EMBEDDING THE SET OF NON-DIVISORIAL IDEALS OF A NUMERICAL SEMIGROUP INTO $\mathbb{N}^n$

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ABSTRACT. The set $\mathcal{G}_0(S)$ of the classes of non-divisorial ideals of a numerical semigroup S can be endowed with a natural partial order induced by the set of star operations on S. We study embeddings of $\mathcal{G}_0(S)$ into $\mathbb{N}^n$, specializing on three families of numerical semigroups with radically different behaviour.

1. Introduction

The concept of star operation was born in the setting of integral domains as a way to generalize the properties of the divisorial closure [10, 6]; it admits a natural extension to numerical semigroups, allowing to define semigroups with properties similar to Krull domains [9]. In [15], the main properties of star operations on numerical semigroups were studied: in particular, it was proven that the number of star operations on a numerical semigroup is always finite and that, for $n > 1$, there is only a finite number of numerical semigroups S with exactly n star operations.

A deeper examination of the questions tackled in [15] led in [16] and [17] to the introduction of a partial order on the set $\mathcal{G}_0(S)$ of the classes of non-divisorial ideals of a numerical semigroup S: in particular, it was shown that there is a strong link between the set $\mathrm{Star}(S)$ of star operations on S and the antichains of $\mathcal{G}_0(S)$ (an *antichain* of a partially ordered set $\mathcal{P}$ is a subset of $\mathcal{P}$ composed by pairwise incomparable elements). In particular, [16] provided a full analysis of the case when the multiplicity of the numerical semigroup is 3, while [17] presented some estimates on the number of star operations on a numerical semigroup and, consequently, on the cardinality of the set of semigroups with some fixed number of star operations.

In this paper, we concentrate on the order on the set $\mathcal{G}_0(S)$. In particular, we are interested in ways to embed it into a product of chains, or, rather, in $\mathbb{N}^n$, for some (possibly small) integer n; this lead to the question of finding the dimension and the tight dimension of $\mathcal{G}_0(S)$ (see Section 3 for the definitions). While we are not able to prove

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2010 *Mathematics Subject Classification.* 06A07, 20M12, 20M14.

Key words and phrases. Numerical semigroups; Star operations; Divisorial ideals; Order dimension; Tight embedding.

general results, we analyze three different families of numerical semigroups with rather different properties: pseudosymmetric semigroups with multiplicity 4 (Section 4), semigroups of the form $\langle 4, 6, x, x+2\rangle$ (with $x \geq 9$ odd; Section 5), and pseudosymmetric semigroups with $F(S) \leq 2\mu(S) + 2$ (where $F(S)$ is the Frobenius number and $\mu(S)$ the multiplicity of S; Section 6). In all cases, we determine an embedding of $\mathcal{G}_0(S)$ into $\mathbb{N}^n$, and determine the dimension and the tight dimension of $\mathcal{G}_0(S)$; we also use these embeddings to estimate the number of star operations on members of these families. Finally, in Section 7, we give two examples of individual numerical semigroups with even different behaviour.

2. Notation and preliminaries

For further information about numerical semigroups, the interested reader may consult [14].

A *numerical semigroup* is a subset S of $\mathbb{N}$ such that:

- $a + b \in S$ for every $a, b \in S$;
- $0 \in S$;
- $\mathbb{N} \setminus S$ is finite.

The notation $S = \{0, x_1, \ldots, x_n, \rightarrow\}$ indicates that S contains the elements $0, x_1, \ldots, x_n$ and every integer bigger than x_n.

The greatest element in $\mathbb{Z} \setminus S$ is the *Frobenius number* of S, and is denoted by $F(S)$; the least element of $S \setminus \{0\}$ is the *multiplicity* of S, and is denoted by $\mu(S)$.

A *fractional ideal* (or simply an *ideal*) of S is a subset $I \subsetneq \mathbb{Z}$ such that $i + s \in I$ for every $i \in I$, $s \in S$; in particular, each fractional ideal is bounded below, and there is a unique $t \in \mathbb{Z}$ such that $I + t := \{i + t \mid i \in I\}$ has minimum 0. We denote by $\mathcal{F}(S)$ the set of fractional ideals of S, and by $\mathcal{F}_0(S)$ the set of fractional ideals I of S such that $\min I = 0$; equivalently, $\mathcal{F}_0(S)$ is the set of fractional ideals I of S such that $S \subseteq I \subseteq \mathbb{N}$. If $I \in \mathcal{F}_0(S)$ and $I \neq \mathbb{N}$, we set $\eta(I) := \max(\mathbb{N} \setminus I)$. Moreover, if $I \in \mathcal{F}_0(S)$ and $k \in I$, we define the *k-shift* of I as the fractional ideal $\rho_k(I) := (-k + I) \cap \mathbb{N} \in \mathcal{F}_0(S)$.

The set $M := S \setminus \{0\}$ is an ideal of S, called the *maximal ideal* of S.

If I is an ideal of S and x a positive integer, the *Apéry set* of I with respect to x is

$$\mathrm{Ap}(I, x) := \{i \in I \mid i - x \notin I\}.$$

In particular, $\mathrm{Ap}(I, x)$ has cardinality x.

If I and J are two fractional ideals, $(I - J) := \{t \in \mathbb{Z} \mid t + J \subseteq I\}$ is again an ideal. The set $(S - M) \setminus S$ is denoted by $T(S)$, and its cardinality, denoted by $t(S)$, is called the *type* of S.

A semigroup S is *symmetric* if $F(S) - a \in S$ for every $a \in \mathbb{N} \setminus S$, and it is *pseudosymmetric* if $F(S)$ is even and $F(S) - a \in S$ for every

$a \in \mathbb{N} \setminus S$, $a \neq F(S)/2$. A semigroup S is symmetric if and only if $t(S) = 1$ [4, Proposition 2].

A *star operation* on S [15] is a map $* : \mathcal{F}(S) \longrightarrow \mathcal{F}(S)$ such that, for every $I, J \in \mathcal{F}(S)$, $x \in \mathbb{Z}$,

- $I \subseteq I^*$ ($*$ is *extensive*);
- $(I^*)^* = I^*$ ($*$ is *idempotent*);
- if $I \subseteq J$, then $I^* \subseteq J^*$ ($*$ is *order-preserving*);
- $x + I^* = (x + I)^*$;
- $S = S^*$.

In particular, since $\mathbb{N}^* = \mathbb{N}$ for every star operation $*$ (this follows from [15, Lemma 3.3]), a star operation $*$ restricts to a map $*_0 : \mathcal{F}_0(S) \longrightarrow \mathcal{F}_0(S)$, and $*$ is completely determined by $*_0$. An ideal I such that $I = I^*$ is said to be a *$*$-closed ideal.*

The set of star operations on S is denoted by $\mathrm{Star}(S)$, and is always a finite set; moreover, if $n > 1$, there is only a finite number of numerical semigroups S such that $|\mathrm{Star}(S)| = n$ [15, Theorem 4.15]. If $*_1, *_2 \in \mathrm{Star}(S)$, we set $*_1 \leq *_2$ if $I^{*_1} \subseteq I^{*_2}$ for every $I \in \mathcal{F}(S)$, or equivalently if $I = I^{*_2}$ implies that $I = I^{*_1}$.

The maximum of $\mathrm{Star}(S)$ under this order is the *divisorial closure*, defined by $I^v := (S - (S - I))$ [15, Section 2]; if $I = I^v$ then I is said to be *divisorial.* The divisorial closure coincides with the identity (i.e., $|\mathrm{Star}(S)| = 1$) if and only if S is symmetric [1, Proposition I.1.15]. The set of non-divisorial ideals I such that $\min I = 0$ is denoted by $\mathcal{G}_0(S)$.

Every non-divisorial ideal I generates a star operation $*_I$, defined, for every $J \in \mathcal{F}(S)$, by (see [15, Proposition 3.6])

$$J^{*_I} := J^v \cap (I - (I - J)) = J^v \cap \bigcap_{\alpha \in (I-J)} (-\alpha + I).$$

Alternatively, $*_I$ is the biggest star operation $*$ such that I is $*$-closed. We have $*_I = *_{I'}$ if and only if $I = x + I'$ for some integer x. In particular, the map

$$\begin{aligned} *\colon \mathcal{G}_0(S) &\longrightarrow \mathrm{Star}(S) \\ I &\longmapsto *_I \end{aligned}$$

is injective, and it can be used to define an order on $\mathcal{G}_0(S)$. We define the *$*$-order* $\leq_*$ to be the *opposite* order with respect to the one induced by the map above: more explicitly, given $I, J \in \mathcal{G}_0(S)$, we have

$$I \leq_* J \iff *_I \geq *_J \iff I = I^{*_J}.$$

An *antichain* of a partially ordered set $(\mathcal{P}, \leq)$ is a (possibly empty) subset $X \subseteq \mathcal{P}$ such that no two different elements of X are comparable under $\leq$. We denote the cardinality of the set of antichains on $(\mathcal{P}, \leq)$ by $\omega(\mathcal{P}, \leq)$ (or simply $\omega(\mathcal{P})$ if there is no danger of confusion). The *n-th Dedekind number*, denoted by $\omega(n)$, is the number of antichains of the power set of a set with n elements, under the order given by the set-theoretic containment. Every antichain Δ of $(\mathcal{G}_0(S), \leq_*)$ induces a star operation $*_\Delta$ on S, defined by

$$J^{*_\Delta} := \bigcap_{I\in\Delta} J^{*_I} = J^v \cap \bigcap_{I\in\Delta} \bigcap_{\alpha\in(I-J)} (-\alpha + I)$$

for every $J \in \mathcal{F}(S)$; moreover, every star operation on S is in the form $*_\Delta$, for some antichain Δ of $(\mathcal{G}_0(S), \leq_*)$ [17, Section 3].

If $x \in \mathbb{N} \setminus S$, we denote by $\mathcal{Q}_x$ the set of ideals $I \in \mathcal{G}_0(S)$ such that $\eta(I) = x$ and $x \in I^v$. If also $F(S) - x \notin S$, or if $x = F(S)$, then $I \in \mathcal{Q}_x$ if and only if $\eta(I) = x$ [17, Proposition 5.2(e)]. The $*$-order on $\mathcal{Q}_x$ is coarser then the set-theoretic containment (i.e., if $I \leq_* J$, then $I \subseteq J$) [17, Proposition 5.7(c)]; if $\mathcal{Q}_x \neq \emptyset$, then the $*$-maximum of $\mathcal{Q}_x$ is the ideal [17, Proposition 5.2(b)]

$$M_x := \bigcup \{I \in \mathcal{F}_0(S) \mid x \notin I\} = \{y \in \mathbb{N} \mid x - y \notin S\}.$$

If $x < y$ and $\mathcal{Q}_x \neq \emptyset$, then also $\mathcal{Q}_y \neq \emptyset$, and $M_x <_* M_y$ [17, Proposition 5.2(c)]. When S is not symmetric, the ideal $M_{F(S)}$ generates the identity star operation [15, Corollary 4.5] (see also [8, Satz 4 and Hillsatz 5]), and thus it is the maximum of $(\mathcal{G}_0(S), \leq_*)$; it is called the *canonical ideal* of S.

An *atom* of S (or of $\mathcal{G}_0(S)$) is an $I \in \mathcal{G}_0(S)$ such that, whenever $*_I \geq *_1 \wedge *_2$ for some $*_1, *_2 \in \mathrm{Star}(S)$, then $*_I \geq *_1$ or $*_I \geq *_2$ (here $*_1 \wedge *_2$ indicates the infimum of $*_1$ and $*_2$) [17, Definitions 4.1 and 4.3]. Sufficient conditions for $I \in \mathcal{G}_0(S)$ to be an atom are that $|I^v \setminus I| = 1$ [17, Proposition 4.8] and that I is an element of $\mathcal{Q}_x$ such that $|M_x \setminus I| \leq 1$ [17, Proposition 5.3]. If every non-divisorial ideal I is an atom, then the number of star operations on S is equal to the number of antichains of $(\mathcal{G}_0(S), \leq_*)$, and conversely [17, Proposition 4.9].

We state explicitly two results which will be useful in the rest of the paper.

Lemma 2.1. *Let S be a numerical semigroup, $I, J \in \mathcal{G}_0(S)$. Let $x \in J^v \setminus J$. If $J \leq_* I$, there is a $t \in I$ such that $J \subseteq \rho_t(I)$ and $t + x \notin I$.*

Proof. By [15, Proposition 3.6], we have

$$J = J^{*_I} = J^v \cap \bigcap_{\alpha\in(I-J)} (-\alpha + I).$$

In particular, there must be a $t \in (I - J)$ such that $x \notin -t + I$; hence, $t + J \subseteq I$ (equivalently, $J \subseteq (-t + I) \cap \mathbb{N} = \rho_t(I)$) and $x + t \notin I$. □

Lemma 2.2. *Let S be a pseudosymmetric semigroup and $I \in \mathcal{G}_0(S)$; let $\tau := F(S)/2$. Then, $\eta(I) \geq \tau$.*

Proof. If $\eta(I) < \tau$, then each element bigger than τ is in I. Hence, $\tau + i \geq \tau$ for every $i \in I$, and $\tau + I \subseteq I$, i.e., $\tau \in (I - I)$. Thus, $I = I^v$ by [1, Proposition I.1.16], and $I \notin \mathcal{G}_0(S)$. □

3. Embeddings and dimensions

Due to the link between antichains of $\mathcal{G}_0(S)$ and star operations, a natural way to find the cardinality of $\mathrm{Star}(S)$ is to study the $*$-order on $\mathcal{G}_0(S)$, calculate the number of antichains and then trying to determine which ones give star operations. This program was carried out in [16] for the case of numerical semigroups of multiplicity 3. More precisely, the following result was proved.

Proposition 3.1. [16, Theorem 7.4] *Let $S := \langle 3, 3\alpha+1, 3\beta+2\rangle$, with α, β positive integers. Then, $(\mathcal{G}_0(S), \leq_*)$ is order-isomorphic to $C(2\alpha-\beta) \times C(2\beta-\alpha+1)$, where $C(k)$ denotes a chain of k elements.*

In a more geometrical way, this proposition can be rephrased by saying that $\mathcal{G}_0(S)$ is isomorphic to a rectangle with sides of length $2\alpha-\beta$ and $2\beta-\alpha+1$.

In particular, the proposition asserts that $\mathcal{G}_0(S)$ can be embedded into $\mathbb{N}^2$, where an *embedding* of partially ordered sets is a map $f : \mathcal{P} \longrightarrow \mathcal{Q}$ such that, for every $p_1, p_2 \in \mathcal{P}$, $p_1 \leq p_2$ if and only if $f(p_1) \leq f(p_2)$. Similarly, an *order-reversing embedding* is a map $f : \mathcal{P} \longrightarrow \mathcal{Q}$ such that $p_1 \leq p_2$ if and only if $f(p_1) \geq f(p_2)$.

The *dimension* of a finite partially ordered set $\mathcal{P}$ (denoted by $\dim(\mathcal{P})$) is the smallest n such that $\mathcal{P}$ can be embedded into $\mathbb{N}^n$ endowed with the the product order (that is, $(a_1, \dots, a_n) \leq (b_1, \dots, b_n)$ if and only if $a_i \leq b_i$ for every i); note that this is not the original definition introduced in [3], but it is equivalent [12, Section 10.4]. Such an n always exists; indeed, $\dim(\mathcal{P}) \leq |\mathcal{P}|$ [7]. Clearly, $\dim(\mathcal{P})$ is also the smallest integer n such that there is an order-reversing embedding of $\mathcal{P}$ into $\mathbb{N}^n$.

It is natural to ask for a characterization of all numerical semigroups S such that $\mathcal{G}_0(S)$ has a fixed dimension; the first cases are not difficult.

Proposition 3.2. *Let S be a numerical semigroup.*

(a) $\dim(\mathcal{G}_0(S)) = 0$ *if and only if S is symmetric.*
(b) $\dim(\mathcal{G}_0(S)) = 1$ *if and only if $\mu(S) = 3$ and S is pseudosymmetric.*

Proof. (a) follows immediately from the fact that $\dim \mathcal{P} = 0$ if and only if $|\mathcal{P}| \leq 1$, and that $|\mathcal{G}_0(S)| \geq 2$ as soon as S is not symmetric.

(b) By the previous point, we can suppose that S is not symmetric; let $\mu := \mu(S)$. If $\mu = 3$, then the claim is essentially [16, Proposition 7.8]. Suppose $\mu \geq 4$: we want to find two ideals I and J such that $I \not\leq_* J$ and $J \not\leq_* I$. Since S is not symmetric, there is an $a \in \mathbb{N}$ such that $a, F(S) - a \notin S$ and $a \leq F(S)/2$. We distinguish four cases.

If $a \geq 3$, then by [15, Lemma 4.13] there are $x_1, x_2 \in \mathbb{N} \setminus S$ such that $a - \mu < x_i < a$; consider $I_i := S \cup \{x \in \mathbb{N} \mid x > a\} \cup \{x_i\}$. By [15, Lemma 4.7], both I_i are in $\mathcal{Q}_a$; moreover, $I_1 \not\subseteq I_2$ and $I_2 \not\subseteq I_1$. Hence, I_1 and I_2 are not comparable in the $*$-order [17, Proposition 5.7(c)].

If $a < 3$ and $\mu \geq 5$, let x_1, x_2 be two different elements in $\{1, 2, 3\} \setminus \{3 - a\}$, and define $I_i := S \cup \{x \in \mathbb{N} \mid x > 4\} \cup \{3 - a, x_i\}$. By the proof of [17, Proposition 5.20], I_1 and I_2 are in $\mathcal{Q}_4$, and again by [17, Proposition 5.7(c)] they are not comparable in the $*$-order.

If $\mu = 4$ and $a = 2$, consider the ideals $I_1 := S \cup \{F(S) - 2\}$ and $I_2 := S \cup (2 + S)$. Then, they are both elements of $\mathcal{Q}_{F(S)}$. We have $2 \notin I_1$ (otherwise $2 = F(S) - 2$ and $F(S) = 4$, against $\mu(S) = 4$) and $F(S) - 2 \notin I_2$ (otherwise $F(S) - 2 \in 2 + S$, i.e., $F(S) - 4 \in S$, against $4 \in S$ and $F(S) \notin S$). Hence, we can apply again [17, Proposition 5.7(c)], and I and J are not $*$-comparable.

If $\mu = 4$ and $a = 1$, then $F(S) - 1 \notin S$. Let $I := S \cup \{F(S) - 1\}$ and $J := M_{F(S)-1}$. By [17, Lemma 5.10], since $I \in \mathcal{Q}_{F(S)}$ and $J \in \mathcal{Q}_{F(S)-1}$, we have $I \not\leq_* J$. By Lemma 2.1, if $J \leq_* I$ there is a $t \in I$ such that $J \subseteq \rho_t(I)$ and $t + F(S) - 1 \notin I$. The latter condition implies that $t = 1$; however, the definition of I now implies that $1 = F(S) - 1$, i.e., $F(S) = 2$. This is impossible when $\mu = 4$, and thus I and J are not $*$-comparable. $\square$

Already in dimension 2, however, it does not seem so easy to obtain a characterization.

Example 3.3. Let $S := \langle 4, 5, 6, 7 \rangle$. By [17, Example 5.21], putting $I(t) := S \cup \{t\}$ and $I(s, t) := S \cup \{s, t\}$, the Hasse diagram of $\mathcal{G}_0(S)$ is the following:

$$\begin{tikzcd} & I(1,2) \arrow[dl] \arrow[d] \arrow[dr] & \\ I(2) \arrow[dr] & I(1,3) \arrow[d] \arrow[dr] & I(1) \arrow[d] \\ & I(2,3) & I(3). \end{tikzcd}$$

It is not hard to embed $\mathcal{G}_0(S)$ into $\mathbb{N}^2$, by sending (see the left of Figure 1)

$$\begin{array}{lll} I(1,2) \mapsto (2,2) & I(2) \mapsto (2,0) & I(1) \mapsto (0,2) \\ I(1,3) \mapsto (1,1) & I(2,3) \mapsto (1,0) & I(3) \mapsto (0,1). \end{array}$$

Hence, $\dim(\mathcal{G}_0(S)) = 2$.

Given $x, y \in \mathcal{P}$ (where $\mathcal{P}$ is a partially ordered set) we say that x *covers* y, and we write $x \prec y$, if $x < y$ and there is no $z \in \mathcal{P}$ such that $x < z < y$. An embedding $f : \mathcal{P} \longrightarrow \mathcal{Q}$ is *tight* if $x \prec y$ implies $f(x) \prec f(y)$ [11]. Any distributive lattice can be tightly embedded into a product of chains [11, Proposition 1]; on the other hand, a non-distributive lattice must have a sublattice isomorphic to M_3 or N_5 (see Figure 2), and it is easily seen that neither M_3 nor N_5 can be

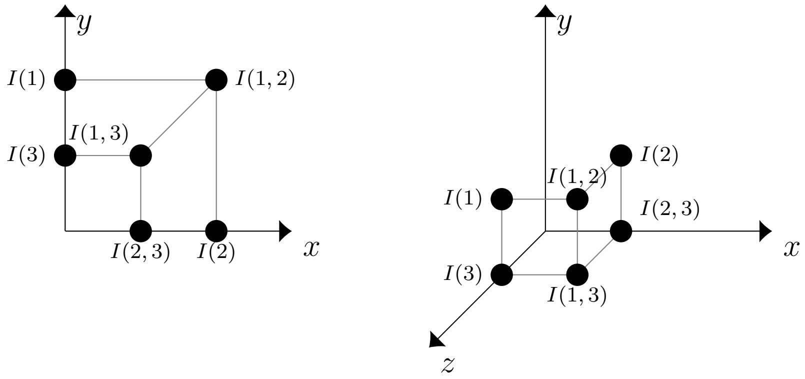


FIGURE 1. Two possible embeddings of $\mathcal{G}_0(\langle 4,5,6,7\rangle)$ into $\mathbb{N}^2$ and $\mathbb{N}^3$.

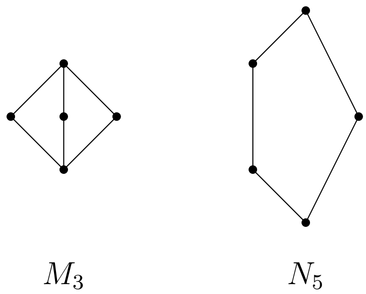


FIGURE 2. The two lattices M_3 (on the left) and N_5 (on the right).

tightly embedded into a product of chains. Note that it is not known if $(\mathcal{G}_0(S), \leq_*)$ is always a distributive lattice.

With this terminology, the embedding given in Example 3.3 is not tight and, indeed, there is no tight embedding of $\mathcal{G}_0(\langle 4,5,6,7\rangle)$ into $\mathbb{N}^2$. On the other hand, we can tightly embed it into $\mathbb{N}^3$ (see the right of Figure 1):

$$\begin{array}{rrr} I(1,2) \mapsto (1,1,1) & I(2) \mapsto (1,1,0) & I(1) \mapsto (0,1,1) \\ I(1,3) \mapsto (1,0,1) & I(2,3) \mapsto (1,0,0) & I(3) \mapsto (0,0,1). \end{array}$$

If $\mathcal{P}$ can be tightly embedded into $\mathbb{N}^n$, but not in $\mathbb{N}^{n-1}$, we call n the *tight dimension* of $\mathcal{P}$, and we denote it by $\dim_t(\mathcal{P})$. If $\mathcal{P}$ cannot be embedded into any $\mathbb{N}^n$, we say that the tight dimension is infinite. Clearly, $\dim_t(\mathcal{P}) \geq \dim(\mathcal{P})$.

The previous results shows that, if S ha multiplicity 3 and is not pseudosymmetric, then $\dim(\mathcal{G}_0(S)) = \dim_t(\mathcal{G}_0(S)) = 2$, while if $S = \langle 4,5,6,7\rangle$ then $\dim(\mathcal{G}_0(S)) = 2$ and $\dim_t(\mathcal{G}_0(S)) = 3$.

4. Pseudosymmetric semigroups of multiplicity 4

In this section we analyze the case of pseudosymmetric semigroups with multiplicity 4. By [13, Theorem 8], such a semigroup must be of the form $S := \langle 4, x, x+2\rangle$, where $x \geq 5$ is an odd number. In particular, the smallest element of S congruent to 2 modulo 4 will be $2x$; hence, $F(S) = 2x - 4$. We set $\tau := F(S)/2 = x - 2$.

In the following, the calculations take a slightly different shape according to whether $x \equiv 1 \bmod 4$ or $x \equiv 3 \bmod 4$, but are essentially analogous. For the sake of simplicity, we shall only carry them out for the case $x \equiv 1 \bmod 4$.

Let thus $x := 4k + 1$; then, $\tau = 4k - 1$, and the Apéry set of S with respect to 4 is $\{0, 4k+1, 4k+3, 8k+2\}$; in particular, each $t > \tau$, $t \notin S$ is congruent to 2 modulo 4. Given integers α, β, γ, we set

$$[\alpha, \beta, \gamma] := 4\mathbb{N} \cup (4\alpha + 1 + 4\mathbb{N}) \cup (4\beta + 2 + 4\mathbb{N}) \cup (4\gamma + 3 + 4\mathbb{N}).$$

It is straightforward to check that $[\alpha, \beta, \gamma]$ is an ideal in $\mathcal{F}_0(S)$ if and only if

$$(1) \qquad \begin{cases} 0 \leq 4\alpha + 1 \leq 4k + 1 \\ 0 \leq 4\beta + 2 \leq 8k + 2 \\ 0 \leq 4\gamma + 3 \leq 4k + 3 \\ 4\alpha + 1 + x \geq 4\beta + 2 \\ 4\gamma + 3 + x + 2 \geq 4\beta + 2 \end{cases} \quad \text{that is,} \quad \begin{cases} 0 \leq \alpha, \gamma \leq k \\ 0 \leq \beta \leq 2k \\ \alpha + k \geq \beta \\ \gamma + k + 1 \geq \beta. \end{cases}$$

Suppose $I := [\alpha, \beta, \gamma] \in \mathcal{F}_0(S)$. For I to be non-divisorial, we must have $\tau \notin (I - I)$ [1, Proposition I.1.16]; hence, at least one between the following must hold:

$$(2) \qquad \begin{cases} 4\beta + 2 + \tau < 4\alpha + 1 \\ 4\gamma + 3 + \tau < 4\beta + 2 \\ \tau < 4\gamma + 3 \end{cases} \quad \text{that is,} \quad \begin{cases} \beta + k < \alpha \\ \gamma + k < \beta \\ k < \gamma + 1. \end{cases}$$

Note that the first condition can never hold, since $\alpha \leq k$.

We divide $\mathcal{G}_0(S)$ into two classes:

- $\mathcal{X} := \{I \in \mathcal{G}_0(S) \mid \tau \in I\}$;
- $\mathcal{Y} := \{I \in \mathcal{G}_0(S) \mid \tau \notin I\}$.

We note that $\mathcal{Y}$ can be further subdivided, in a natural way, into $\mathcal{Q}_\tau$ (the ideals with $\eta(I) = \tau$) and the ideals with $\eta(I) > \tau$ (since, by Lemma 2.2, $\eta(I) \geq \tau$); however, this is not necessary for our analysis.

We first analyze the two classes separately.

Suppose $I := [\alpha, \beta, \gamma] \in \mathcal{X}$. Since $\tau = 4k - 1 \in I$, we have $\gamma \leq k - 1$, so the third condition of (2) never holds; hence, $\gamma + k < \beta$, i.e., $\gamma + k + 1 \leq \beta$. By (1), we have $\gamma + k + 1 \geq \beta$, and thus $\beta = \gamma + k + 1$. The condition $\beta \leq \alpha + k$ thus becomes $\gamma + 1 \leq \alpha$; hence, the ideals of this class have the form

$$x(\omega,\lambda) := [\omega, \lambda+k+1, \lambda]$$

with $0 \le \omega \le k$ and $0 \le \lambda \le \omega - 1$. Moreover, no $x(\omega,\lambda)$ is divisorial, and

$$x(\omega,\lambda)^v = [\omega, \lambda+k, \lambda] = x(\omega,\lambda) \cup \{4\lambda+4k+2\}.$$

We now want to find the $*$-order on $\mathcal{X}$. We claim that $x_1 := x(\omega_1,\lambda_1) \le_* x(\omega_2,\lambda_2) =: x_2$ if and only if $\lambda_1 \le \lambda_2$ and $\lambda_1-\omega_1 \le \lambda_2-\omega_2$. Without loss of generality, we may suppose that $x_1 \ne x_2$.

Suppose the two inequalities hold. Since $\lambda_1 \le \lambda_2$, we have

$$x' := \rho_{4(\lambda_2-\lambda_1)}(x_2) = [\omega_2 - \lambda_2 + \lambda_1, \lambda_1 + k + 1, \lambda_1]$$

with $\omega_2 - \lambda_2 + \lambda_1 \le \omega_1 - \lambda_1 + \lambda_1 = \omega_1$; hence, $x_1 \subseteq x'$. However, x' does not contain $4\lambda_1 + 4k + 2$; hence, $x_1 = x_1^v \cap x' = x_1^v \cap \rho_{4(\lambda_2-\lambda_1)}(x_2)$, and thus $x_1 \le_* x_2$.

Conversely, suppose $x_1 \le_* x_2$. Since $\eta(x_1) = 4\lambda_1 + 4k + 2 \in I^v \setminus I$, we must have $\eta(x_2) \ge \eta(x_1)$, and thus $\lambda_2 \ge \lambda_1$. Since every element out of x_2 bigger than τ is congruent to 2 modulo 4, we must have $x_1 \subseteq \rho_{4(\lambda_2-\lambda_1)}(x_2)$, which by the previous calculation is equivalent to $\lambda_1 - \omega_1 \le \lambda_2 - \omega_2$.

Let now $I := [\alpha,\beta,\gamma] \in \mathcal{Y}$. Since $\tau \notin I$, we have $\gamma = k$. Hence, I is in the form

$$y(\omega,\lambda) := [\omega,\lambda,k]$$

with $0 \le \omega \le k$ and $0 \le \lambda \le \omega + k$. Moreover, these ideals are non-divisorial unless $\omega = k$ and $\lambda = 2k$ (since $[k,2k,k] = S$ is divisorial), so we must exclude this case.

We claim that $y_1 := y(\omega_1,\lambda_1) \le_* y(\omega_2,\lambda_2) =: y_2$ if and only if $\omega_1 \ge \omega_2$ and $\lambda_1 \ge \lambda_2$, i.e., if and only if $y_1 \subseteq y_2$.

Indeed, $y(\omega,\lambda)^v = y(\omega,\lambda) \cup \{\tau\}$; hence, if $y_1 \subseteq y_2$ then $y_1 = y_1^v \cap y_2$ and $y_1 \le_* y_2$.

Conversely, suppose $y_1 \le_* y_2$ but $y_1 \nsubseteq y_2$. Then, there must be a $t \in y_2$ such that $y_1 \subseteq \rho_t(y_2)$ but $\tau + t \notin y_2$. Since $y_1 \nsubseteq y_2$, $t > 0$; hence, $\tau + t \equiv 2 \bmod 4$, and since $\tau \equiv 3 \bmod 4$ also $t \equiv 3 \bmod 4$. Since $t \in y_2$, it must be $t \ge \tau + 4$; but this implies $t + \tau \ge \tau + 4 + \tau \ge F(S) + 4$, which belongs to S and, *a fortiori*, to y_2. This is a contradiction, and we must have $y_1 \subseteq y_2$.

Theorem 4.1. *Let $S := \langle 4, 4k+1, 4k+3\rangle$. Then, the map*

$$\begin{aligned} \Psi\colon \mathcal{G}_0(S) &\longrightarrow \mathbb{N}^2 \\ x(\omega,\lambda) &\longmapsto (2k-1+\lambda-\omega, k+1+\lambda) \\ y(\omega,\lambda) &\longmapsto (2k-\lambda, k-\omega) \end{aligned}$$

is a tight embedding.

Proof. The fact that Ψ is an embedding when restricted to $\mathcal{X}$ or to $\mathcal{Y}$ is a consequence of the previous reasoning. Moreover, $k+1+\lambda > k \geq k-\omega$ for every $\lambda, \omega \geq 0$, and thus Ψ is injective.

Let now $I := x(\omega_1, \lambda_1) \in \mathcal{X}$ and $J := y(\omega_2, \lambda_2) \in \mathcal{Y}$: we have to prove that $I \leq_* J$ if and only if $\Psi(I) \leq \Psi(J)$, and that $I \geq_* J$ if and only if $\Psi(I) \geq \Psi(J)$.

For the former case, $\Psi(I) \leq \Psi(J)$ never happens; suppose $I \leq_* J$. Since $I^v \setminus I = \{4\lambda_1 + 4k + 2\}$, there is a $t \in J$ such that $I \subseteq \rho_t(J)$ and $4\lambda_1 + 4k + 2 + t \nsubseteq J$. Since $I \nsubseteq J$ (being $\tau \in I \setminus J$) and all elements bigger than τ and out of S are congruent to 2 modulo 4, we must have $t \equiv 0 \bmod 4$, i.e., $t = 4s$ for some integer s. Then,

$$\rho_t(J) = [\omega_2 - s, \lambda_2 - s, k - s]$$

and thus we must have

$$\begin{cases} \omega_2 - s \leq \omega_1 \\ \lambda_2 - s = \lambda_1 + k + 1 \\ k - s \leq \lambda_1. \end{cases}$$

The second condition implies $\lambda_1 = \lambda_2 - s - k - 1$; plugging it in the third condition we have

$$k - s \leq \lambda_2 - s - k - 1 \Longrightarrow \lambda_2 \geq 2k + 1.$$

However, $\lambda_2 \leq 2k$ by construction; hence, $I \nleq_* J$, as claimed.

For the second case, we have $\Psi(I) \geq \Psi(J)$ if and only if

$$\begin{cases} 2k - 1 + \lambda_1 - \omega_1 \geq 2k - \lambda_2 \\ k + 1 + \lambda_1 \geq k - \omega_2 \end{cases} \quad \text{that is,} \quad \begin{cases} \lambda_1 - \omega_1 - 1 \geq -\lambda_2 \\ \lambda_1 + 1 \geq -\omega_2; \end{cases}$$

since the second condition is always satisfied, $\Psi(I) \geq \Psi(J)$ if and only if $\omega_1 - \lambda_1 - 1 \geq \lambda_2$.

Suppose this inequality holds. The Apéry set $\mathrm{Ap}(I, 4)$ is equal to $\{0, 4\omega_1 + 1, 4(\lambda_1 + k + 1) + 2, 4\lambda_1 + 3\}$; hence,

$$\mathrm{Ap}(-(4\lambda_1 + 3) + I, 4) = \{-(4\lambda_1 + 3), 4(\omega_1 - \lambda_1 - 1) + 2, 4k + 3, 0\}$$

and thus

$$\rho_{4\lambda_1+3}(I) = [0, \omega_1 - \lambda_1 - 1, k] = y(0, \omega_1 - \lambda_1 - 1).$$

However, by hypothesis $\omega_1 - \lambda_1 - 1 \geq \lambda_2$; by the analysis of the $*$-order in $\mathcal{Y}$, it follows that

$$I \geq_* y(0, \omega_1 - \lambda_1 - 1) \geq_* y(0, \lambda_2) \geq_* y(\omega_2, \lambda_2),$$

as claimed.

Conversely, suppose $I \geq_* J$. Since $\tau \in J^v \setminus J$, there must be a $t \in I$ such that $\rho_t(I)$ contains J but not τ. Since $\tau \in I$, $t > 0$; hence, $t + \tau \equiv 2 \bmod 4$, and thus $t \equiv 3 \bmod 4$. Since $I = [\omega_1, \lambda_1 + k + 1, \lambda_1]$, we must have $t = 4(\lambda_1 + s) + 3$ for some $s \geq 0$; hence, applying the

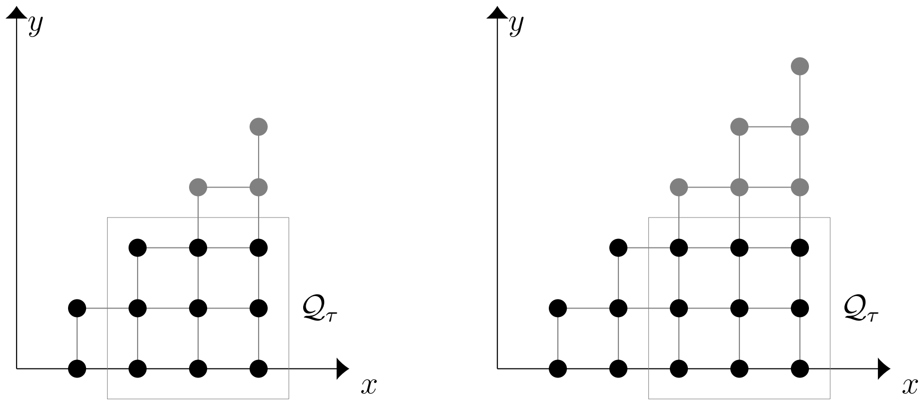


FIGURE 3. The image of $\Psi(\mathcal{G}_0(S))$ for the semigroups $\langle 4,9,11\rangle$ (on the left) and $\langle 4,11,13\rangle$ (on the right). Black circles represent ideals of $\mathcal{Y}$, gray circles are ideals of $\mathcal{X}$.

same reasoning of the previous case, $\rho_t(I) = [0, \omega_1 - \lambda_1 - 1, k]$. Since $J \subseteq I$, this implies $\lambda_2 \leq \omega_1 - \lambda_1 - 1$, as claimed.

The fact that Ψ is tight follows directly from the previous part of the proof. The theorem is proved. $\square$

The case $x = 4k+3$ is essentially analogous: the set $\mathcal{G}_0(S)$ can be divided into the two classes $\mathcal{X}$ and $\mathcal{Y}$, with $\mathcal{X}$ containing the ideals

$$x(\omega,\lambda) := [\lambda, \lambda+k+1, \omega] \quad \text{for} \quad \begin{cases} 0 \leq \omega \leq k \\ 0 \leq \lambda \leq \omega, \end{cases}$$

while $\mathcal{Y}$ contains the ideals

$$y(\omega,\lambda) := [k+1, \lambda, \omega] \quad \text{for} \quad \begin{cases} 0 \leq \omega \leq k \\ 0 \leq \lambda \leq \omega+k+1 \\ (\omega,\lambda) \neq (k, 2k+1). \end{cases}$$

The analogue of Theorem 4.1 is the following.

Theorem 4.2. *Let $S := \langle 4, 4k+3, 4k+5\rangle$. Then, the map*

$$\begin{aligned} \Psi\colon \mathcal{G}_0(S) &\longrightarrow \mathbb{N}^2 \\ x(\omega,\lambda) &\longmapsto (2k+1+\lambda-\omega, k+1+\lambda) \\ y(\omega,\lambda) &\longmapsto (2k+1-\lambda, k-\omega) \end{aligned}$$

is a tight embedding.

The range of Ψ in these two cases is pictured in Figure 3.

An immediate consequence is the following.

Theorem 4.3. *Let S be a pseudosymmetric semigroup of multiplicity 4. Then,* $\dim(\mathcal{G}_0(S)) = \dim_t(\mathcal{G}_0(S)) = 2$.

Proof. By Theorems 4.1 and 4.2 we have $\dim(\mathcal{G}_0(S)) \leq \dim_t(\mathcal{G}_0(S)) \leq 2$. However, $\dim(\mathcal{G}_0(S)) \geq 2$ by Proposition 3.2(b). The claim is proved. $\square$

This embedding allows also to determine the number of star operations on these semigroups S. Indeed, we have seen in the analysis of $\mathcal{X}$ and $\mathcal{Y}$ that $|I^v \setminus I| = 1$ for every $I \in \mathcal{G}_0(S)$; hence, by [17, Propositions 4.8 and 4.9], the cardinality of $\mathrm{Star}(S)$ is exactly the number of antichains of $\mathcal{G}_0(S)$.

Let now $\mathcal{H}(n)$ be the set of points $(x, y) \in \mathbb{N}^2$ such that $0 \leq x \leq n$ and $0 \leq y \leq x$: then, $\mathcal{H}(n)$ is just an "half-square". By Theorems 4.1 and 4.2, the image of $\mathcal{G}_0(S)$, for S a semigroup of multiplicity 4, is exactly $\mathcal{H}(n) \setminus \{(0,0)\}$ (for an appropriate n).

Proposition 4.4. *The partially ordered set $\mathcal{H}(n)$ has 2^{n+1} antichains.*

Proof. If $n = 0$, then $\mathcal{H}(n)$ is just a point and has two antichains (the empty one and the one formed by the point).

Suppose the claim holds for $n-1$. We can divide the set of antichain of $\mathcal{H}(n)$ into the $n+2$ sets $A^n_0, \ldots, A^n_n, A^n_\infty$, where an antichain belongs to A^n_i if it contains the point (n, i), and to A^n_∞ if it does not contain any point with first coordinate n. (Note that an antichain cannot have two points with first coordinate n, and thus these sets are disjoint.)

If now R is the bottom row of $\mathcal{H}(n)$, the difference $\mathcal{H}(n) \setminus R$ is order-isomorphic to $\mathcal{H}(n-1)$, and (for $i > 0$) the antichains in A^n_i correspond bijectively to antichains in A^{n-1}_{i-1}; moreover, the antichain of A^n_0 correspond to the antichains in A^{n-1}_∞. On the other hand, antichains in A^n_∞ correspond to the antichains of $\mathcal{H}(n) \setminus C$, where C is the rightmost column; however, $\mathcal{H}(n) \setminus C \simeq \mathcal{H}(n-1)$. Therefore,

$$\begin{aligned}\omega(\mathcal{H}(n)) = |A^n_\infty| + \sum_{i=0}^{n} |A^n_i| = \omega(\mathcal{H}(n-1)) + |A^{n-1}_\infty| + \sum_{j=0}^{n-1} |A^{n-1}_j| = \\ = \omega(\mathcal{H}(n-1)) + \omega(\mathcal{H}(n-1)) = 2^{n-1+1} + 2^{n-1+1} = 2^{n+1},\end{aligned}$$

as claimed. By induction, the claim holds for every n. $\square$

Theorem 4.5. *Let S be a pseudosymmetric numerical semigroup of multiplicity 4, and let $\tau := F(S)/2$. Then,*

$$|\mathrm{Star}(S)| = 2^{\frac{\tau+3}{2}} - 1 = 2^{\frac{F(S)+6}{4}} - 1.$$

Proof. By the discussion before Proposition 4.4, $|\mathrm{Star}(S)|$ is exactly the number of antichains of $\mathcal{G}_0(S)$, or, equivalently, the number of antichains of $\Psi(\mathcal{G}_0(S))$ (where Ψ is the embedding defined in Theorem 4.1 or Theorem 4.2, according to the equivalence class of x modulo 4). Since $\Psi(\mathcal{G}_0(S)) = \mathcal{H}(n) \setminus \{(0,0)\}$ for some n, and since $(0,0)$ is a minimal element of $\mathcal{H}(n)$, we have $|\mathrm{Star}(S)| = 2^{n+1} - 1$; hence, we just have to find n in function of S.

If $S = \langle 4, 4k+1, 4k+3\rangle$, then the rightmost element of $\Psi(\mathcal{G}_0(S))$ on the x-axis is $(2k, 0)$. Since $\tau = 4k-1$, it follows that $n = \frac{\tau+1}{2}$, and thus $|\mathrm{Star}(S)| = 2^{\frac{\tau+3}{2}} - 1$.

Analogously, if $S = \langle 4, 4k+3, 4k+5\rangle$ then the rightmost element is $(2k+1, 0)$, and $\tau = 4k+1$; hence, $n = \frac{\tau+1}{2}$, and thus $|\mathrm{Star}(S)| = 2^{\frac{\tau+3}{2}} - 1$.

The other expression follows, since $\tau = F(S)/2$. The claim is proved. □

In particular, since there is exactly one pseudosymmetric semigroup of multiplicity 4 for every Frobenius number $F(S)$, we see that the number of such semigroups with $|\mathrm{Star}(S)| \leq n$, where $n \geq 7$, is exactly $\lfloor \log_2(n+1)\rfloor - 2$.

5. A linear family

In this section we analyze semigroups of the form $\langle 4, 6, x, x+2\rangle$, where $x \geq 9$ is an odd number. As in the previous section, the calculations for the cases $x \equiv 1 \bmod 4$ and $x \equiv 3 \bmod 4$ are slightly different, but essentially equivalent; for the sake of simplicity, we shall do the full analysis only of the former case.

Let thus $k \geq 2$ and let $S := \langle 4, 6, 4k+1, 4k+3\rangle = \{0, 4, 6, 8, \ldots, 4k, \rightarrow\}$. Let

$$[\alpha, \beta, \gamma] := 4\mathbb{N} \cup (4\alpha + 1 + 4\mathbb{N}) \cup (4\beta + 2 + 4\mathbb{N}) \cup (4\gamma + 3 + 4\mathbb{N}).$$

Then, $[\alpha, \beta, \gamma]$ is an ideal in $\mathcal{F}_0(S)$ if and only if

$$\begin{cases} 0 \leq \alpha, \gamma \leq k \\ \alpha - 2 \leq \gamma \leq \alpha + 1 \\ 0 \leq \beta \leq 1; \end{cases}$$

therefore, $\mathcal{F}_0(S)$ can be divided into the following eight classes:

- $\mathcal{R}_{0,-2} := \{[\alpha, 0, \alpha - 2] \mid 2 \leq \alpha \leq k\}$
- $\mathcal{R}_{0,-1} := \{[\alpha, 0, \alpha - 1] \mid 1 \leq \alpha \leq k\}$
- $\mathcal{R}_{0,0} := \{[\alpha, 0, \alpha] \mid 0 \leq \alpha \leq k\}$
- $\mathcal{R}_{0,1} := \{[\alpha, 0, \alpha + 1] \mid 0 \leq \alpha \leq k-1\}$
- $\mathcal{R}_{1,-2} := \{[\alpha, 1, \alpha - 2] \mid 2 \leq \alpha \leq k\}$
- $\mathcal{R}_{1,-1} := \{[\alpha, 1, \alpha - 1] \mid 1 \leq \alpha \leq k\}$
- $\mathcal{R}_{1,0} := \{[\alpha, 1, \alpha] \mid 0 \leq \alpha \leq k\}$
- $\mathcal{R}_{1,1} := \{[\alpha, 1, \alpha + 1] \mid 0 \leq \alpha \leq k-1\}$.

In particular, $S = [k, 1, k]$. The ideals $\rho_t(S)$ different from $\mathbb{N}$ are $\rho_{4l}(S) = [k-l, 0, k-l]$ and $\rho_{4l+2}(S) = [k-l, 0, k-l-1]$ (both cases for $0 < l < k$), and these ideals are all comparable; thus, this class is closed by intersections. Therefore, the divisorial ideals are exactly the families $\mathcal{R}_{0,0}$ and $\mathcal{R}_{0,-1}$, with the exception of $[k, 0, k]$ and $[k, 0, k-1]$.

We shall consider three subsets of $\mathcal{G}_0(S)$ separately; the first one is $\mathcal{A} := \mathcal{R}_{0,1} \cup \mathcal{R}_{1,1} \cup \mathcal{R}_{0,-2} \cup \mathcal{R}_{1,-2}$.

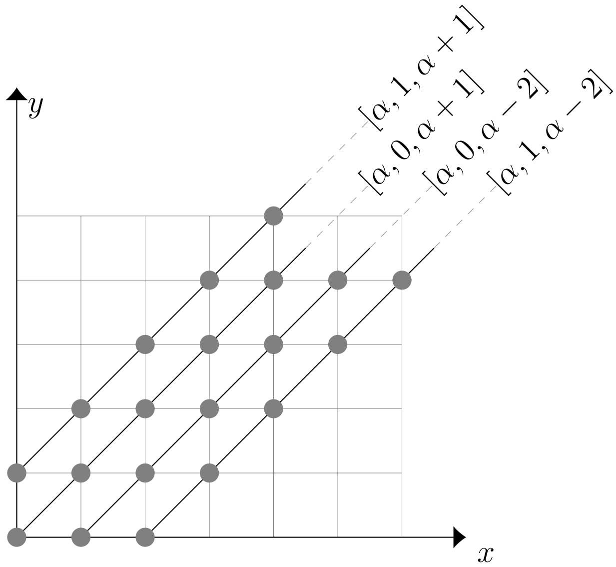


FIGURE 4. The embedding of $\mathcal{A}$ into $\mathbb{N}^2$.

Proposition 5.1. *Let S and $\mathcal{A}$ be as above. The map*

$$\begin{aligned} \Psi_{\mathcal{A}} \colon \mathcal{A} &\longrightarrow \mathbb{N}^2 \\ [\alpha, 1, \alpha + 1] &\longmapsto (k - \alpha - 1, k - \alpha) \\ [\alpha, 0, \alpha + 1] &\longmapsto (k - \alpha - 1, k - \alpha - 1) \\ [\alpha, 0, \alpha - 2] &\longmapsto (k - \alpha + 1, k - \alpha) \\ [\alpha, 1, \alpha - 2] &\longmapsto (k - \alpha + 2, k - \alpha) \end{aligned}$$

is an order-reversing embedding.

From a graphical point of view, this map sends each class into a segment parallel to the bisector of the first and the third quadrant (see Figure 4).

Proof. Let $I := [\alpha_1, \beta_1, \gamma_1]$ and $J := [\alpha_2, \beta_2, \gamma_2]$ be two ideals. We must show that $I \geq_* J$ if and only if $\Psi(I) \leq \Psi(J)$ (where, for simplicity, $\Psi := \Psi_{\mathcal{A}}$). We shall divide the proof according to which class I and J belong. Let $\mathcal{R}_1$ be the class of I and $\mathcal{R}_2$ be the class of J.

Let $\mathcal{S}_i := \Psi(\mathcal{R}_i)$, and suppose first that the two segments $\mathcal{S}_1, \mathcal{S}_2$ are contiguous (i.e., the other two segments are outside the strip between $\mathcal{S}_1$ and $\mathcal{S}_2$). Then, $\mathcal{S}_1 \cup \mathcal{S}_2$ is linearly ordered (it suffices to go alternatively one step to the right and one step up in the grid); thus, the embedding condition is equivalent to the condition that $\mathcal{R}_1 \cup \mathcal{R}_2$ is linearly ordered (in the "right" way).

Suppose $I \in \mathcal{R}_{1,1}$ and $J \in \mathcal{R}_{0,1}$. Then, $\Psi(I) \leq \Psi(J)$ if and only if $\alpha_1 \geq \alpha_2$ and $\alpha_1 \geq \alpha_2 + 1$, i.e., if and only if the latter holds. Similarly, $\Psi(I) \geq \Psi(J)$ if and only if $\alpha_1 \leq \alpha_2$. Hence, we must prove that, for every α,

$$[\alpha, 0, \alpha + 1] \geq_* [\alpha, 1, \alpha + 1] \geq_* [\alpha - 1, 0, \alpha].$$

Indeed, $\rho_{4\alpha+1}([\alpha, 0, \alpha + 1]) = [0, 1, 0]$, and thus

$$[\alpha, 1, \alpha + 1] = [\alpha, 0, \alpha + 1] \cap [0, 1, 0]$$

is $*_{[\alpha,0,\alpha+1]}$-closed; analogously, $\rho_4([\alpha, 1, \alpha+1]) = [\alpha-1, 0, \alpha]$, and thus the latter is $\leq_* [\alpha, 1, \alpha + 1]$.

If $I \in \mathcal{R}_{0,1}$ and $J \in \mathcal{R}_{0,-2}$, then we must prove that

$$[\alpha, 0, \alpha + 1] \geq_* [\alpha + 1, 0, \alpha - 1] \geq_* [\alpha - 1, 0, \alpha],$$

which is true since each term is equal to the 2-shift of the previous one.

If $I \in \mathcal{R}_{0,-2}$ and $J \in \mathcal{R}_{1,-2}$, then we must prove that

$$[\alpha, 0, \alpha - 2] \geq_* [\alpha, 1, \alpha - 2] \geq_* [\alpha - 1, 0, \alpha - 3],$$

which follows from the fact that $\rho_4([\alpha, 1, \alpha - 2]) = [\alpha - 1, 0, \alpha - 3]$ and that $[\alpha, 1, \alpha - 2] = \rho_{4\alpha-5}([\alpha, 0, \alpha - 2]) \cap [\alpha, 0, \alpha - 2]$.

In particular, these cases also provide a proof for the case where the class of I is equal to the class of J.

Suppose now that the segments are not contiguous.

If $I \in \mathcal{R}_{1,1}$ and $J \in \mathcal{R}_{0,-2}$, then $\Psi(I) \leq \Psi(J)$ if and only if $\alpha_1 \geq \alpha_2$. If the latter condition is true, then

$$[\alpha_2, 1, \alpha_2 + 1] \geq_* [\alpha_2 - 1, 0, \alpha_2] \geq_* [\alpha_2, 0, \alpha_2 - 2]$$

by the previous part of the proof; since $\alpha_2 \geq \alpha_1$, moreover, $[\alpha_2, 0, \alpha_2 - 2] \geq_* [\alpha_1, 0, \alpha_1 - 2]$, and $I \geq_* J$.

Conversely, suppose that $\alpha_1 \not\geq \alpha_2$. We want to show that $I \not\geq_* J$, and thus it suffices to consider the case $\alpha_2 = \alpha_1 + 1$; i.e., we must show that $I = [\alpha, 1, \alpha+1] \not\geq_* [\alpha+1, 0, \alpha-1] = J$ for every α. The biggest element in $\mathbb{N} \setminus J$ is $4\alpha + 1$, and it does belong to $[\alpha + 1, 0, \alpha - 1]^v = [\alpha, 0, \alpha - 1]$. Thus, if $I \geq_* J$ there must be a $t \in I$ such that $J \subseteq \rho_t(I)$ but $4\alpha+1+t \notin I$. The only element of $\mathbb{N} \setminus I$ greater or equal than $4\alpha+1$ is $4\alpha + 3$, so t should be 2; however, $2 \notin I$. Hence, $I \not\geq_* J$, as requested.

All the other cases follow by using the same technique. $\square$

Let us now consider the other two classes of non-divisorial ideals, namely $\mathcal{R}_{1,0}$ and $\mathcal{R}_{1,-1}$. For reasons that will be clear shortly, we define

$$\mathcal{B} := (\mathcal{R}_{1,0} \cup \mathcal{R}_{1,-1}) \setminus \{[k, 1, k - 1]\}.$$

Proposition 5.2. *$\mathcal{B}$ is linearly ordered (in the $*$-order).*

Proof. We claim that

$$[\alpha, 1, \alpha] \geq_* [\alpha + 1, 1, \alpha] \quad \text{for } \alpha \in [0, k - 1]$$

and

$$[\alpha, 1, \alpha - 1] \geq_* [\alpha, 1, \alpha] \quad \text{for } \alpha \in [1, k - 1].$$

Indeed, $[t,0,t-1]$ is divisorial for every $t\in[0,k-1]$, and thus
$$[\alpha+1,1,\alpha]=[\alpha,1,\alpha]\cap[\alpha+1,0,\alpha-1],$$
i.e., $[\alpha+1,1,\alpha]\leq_*[\alpha,1,\alpha]$. In the same way, $[t,0,t]$ is divisorial for every $t\in[1,k-1]$, and thus
$$[\alpha,1,\alpha]=[\alpha+1,1,\alpha]\cap[\alpha,0,\alpha]$$
so that $[\alpha,1,\alpha]\leq_*[\alpha,0,\alpha]$. $\square$

How do the ideals of $\mathcal{B}$ compare with the ideals of $\mathcal{A}$? The upper right corner of the image of $\Psi_{\mathcal{A}}$ is the following (arrows go from bigger to smaller in the $*$-order):
$$\begin{tikzcd}
{[0,1,1]} & \\
{[0,0,1]} \arrow[u] & \\
{[2,0,0]} \arrow[u] \arrow[r] & {[2,1,0]}\,.
\end{tikzcd}$$
We claim that $\Psi_{\mathcal{A}}$ can be extended to $\mathcal{B}$ in the following way:
$$\begin{tikzcd}
{[0,1,1]} \arrow[r] & {[1,1,0]} \arrow[r] & {[1,1,1]} \arrow[r] & {[2,1,0]} \arrow[r, dashed] & \cdots \\
{[0,0,1]} \arrow[u] \arrow[r] & {[0,1,0]} \arrow[u] & & & \\
{[2,0,0]} \arrow[u] \arrow[r] & {[2,1,0]}\,. \arrow[u] & & &
\end{tikzcd}$$
We first show the positive:

- $[0,0,1]\geq_*[0,1,0]$ since $\rho_1([0,0,1])=[0,1,0]$;
- $[0,1,1]\geq_*[1,1,0]$ since $\rho_1([0,1,1])=[1,1,0]$.

For the negative, it is enough to show that $[0,1,1]\not\geq_*[0,1,0]$ and that $[0,1,0]$ is not bigger than any element of $\mathcal{A}$.

For the former case, we note that $\eta([0,1,0])=2\in[0,1,0]^v$, and thus there must be a $t\in[0,1,1]$ such that $[0,1,0]\subseteq\rho_t([0,1,1])$ but $t+2\notin[0,1,1]$. However, $\mathbb{N}\setminus([0,1,1])=\{2,3\}$, and thus t can only be 0 or 1: in both cases, $\rho_t([0,1,1])$ does not contain $[0,1,0]$, and $[0,1,1]\not\geq_*[0,1,0]$.

Suppose now that $J\leq_*[0,1,0]$ for some $J\in\mathcal{A}$. We first note that the only shifts of $[0,1,0]$ that are not equal to $\mathbb{N}$ are the 0-shift (i.e., the identity) and the 1-shift, which gives $[1,0,0]$ (which is a divisorial ideal). Hence, any element smaller than $[0,1,0]$ must also be smaller than $[0,1,0]\cap I$, where I is a divisorial ideal. Now I is either in the form $[\alpha,0,\alpha]$ or $[\alpha,0,\alpha-1]$; the intersection of these ideals with $[0,1,0]$ gives only elements of $\mathcal{B}$. Since all the elements of $\mathcal{B}$ (except $[0,1,0]$)

are smaller than $[1,1,0]$, and $[1,1,0]$ is smaller than $[0,1,1]$, we have $J \leq_* [0,1,1]$. However, by Proposition 5.1, $[0,1,1]$ is a minimal element of $\mathcal{A}$; since also $[0,1,1] \geq_* [0,1,0]$, J cannot exist.

There is a lot of freedom in the choice of the embedding on $\mathcal{B}$: for example, the image of $\mathcal{B}$ can be chosen to be a segment parallel to the x-axis, or a segment parallel to the y-axis. We choose a middle ground, with the image extending over two diagonals: more specifically, we define

$$\begin{aligned} \Psi_{\mathcal{B}} \colon \mathcal{B} &\longrightarrow \mathbb{N}^2 \\ [\alpha,0,\alpha] &\longmapsto (k+\alpha, k+\alpha) \\ [\alpha,0,\alpha+1] &\longmapsto (k+\alpha, k+\alpha+1). \end{aligned}$$

The only ideals to which we still have to find a place are now $[k,0,k]$, $[k,0,k-1]$ and $[k,1,k-1]$. It is worthwhile to note that, up to now, we were able to construct a tight embedding of $\mathcal{A}\cup\mathcal{B}$ into $\mathbb{N}^2$.

We start with considering $[k,0,k]$ and $[k,0,k-1]$. We claim that (as above, arrows go from bigger to smaller)

$$\begin{array}{ccc} [k-1,0,k] & \longrightarrow & [k,0,k-2] \\ \downarrow & & \downarrow \\ [k,0,k] & \longrightarrow & [k,0,k-1]\,. \end{array}$$

Indeed:

- $[k-1,0,k] \geq_* [k,0,k-2]$ follows from Proposition 5.1;
- $\rho_2([k,0,k]) = [k,0,k-1]$, so $[k,0,k] \geq_* [k,0,k-1]$;
- $[k-1,0,k]$ is the canonical ideal, so $[k-1,0,k] \geq_* [k,0,k]$;
- $[k,0,k-1] = [k-1,0,k-1] \cap [k,0,k-2]$, and since $[k-1,0,k-1]$ is divisorial we have $[k,0,k-2] \geq_* [k,0,k-1]$.

We claim that these are the unique relationships between $[k,0,k]$, $[k,0,k-1]$ and the other ideals.

Proposition 5.3. *Let $I \in \mathcal{G}_0(S)$ be an ideal that is comparable, in the $*$-order, with $[k,0,k]$ or with $[k,0,k-1]$. Then, $I \in \{[k,0,k],[k,0,k-1],[k-1,0,k],[k,0,k-2]\}$.*

Proof. Since $[k,0,k] \geq_* [k,0,k-1]$, it is enough to show that, if $[k,0,k] \geq_* I$ or if $I \geq_* [k,0,k-1]$ then I is one of these four ideals.

Consider the shifts of $[k,0,k]$: then,

- $\rho_{4l}([k,0,k]) = [k-l,0,k-l]$, which is divisorial when $l>0$, and equal to $[k,0,k]$ when $l=0$;
- $\rho_{4l+2}([k,0,k]) = [k-l,0,k-l-1]$, which is divisorial when $l>0$, and equal to $[k,0,k-1]$ when $l=0$.

Moreover, the intersection of $[k,0,k]$ or $[k,0,k-1]$ with a divisorial ideal is either divisorial or one of these two ideals; hence, no other ideal is $*$-smaller than $[k,0,k]$.

Suppose now that $I \geq_* [k,0,k-1]$, and let $I=[\alpha,\beta,\gamma]$. We have $\eta([k,0,k-1])=4k-3$, and $4k-3$ belongs to $[k,0,k-1]^v=[k-1,0,k]$; hence, there is $t\in I$ such that $[k,0,k-1]\subseteq\rho_t(I)$ and $4k-3+t\notin I$. Since $F(S)=4k-1$, and since $4k-2\in S\subseteq I$ (since $6\in S$ and $k\geq 2$), t can be either 0 or 2.

If $t=0$, then $\alpha=k$ and $\beta=0$; hence, I is one of $[k,0,k]$, $[k,0,k-1]$ and $[k,0,k-2]$, and we have nothing new.

If $t=2$, then $\gamma=k$; hence, I should be one of $[k,0,k]$, $[k-1,0,k]$, $[k,1,k]$, and $[k-1,1,k]$. The first two are known, while the other two are divisorial, and thus they cannot give anything new. The claim is proved. □

In particular, the previous proposition shows a way to extend Ψ, keeping it a tight embedding, by adding a new dimension: we set

$$\Psi([k,0,k]):=(0,0,1) \quad\text{and}\quad \Psi([k,0,k-1]):=(1,0,1),$$

while elements of $\mathcal{A}\cup\mathcal{B}$ go into the xy-plane.

The last ideal to consider is $[k,1,k-1]$.

Proposition 5.4. $[k,1,k-1]$ *is a minimal element of* $(\mathcal{G}_0(S),\leq_*)$. *Moreover, if* $J>_*[k,1,k-1]$, *then* $J\in\{[k-1,0,k],[k,0,k-2],[k,1,k-2]\}$.

Proof. Let $I:=[k,1,k-1]=\{0,4,6,8,\dots,4(k-1)+2,\rightarrow\}$. If $t\in I$, $t>0$, then $\rho_t(I)$ is divisorial. Moreover, I is contained in any divisorial ideal J of $\mathcal{F}_0(S)$ (with the exception of S); therefore, I is minimal in $(\mathcal{G}_0(S),\leq_*)$.

We have $[k-1,0,k]\geq_*[k,0,k-2]\geq_*[k,1,k-2]$; furthermore, $I^v=[k-1,0,k-1]$, and thus $I=I^v\cap[k,1,k-2]$, and hence $I\leq_*[k,1,k-2]$.

Suppose now that $J\geq_* I$ for some $J=[\alpha,\beta,\gamma]$. Then, there is a $t\in J$ such that $I\subseteq\rho_t(J)$ but $4(k-1)+1+t\notin J$ (since $4(k-1)+1\in I^v\setminus I$). Hence, there must be an element of $\mathbb{N}\setminus J$ that is at least $4(k-1)+1$; since the greatest element of $\mathbb{N}\setminus S$ is $4k-1$, this means, as in Proposition 5.3, that $\mathbb{N}\setminus J$ contains at least one between $4k-3$ and $4k-1$, and $t\in\{0,2\}$.

If $t=2$, then $\gamma=k$, and so J is $[k,0,k]$ or $[k-1,k,0]$; however, Proposition 5.3 shows that these ideals are not $*$-bigger than I.

If $t=0$, then $\alpha=k$ and we have six possibilities: $[k,0,k-2]$, $[k,0,k-1]$, $[k,0,k]$, $[k,1,k-2]$, $[k,1,k-1]$ and $[k,1,k]$. However, this ideals are either divisorial or have already been considered, and thus we don't get anything new. The claim is proved. □

The images of $[k-1,0,k]$, $[k,0,k-2]$ and $[k,1,k-2]$ under Ψ lie on the x-axis; thus, the natural way to extend Ψ to $[k,1,k-1]$ is by putting $\Psi([k,1,k-1])=(L,0,0)$, with L chosen bigger than the first coordinate of $\Psi(I)$, for every $I\in\mathcal{A}\cup\mathcal{B}$. In particular, $L=2k+1$

suffices. Another possibility is to introduce a new dimension, sending $[k, 1, k-1]$ to $(2, 0, 0, 1)$ (and leaving the rest in the space where the fourth coordinate is 0).

Putting together all the results of this section, we get the following.

Theorem 5.5. *Let $S := \langle 4, 6, 4k+1, 4k+3\rangle$, with $k \geq 2$. Then, the map*

$$\begin{aligned}
\Psi\colon \mathcal{G}_0(S) &\longrightarrow \mathbb{N}^3 & \\
[\alpha, 1, \alpha+1] &\longmapsto (k-\alpha-1, k-\alpha, 0) & \text{for } 0 \leq \alpha \leq k-1 \\
[\alpha, 0, \alpha+1] &\longmapsto (k-\alpha-1, k-\alpha-1, 0) & \text{for } 0 \leq \alpha \leq k-1 \\
[\alpha, 0, \alpha-2] &\longmapsto (k-\alpha+1, k-\alpha, 0) & \text{for } 2 \leq \alpha \leq k \\
[\alpha, 1, \alpha-2] &\longmapsto (k-\alpha+2, k-\alpha, 0) & \text{for } 2 \leq \alpha \leq k \\
[\alpha, 1, \alpha] &\longmapsto (k+\alpha, k+\alpha, 0) & \text{for } 0 \leq \alpha \leq k-1 \\
[\alpha, 1, \alpha+1] &\longmapsto (k+\alpha, k+\alpha+1, 0) & \text{for } 0 \leq \alpha \leq k-1 \\
[k, 0, k] &\longmapsto (0, 0, 1) & \\
[k, 0, k-1] &\longmapsto (1, 0, 1) & \\
[k, 1, k-1] &\longmapsto (2k+1, 0, 0) &
\end{aligned}$$

is an order-reversing embedding. Moreover, the map

$$\begin{aligned}
\Psi'\colon \mathcal{G}_0(S) &\longrightarrow \mathbb{N}^4 \\
I &\longmapsto (\Psi(I), 0) \quad \text{if } I \neq [k, 1, k-1] \\
[k, 1, k-1] &\longmapsto (2, 0, 0, 1)
\end{aligned}$$

is an order-reversing tight embedding.

The version for $x \equiv 3 \bmod 4$ is structurally analogous.

Theorem 5.6. *Let $S := \langle 4, 6, 4k-1, 4k+1\rangle$, with $k \geq 3$. Then, the map*

$$\begin{aligned}
\Psi\colon \mathcal{G}_0(S) &\longrightarrow \mathbb{N}^3 & \\
[\alpha, 1, \alpha-2] &\longmapsto (k-\alpha, k-\alpha+1, 0) & \text{for } 2 \leq \alpha \leq k \\
[\alpha, 0, \alpha-2] &\longmapsto (k-\alpha, k-\alpha, 0) & \text{for } 2 \leq \alpha \leq k \\
[\alpha, 0, \alpha+1] &\longmapsto (k-\alpha+3, k-\alpha+2, 0) & \text{for } 0 \leq \alpha \leq k-2 \\
[\alpha, 1, \alpha+1] &\longmapsto (k-\alpha+4, k-\alpha+2, 0) & \text{for } 0 \leq \alpha \leq k-2 \\
[\alpha, 1, \alpha] &\longmapsto (k+\alpha+2, k+\alpha, 0) & \text{for } 0 \leq \alpha \leq k-2 \\
[\alpha, 1, \alpha+1] &\longmapsto (k+\alpha, k+\alpha+1, 0) & \text{for } 0 \leq \alpha \leq k-2 \\
[k, 0, k-1] &\longmapsto (0, 0, 1) & \\
[k-1, 0, k-1] &\longmapsto (1, 0, 1) & \\
[k-1, 1, k-1] &\longmapsto (2k+1, 0, 0) &
\end{aligned}$$

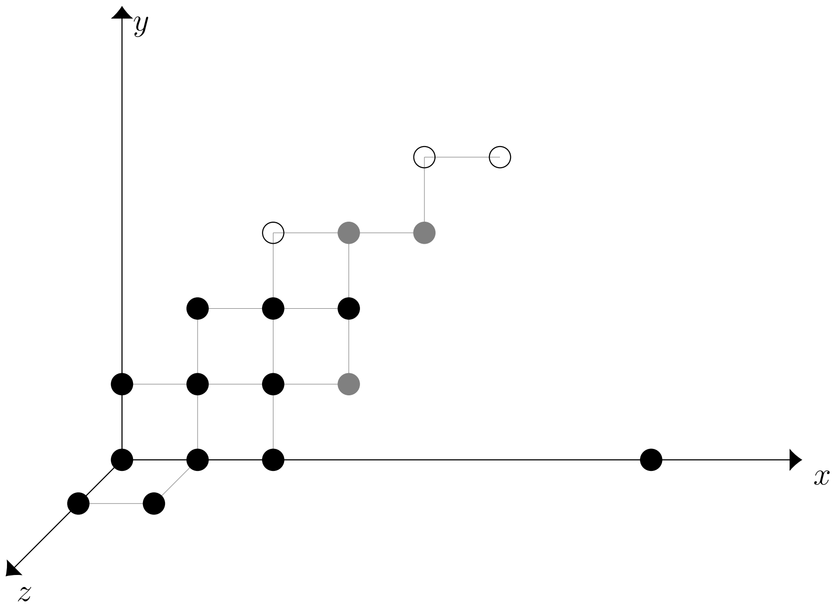


FIGURE 5. Growth of $\Psi(\mathcal{G}_0(S))$ in the family $\langle 4,6,x,x+2\rangle$. Black circles are the image for $\langle 4,6,9,11\rangle$, gray circles are the ideals added by $\langle 4,6,11,13\rangle$ and white circles are the ideals added by $\langle 4,6,13,15\rangle$. The black circle on the far right "moves" to be always on the right of the rest of the image.

is an order-reversing embedding. Moreover, the map

$$\begin{aligned}\Psi'\colon \mathcal{G}_0(S) &\longrightarrow \mathbb{N}^4\\ I &\longmapsto (\Psi(I),0) \quad \text{if } I\neq[k-1,1,k-1]\\ [k-1,1,k-1] &\longmapsto (2,0,0,1)\end{aligned}$$

is an order-reversing tight embedding.

The existence of Ψ and Ψ' allows to find the dimensions of $\mathcal{G}_0(S)$.

Theorem 5.7. *Let $S:=\langle 4,6,x,x+2\rangle$, where $x\geq 9$ is an odd number.*

(a) $\dim(\mathcal{G}_0(S))=2$.
(b) $\dim_t(\mathcal{G}_0(S))=4$.

Proof. (a) By Proposition 3.2(b) we have $\dim(\mathcal{G}_0(S))\geq 2$. Let $n\geq 2k+1$; then, $\Psi(\mathcal{G}_0(S))$ is contained into $\mathcal{R}:=\mathcal{R}_1\cup\mathcal{R}_2$, where

$$\mathcal{R}_1:=\{(x,y,0)\mid 0\leq x,y\leq n\}$$

and

$$\mathcal{R}_2:=\{(x,0,z)\mid 0\leq x,z\leq n\}.$$

We claim that $\mathcal{R}$ can be embedded into $\mathbb{N}^2$. Indeed, define

$$\begin{aligned}\Theta\colon \mathcal{R} &\longrightarrow \mathbb{N}^2\\ (x,y,0) &\longmapsto (x,(n+1)y+x)\\ (x,0,z) &\longmapsto ((n+1)z+x,x).\end{aligned}$$

Note that Θ is well-defined since $\mathcal{R}_1\cap\mathcal{R}_2=\{(x,0,0)\mid 0\leq x\leq n\}$ and $(x,0,0)$ get sent to (x,x) under both definitions.

Since $0\leq x\leq n$, we have $(n+1)y_1+x_1\leq(n+1)y_2+x_2$ if and only if $y_1\leq y_2$; hence, $\Theta|_{\mathcal{R}_1}$ is an embedding. Analogously, $\Theta|_{\mathcal{R}_2}$ is an embedding. If now $P:=(x_1,y,0)\in\mathcal{R}_1\setminus\mathcal{R}_2$ and $Q:=(x_2,0,z)\in\mathcal{R}_2\setminus\mathcal{R}_1$, then $y\neq 0\neq z$, and thus P and Q are not comparable; moreover, $(n+1)z+x>x$ and $(n+1)y+x>x$, and thus neither $\Theta(P)$ and $\Theta(Q)$ are comparable.

Therefore, Θ is an embedding of $\mathcal{R}$ into $\mathbb{N}^2$, and $\Theta\circ\Psi$ embeds $\mathcal{G}_0(S)$ into $\mathbb{N}^2$. Hence, $\dim(\mathcal{G}_0(S))\leq 2$, and thus $\dim(\mathcal{G}_0(S))=2$, as claimed.

(b) The existence of Ψ' shows that $\dim_t(\mathcal{G}_0(S))\leq 4$. For the sake of simplicity, we shall give the proof only in the case $x=4k+1$ (but the case $x=4k+3$ is completely analogous). Let $\mathcal{X}:=\mathcal{A}\cup\{[k,0,k],[k,0,k-1]\}$. We claim that, up to permutation of coordinates, any order-reversing tight embedding Φ of $\mathcal{X}$ into $\mathbb{N}^3$ coincides with the Ψ defined in Theorem 5.5.

Without loss of generality, we can put $\Phi([k-1,0,k])=(0,0,0)$. The ideal $[k-1,0,k]$ covers three ideals, namely $[k-1,1,k]$, $[k,0,k-2]$ and $[k,0,k]$; since Φ is tight, they must be sent (in some order) to $(0,1,0)$ and $(1,0,0)$ and $(0,0,1)$; again without loss of generality, we can suppose that Φ and Ψ coincide on them.

The ideal $[k,0,k-1]$ is covered, in the $*$-order, by $[k,0,k-2]$ and $[k,0,k]$; to respect the tightness of Φ, therefore, it must be sent to $(1,0,1)$. Likewise, $[k-2,0,k-1]$ is covered by $[k,0,k-2]$ and $[k-1,1,k]$ and thus must be sent to $(1,1,0)$.

Consider now $[k,1,k-2]$: it is covered by $[k,0,k-2]$, and thus its image has distance one from $\Phi([k,0,k-2])=(1,0,0)$: hence, it must be one between $(2,0,0)$, $(1,1,0)$ and $(1,0,1)$. The second and the third ones are impossible since Φ must be injective; hence, $\Phi([k,1,k-2])=(2,0,0)$. Similarly, $[k-2,1,k-1]$ is covered only by $[k-1,0,k-3]$ and thus its image has distance 1 from $(1,1,0)$; hence, it must be $(2,1,0)$, $(1,2,0)$ or $(1,1,1)$. The first one is impossible because $(1,1,0)=\Phi([k-1,1,k])\leq(2,1,0)$ while $[k-1,1,k]\not\geq_*[k-2,1,k]$; the third one is impossible because $(1,0,1)=\Phi([k,0,k-1])\leq(1,1,1)$ while $[k,0,k-1]\not\geq_*[k-2,1,k]$; hence $\Phi([k-2,1,k])=(1,2,0)$.

Step by step, we can thus construct all the image of $\mathcal{X}$ without any choice (except the starting ones). Hence, Ψ is essentially the unique tight embedding of $\mathcal{X}$ into $\mathbb{N}^3$.

Finally, consider $[k,1,k-1]$: it is covered only by $[k,1,k-2]$, and thus its image must be at distance one from $\Psi([k,1,k-2])=(2,0,0)$;

hence, it can be only $(3,0,0)$, $(2,1,0)$ or $(2,0,1)$. The first one would imply that $[k,1,k-1] \geq_* [k-1,1,k-3]$, which is false; the second one is already the image of $[k-1,0,k-3]$; the third one would imply $[k,0,k-1] \geq_* [k,1,k-1]$. Since all these cases are impossible, we cannot embed tightly $\mathcal{X} \cup \{[k,1,k-1]\}$ into $\mathbb{N}^3$; hence, $\dim_t(\mathcal{G}_0(S)) \geq \dim_t(\mathcal{X} \cup \{[k,1,k-1]\}) > 3$. It follows that $\dim_t(\mathcal{G}_0(S)) = 4$, as claimed. $\square$

Unlike what happens in the previous section, this representation does not lead directly to the calculation of the number of star operations on these semigroups: the reason is that not all ideals are atoms. For example, $[0,1,0] = [0,1,1] \cap [1,1,0]$, but $[0,1,0] \not\leq_* [0,1,1]$ and $[0,1,0] \not\leq_* [1,1,0]$, so $[0,1,0]$ is not an atom of S [17, Proposition 4.4].

However, we can at least recursively count the number of antichains of $\mathcal{G}_0(S)$. Let $S_l := \langle 4, 6, 9+2\cdot l, 11+2\cdot l\rangle$; then, an application of Theorems 5.5 and 5.6 (or a look at Figure 5) shows that

$$\omega(\mathcal{G}_0(S_{l+1})) = \omega(\mathcal{G}_0(S_l)) + 30$$

for every $l \geq 0$, and thus that

$$\omega(\mathcal{G}_0(S_l)) = \omega(\mathcal{G}_0(S_0)) + 30 \cdot l = 65 + 30 \cdot l.$$

In particular, we have the following.

Proposition 5.8. *Let $S_l := \langle 4, 6, 9+2\cdot l, 11+2\cdot l\rangle$, where $l \geq 0$ is an integer.*

- *(a)* $|\mathrm{Star}(S_l)| \leq 65 + 30 \cdot l$.
- *(b) Let $\widetilde{\xi}(n)$ be the number of semigroups S_l such that $|\mathrm{Star}(S_l)| \leq n$. Then, $\widetilde{\xi}(n) \geq \dfrac{n-65}{30} \geq \dfrac{n}{30} - 2$.*

It is reasonable to think that, like for the number of antichains, also the cardinality of $\mathrm{Star}(S)$ verifies a similar linear growth; indeed, a computer calculation (obtained by checking the image of the map $\mathcal{A}$ defined in [17, discussion after Definition 3.2], implemented using GAP's package `numericalsgps` [5, 2]) shows that

$$|\mathrm{Star}(S_l)| = 51 + 20 \cdot l \quad \text{for } 0 \leq l \leq 20.$$

However, we are not yet able to prove this formula.

6. Two pseudosymmetric families

In this section, we analyze two families of pseudosymmetric semigroups whose Frobenius number is small compared to the multiplicity.

Lemma 6.1. *Let $\mu \geq 3$ be an integer.*

- *(a) There is a unique pseudosymmetric semigroup S such that $\mu(S) = \mu$ and $F(S) = 2\mu - 2$.*
- *(b) There is a unique pseudosymmetric semigroup T such that $\mu(T) = \mu$ and $F(T) = 2\mu + 2$.*

Proof. Let S be a pseudosymmetric semigroup such that $\mu(S) = \mu$ and $F(S) = 2\mu - 2$. None of the integers $1, \dots, \mu - 2$ belong to S, and each one is smaller than $F(S)/2 = \mu - 1$; hence, $F(S) - 1, \dots, F(S) - (\mu - 2)$ are in S. Since $F(S) = 2\mu - 2$, these numbers are exactly $\mu, \dots, 2\mu - 3$, and $S = \{0, \mu, \mu + 1, \dots, 2\mu - 3, 2\mu - 1, \rightarrow\}$, which is pseudosymmetric.

Analogously, if T is pseudosymmetric, $\mu(T) = \mu$, and $F(T) = 2\mu + 2$, then T must be equal to $\{0, \mu, \mu + 3, \dots, 2\mu + 1, 2\mu + 3, \rightarrow\}$. $\square$

Before starting with the analysis of the two cases, we fix a notation. Let $X := \{x_1, \dots, x_n\}$ be an ordered set of n elements. The *standard embedding* of the power set $\mathcal{P}(X)$ of X is

$$\begin{aligned} \Psi_n \colon \mathcal{P}(X) &\longrightarrow \mathbb{N}^n \\ A &\longmapsto (\delta(A, x_1), \dots, \delta(A, x_n)) \end{aligned}$$

where $\delta(A, x_i) = 1$ if $x_i \in A$ and $\delta(A, x_i) = 0$ if $x_i \notin A$. Clearly, Ψ_n is a tight embedding.

6.1. **The case $F(S) = 2\mu(S) - 2$.** Let now $S := \{0, \mu, \mu + 1, \dots, 2\mu - 3, 2\mu - 1, \rightarrow\}$. Let $F := F(S) = 2\mu - 2$ and let $\tau := F/2 = \mu - 1$.

This case was treated in [17, Proposition 6.3]: the elements of $\mathcal{G}_0(S)$ are $J := S \cup \{\tau\}$ and $I_A := S \cup \{F\} \cup A$, for any $A \subseteq \{1, \dots, \mu - 2\}$. Moreover, J is the canonical ideal of S, while $I_A \geq_* I_B$ if and only if $A \supseteq B$.

It follows that $(\mathcal{G}_0(S), \leq_*) \setminus \{J\}$ is order-isomorphic to the power set of $\{1, \dots, \mu - 2\}$; hence, the map

$$\begin{aligned} \mathcal{G}_0(S) &\longrightarrow \mathbb{N}^{\mu-2} \\ J &\longmapsto P \\ I_A &\longmapsto \Psi_{\mu-2}(A), \end{aligned}$$

where $\Psi_{\mu-2}$ is the standard embedding of $\{1, \dots, \mu - 2\}$ and P is any point such that $P > (1, \dots, 1)$, is an embedding. In particular, choosing $P = (2, 1, \dots, 1)$ the embedding is also tight, and we have the following.

Theorem 6.2. *Let $S := \{0, \mu, \mu + 1, \dots, 2\mu - 3, 2\mu - 1, \rightarrow\}$. Then, $\dim(\mathcal{G}_0(S)) = \dim_t(\mathcal{G}_0(S)) = \mu - 2$.*

Proof. The existence of Ψ shows that $\dim(\mathcal{G}_0(S)) \leq \dim_t(\mathcal{G}_0(S)) \leq \mu - 2$. Moreover, $\dim(\mathcal{G}_0(S)) \geq \dim(\mathcal{G}_0(S) \setminus \{J\})$, which is $\mu - 2$ by [12, Theorem 10.4.4], since $\mathcal{G}_0(S) \setminus \{J\}$ is isomorphic to the power set of $\{1, \dots, \mu - 2\}$. The claim is proved. $\square$

The description of $\mathcal{G}_0(S)$ also allows to prove that $|\mathrm{Star}(S)| = 1 + \omega(\mu - 2)$ (see [17, Proposition 6.3]).

6.2. **The case $F(S) = 2\mu(S) + 2$.** Let now $S := \{0, \mu, \mu+3, \ldots, 2\mu+1, 2\mu+3, \rightarrow\}$. Let $F := F(S) = 2\mu + 2$ and let $\tau := F/2 = \mu + 1$. Clearly, $\mathbb{N} \setminus S = \{1, \ldots, \mu-1, \tau, \tau+1, F\}$.

By Lemma 2.2, $\eta(I) \in \{\tau, \tau+1, F\}$ for every $I \in \mathcal{G}_0(S)$. We distinguish four classes of ideals:

(1) $\eta(I) = F$. Then, $I = S \cup \{\tau\} = M_F$.
(2) $\eta(I) = \tau$. Then, $I = S \cup \{\tau+1, F\} \cup X$, where $X \subseteq \{2, \ldots, \mu-1\}$ ($1 \notin I$ since otherwise $1 + \mu = \tau \in I$). Define $A_X := S \cup \{\tau+1, F\} \cup X$: then, $A_X^v = A_X \cup \{\tau\}$, so that each A_X is non-divisorial.
(3) $\eta(I) = \tau + 1$ and $\tau \notin I$. Since $\tau = \mu + 1$ and $\tau + 1 = \mu + 2$ are not in I, we have $1, 2 \notin I$. Therefore, $I = S \cup \{F\} \cup Y$, with $Y \subseteq \{3, \ldots, \mu-1\}$. Each subset Y defines a non-divisorial ideal; if $B_Y := S \cup \{F\} \cup Y$ then by [17, Proposition 6.2(a)] we have $B_Y^v = B_Y \cup \{\tau\}$.
(4) $\eta(I) = \tau + 1$ and $\tau \in I$. Then, $2 \notin I$; moreover, if $1 \notin I$ then $I + \tau \subseteq I$, and I would be divisorial [1, Proposition I.1.16]. Therefore, $I = S \cup \{1, \tau, F\} \cup Z$, with $Z \subseteq \{3, \ldots, \mu-1\}$; each Z defines a non-divisorial ideal, and if $C_Z := S \cup \{1, \tau, F\} \cup Z$ then $C_Z^v = C_Z \cup \{\tau+1\}$ (since $C_Z \cup \{\tau+1\}$ is divisorial by Lemma 2.2).

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be the sets of ideals A_X, B_Y and C_Z, respectively. Note that $\mathcal{A} = \mathcal{Q}_\tau$ and $\mathcal{C} = \mathcal{Q}_{\tau+1}$.

Clearly, if $X \subseteq X'$ then $A_X = A_X^v \cap A_{X'}$, and thus $A_X \leq_* A_{X'}$. Conversely, if $X \nsubseteq X'$ and $A_X \subseteq \rho_t(A_{X'})$, then $t > 0$, and thus $\tau \in \rho_t(A_{X'})$; this implies that the $*_{A_{X'}}$-closure of A_X contains τ, and thus it cannot be A_X. Hence, $A_X \not\leq_* A_{X'}$. Therefore, $\mathcal{A}$ (with the $*$-order) is isomorphic to the power set of $\{2, \ldots, \mu-1\}$. With the same reasoning, we see that the same happens for $\mathcal{B}$ and $\mathcal{C}$. In particular, we have the standard embeddings $\Psi_{\mu-2} : \mathcal{A} \longrightarrow \mathbb{N}^{\mu-2}$, $\Psi_{\mu-3} : \mathcal{B} \longrightarrow \mathbb{N}^{\mu-3}$ and $\Psi_{\mu-3} : \mathcal{C} \longrightarrow \mathbb{N}^{\mu-3}$.

To obtain the full picture of $\mathcal{G}_0(S)$, we need now to show how ideals of different classes compare under the $*$-order.

Suppose $A_X \leq_* B_Y$. Since $\tau \in A_X^v \setminus A_X$, there must be a $t \in B_Y$ such that $I \subseteq \rho_t(B_Y)$ and $t + \tau \notin B_Y$. The latter condition implies $t = 0$ or $t = 1$; however, $A_X \nsubseteq B_Y$ (since $\tau+1 \in A_X \setminus B_Y$) and $1 \notin B_Y$. Hence, $A_X \not\leq_* B_Y$. In a similar way, we can see that, for every X, Y, Z, we have $C_Z \not\leq_* B_Y$ and $C_Z \not\leq_* A_X$.

On the other hand, we claim that $B_Y \leq_* A_X$ if and only if $Y \subseteq X$. Indeed, if $Y \subseteq X$ then $B_Y = B_Y^v \cap A_X$; however, if $B_Y \subseteq \rho_t(A_X)$, then either $t = 0$ (and $Y \subseteq X$) or $t > 0$ (in which case $B_Y^v \subseteq \rho_t(A_X)$ would not be significant for the calculation of $B_Y^{*_{A_X}}$).

To study the case of $\mathcal{C}$, we need a notation. For each $Z \subseteq \{3, \ldots, \mu-1\}$, let $\nu_1(Z) := \{z - 1 : z \in Z \cup \{\mu\}\}$. Then, we claim that $A_X \leq_* C_Z$

if and only if $X \subseteq \nu_1(Z)$: indeed, if the latter is true then $A_X = \rho_1(C_Z) \cap A_X^v$. On the other hand, if $A_X \leq_* C_Z$, let $t \in C_Z$ be such that $t + A_X \subseteq C_Z$ and $t + \tau \notin C_Z$: the latter condition implies $t = 1$, which forces $A_X \subseteq \rho_1(C_Z)$ and $X \subseteq \nu_1(Z)$. The same reasoning shows that $B_Y \leq_* C_Z$ if and only if $Y \subseteq \nu_1(Z)$.

We can now construct an embedding Ψ of $\mathcal{G}_0(S)$ into $\mathbb{N}^{\mu-2}$. The "smallest" elements are the ideals in $\mathcal{B}$: we put

$$\Psi(B_Y) := (0, \Psi_{\mu-3}(Y)).$$

Then, there are the elements of $\mathcal{A}$: we move them along the first dimension, and define

$$\Psi(A_X) := \Psi_{\mu-2}(X) + (1, 0, \ldots, 0).$$

Clearly, $\Psi(A_X) \geq \Psi(B_Y)$ if and only if $X \supseteq Y$, and so if and only if $A_X \geq_* B_Y$. The image of $\mathcal{A}$ is a cube extending from $(1, 0, \ldots, 0)$ to $(2, 1, \ldots, 1)$; the class $\mathcal{C}$ will need to be parallel to another face of the cube. For simplicity of notation, we choose

$$\Psi(C_Z) := (\Psi_{\mu-3}(Z), 2) + (1, 0, \ldots, 0).$$

Then, the i-th component of $\Psi_{\mu-3}(Z)$ is exactly $\delta(\nu_1(Z), i)$, i.e., $\Psi_{\mu-3}(Z)_i = 1$ if and only if $i \in \nu_1(Z)$, so that $\Psi(A_X) \subseteq \Psi(C_Z)$ if and only if $X \subseteq \nu_1(Z)$, and analogously $\Psi(B_Y) \subseteq \Psi(C_Z)$ if and only if $Y \subseteq \nu_1(Z)$.

Hence, if P is any point such that $P > (2, 1, \ldots, 1, 2)$, the map

$$\begin{aligned} \Psi\colon \mathcal{G}_0(S) &\longrightarrow \mathbb{N}^{\mu-2} \\ A_X &\longmapsto \Psi_{\mu-2}(X) + (1, 0, \ldots, 0) \\ B_Y &\longmapsto (0, \Psi_{\mu-3}(Y)) \\ C_Z &\longmapsto (\Psi_{\mu-3}(Z), 2) + (1, 0, \ldots, 0) \\ M_F &\longmapsto P \end{aligned}$$

will be an embedding, and choosing (for example) $P = (2, 2, 1, \ldots, 1, 2)$ it will also be tight.

Theorem 6.3. . *Let* $S := \{0, \mu, \mu+3, \ldots, 2\mu+1, 2\mu+3, \rightarrow\}$. *Then,* $\dim(\mathcal{G}_0(S)) = \dim_t(\mathcal{G}_0(S)) = \mu - 2$.

Proof. It is enough to repeat the proof of Theorem 6.2. □

We further note that every ideal of S is an atom (since $|I^v \setminus I| = 1$ for every I [17, Proposition 4.8]); hence, this description allow, in principle, to calculate the number of star operations on T. Since this seems to be quite hard to do, we just note that, since $\mathcal{A} = \mathcal{Q}_\tau$ and $\mathcal{C} = \mathcal{Q}_{\tau+1}$, by [17, Proposition 5.11] we have

$$|\mathrm{Star}(S)| \geq \omega(\mu - 2) + \omega(\mu - 3),$$

and in particular $|\mathrm{Star}(S)|$ grows super-exponentially in μ.

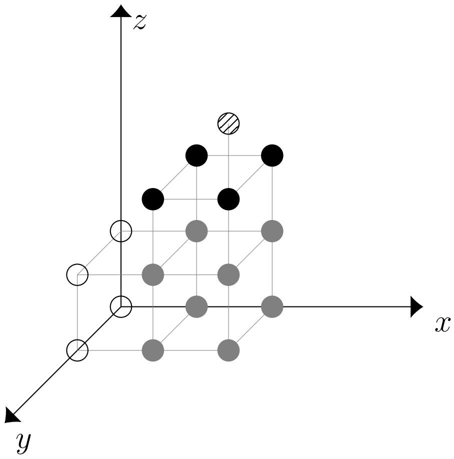


FIGURE 6. The embedding of $\mathcal{G}_0(\{0,5,8,9,10,11,13,\rightarrow\})$ into $\mathbb{N}^3$. White circle ares elements of $\mathcal{B}$, gray circles are elements of $\mathcal{A}$, and black circles are elements of $\mathcal{C}$. The striped circle is $M_{F(S)}$.

7. Two individual examples

In this section we collect two examples of $\mathcal{G}_0(S)$ for numerical semigroups S outside the families considered in the previous sections; both highlight behaviour not found in the other cases we considered. Both calculations – of which we present only the end result – were carried out by using GAP, and in particular its package `numericalsgps`.

Example 7.1. Let $S := \langle 5,6,13\rangle = \{0,5,6,10,11,12,13,15,\rightarrow\}$. Then, S is a pseudosymmetric numerical semigroup with multiplicity 5 and Frobenius number 14. If $x_1,\dots,x_k \in \mathbb{N}\setminus S$, denote by $I(x_1,\dots,x_k)$ the ideal $S\cup\{x_1,\dots,x_k\}$. Then, the $*$-order on $\mathcal{G}_0(S)$ is the following (arrows goes from bigger to smaller):

$$\begin{tikzcd}
 & & & I(7) \arrow[d] \\
 & & I(2,7,8,14) \arrow[d] & I(1,2,7,8,14) \arrow[l] \arrow[d] \\
 & I(3,8,9,14) \arrow[d] & I(3,4,8,9,14) \arrow[l] \arrow[d] & I(1,4,7,9,14) \arrow[l] \arrow[d] \\
I(8,14) \arrow[d] & I(8,9,14) \arrow[l] \arrow[d] & I(4,8,9,14) \arrow[l] \arrow[d] & I(1,7,9,14) \arrow[l] \arrow[d] \\
I(14) & I(9,14) \arrow[l] & I(4,9,14) \arrow[l] & I(1,7,14) \arrow[l]
\end{tikzcd}$$

In particular, $\dim(\mathcal{G}_0(S)) = \dim_t(\mathcal{G}_0(S)) = 2$.

We also note that $\mathcal{G}_0(S)$ is isomorphic (as a partially ordered set) to $\mathcal{G}_0(\langle 4,9,11\rangle)$ (see Figure 3). Moreover, as in that case the ideals of $\mathcal{Q}_\tau$ (where $\tau := F(S)/2$) form a square: in this case, it is composed by the ideals $I(3,4,8,9,14)$, $I(4,8,9,14)$, $I(3,8,9,14)$ and $I(8,9,14)$. However, the dimension and the position of the square differs in the two cases.

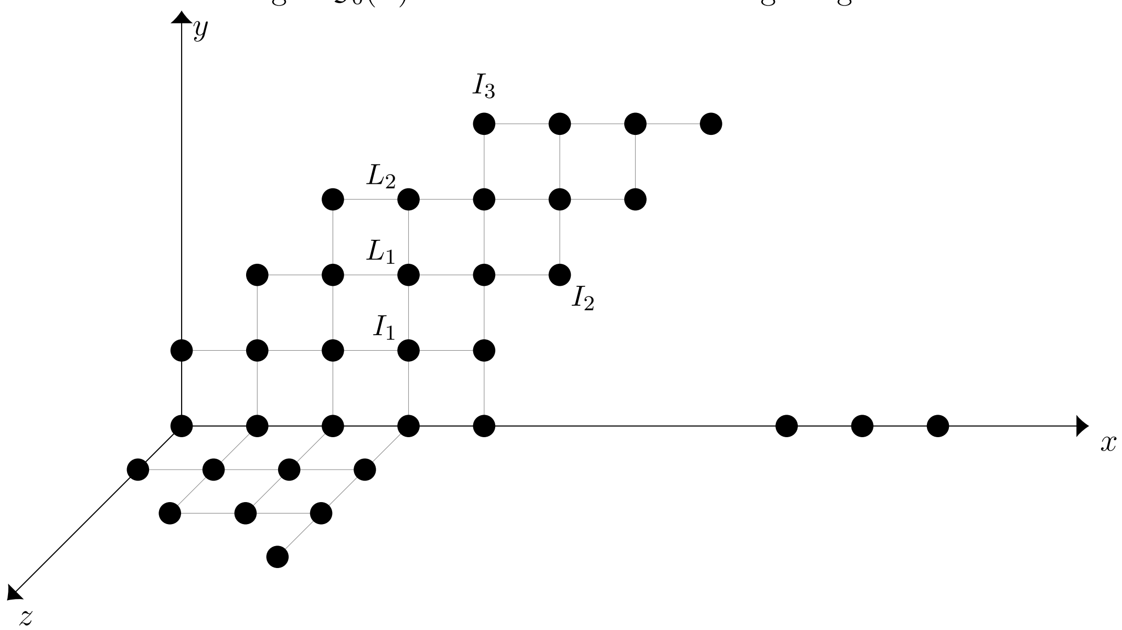
Example 7.2. Let $S := \langle 4,10,15,21\rangle$. Then, S is the prototype of the family of semigroups of the form $S_l := \langle 4,10,15+4\cdot l,21+4\cdot l\rangle$, whose behaviour seems similar to the family considered in Section 5: for example, experimentally we have

$$|\mathrm{Star}(S_l)| = 1368 + 400\cdot l \quad \text{for } 0 \le l \le 8.$$

Let

$$J := [0,2,2] := S\cup(1+4\mathbb{N})\cup(11+4\mathbb{N}) = 4\mathbb{N}\cup(1+4\mathbb{N})\cup(10+4\mathbb{N})\cup(11+4\mathbb{N}).$$

There is an embedding of $\mathcal{G}_0(S)$ into $\mathbb{N}^3$ with the following image:



In particular, using the same proof of Theorem 5.7, we have $\dim(\mathcal{G}_0(S)\setminus\{J\}) = 2$ and $\dim_t(\mathcal{G}_0(S)\setminus\{J\}) = 4$.

On the other hand, J cannot be easily put in this picture: we have $I_1 \ge_* J \ge_* I_2$ and $J \ge_* I_3$, and in the $*$-order there are no ideals between I_1 and J, nor between J and I_2 or J and I_3.

Indeed, $\mathcal{G}_0(S)$ cannot even be tightly embedded in any $\mathbb{N}^n$: the subset $\{I_1,L_1,L_2,J,I_3\}$ is a sublattice of $\mathcal{G}_0(S)$ that is isomorphic to N_5 (see Figure 2), and thus the tight dimension of $\mathcal{G}_0(S)$ is infinite.

8. Further questions

The families we analyzed in the previous sections are, obviously, a rather tiny section of all the numerical semigroups, and the two examples in Section 7 show more pathological behaviour. There are a number of questions that can be posed on the general properties of the $*$-order of $\mathcal{G}_0(S)$ with respect to S; the following list collects some.

- Is the dimension (and the tight dimension, when not infinite) of $\mathcal{G}_0(S)$ bounded by some function of the multiplicity?
- Which semigroups S satisfy $\dim_t(\mathcal{G}_0(S)) < \infty$?
- Beyond semigroups of multiplicity 3 and pseudosymmetric semigroups of multiplicity 4, which are the other semigroups S with $\dim_t(\mathcal{G}_0(S)) = 2$?
- Which semigroups S satisfy $\dim(\mathcal{G}_0(S)) = 2$?
- If S is pseudosymmetric, is it true that $\dim(\mathcal{G}_0(S)) = \dim_t(\mathcal{G}_0(S))$?
- If S is pseudosymmetric, does the inequality $\dim_t(\mathcal{G}_0(S)) \leq \mu - 2$ hold?
- Let $\mu \geq 4$. For which integers k there is a numerical semigroup with multiplicity μ and $\dim_t(\mathcal{G}_0(S)) = k$? For which k it can be chosen to be pseudosymmetric?

9. Acknowledgements

The author wishes to thank the referee for his/her thorough reading of the manuscript and for his/her suggestions.

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