal valuation overring of D: we show in Section 3 that such a space is compact only if V can be obtained from D in a very specific way (more precisely, as the integral closure of a localization of a finitely generated algebra over D), and we follow up in Sections 4 and 5 by showing that this condition implies a bound on the dimension of V in relation with the dimension of D (Proposition 4.3) and a quite strict condition on the intersection of sets of prime ideals of D (Theorem 5.1). Section 6 is dedicated to a brief application of these criteria to the study of Kronecker function rings (the definition will be recalled later).

In Section 7, we consider the set $\mathrm{Over}(D)$ of overrings of D (which is known to be itself a spectral space [7, Proposition 3.5]). Using the result proved in the previous sections, we show that, when D is a Noetherian domain, some distinguished subspaces of $\mathrm{Over}(D)$ (for example, the subspace of overrings of D that are Noetherian) are not spectral.

2. Preliminaries and notation

2.1. Spectral spaces. A topological space X is a *spectral space* if there is a ring R such that X is homeomorphic to the prime spectrum $\mathrm{Spec}(R)$, endowed with the Zariski topology. Spectral spaces can be characterized in a purely topological way as those spaces that are T_0, compact, with a basis of open and compact subset that is closed by finite intersections and such that every irreducible closed subset has a generic point (i.e., it is the closure of a single point) [17, Proposition 4].

On a spectral space X it is possible to define two new topologies: the *inverse* and the *constructible* topology.

The *inverse topology* is the topology on X having, as a basis of closed sets, the family of open and compact subspaces of X. Endowed with the inverse topology, X is again a spectral space [17, Proposition 8]; moreover, a subspace $Y\subseteq X$ is closed in the inverse topology if and only if Y is compact (in the original topology) and $Y=Y^{\mathrm{gen}}$ [8, Remark 2.2 and Proposition 2.6], where

$$\begin{aligned} Y^{\mathrm{gen}} &:= \{z\in X \mid z\leq y \text{ for some } y\in Y\} \\ &= \{z\in X \mid y\in \mathrm{Cl}(z) \text{ for some } y\in Y\}, \end{aligned}$$

with $\mathrm{Cl}(z)$ denoting the closure of the singleton $\{z\}$ (again, in the original topology) and $\leq$ is the order induced by the original topology [22, d-1], which coincides on $\mathrm{Spec}(R)$ with the set-theoretic inclusion.

The *constructible topology* on X (also called *patch topology*) is the coarsest topology such that the open and compact subsets of X are both open and closed. Endowed with the constructible topology, X is a spectral space that is also Haussdorff (see [29, Propositions 3 and 5], [30] or [14, Proposition 5]), and the constructible topology is finer than both the original and the inverse topology. A subset of X closed in the constructible topology is said to be a *proconstructible subset* of X; if Y is proconstructible, then it is a spectral space when endowed with the topology induced by the original spectral topology of X, and the constructible topology on Y is exactly the topology induced by the constructible topology on X (this follows from [3, 1.9.5(vi–vii)]).

2.2. Noetherian spaces. A topological space X is *Noetherian* if X verifies the ascending chain condition on the open subsets, or equivalently if every subspace of X is compact. Examples of Noetherian spaces are finite spaces and the prime spectra of Noetherian rings. If $\mathrm{Spec}(R)$ is a Noetherian space, then every proper ideal of R has only finitely many minimal primes (see, e.g., the proof of [2, Chapter 4, Corollary 3, p.102] or [1, Chapter 6, Exercises 5 and 7]).

2.3. Overrings and the Zariski space. Let $D \subseteq K$ be an extension of integral domains. We denote the set of all rings contained between D and K by $\mathrm{Over}(K|D)$; if K is a field (not necessarily the quotient field of D), the set of all valuation rings containing D with quotient field K is denoted by $\mathrm{Zar}(K|D)$, and it is called the *Zariski space* (or the *Zariski-Riemann space*) of D.

The *Zariski topology* on $\mathrm{Over}(K|D)$ is the topology having, as a subbasis, the sets of the form

$$B(x_1, \ldots, x_n) := \{T \in \mathrm{Over}(K|D) \mid x_1, \ldots, x_n \in T\},$$

as $\{x_1, \ldots, x_n\}$ ranges among the finite subsets of K. Under this topology, both $\mathrm{Over}(K|D)$ [7, Proposition 3.5] and its subspace $\mathrm{Zar}(K|D)$ ([5], [4]) are spectral spaces, and the order induced by this topology is the inverse of the set-theoretic inclusion. In particular, every $Y \subseteq \mathrm{Over}(K|D)$ with a minimum element is compact, and, if Z is an arbitrary subset of $\mathrm{Over}(K|D)$, then $Z^{\mathrm{gen}} = \{T \in \mathrm{Over}(K|D) \mid T \supseteq A \text{ for some } A \in Z\}$.

We denote by $\mathrm{Zar}_{\min}(D)$ the set of minimal elements of $\mathrm{Zar}(D)$; since $\mathrm{Zar}(D)$ is a spectral space, every $V \in \mathrm{Zar}(D)$ contains an element $W \in \mathrm{Zar}_{\min}(D)$.

If K is the quotient field of D, then we set $\mathrm{Over}(K|D) =: \mathrm{Over}(D)$ and $\mathrm{Zar}(K|D) =: \mathrm{Zar}(D)$. Elements of $\mathrm{Over}(D)$ are called *overrings* of D, elements of $\mathrm{Zar}(D)$ are the *valuation overrings* of D and elements of $\mathrm{Zar}_{\min}(D)$ are the *minimal valuation overrings* of D.

The *center map* is the application

$$\begin{aligned}\gamma\colon \operatorname{Zar}(K|D) &\longrightarrow \operatorname{Spec}(D),\\ V &\longmapsto \mathfrak{m}_V\cap D,\end{aligned}$$

where $\mathfrak{m}_V$ is the maximal ideal of V. When $\operatorname{Zar}(K|D)$ and $\operatorname{Spec}(D)$ are endowed with the respective Zariski topologies, the map γ is continuous ([33, Chapter VI, §17, Lemma 1] or [4, Lemma 2.1]), surjective (this follows, for example, from [1, Theorem 5.21] or [15, Theorem 19.6]) and closed [4, Theorem 2.5].

2.4. Semistar operations. Let D be a domain with quotient field K. Let $\mathbf{F}(D)$ be the set of D-submodules of K, $\mathcal{F}(D)$ be the set of fractional ideals of D, and $\mathcal{F}_f(D)$ be the set of finitely generated fractional ideals of D.

A *semistar operation* on D is a map $\star\colon \mathbf{F}(D)\longrightarrow\mathbf{F}(D)$, $I\mapsto I^\star$, such that, for every $I,J\in\mathbf{F}(D)$ and every $x\in K$,

$I\subseteq I^\star$;
if $I\subseteq J$, then $I^\star\subseteq J^\star$;
$(I^\star)^\star=I^\star$;
$x\cdot I^\star=(xI)^\star$.

Given a semistar operation $\star$, the map $\star_f$ is defined on every $E\in\mathbf{F}(D)$ by

$$E^{\star_f}=\bigcup\{F^\star\mid F\in\mathcal{F}_f(D), F\subseteq E\}.$$

The map $\star_f$ is always a semistar operation; if $\star=\star_f$, then $\star$ is said to be *of finite type*. Two semistar operations of finite type $\star_1,\star_2$ are equal if and only if $I^{\star_1}=I^{\star_2}$ for every $I\in\mathcal{F}_f(D)$. See [25] for general information about semistar operations.

If $\Delta\subseteq\operatorname{Zar}(D)$, then $\wedge_\Delta$ is defined as the semistar operation on D such that

$$I^{\wedge_\Delta}:=\bigcap\{IV\mid V\in\Delta\}$$

for every D-submodule I of K; a semistar operation of type $\wedge_\Delta$ is said to be a *valuative semistar operation*. By [11, Proposition 4.5], $\wedge_\Delta$ is of finite type if and only if Δ is compact (in the Zariski topology of $\operatorname{Zar}(D)$). If $\Delta,\Lambda\subseteq\operatorname{Zar}(D)$, then $\wedge_\Delta=\wedge_\Lambda$ if and only if $\Delta^{\mathrm{gen}}=\Lambda^{\mathrm{gen}}$ [10, Lemma 5.8(1)], while $(\wedge_\Delta)_f=(\wedge_\Lambda)_f$ if and only if Δ and Λ have the same closure with respect to the inverse topology [8, Theorem 4.9]. The semistar operation $\wedge_{\operatorname{Zar}(D)}$ is usually denoted by b and called the *b-operation*.

3. The use of minimal valuation domains

The starting point of this paper is the following well-known result.

Proposition 3.1 (see, e.g., [19, Proposition 6.8.2]). *Let I be an ideal of an integral domain D; let $x \in D$. Then, $x \in IV$ for every $V \in \mathrm{Zar}(D)$ if and only if there are $n \geq 1$ and $a_1, \dots, a_n \in D$ such that $a_i \in I^i$ and*

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0. \tag{1}$$

An inspection of the proof of the previous proposition given in [19] shows that this result does not really rely on the fact that I is an ideal of D, or on the fact that $x \in D$; indeed, it applies to every D-submodule I of the quotient field K, and to every $x \in K$. In the terminology of semistar operations, this means that, for each $I \in \mathbf{F}(D)$, $I^b = I^{\wedge_{\mathrm{Zar}(D)}}$ is exactly the set of elements $x \in K$ that verify an equation like (1), with $a_i \in I^i$. We are interested in generalizing that proof in a different way; we need the following definitions.

Definition 3.2. Let D be an integral domain and let $\Delta, \Lambda \subseteq \mathrm{Over}(D)$. We say that Λ *dominates* Δ if, for every $T \in \Delta$ and every $M \in \mathrm{Max}(T)$, there is a $A \in \Lambda$ such that $T \subseteq A$ and $1 \notin MA$.

For example, $\mathrm{Zar}(D)$ dominates every subset of $\mathrm{Over}(D)$, while the set of localizations of D dominates $\{D\}$.

Definition 3.3. Let D be an integral domain domain. We denote by $D[\mathcal{F}_f]$ the set of finitely generated D-algebras of $\mathrm{Over}(D)$, or equivalently

$$D[\mathcal{F}_f] := \big\{D[I] : I \in \mathcal{F}_f(R)\big\}.$$

Even if the proof of the following result essentially repeats the proof of [19, Proposition 6.8.2], we replay it here for clarity.

Proposition 3.4. *Let D be an integral domain, and suppose that $\Delta \subseteq \mathrm{Zar}(D)$ dominates $D[\mathcal{F}_f]$. Then, for every finitely generated ideal I of D, $I^{\wedge_\Delta} = I^b$.*

Proof. Clearly, $I^b \subseteq I^{\wedge_\Delta}$. Suppose thus that $x \in I^{\wedge_\Delta}$, $x \neq 0$, and let $I = (i_1, \dots, i_k)D$. Define $J := x^{-1}I \in \mathcal{F}_f(D)$, and let $A := D[J] = D[x^{-1}i_1, \dots, x^{-1}i_k]$; by definition, $J \subseteq A$.

If $JA \neq A$, then there is a maximal ideal M of A containing J, and thus, by domination, there is a valuation domain $V \in \Delta$ containing A whose maximal ideal $\mathfrak{m}_V$ is such that $JV \subseteq \mathfrak{m}_V$, and thus $IV \subseteq x\mathfrak{m}_V$. However, $x \in I^b \subseteq IV$, which implies $x \in x\mathfrak{m}_V$, a contradiction.

Hence, $JA = A$, i.e., $1 = j_1 a_1 + \cdots + j_n a_n$ for some $j_t \in J$, $a_t \in A$; writing explicitly the elements of A as elements of $D[J]$ and using $J = x^{-1}I$, we find that there must be an $N \in \mathbb{N}$ and elements $i_t \in I^t$ such that $x^N = i_1 x^{N-1} + \cdots + i_{N-1}x + i_N$, which gives an equation of integral dependence of x over I. Therefore, $x \in I^b$, as requested. □

We can now use the properties of valuative semistar operations to study compactness.

PROPOSITION 3.5. *Let D be an integral domain, and let $\Delta \subseteq \mathrm{Zar}(D)$ be a set that dominates $D[\mathcal{F}_f]$. Then, Δ is compact if and only if it contains* $\mathrm{Zar}_{\min}(D)$.

Proof. If Δ contains $\mathrm{Zar}_{\min}(D)$, then $\mathcal{U}$ is an open cover of Δ if and only if it is an open cover of $\mathrm{Zar}(D)$; thus, Δ is compact since $\mathrm{Zar}(D)$ is.

Conversely, suppose Δ is compact. By Proposition 3.4, $I^{\wedge_\Delta} = I^b$ for every finitely generated ideal I; hence, $(\wedge_\Delta)_f = b_f = b$. By [10, Lemma 5.8(1)], it follows that the closure of Δ with respect to the inverse topology of $\mathrm{Zar}(D)$ is the whole $\mathrm{Zar}(D)$; however, since Δ is compact, its closure in the inverse topology is exactly $\Delta^{\mathrm{gen}} = \Delta^{\uparrow} = \{W \in \mathrm{Zar}(D) \mid W \supseteq V \text{ for some } V \in \Delta\}$. Hence, Δ must contain $\mathrm{Zar}_{\min}(D)$. □

Thus, to find a subset of $\mathrm{Zar}(D)$ that is not compact, it is enough to find a Δ that dominates $D[\mathcal{F}_f]$ but that does not contain $\mathrm{Zar}_{\min}(D)$. The easiest case where this criterion can be applied is when $\Delta = \mathrm{Zar}(D) \setminus \{V\}$ for some $V \in \mathrm{Zar}_{\min}(D)$.

THEOREM 3.6. *Let D be an integral domain and let $V \in \mathrm{Zar}_{\min}(D)$. If $\mathrm{Zar}(D) \setminus \{V\}$ is compact, then V is the integral closure of $D[x_1, \dots, x_n]_M$ for some $x_1, \dots, x_n \in K$ and some $M \in \mathrm{Max}(D[x_1, \dots, x_n])$.*

Proof. If $\Delta := \mathrm{Zar}(D) \setminus \{V\}$ is compact, then by Proposition 3.5 it cannot dominate $D[\mathcal{F}_f]$. Hence, there is a finitely generated fractional ideal I such that Δ does not dominate $A := D[I]$, and so a maximal ideal M of A such that $1 \in MW$ for every $W \in \Delta$. In particular, $A \neq K$ (otherwise M would be (0)).

However, there must be a valuation ring containing A_M whose center (on A_M) is MA_M, and the unique possibility for this valuation ring is V: it follows that V is the unique valuation ring centered on MA_M. However, the integral closure of A_M is the intersection of the valuation rings with center MA_M (since every valuation ring containing A_M contains a valuation ring centered on MA_M [15, Corollary 19.7]); thus, V is the integral closure of A_M. □

4. The dimension of V

Before embarking on using Theorem 3.6, we prove a simple yet general result.

PROPOSITION 4.1. *Let D be an integral domain. If $\mathrm{Zar}(D)$ is a Noetherian space, so is* $\mathrm{Spec}(D)$.

Proof. The claim follows from the fact that $\mathrm{Spec}(D)$ is the continuous image of $\mathrm{Zar}(D)$ through the center map γ, and that the image of a Noetherian space is still Noetherian. □

Note that the converse of this proposition is far from being true (this is, for example, a consequence of Proposition 5.4 or of Proposition 7.1).

The problem in using Theorem 3.6 is that it is usually difficult to control the behaviour of finitely generated algebras over D. We can, however, control the behaviour of the prime spectrum of D.

LEMMA 4.2. *Let D be an integral domain, and let $V \in \mathrm{Zar}(D)$ be the integral closure of D_M, for some $M \in \mathrm{Spec}(D)$. Then, the set of prime ideals of D contained in M is linearly ordered.*

Proof. Let P, Q be two prime ideals of D contained in M; then, $PD_M, QD_M \in \mathrm{Spec}(D_M)$. Since $D_M \subseteq V$ is an integral extension, $PD_M = P' \cap D_M$ and $QD_M = Q' \cap D_M$ for some $P', Q' \in \mathrm{Spec}(V)$; however, V is a valuation domain, and thus (without loss of generality) $P' \subseteq Q'$. Hence, $PD_M \subseteq QD_M$ and $P \subseteq Q$, as requested. □

PROPOSITION 4.3. *Let D be an integral domain, let $V \in \mathrm{Zar}_{\min}(D)$ and suppose that $\mathrm{Zar}(D) \setminus \{V\}$ is compact. Let $\iota_V : \mathrm{Spec}(V) \longrightarrow \mathrm{Spec}(D)$ be the canonical spectral map associated to the inclusion $D \hookrightarrow V$. For every $P \in \mathrm{Spec}(D)$, $|\iota_V^{-1}(P)| \leq 2$; in particular, $\dim(V) \leq 2\dim(D)$.*

Proof. Suppose $|\iota_V^{-1}(P)| > 2$: then, there are prime ideals $Q_1 \subsetneq Q_2 \subsetneq Q_3$ of V such that $\iota_V(Q_1) = \iota_V(Q_2) = \iota_V(Q_3) =: P$. If $\mathrm{Zar}(D) \setminus \{V\}$ is compact, by Theorem 3.6 there is a finitely generated D-algebra $A := D[a_1, \dots, a_n]$ such that V is the integral closure of A_M, for some maximal ideal M of A. We can write A_M as a quotient $\frac{D[X_1,\dots,X_n]_{\mathfrak{a}}}{\mathfrak{b}}$, where $X_1, \dots, X_n$ are independent indeterminates and $\mathfrak{a}, \mathfrak{b} \in \mathrm{Spec}(D[X_1, \dots, X_n])$. Since $A_M \subseteq V$ is an integral extension, $Q_i \cap A \neq Q_j \cap A$ if $i \neq j$.

For $i \in \{1, 2, 3\}$, let $\mathfrak{q}_i$ be the prime ideal of $D[X_1, \dots, X_n]$ whose image in A is Q_i; then, $\mathfrak{q}_1$, $\mathfrak{q}_2$ and $\mathfrak{q}_3$ are distinct, $\mathfrak{q}_i \cap D = P$ for each i, and the set of ideals between $\mathfrak{q}_1$ and $\mathfrak{q}_3$ is linearly ordered (by Lemma 4.2). However, the prime ideals of $D[X_1, \dots, X_n]$ contracting to P are in a bijective and order-preserving correspondence with the prime ideals of $F[X_1, \dots, X_n]$, where F is the quotient field of D/P; since $F[X_1, \dots, X_n]$ is a Noetherian ring, there are an infinite number of prime ideals between the ideals corresponding to $\mathfrak{q}_1$ and $\mathfrak{q}_3$. This is a contradiction, and $|\iota_V^{-1}(P)| \leq 2$.

For the "in particular" statement, take a chain $(0) \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_k$ in $\mathrm{Spec}(V)$. Then, the corresponding chain of the $P_i := Q_i \cap D$ has length at most $\dim(D)$, and moreover $\iota^{-1}((0)) = \{(0)\}$. Hence, $k + 1 \leq 2\dim(D) + 1$ and $\dim(V) \leq 2\dim(D)$. □

The *valuative dimension* of D, indicated by $\dim_v(D)$, is defined as the supremum of the dimensions of the valuation overrings of D; we have always $\dim(D) \leq \dim_v(D)$, and $\dim_v(D)$ can be arbitrarily large with respect to $\dim(D)$ [15, Section 30, Exercises 16 and 17]. In particular, with the notation

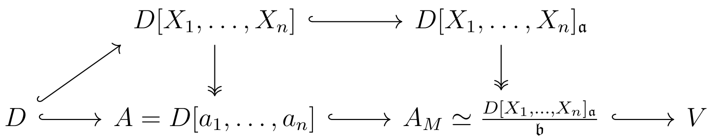

FIGURE 1. Rings involved in the proof of Proposition 4.3.

of the previous proposition, the cardinality of $\iota_V^{-1}(P)$ can be arbitrarily large: for example, if $(D,\mathfrak{m})$ is local and one-dimensional, then $|\iota_V^{-1}(\mathfrak{m})| = \dim_v(D)$.

COROLLARY 4.4. *Let D be an integral domain such that* $\mathrm{Zar}(D)$ *is Noetherian. Then,* $\dim_v(D) \leq 2\dim(D)$.

Proof. If $\mathrm{Zar}(D)$ is Noetherian, then in particular $\mathrm{Zar}(D)\setminus\{V\}$ is compact for every $V \in \mathrm{Zar}_{\min}(D)$. Hence, $\dim(V) \leq 2\dim(D)$ for every $V \in \mathrm{Zar}_{\min}(D)$, by Proposition 4.3; since, if $W \supseteq V$ are valuation domain, $\dim(W) \leq \dim(V)$, the claim follows. □

PROPOSITION 4.5. *Let D be an integral domain, and let $V \in \mathrm{Zar}_{\min}(D)$ be such that $\mathrm{Zar}(D)\setminus\{V\}$ is compact; let $(0) \subsetneq P_1 \subsetneq \cdots \subsetneq P_k$ be the chain of prime ideals of V and let $Q_i := P_i \cap D$. Denote by $\mathrm{ht}(P)$ the height of the prime ideal P. Then:*

(a) *for every $0 \leq t \leq \dim(D)$, we have*

$$\dim(V) \leq \dim_v(D_{Q_t}) + 2\big(\dim(D) - \mathrm{ht}(Q_t)\big);$$

(b) *if D_{Q_t} is a valuation domain, then*

$$\dim(V) \leq 2\dim(D) - \mathrm{ht}(Q_t).$$

Proof. (a) Let $(0) \subsetneq Q^{(1)} \subsetneq Q^{(2)} \subsetneq \cdots \subsetneq Q^{(s)}$ be the chain $(0) \subseteq Q_1 \subseteq \cdots \subseteq Q_k$ without the repetitions, and let a be the index such that $Q^{(a)} = Q_t$. For every $b > a$, by the proof of Proposition 4.3 there can be at most two prime ideals of V over $Q^{(b)}$; on the other hand, V_{P_t} is a valuation overring of D_{Q_t}, and thus $t = \dim(V_{P_t}) \leq \dim_v(D_{Q_t})$. Therefore,

$$\dim(V) \leq t + 2(s-a) \leq \dim_v(D_{Q_t}) + 2\big(\dim(D) - \mathrm{ht}(Q_t)\big)$$

since each ascending chain of prime ideals starting from Q_t has length at most $\dim(D) - \mathrm{ht}(Q_t)$.

Point (b) follows, since $\dim(V) = \dim_v(V)$ for every valuation domain V. □

EXAMPLE 4.6. A class of integral domain whose Zariski space is Noetherian is constituted by the class of Prüfer domains with Noetherian spectrum. Indeed, if D is a Prüfer domain then the valuation overrings of D are exactly the localizations of D at prime ideals; thus, the center map γ establishes a homeomorphism between $\mathrm{Zar}(D)$ and $\mathrm{Spec}(D)$. Thus, if the latter is Noetherian also the former is Noetherian.

In this case, $\dim(D)=\dim_v(D)$.

EXAMPLE 4.7. It is also possible to construct domains whose Zariski space is Noetherian but with $\dim(D)\neq\dim_v(D)$. For example, let L be a field, and consider the ring $A:=L+YL(X)[[Y]]$, where X and Y are independent indeterminates. Then, the valuation overrings of A different from $F:=L(X)((Y))$ are the rings in the form $V+YL(X)[[Y]]$, as V ranges among the valuation rings containing L and having quotient field $L(X)$; that is, $\mathrm{Zar}(A)\setminus\{F\}\simeq\mathrm{Zar}(L(X)|L)$. By the following Corollary 5.5, $\mathrm{Zar}(A)$ is a Noetherian space.

From this, we can construct analogous examples of arbitrarily large dimension. Indeed, if R is an integral domain with quotient field K, and $T:=R+XK[[X]]$, then as above $\mathrm{Zar}(T)$ is composed by $K((X))$ and by rings of the form $V+XK[[X]]$, as V ranges in $\mathrm{Zar}(R)$; in particular, $\mathrm{Zar}(T)=\{K((X))\}\cup\mathcal{X}$, where $\mathcal{X}\simeq\mathrm{Zar}(R)$. Thus, $\mathrm{Zar}(T)$ is Noetherian if $\mathrm{Zar}(R)$ is. Moreover, $\dim(T)=\dim(R)+1$ and $\dim_v(T)=\dim_v(R)+1$.

Consider now the sequence of rings $R_1:=L+YL(X)[[Y]]$, $R_2:=R_1+Y_2Q(R_1)[[Y_2]],\ldots,R_n:=R_{n-1}+Y_nQ(R_{n-1})[[Y_n]]$, where $Q(R)$ indicates the quotient field of R and each Y_i is an indeterminate over $Q(R_{i-1})((Y_{i-1}))$. Recursively, we see that each $\mathrm{Zar}(R_n)$ is Noetherian, while $\dim(R_n)=n\neq n+1=\dim_v(R_n)$.

5. Intersections of prime ideals

The results of the previous sections, while very general, are often difficult to apply, because it is usually not easy to determine the valuative dimension of a domain D. More applicable criteria, based on the prime spectrum of D, are the ones that we will prove next.

THEOREM 5.1. *Let D be a local integral domain, and suppose there is a set $\Delta\subseteq\mathrm{Spec}(D)$ and a prime ideal Q such that:*

$Q\notin\Delta$;
no two members of Δ are comparable;
$\bigcap\{P\mid P\in\Delta\}=Q$;
D_Q is a valuation domain.

Then, for any minimal valuation overring V of D contained in D_Q, $\mathrm{Zar}(D)\setminus\{V\}$ is not compact; in particular, $\mathrm{Zar}(D)$ is not Noetherian.

Proof. Note first that, since V is a minimal valuation overring, its center M on D must be the maximal ideal of D [15, Corollary 19.7]. Suppose that

$\mathrm{Zar}(D) \setminus \{V\}$ is compact: by Theorem 3.6, there is a finitely generated D-algebra $A := D[x_1, \dots, x_n]$ such that V is the integral closure of A_M for some $M \in \mathrm{Max}(A)$.

Let $I := x_1^{-1}D \cap \cdots \cap x_n^{-1}D \cap D = (D :_D x_1) \cap \cdots \cap (D :_D x_n)$. If $I \subseteq Q$, then $(D :_D x) \subseteq Q$ for some $x_i := x$; then, since D_Q is flat over D,

$$(D_Q :_{D_Q} x) = (D :_D x)D_Q \subseteq QD_Q,$$

and in particular $x \notin D_Q$. However, $V \subseteq D_Q$, and thus $x \notin V$, a contradiction. Hence, we must have $I \nsubseteq Q$.

In this case, there must be a prime ideal $P_1 \in \Delta$ not containing I. Moreover, $I \cap P_1 \nsubseteq Q$ too, and thus there is another prime $P_2 \in \Delta$, $P_1 \neq P_2$, not containing I. By Lemma 4.2, the prime ideals of A inside M are linearly ordered; in particular, we can suppose without loss of generality that $\mathrm{rad}(P_2A) \subseteq \mathrm{rad}(P_1A)$.

Let now $t \in P_2 \setminus P_1$; then, $t \in \mathrm{rad}(P_1A)$, and thus there are $p_1, \dots, p_k \in P_1$, $a_1, \dots, a_n \in A$ such that $t^e = p_1a_1 + \cdots + p_ka_k$ for some positive integer e. For each i, $a_i = B_i(x_1, \dots, x_n)$, where B_i is a polynomial over D of total degree d_i; let $d := \sup\{d_1, \dots, d_k\}$, and take an $r \in I \setminus P_1$ (recall that $I \nsubseteq P_1$). Then, $r^dB_i(x_1, \dots, x_n) \in D$ for each i; therefore,

$$r^dt^e = p_1r^da_1 + \cdots + p_kr^da_k \in p_1D + \cdots + p_kD \subseteq P_1.$$

However, by construction, both r and t are out of P_1; since P_1 is prime, this is impossible. Hence, $\mathrm{Zar}(D) \setminus \{V\}$ is not compact, and $\mathrm{Zar}(D)$ is not Noetherian. □

The first corollaries of this result can be obtained simply by putting $Q = (0)$. Recall that a *G-domain* (or *Goldman domain*) is an integral domain such that the intersection of all nonzero prime ideals is nonzero. They were introduced by Kaplansky for giving a new proof of Hilbert's Nullstellensatz (see for example [21, Section 1.3]).

Corollary 5.2. *Let D be a local domain of finite dimension, and suppose that D is not a G-domain. Then,* $\mathrm{Zar}(D) \setminus \{V\}$ *is not compact for every* $V \in \mathrm{Zar}_{\min}(D)$.

Proof. Since D is finite-dimensional, every prime ideal of D contains a prime ideal of height 1; since D is not a G-domain, it follows that the intersection of the set $\mathrm{Spec}^1(D)$ of the height-1 prime ideals of D is (0). The localization $D_{(0)}$ is the quotient field of D, and thus a valuation domain; therefore, we can apply Theorem 5.1 to $\Delta := \mathrm{Spec}^1(D)$. □

Corollary 5.3. *Let D be a local domain. If D has infinitely many height-1 primes, then* $\mathrm{Zar}(D)$ *is not Noetherian.*

Proof. Let I be the intersection of all height-1 prime ideals. If $I \neq (0)$, every height-one prime of D would be minimal over I; since there is an infinite number of them, $\mathrm{Spec}(D)$ would not be Noetherian, and by Proposition 4.1 neither $\mathrm{Zar}(D)$ would be Noetherian. Hence, $I = (0)$. But then we can apply Theorem 5.1 (for $Q = I$). □

Note that the hypothesis that D is local is needed in Theorem 5.1 and in Corollary 5.3: for example, $\mathbb{Z}$ has infinitely many height-1 primes, and $\bigcap\{P \mid P \in \mathrm{Spec}^1(D)\} = (0)$, but $\mathrm{Zar}(\mathbb{Z}) \simeq \mathrm{Spec}(\mathbb{Z})$ is a Noetherian space.

PROPOSITION 5.4. *Let D be an integral domain. If D is not a field, then $\mathrm{Zar}(D[X])$ is a not a Noetherian space.*

Proof. Since D is not a field, there exist a nonzero prime ideal P of D. For any $a \in P$, let $\mathfrak{p}_a$ be the ideal of $D[X]$ generated by $X - a$; then, each $\mathfrak{p}_a$ is a prime ideal of height 1, $\mathfrak{p}_a \neq \mathfrak{p}_b$ if $a \neq b$, and $\bigcap\{\mathfrak{p}_a \mid a \in P\} = (0)$.

The prime ideal $\mathfrak{m} := PD[X] + XD[X]$ contains every $\mathfrak{p}_a$; by Corollary 5.3, $\mathrm{Zar}(D[X]_{\mathfrak{m}})$ is not Noetherian. Therefore, neither $\mathrm{Zar}(D[X])$ is Noetherian. □

COROLLARY 5.5. *Let $F \subseteq L$ be a transcendental field extension.*

(a) *If $\mathrm{trdeg}_F(L) = 1$ and L is finitely generated over F then $\mathrm{Zar}(L|F)$ is Noetherian.*
(b) *If $\mathrm{trdeg}_F(L) > 1$ then $\mathrm{Zar}(L|F)$ is not Noetherian.*

Proof. (a) Let $L = F(\alpha_1, \dots, \alpha_n)$; without loss of generality we can suppose that α_1 is transcendental over F. Then, the extension $F(\alpha_1) \subseteq L$ is algebraic and finitely generated, and thus finite.

Each $V \in \mathrm{Zar}(L|F)$ must contain either α_1 or α_1^{-1}; therefore, $\mathrm{Zar}(L|F) = \mathrm{Zar}(L|F[\alpha_1]) \cup \mathrm{Zar}(L|F[\alpha_1^{-1}])$. However, $\mathrm{Zar}(L|A) = \mathrm{Zar}(A')$ for every domain A, where we denote by A' is the integral closure of A in L; since $F[\alpha_1]$ (respectively, $F[\alpha_1^{-1}]$) is a principal ideal domain and $F(\alpha_1) \subseteq L$ is finite, the integral closure of $F[\alpha_1]$ (resp., $F[\alpha_1^{-1}]$) is a Dedekind domain, and thus $\mathrm{Zar}(L|F[\alpha_1]) = \mathrm{Zar}(F[\alpha_1]') \simeq \mathrm{Spec}(F[\alpha_1]')$ is Noetherian. Being the union of two Noetherian spaces, $\mathrm{Zar}(L|F)$ is itself Noetherian.

(b) Suppose $\mathrm{trdeg}_F(L) > 1$. Then, there are $X, Y \in L$ such that $\{X, Y\}$ is an algebraically independent set over F; in particular, we have a continuous surjective map $\mathrm{Zar}(L|F) \longrightarrow \mathrm{Zar}(F(X,Y)|F)$ given by $V \mapsto V \cap F(X,Y)$. However, $\mathrm{Zar}(F(X,Y)|F)$ contains $\mathrm{Zar}(F[X,Y])$; by Proposition 5.4, the latter is not Noetherian, since $F[X,Y]$ is the polynomial ring over $F[X]$, a domain of dimension 1. Thus, $\mathrm{Zar}(L|F)$ is not Noetherian. □

The condition that $\bigcap\{P \mid P \in \Delta\} = Q$ of Theorem 5.1 can be slightly generalized, requiring only that the intersection is contained in Q. However, doing so we can only prove that $\mathrm{Zar}(D)$ is not Noetherian, without always finding a specific V such that $\mathrm{Zar}(D) \setminus \{V\}$ is not compact.

Proposition 5.6. *Let D be a local integral domain, and suppose there is a set $\Delta\subseteq\mathrm{Spec}(D)$ and a prime ideal Q such that*:

$Q\notin\Delta$;
no two members of Δ *are comparable*;
$\bigcap\{P\mid P\in\Delta\}\subseteq Q$;
D_Q *is a valuation domain.*

Then, $\mathrm{Zar}(D)$ *is not Noetherian.*

Proof. If $\mathrm{Spec}(D)$ is not Noetherian, by Proposition 4.1 neither is $\mathrm{Zar}(D)$; suppose that $\mathrm{Spec}(D)$ is Noetherian.

Let $I:=\bigcap\{P\mid P\in\Delta\}$; since an overring of a valuation domain is still a valuation domain, we can suppose that Q is a minimal prime of I. Since D has Noetherian spectrum, the radical ideal I has only a finite number of minimal primes, say $Q=:Q_1,Q_2,\ldots,Q_n$; let $\Delta_i:=\{\mathfrak{p}\in\Delta\mid Q_i\subseteq\mathfrak{p}\}$ and $I_i:=\bigcap\{\mathfrak{p}\mid\mathfrak{p}\in\Delta_i\}$. By standard properties of minimal primes, $\Delta=\Delta_1\cup\cdots\cup\Delta_n$ and $I=I_1\cap\cdots\cap I_n$.

In particular, $I_1\cap\cdots\cap I_n\subseteq Q$; hence, $I_k\subseteq Q$ for some k. However, $Q_k\subseteq I_k$, and thus $Q_k\subseteq Q$; since different minimal primes of the same ideal are not comparable, $k=1$ and $Q\subseteq I_1\subseteq Q$, i.e., $I_1=Q$. Then, Δ_1 is a family of primes satisfying the hypothesis of Theorem 5.1; in particular, $\mathrm{Zar}(D)$ is not Noetherian. □

An *essential prime* of a domain D is a $P\in\mathrm{Spec}(D)$ such that D_P is a valuation domain. D is an *essential domain* if it is equal to the intersection of the localizations of D at the essential primes. If, moreover, the family of the essential primes is compact, then D can be called a *Prüfer v-multiplication domain* (*PvMD* for short) [12, Corollary 2.7]; note that the original definition of PvMDs was given through star operations (more precisely, D is a PvMD if and only if D_P is a valuation ring for every t-maximal ideal P [16], [20]).

Proposition 5.7. *Let D be an essential domain. Then*, $\mathrm{Zar}(D)$ *is Noetherian if and only if D is a Prüfer domain with Noetherian spectrum.*

Proof. If D is a Prüfer domain with Noetherian spectrum, then $\mathrm{Zar}(D)\simeq\mathrm{Spec}(D)$ is Noetherian (see Example 4.6). Conversely, suppose $\mathrm{Zar}(D)$ is Noetherian: by Proposition 4.1, $\mathrm{Spec}(D)$ is Noetherian. Let $\mathcal{E}$ be the set of essential prime ideals of D: since $\mathrm{Spec}(D)$ is Noetherian, $\mathcal{E}$ is compact, and thus D is a PvMD.

Suppose by contradiction that D is not a Prüfer domain. Then, there is a maximal ideal M of D such that D_M is not a valuation domain; since the localization of a PvMD is a PvMD [20, Theorem 3.11], and $\mathrm{Zar}(D_M)$ is a subspace of $\mathrm{Zar}(D)$, without loss of generality we can suppose $D=D_M$, that is, we can suppose that D is local.

Since $\mathcal{E}$ is compact, every $P\in\mathcal{E}$ is contained in a maximal element of $\mathcal{E}$; let Δ be the set of such maximal elements. Clearly, $D=\bigcap\{D_P\mid P\in\Delta\}$. If

Δ were finite, D would be an intersection of finitely many valuation domains, and thus it would be a Prüfer domain [15, Theorem 22.8]; hence, we can suppose that Δ is infinite. Let $I := \bigcap\{P \mid P \in \Delta\}$.

Each $P \in \Delta$ contains a minimal prime of I; however, since $\mathrm{Spec}(D)$ is Noetherian, I has only finitely many minimal primes. It follows that there is a minimal prime Q of I that is not contained in Δ; in particular, $\bigcap\{P \mid P \in \Delta\} \subseteq Q$, and thus we can apply Proposition 5.6. Hence, $\mathrm{Zar}(D)$ is not Noetherian, which is a contradiction. □

REMARK 5.8. The previous proof can be interpreted using the terminology of the theory of star operations. Indeed, any essential prime P is a t-ideal, i.e., $P = P^t$, where (for any ideal J of D) $J^t := \bigcup\{(D : (D : I)) \mid I \subseteq J \text{ is finitely generated}\}$ [20, Lemma 3.17] and if D is a PvMD then the set Δ of the maximal elements of $\mathcal{E}$ is exactly the set of *t-maximal ideals*, that is, the set of the ideals I such that $I = I^t$ and $J \neq J^t$ for every proper ideal $I \subsetneq J$.

COROLLARY 5.9. *Let D be a Krull domain. Then, $\mathrm{Zar}(D)$ is Noetherian if and only if $\dim(D) = 1$, that is, if and only if D is a Dedekind domain.*

Proof. If $\dim(D) = 1$, then D is Noetherian and so is $\mathrm{Zar}(D)$. If $\dim(D) > 1$, then D is not a Prüfer domain; since each Krull domain is a PvMD, we can apply Proposition 5.7. □

Note that this corollary can also be proved directly from Corollary 5.3 since, if D is Krull, and $P \in \mathrm{Spec}(D)$ has height 2 or more, then D_P has infinitely many height-1 primes.

6. An application: Kronecker function rings

Let D be an integrally closed integral domain with quotient field K. For every $V \in \mathrm{Zar}(D)$, let $V(X) := V[X]_{\mathfrak{m}_V[X]} \subseteq K(X)$, where $\mathfrak{m}_V$ is the maximal ideal of V. If $\Delta \subseteq \mathrm{Zar}(D)$, the *Kronecker function ring* of D with respect to Δ is

$$\mathrm{Kr}(D,\Delta) := \bigcap\{V(X) \mid V \in \Delta\};$$

equivalently,

$$\mathrm{Kr}(D,\Delta) = \{f/g \mid f,g \in D[X], g \neq 0, \mathbf{c}(f) \subseteq (\mathbf{c}(g))^{\wedge_\Delta}\},$$

where $\mathbf{c}(f)$ is the content of f and $\wedge_\Delta$ is the semistar operation defined in Section 2.4. See [15, Chapter 32] or [13] for general properties of Kronecker function rings.

The set of Kronecker function rings it exactly the set of overrings of the basic Kronecker function ring $\mathrm{Kr}(D, \mathrm{Zar}(D))$; this set is in bijective correspondence with the set of finite-type valuative semistar operations [15, Remark 32.9], or equivalently with the set of nonempty subsets of $\mathrm{Zar}(D)$ that are closed in the inverse topology [8, Theorem 4.9].

Let $\mathcal{K}(D)$ be the set of Kronecker function rings T of D such that $T \cap K = D$. Then, $\mathcal{K}(D)$ is in bijective correspondence with the set of finite-type valuative *star* operations, or equivalently with the set of inverse-closed representation of D through valuation rings, that is, the sets $\Delta \subseteq \mathrm{Zar}(D)$ that are closed in the inverse topology and such that $\bigcap\{V \mid V \in \Delta\} = D$ [27, Proposition 5.10].

It has been conjectured [23] that $\mathcal{K}(D)$ is either a singleton (in which case D is said to be a *vacant domain*; see [6]) or infinite, and this has been proved to be the case for a wide class of pseudo-valuation domains [6, Theorem 4.10]. As a consequence of the following proposition, we will prove this conjecture for another class of domains.

Proposition 6.1. *Let D be an integrally closed integral domain such that $1 < |\mathcal{K}(D)| < \infty$. Then, there is a minimal valuation overring V of D such that $\mathrm{Zar}(D) \setminus \{V\}$ is compact.*

Proof. Suppose $|\mathcal{K}(D)| > 1$. Then, there is an inverse-closed representation Δ of D different from $\mathrm{Zar}(D)$; let $\Lambda := \mathrm{Zar}(D) \setminus \Delta$. For each $W \in \Lambda$, let $\Delta(W) := \Delta \cup \{W\}^{\uparrow}$; then, every $\Delta(W)$ is an inverse-closed representation of D, and $\Delta(W) \neq \Delta(W')$ if $W \neq W'$ (since, without loss of generality, $W \not\supseteq W'$, and thus $W \notin \Delta(W')$). Hence, each $W \in \Lambda$ give rise to a different member of $\mathcal{K}(D)$; since $|\mathcal{K}(D)| < \infty$, it follows that Λ is finite.

If now V is minimal in Λ, then $\mathrm{Zar}(D) \setminus \{V\} = \Delta \cup (\Lambda \setminus \{V\})$ is closed by generalizations; since Λ is finite, it follows that $\mathrm{Zar}(D) \setminus \{V\}$ is the union of two compact subspaces, and thus it is itself compact. □

Corollary 6.2. *Let D be an integrally closed local integral domain, and suppose there exist a set $\Delta \subseteq \mathrm{Spec}(D)$ of incomparable nonzero prime ideals such that $\bigcap\{P \mid P \in \Delta\} = (0)$. Then, $|\mathcal{K}(D)| \in \{1, \infty\}$.*

Proof. By Theorem 5.1, each $\mathrm{Zar}(D) \setminus \{V\}$ is noncompact. The claim now follows from Proposition 6.1. □

7. Overrings of Noetherian domains

If D is a Noetherian domain, Theorem 3.6 admits a direct application, without using any of the results proved in Sections 4 and 5. Indeed, if D is Noetherian with quotient field K, then it is the same for any localization of $D[x_1, \dots, x_n]$, for arbitrary $x_1, \dots, x_n \in K$; thus, the integral closure of $D[x_1, \dots, x_n]_M$ is a Krull domain for each maximal ideal M of $D[x_1, \dots, x_n]$ ([24, (33.10)] or [19, Theorem 4.10.5]). Since a domain that is both Krull and a valuation ring must be a field or a discrete valuation ring, Theorem 3.6 implies that $\mathrm{Zar}(D) \setminus \{V\}$ is not compact as soon as V is a minimal valuation overring of dimension 2 or more.

We can actually say more than this; the following is a proof through Proposition 3.5 of an observation already appeared in [9, Example 3.7].

PROPOSITION 7.1. *Let D be a Noetherian domain with quotient field K, and let Δ be the set of valuation overrings of D that are Noetherian (i.e., Δ is the union of $\{K\}$ with the set of discrete valuation overrings of D). Then, Δ is compact if and only if* $\dim(D)=1$.

Proof. If $\dim(D)=1$, then $\Delta=\mathrm{Zar}(D)$, and thus it is compact.

On the other hand, for every ideal I of D, $I^{\wedge_\Delta}=I^b$ [19, Proposition 6.8.4]; however, if $\dim(D)>1$, then $\mathrm{Zar}(D)$ contains elements of dimension 2, and thus Δ cannot contain $\mathrm{Zar}_{\min}(D)$. The claim now follows from Proposition 3.5. □

REMARK 7.2.

(1) The equality $I^{\wedge_\Delta}=I^b$ holds also if we restrict Δ to be the set of discrete valuation overrings of D whose center is a maximal ideal of D [19, Proposition 6.8.4]. For each prime ideal of height 2 or more, by passing to D_P, we can thus prove that the set of discrete valuation overrings of D with center P is not compact (and in particular it is infinite).

(2) The previous proposition also allows a proof of the second part of Corollary 5.5 without using Theorem 5.1, since $F[X,Y]$ is a Noetherian domain of dimension 2.

By Proposition 7.1, in particular, the space Δ of Noetherian valuation overrings of D (where D is Noetherian and $\dim(D)\geq 2$) is not a spectral space, since it is not compact. Our next purpose is to see Δ as an intersection $X\cap\mathrm{Zar}(D)$, for some subset X of $\mathrm{Over}(D)$, and use this representation to prove facts about X. We start with using the inverse topology.

PROPOSITION 7.3. *Let D be a Noetherian domain with quotient field K, and let*:

- *X_1 be the set of all overrings of D that are Noetherian and of dimension at most* 1;
- *X_2 be the set of all overrings of D that are Dedekind domains* (*K included*).

For $i\in\{1,2\}$, the following are equivalent:

(i) *X_i is compact*;
(ii) *X_i is spectral*;
(iii) *X_i is proconstructible in* $\mathrm{Over}(D)$;
(iv) $\dim(D)=1$.

Proof. (i) $\Longrightarrow$ (iii). In both cases, $X=X^{\mathrm{gen}}$: for X_1 see [21, Theorem 93], while for X_2 see, for example, [15, Theorem 40.1] (or use the previous result and [15, Corollary 36.3]). (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) always holds.

(iv) $\Longrightarrow$ (i). If $\dim(D)=1$, then $X_1=\mathrm{Over}(D)$, while $X_2=\mathrm{Over}(D')$, where D' is the integral closure of D, and both are compact since they have a minimum.

(iii) $\Longrightarrow$ (iv). If X_i is proconstructible, so is $X_i \cap \mathrm{Zar}(D)$ (since $\mathrm{Zar}(D)$ is also proconstructible), and in particular $X_i \cap \mathrm{Zar}(D)$ is compact. However, in both cases, $X_i \cap \mathrm{Zar}(D)$ is exactly the set of Noetherian valuation overrings of D; by Proposition 7.1, $\dim(D) = 1$. ☐

REMARK 7.4. The equivalence between the first three conditions of Proposition 7.3 holds for every subset $X \subseteq \mathrm{Over}(D)$ such that $X = X^{\mathrm{gen}}$ (and every domain D). In particular, it holds if X is the set of overrings of D that are principal ideal domains, and, with the same proof of the other cases, we can show that if D is Noetherian and these conditions hold, then $\dim(D) = 1$. However, it is not clear if, when D is Noetherian and $\dim(D) = 1$, this set is actually compact.

Another immediate consequence of Proposition 7.1 is that the set $\mathrm{NoethOver}(D)$ of Noetherian overrings of D is not proconstructible as soon as D is Noetherian and $\dim(D) \geq 2$: indeed, if it were, then $\mathrm{NoethOver}(D) \cap \mathrm{Zar}(D) = \Delta$ would be proconstructible, against the fact that Δ is not compact. However, this is also a consequence of a more general result. We need a topological lemma.

LEMMA 7.5. *Let $Y \subseteq X$ be spectral spaces. Suppose that there is a subbasis $\mathcal{B}$ of X such that, for every $B \in \mathcal{B}$, both B and $B \cap Y$ are compact. Then, Y is a proconstructible subset of X.*

Proof. The hypothesis on $\mathcal{B}$ implies that the inclusion map $Y \hookrightarrow X$ is a spectral map; by [3, 1.9.5(vii)], it follows that Y is a proconstructible subset of X. ☐

PROPOSITION 7.6. *Let D be an integral domain with quotient field K, and let $D[\mathcal{F}_f]$ be the set of finitely generated D-algebras contained in K.*

(a) *$D[\mathcal{F}_f]$ is dense in* $\mathrm{Over}(D)$, *with respect to the constructible topology.*
(b) *Let X such that $D[\mathcal{F}_f] \subseteq X \subseteq \mathrm{Over}(D)$. Then, X is spectral in the Zariski topology if and only if $X = \mathrm{Over}(D)$.*

Proof. (a) A basis of the constructible topology is given by the sets of type $U \cap (X \setminus V)$, as U and V ranges in the open and compact subsets of $\mathrm{Over}(D)$. Such an U can be written as $B_1 \cup \cdots \cup B_n$, where each $B_i = B(x_1^{(i)}, \dots, x_n^{(i)})$ is a basic open set of $\mathrm{Over}(D)$; thus, we can suppose that $U = B(x_1, \dots, x_n)$. Suppose $\Omega := U \cap (X \setminus V)$ is nonempty; we claim that $A := D[x_1, \dots, x_n] \in \Omega \cap D[\mathcal{F}_f]$. Clearly $A \in D[\mathcal{F}_f]$ and $A \in U$; let $T \in \Omega$. Then, $T \in U$, and thus $A \subseteq T$; therefore, A is in the closure $\mathrm{Cl}(T)$ of T, with respect to the Zariski topology. But $X \setminus V$ is closed, and thus $\mathrm{Cl}(T) \subseteq X \setminus V$; that is, $A \in X \setminus V$. Hence, $A \in \Omega \cap D[\mathcal{F}_f]$, which in particular is nonempty, and $D[\mathcal{F}_f]$ is dense.

(b) Suppose X is spectral. For every $x_1, \dots, x_n$, the set $X \cap B(x_1, \dots, x_n)$ has a minimum (i.e., $D[x_1 \dots, x_n]$), so it is compact. Since the family of all

$B(x_1, \dots, x_n)$ is a basis, by Lemma 7.5 it follows that X is proconstructible. By the previous point, we must have $X = \mathrm{Over}(D)$. □

COROLLARY 7.7. *Let D be a Noetherian domain. The spaces*
- $\mathrm{NoethOver}(D) := \{T \in \mathrm{Over}(D) \mid T$ *is Noetherian*$\}$, *and*
- $\mathrm{KrullOver}(D) := \{T \in \mathrm{Over}(D) \mid T$ *is a Krull domain*$\}$

are spectral if and only if $\dim(D) = 1$.

Proof. If $\dim(D) = 1$, then the claim follows by Proposition 7.3.

If $\dim(D) \geq 2$, then $\mathrm{NoethOver}(D)$ is not spectral by Proposition 7.6(b) and the Hilbert Basis Theorem; the case of $\mathrm{KrullOver}(D)$ follows in the same way, since $\mathrm{KrullOver}(D) \cap B(x_1, \dots, x_n)$ has always a minimum (i.e., the integral closure of $D[x_1, \dots, x_n]$). □

More generally, consider a property $\mathcal{P}$ of Noetherian domains such that every field and every discrete valuation ring satisfies $\mathcal{P}$; for example, $\mathcal{P}$ may be the property of being regular, Gorenstein or Cohen-Macaulay. Let $X_{\mathcal{P}}(D)$ be the set of overrings of D satisfying $\mathcal{P}$; then, $X_{\mathcal{P}}(D) \cap \mathrm{Zar}(D)$ is not compact, and thus $X_{\mathcal{P}}(D)$ is not proconstructible. On the other hand, if $X_{\mathcal{P}}(T)$ is compact for every overring of D that is finitely generated as a D-algebra, then by Lemma 7.5 it follows that $X_{\mathcal{P}}(D)$ cannot be a spectral space. Thus, the assignment $D \mapsto X_{\mathcal{P}}(D)$ cannot be "too good": either some $X_{\mathcal{P}}(T)$ is not compact, or $X_{\mathcal{P}}(D)$ is not spectral.

QUESTION. Let $\mathcal{P}$ be the property of being regular, the property of being Gorenstein or the property of being Cohen-Macaulay. Is it possible to characterize for which Noetherian domains D there is a $T \in \mathrm{Over}(D)$ such that $X_{\mathcal{P}}(T)$ is not compact and for which $X_{\mathcal{P}}(D)$ is not spectral?

Acknowledgments. Section 6 was inspired by a talk given by Daniel McGregor at the congress "Recent Advances in Commutative ring and Module Theory" in Bressanone (June 13–17, 2017). I also thank the referee for his numerous suggestions, which improved the paper.

REFERENCES

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, MA–London–Don Mills, ON, 1969. MR 0242802

[2] N. Bourbaki, *Commutative algebra. Chapters 1–7*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1989. Translated from the French, Reprint of the 1972 edition. MR 0979760

[3] J. Dieudonné and A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I*, Publ. Math. Inst. Hautes Études Sci. **20** (1964), 1–259. MR 0173675

[4] D. E. Dobbs, R. Fedder and M. Fontana, *Abstract Riemann surfaces of integral domains and spectral spaces*, Ann. Mat. Pura Appl. (4) **148** (1987), 101–115. MR 0932760

[5] D. E. Dobbs and M. Fontana, *Kronecker function rings and abstract Riemann surfaces*, J. Algebra **99** (1986), no. 1, 263–274. MR 0836646
[6] A. Fabbri, *Integral domains having a unique Kronecker function ring*, J. Pure Appl. Algebra **215** (2011), no. 5, 1069–1084. MR 2747239
[7] C. A. Finocchiaro, *Spectral spaces and ultrafilters*, Comm. Algebra **42** (2014), no. 4, 1496–1508. MR 3169645
[8] C. A. Finocchiaro, M. Fontana and K. A. Loper, *The constructible topology on spaces of valuation domains*, Trans. Amer. Math. Soc. **365** (2013), no. 12, 6199–6216. MR 3105748
[9] C. A. Finocchiaro, M. Fontana and D. Spirito, *New distinguished classes of spectral spaces: A survey*, Multiplicative ideal theory and factorization theory: Commutative and non-commutative perspectives, Springer Verlag, 2016. MR 3565806
[10] C. A. Finocchiaro, M. Fontana and D. Spirito, *Spectral spaces of semistar operations*, J. Pure Appl. Algebra **220** (2016), no. 8, 2897–2913. MR 3471195
[11] C. A. Finocchiaro and D. Spirito, *Some topological considerations on semistar operations*, J. Algebra **409** (2014), 199–218. MR 3198840
[12] C. A. Finocchiaro and F. Tartarone, *On a topological characterization of Prüfer v-multiplication domains among essential domains*, J. Commut. Algebra **8** (2016), no. 4, 513–536. MR 3566528
[13] M. Fontana and K. A. Loper, *An historical overview of Kronecker function rings, Nagata rings, and related star and semistar operations*, Multiplicative ideal theory in commutative algebra, Springer, New York, 2006, pp. 169–187. MR 2265808
[14] M. Fontana and K. A. Loper, *The patch topology and the ultrafilter topology on the prime spectrum of a commutative ring*, Comm. Algebra **36** (2008), no. 8, 2917–2922. MR 2440291
[15] R. Gilmer, *Multiplicative ideal theory*, Pure and Applied Mathematics, vol. 12, Marcel Dekker Inc., New York, 1972. MR 0427289
[16] M. Griffin, *Some results on v-multiplication rings*, Canad. J. Math. **19** (1967), 710–722. MR 0215830
[17] M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. **142** (1969), 43–60. MR 0251026
[18] R. Huber and M. Knebusch, *On valuation spectra*, Recent advances in real algebraic geometry and quadratic forms (Berkeley, CA, 1990/1991; San Francisco, CA, 1991), Contemp. Math., vol. 155, Amer. Math. Soc., Providence, RI, 1994, pp. 167–206. MR 1260707
[19] C. Huneke and I. Swanson, *Integral closure of ideals, rings, and modules*, London Mathematical Society Lecture Note Series, vol. 336, Cambridge University Press, Cambridge, 2006. MR 2266432
[20] B. G. Kang, *Prüfer v-multiplication domains and the ring* $R[X]_{N_v}$, J. Algebra **123** (1989), no. 1, 151–170. MR 1000481
[21] I. Kaplansky, *Commutative rings*, revised ed., The University of Chicago Press, Chicago, IL–London, 1974. MR 0345945
[22] P. Klaas, Hart, J. Nagata and J. E. Vaughan, eds., *Encyclopedia of general topology*, Elsevier Science Publishers, B.V., Amsterdam, 2004. MR 2049453
[23] D. J. McGregor, Personal communication.
[24] M. Nagata, *Local rings*, Interscience Tracts in Pure and Applied Mathematics, vol. 13, Interscience Publishers a division of John Wiley & Sons, New York–London, 1962. MR 0155856
[25] A. Okabe and R. Matsuda, *Semistar-operations on integral domains*, Math. J. Toyama Univ. **17** (1994), 1–21. MR 1311837
[26] B. Olberding, *Noetherian spaces of integrally closed rings with an application to intersections of valuation rings*, Comm. Algebra **38** (2010), no. 9, 3318–3332. MR 2724221

[27] B. Olberding, *Affine schemes and topological closures in the Zariski–Riemann space of valuation rings*, J. Pure Appl. Algebra **219** (2015), no. 5, 1720–1741. MR 3299704

[28] B. Olberding, *Topological aspects of irredundant intersections of ideals and valuation rings*, Multiplicative ideal theory and factorization theory: Commutative and non-commutative perspectives, Springer Verlag, 2016. MR 3565813

[29] J.-P. Olivier, *Anneaux absolument plats universels et épimorphismes d'anneaux*, C. R. Acad. Sci. Paris Sér. A-B **266** (1968), A317–A318. MR 0238836

[30] J.-P. Olivier, *Anneaux absolument plats universels et épimorphismes à buts réduits*, Séminaire Samuel. Algèbre commutative, 2: Les épimorphismes d'anneaux, pp. 1967–1968.

[31] Z. Oscar, *The reduction of the singularities of an algebraic surface*, Ann. of Math. (2) **40** (1939), 639–689. MR 0000159

[32] Z. Oscar, *The compactness of the Riemann manifold of an abstract field of algebraic functions*, Bull. Amer. Math. Soc. **50** (1944), 683–691. MR 0011573

[33] Z. Oscar and P. Samuel, *Commutative algebra. Vol. II*, Graduate Texts in Mathematics, vol. 29, Springer-Verlag, New York, 1975. Reprint of the 1960 edition. MR 0120249

[34] N. Schwartz, *Compactification of varieties*, Ark. Mat. **28** (1990), no. 2, 333–370. MR 1084021

Dario Spirito, Dipartimento di Matematica e Fisica, Università degli Studi "Roma Tre", Roma, Italy

E-mail address: spirito@mat.uniroma3.it