

# TOPOLOGICAL PROPERTIES OF LOCALIZATIONS, FLAT OVERRINGS AND SUBLOCALIZATIONS

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ABSTRACT. We study the set of localizations of an integral domain from a topological point of view, showing that it is always a spectral space and characterizing when it is a proconstructible subspace of the space of all overrings. We then study the same problems in the case of quotient rings, flat overrings and sublocalizations.

## 1. INTRODUCTION

The Zariski topology on the set  $\text{Over}(D)$  of overrings of an integral domain was introduced as a natural generalization of the Zariski topology on the space  $\text{Zar}(D)$  of valuation overrings of  $D$  (called the *Zariski space* of  $D$ ), which in turn was introduced by Zariski in order to tackle the problem of resolution of singularities [35, 36].

It has been proved that  $\text{Over}(D)$ , like  $\text{Zar}(D)$ , is a *spectral space*, meaning that it is homeomorphic to the prime spectrum of a ring [10, Proposition 3.5]. There are other subspaces of  $\text{Over}(D)$  that are always spectral: for example, this happens for the space of integrally closed overrings [10, Proposition 3.6] and the space of local overrings [12, Corollary 2.14].

In the last two cases, the role of  $D$  in the definition of the space is merely to provide a setting ( $\text{Over}(D)$ ): that is, for an overring, being integrally closed or local (or a valuation domain, for the case of  $\text{Zar}(D)$ ) is a property completely independent from  $D$ . Indeed, with very similar proofs it is possible to generalize these results to the case of the spaces of rings comprised between two fixed rings (see e.g. [10, Propositions 3.5 and 3.6] and [12, Example 2.13]), as well as using these methods to study spaces of modules [31, Example 2.2].

In this paper, we study four subspaces of  $\text{Over}(D)$  that are much more closely related to  $D$ ; more precisely, such that, given an overring  $T$ , the belonging of  $T$  to the space depends not on the properties of  $T$  but rather on the relation between  $D$  and  $T$ . In Section 3 we shall

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start from the space of localizations (at prime ideals); then we will consider the space of quotient rings (Section 4), sublocalizations of  $D$  (i.e., intersection of localizations of  $D$ ; Section 5) and flat overrings (Section 6).

In each case, we will study two questions: under which conditions they are spectral spaces and under which condition they are closed in the constructible topology of  $\text{Over}(D)$ . We shall answer completely these questions in the case of localizations (Theorem 3.2) and quotient rings (Corollary 4.3 and Theorem 4.4); for sublocalizations we will find a sufficient condition (Theorem 5.5), while for flat overrings we will prove a characterization that is, however, very difficult to use (Proposition 6.1). We shall also study the space of flat submodules of an  $R$ -module (for rings  $R$  that are not necessarily integral domains) and the possibility of representing the space of sublocalizations of  $D$  in a more topological way.

## 2. PRELIMINARIES

**2.1. Spectral spaces.** A *spectral space* is a topological space homeomorphic to the prime spectrum of a (commutative, unitary) ring (endowed with the Zariski topology). Spectral spaces can be characterized topologically as those spaces that are  $T_0$  (i.e., such that for every pair of points at least one of them is contained in an open set not containing the other), compact, with a basis of open and compact subsets closed by finite intersections, and such that every nonempty irreducible closed subset has a generic point (i.e., it is the closure of a single point) [25, Proposition 4].

If  $X$  is a spectral space, the *constructible topology* (or *patch topology*) on  $X$  (which we denote by  $X^{\text{cons}}$ ) is the coarsest topology such that the open and compact subspaces of the original topology are both open and closed. The space  $X^{\text{cons}}$  is always a spectral space, that is moreover Hausdorff and totally disconnected [25, Theorem 1].

A subset  $Y \subseteq X$  is said to be *proconstructible* if it is closed, with respect to the constructible topology; in this case, the constructible topology on  $Y$  coincides with the topology induced by the constructible topology on  $X$ , and  $Y$  (with the original topology) is a spectral space (this follows from [6, 1.9.5(vi-vii)]). The converse does not hold, i.e., a subspace  $Y$  of a spectral space  $X$  may be spectral but not proconstructible; however, the following result holds.

**Lemma 2.1.** *Let  $Y \subseteq X$  be spectral spaces. Suppose that there is a subbasis  $\mathcal{B}$  of  $X$  such that, for every  $B \in \mathcal{B}$ , both  $B$  and  $B \cap Y$  are compact. Then,  $Y$  is a proconstructible subset of  $X$ .*

*Proof.* The hypothesis on  $\mathcal{B}$  implies that the inclusion map  $Y \hookrightarrow X$  is a spectral map; by [6, 1.9.5(vii)], it follows that  $Y$  is a proconstructible subset of  $X$ .  $\square$

For further results about the constructible topology and the relation between ultrafilters and the constructible topology, see [19, 11, 10, 12].

**2.2. The space  $\mathcal{X}(X)$ .** Let  $X$  be a spectral space. The *inverse topology* on  $X$  is the space  $X^{\text{inv}}$  having, as a basis of closed sets, the open and compact subspaces of  $X$ ; equivalently, it is the topology having as closed sets the subsets of  $X$  that are compact and closed by generalizations. The space  $X^{\text{inv}}$  is again a spectral space. Following [15], we denote by  $\mathcal{X}(X)$  the space of nonempty subsets of  $X$  that are closed in the inverse topology; this space can be endowed with a topology having, as a basis of open sets, the sets of the form

$$\mathcal{U}(\Omega) := \{Y \in \mathcal{X}(X) \mid Y \subseteq \Omega\},$$

as  $\Omega$  ranges among the open and compact subspaces of  $X$ . Under this topology,  $\mathcal{X}(X)$  is again a spectral space [15, Theorem 3.2(1)].

If  $X = \text{Spec}(R)$  for some ring  $R$ , we set  $\mathcal{X}(R) := \mathcal{X}(\text{Spec}(R))$ .

**2.3. Semistar operations.** Let  $D$  be an integral domain with quotient field  $K$ , and let  $\mathbf{F}(D)$  be the set of  $D$ -submodules of  $K$ . A *semistar operation* on  $D$  is a map  $\star : \mathbf{F}(D) \rightarrow \mathbf{F}(D)$  such that, for every  $I, J \in \mathbf{F}(D)$  and every  $x \in K$ ,

- (1)  $I \subseteq I^\star$ ;
- (2)  $(I^\star)^\star = I^\star$ ;
- (3) if  $I \subseteq J$  then  $I^\star \subseteq J^\star$ ;
- (4)  $x \cdot I^\star = (xI)^\star$ .

A semistar operation is called *spectral* if it is in the form  $s_\Delta$  for some  $\Delta \subseteq \text{Spec}(D)$ , where

$$I^{s_\Delta} := \bigcap \{ID_P \mid P \in \Delta\}$$

for every  $I \in \mathbf{F}(D)$ . If  $\star$  is spectral, then  $(I \cap J)^\star = I^\star \cap J^\star$  for every  $I, J \in \mathbf{F}(D)$ .

Starting from any semistar operations  $\star$ , we can define two maps  $\star_f$  and  $\tilde{\star}$  by putting, for every  $I \in \mathbf{F}(D)$ ,

$$I^{\star_f} = \bigcup \{J^\star \mid J \subseteq I, J \text{ is finitely generated}\}$$

and

$$I^{\tilde{\star}} := \bigcup \{(I : E) \mid 1 \in E^\star, E \text{ is finitely generated}\}.$$

Both  $\star_f$  and  $\tilde{\star}$  are semistar operations, and we always have  $(\star_f)_f = \star_f$  and  $\tilde{\tilde{\star}} = \tilde{\star}$ . If  $\star = \star_f$  then  $\star$  is said to be of *finite type*; on the other hand,  $\star = \tilde{\star}$  if and only if  $\star$  is spectral and of finite type.

If  $\star = s_\Delta$  is a spectral operation, then  $\star$  is of finite type if and only if  $\Delta$  is compact [16, Corollary 4.4].

The space  $\text{SStar}(D)$  of semistar operations on  $D$  can be endowed with a topology having, as a basis of open sets, the sets of the form

$$V_I := \{\star \in \text{SStar}(D) \mid 1 \in I^\star\},$$

as  $I$  ranges in  $\mathbf{F}(D)$ . In the induced topology, both the space  $\text{SStar}_f(D)$  of finite-type operations and the space  $\text{SStar}_{f,sp}(D)$  of finite-type spectral operations are spectral (see [16, Theorem 2.13] for the former and [13, Theorem 4.6] for the latter). Moreover,  $\text{SStar}_{f,sp}(D)$  is homeomorphic to  $\mathcal{X}(D)$  [15, Proposition 5.2].

**2.4. The  $t$ -operation.** Let  $D$  be an integral domain with quotient field  $K$ , and let  $\star$  be a semistar operation on  $D$ . If  $D^\star = D$ , then the restriction of  $\star$  to the set  $\mathcal{F}(D)$  of fractional ideals of  $D$  is said to be a *star operation* on  $D$ . A classical example of a star operation is the *divisorial closure* (or  $v$ -operation), which is defined by  $I^v := (D : (D : I))$ , where  $(I : J) := \{x \in K \mid xJ \subseteq I\}$ ; the divisorial closure is the biggest star operation on  $D$ , in the sense that  $I^\star \subseteq I^v$  for every star operation  $\star$  and every  $I \in \mathcal{F}(D)$ .

The  $t$ -operation is the finite-type operation associated to the  $v$ -operation; that is,  $t := v_f$ . The  $t$ -operation is the biggest finite-type star operation. The  $w$ -operation, defined by  $w := \tilde{t} = \tilde{v}$ , is the biggest spectral star operation of finite type.

If  $\star$  is a star operation on  $D$ , a prime ideal  $P$  of  $D$  such that  $P = P^\star$  is said to be a  $\star$ -prime; the set of all  $\star$ -primes is called the  $\star$ -spectrum and is denoted by  $\text{QSpec}^\star(D)$ . If  $\star = s_\Delta$  is a spectral star operation, then  $\text{QSpec}^\star(D) = \Delta^\downarrow = \{Q \in \text{Spec}(D) \mid Q \subseteq P \text{ for some } P \in \Delta\}$ .

We always have  $D = \bigcap \{D_P \mid P \in \text{QSpec}^t(D)\}$ .

See [20, Chapter 32] for more properties of star operations.

**2.5. Overrings.** Let  $D$  be an integral domain with quotient field  $K$ . An *overring* of  $D$  is a ring comprised between  $D$  and  $K$ . The space  $\text{Over}(D)$  of the overrings of  $D$  can be endowed with a topology having, as a basis of open sets, the sets of the form

$$\mathcal{B}(x_1, \dots, x_n) := \{T \in \text{Over}(D) \mid x_1, \dots, x_n \in T\} = \text{Over}(D[x_1, \dots, x_n]),$$

as  $x_1, \dots, x_n$  range in  $K$ . Under this topology,  $\text{Over}(D)$  is a spectral space [10, Proposition 3.5].

### 3. LOCALIZATIONS

The first space we analyze is the space of localizations of an integral domain  $D$  at its primes ideals, which we denote by  $\text{Loc}(D)$ ; that is,

$$\text{Loc}(D) := \{D_P \mid P \in \text{Spec}(D)\}.$$

**Definition 3.1.** *Let  $D$  be an integral domain. We say that  $D$  is rad-colon coherent if, for every  $x \in K \setminus D$ , there is a finitely generated ideal  $I$  such that  $\text{rad}(I) = \text{rad}((D :_D x))$ , i.e., if and only if  $\mathcal{D}((D :_D x))$  is compact in  $\text{Spec}(D)$  for every  $x \in K$ .*

Obvious examples of rad-colon coherent domains are Noetherian domains or, more generally, domains with Noetherian spectrum. Another

large class of such domains is the class of *coherent domains*, i.e., domains where the intersection of two finitely generated ideals is still finitely generated; this follows from the fact that  $(D :_D x) = D \cap x^{-1}D$ . In particular, this class contains all Prüfer domains [20, Proposition 25.4(1)], or more generally the polynomial rings in finitely many variables over Prüfer domains [22, Corollary 7.3.4]. See the following Example 3.3 for a domain that is not rad-colon coherent.

**Theorem 3.2.** *Let  $D$  be an integral domain.*

- (a)  $\text{Loc}(D)$  is a spectral space.
- (b)  $\text{Loc}(D)$  is proconstructible in  $\text{Over}(D)$  if and only if  $D$  is rad-colon coherent.

*Proof.* (a) By [7, Lemma 2.4], the map

$$\begin{aligned} \lambda: \text{Spec}(D) &\longrightarrow \text{Over}(D) \\ P &\longmapsto D_P. \end{aligned}$$

is a topological embedding whose image is exactly  $\text{Loc}(D)$ . In particular, since  $\text{Spec}(D)$  is a spectral space, so is  $\text{Loc}(D)$ .

(b) We first note that

$$\begin{aligned} \mathcal{B}(x) \cap \text{Loc}(D) &= \{D_P \in \text{Loc}(D) \mid x \in D_P\} \\ &= \{D_P \in \text{Loc}(D) \mid 1 \in (D_P : x) \cap D\} = \\ &= \{D_P \in \text{Loc}(D) \mid 1 \in (D :_D x)D_P\} = \\ &= \{D_P \in \text{Loc}(D) \mid (D :_D x) \not\subseteq P\} = \lambda(\mathcal{D}((D :_D x))). \end{aligned}$$

Suppose  $\text{Loc}(D)$  is proconstructible in  $\text{Over}(D)$ . Since, for any  $x \in K$ ,  $\mathcal{B}(x)$  is also a proconstructible subspace of  $\text{Over}(D)$ , then  $\mathcal{B}(x) \cap \text{Loc}(D)$  is closed in  $\text{Over}(D)^{\text{cons}}$ ; since the Zariski topology is weaker than the constructible topology,  $\mathcal{B}(x) \cap \text{Loc}(D)$  must be compact in the Zariski topology. By the previous calculation,  $\mathcal{B}(x) \cap \text{Loc}(D) = \lambda(\mathcal{D}(D :_D x))$ , and thus  $\mathcal{D}(D :_D x)$  must be compact. Hence,  $D$  is rad-colon coherent.

Conversely, suppose  $D$  is rad-colon coherent. Then, each  $\mathcal{B}(x) \cap \text{Loc}(D)$  is compact, and thus  $\{\mathcal{B}(x) \cap \text{Loc}(D) \mid x \in K\}$  is a subbasis of compact subsets for  $\text{Loc}(D)$ ; applying Lemma 2.1 we see that  $\text{Loc}(D)$  is a proconstructible subset of  $\text{Over}(D)$ .  $\square$

As a first use of this theorem, we give an example of a domain that is not rad-colon coherent.

**Example 3.3.** Let  $D$  be an essential domain that is not a PvMD; that is, suppose that  $D$  is the intersection of a family of valuation rings, each of which is a localization of  $D$ , but suppose that there is a  $t$ -prime ideal  $P$  such that  $D_P$  is not a valuation ring. Such a ring does indeed exist – see [23].

Let  $\mathcal{E}$  be the set of prime ideals  $P$  of  $D$  such that  $D_P$  is a valuation domain. Since  $D$  is not a PvMD, not all  $t$ -primes are in  $\mathcal{E}$ . Since  $\mathcal{E} \subseteq$

$\text{QSpec}^t(D)$  [27, Lemma 3.17], we thus have  $\mathcal{E} \subsetneq \text{QSpec}^t(D)$ . If  $\mathcal{E}$  is compact, then  $s_{\mathcal{E}}$  is a semistar operation of finite type on  $D$ ; however, since  $D$  is essential (and thus, by definition,  $\bigcap\{D_P \mid P \in \mathcal{E}\} = D$ ) we have  $D^{s_{\mathcal{E}}} = D$ , and thus the restriction of  $s_{\mathcal{E}}$  to the fractional ideals of  $D$  is a spectral star operation of finite type, which implies that  $I^{s_{\mathcal{E}}} \subseteq I^w$  for every finite-type operation. In particular,

$$\mathcal{E} = \text{QSpec}^{s_{\mathcal{E}}}(D) \supseteq \text{QSpec}^w(D) \supseteq \text{QSpec}^t(D),$$

and thus  $\mathcal{E} = \text{QSpec}^t(D)$ , a contradiction. Therefore,  $\mathcal{E}$  is not compact.

However,  $\lambda(\mathcal{E}) = \text{Loc}(D) \cap \text{Zar}(D)$ ; if  $\text{Loc}(D)$  were to be proconstructible in  $\text{Over}(D)$ , so would be  $\lambda(\mathcal{E})$  (since  $\text{Zar}(D)$  is always proconstructible). But this would imply that  $\lambda(\mathcal{E})$  is, in particular, compact, a contradiction. Hence  $\text{Loc}(D)$  is not proconstructible in  $\text{Over}(D)$ , and  $D$  is not rad-colon coherent.

There are at least three natural ways to extend  $\text{Loc}(D)$  to non-local overrings of  $D$ .

The first is by considering general localizations of  $D$  (which we will call, for clarity, *quotient rings*), that is, overrings in the form  $S^{-1}D$  for some multiplicatively closed subsets  $S$  of  $D$ . We denote this set by  $\text{Over}_{\text{qr}}(D)$ .

The second is through the set of *flat* overrings of  $D$  (that is, overrings that are flat when considered as  $D$ -modules). We denote this set by  $\text{Over}_{\text{flat}}(D)$ .

The third is by considering *sublocalizations* of  $D$ , i.e., overrings that are intersection of localizations (or, equivalently, quotient rings) of  $D$ . We denote this set by  $\text{Over}_{\text{sloc}}(D)$ .

It is well-known that  $\text{Over}_{\text{qr}}(D) \subseteq \text{Over}_{\text{flat}}(D) \subseteq \text{Over}_{\text{sloc}}(D)$ , and that both inclusions may be strict. For example, any overring of a Prüfer domain is flat, but it need not be a quotient ring: in the case of Dedekind domains, this happens if and only if the class group of  $D$  is torsion [21, Corollary 2.6] (more generally, a Prüfer domain  $D$  such that  $\text{Over}_{\text{qr}}(D) = \text{Over}_{\text{flat}}(D)$  is said to be a *QR-domain* – see [20, Section 27] or [18, Section 3.2]). As for sublocalizations that are not flat, we shall give an example later (Example 6.3); see also [24].

In all three cases, a natural question is to ask if (or when) the spaces are spectral, and if (or when) they are proconstructible in  $\text{Over}(D)$ ; moreover, we could ask if there is some construction through which we can represent them. We shall treat the case of quotient rings in Section 4, the case of sublocalizations in Section 5 and the case of flat overrings in Section 6.

A first result is a relation between their proconstructibility and the proconstructibility of  $\text{Loc}(D)$ .

**Proposition 3.4.** *Let  $D$  be an integral domain. If  $\text{Over}_{\text{qr}}(D)$  or  $\text{Over}_{\text{flat}}(D)$  is proconstructible, then  $D$  is rad-colon coherent.*

*Proof.* Let  $X$  be either  $\text{Over}_{\text{qr}}(D)$  or  $\text{Over}_{\text{flat}}(D)$ , and let  $\text{LocOver}(D)$  be the space of local overrings of  $D$ . Then,  $X \cap \text{LocOver}(D) = \text{Loc}(D)$ ; since  $\text{LocOver}(D)$  is always proconstructible [12, Corollary 2.14], if  $X$  is proconstructible so is  $\text{Loc}(D)$ . By Theorem 3.2(b), it follows that  $D$  is rad-colon coherent.  $\square$

Note that  $\text{Over}_{\text{sloc}}(D) \cap \text{LocOver}(D)$  may not be equal to  $\text{Loc}(D)$  – see Example 6.3.

#### 4. QUOTIENT RINGS

As localizations at prime ideals of  $D$  can be represented through  $\text{Spec}(D)$ , we can represent quotient rings by multiplicatively closed subsets; more precisely, there is a one-to-one correspondence between  $\text{Over}_{\text{qr}}(D)$  and the set of multiplicatively closed subsets that are saturated. For technical reasons, it is more convenient to work with the complements of multiplicatively closed subsets.

**Definition 4.1.** *Let  $R$  be a ring (not necessarily a domain). A semi-group prime on  $R$  is a nonempty subset  $\mathcal{Q} \subseteq R$  such that:*

- (1) for each  $r \in R$  and for each  $\pi \in \mathcal{Q}$ ,  $r\pi \in \mathcal{Q}$ ;
- (2) for all  $\sigma, \tau \in R \setminus \mathcal{Q}$ ,  $\sigma\tau \in R \setminus \mathcal{Q}$ ;
- (3)  $\mathcal{Q} \neq R$ .

By [30, (2.3)], a nonempty  $\mathcal{Q} \subseteq R$  is a semigroup prime of  $R$  if and only if it is a union of prime ideals, if and only if  $R \setminus \mathcal{Q}$  is a saturated multiplicatively closed subset.

Let  $\mathcal{S}(R)$  denote the set of semigroup primes of a ring  $R$ . As in [30] and in [14], we endow  $\mathcal{S}(R)$  with the topology (which we call the *Zariski topology*) whose subbasic closed sets have the form

$$\mathcal{V}_{\mathcal{S}}(x_1, \dots, x_n) := \{\mathcal{Q} \in \mathcal{S}(R) \mid x_1, \dots, x_n \in \mathcal{Q}\},$$

as  $x_1, \dots, x_n$  ranges in  $R$ ; equivalently, we can consider the subbasis of open sets

$$\mathcal{D}_{\mathcal{S}}(x_1, \dots, x_n) := \mathcal{S}(R) \setminus \mathcal{V}_{\mathcal{S}}(x_1, \dots, x_n) = \{\mathcal{Q} \in \mathcal{S}(R) \mid x_i \notin \mathcal{Q} \text{ for some } i\}.$$

We collect the properties of this topology of our interest in the next proposition.

**Proposition 4.2.** [14, Propositions 2.3 and 3.1] *Let  $R$  be a ring and endow  $\mathcal{S}(R)$  with the Zariski topology.*

- (a) *The family  $\{\mathcal{D}_{\mathcal{S}}(x) \mid x \in R\}$  is a basis of compact and open subsets of  $\mathcal{S}(R)$ , which is closed by intersections.*
- (b) *The set-theoretic inclusion  $\text{Spec}(R) \hookrightarrow \mathcal{S}(R)$  is a topological embedding.*
- (c)  *$\mathcal{S}(R)$  is a spectral space.*

(d) Suppose  $D$  is an integral domain. The map

$$\begin{aligned} \lambda_{qr} : \mathbf{S}(D) &\longrightarrow \text{Over}(D) \\ \mathcal{Q} &\longmapsto (R \setminus \mathcal{Q})^{-1}D. \end{aligned}$$

is a topological embedding whose image is  $\text{Over}_{qr}(D)$ .

In particular, by points (c) and (d) of the previous proposition we get immediately the following result.

**Corollary 4.3.**  $\text{Over}_{qr}(D)$  is a spectral space for every integral domain  $D$ .

On the other hand, proconstructibility holds less frequently for  $\text{Over}_{qr}(D)$  than it does for  $\text{Loc}(D)$ .

**Theorem 4.4.** Let  $D$  be an integral domain with quotient field  $K$ . Then,  $\text{Over}_{qr}(D)$  is proconstructible in  $\text{Over}(D)$  if and only if, for every  $x \in K$ , the ideal  $\text{rad}((D :_D x))$  is the radical of a principal ideal.

*Proof.* As in the proof of Theorem 3.2, we see that an overring  $T$  is in  $\mathcal{B}(x) \cap \text{Over}_{qr}(D)$  if and only if  $T = \lambda_{qr}(\mathcal{Q})$  for some semigroup prime  $\mathcal{Q}$  not containing  $(D :_D x)$ . Moreover, we note that a semigroup prime contains an ideal  $I$  if and only if it contains the radical of  $I$ .

Therefore, if each  $\text{rad}((D :_D x))$  is the radical of a principal ideal, then each  $\mathcal{B}(x) \cap \text{Over}_{qr}(D)$  is equal to  $\lambda_{qr}(\mathcal{D}_{\mathbf{S}}(y))$  for some  $y \in D$ . However, by Proposition 4.2(a),  $\mathcal{D}_{\mathbf{S}}(y)$  is compact, and thus so is  $\mathcal{B}(x) \cap \text{Over}_{qr}(D)$ ; by Lemma 2.1,  $\text{Over}_{qr}(D)$  is proconstructible in  $\text{Over}(D)$ .

Conversely, suppose there is a  $x \in K$  such that  $I := \text{rad}((D :_D x))$  is not the radical of a principal ideal.

*Claim 1:* let  $y \in D$ . Then,  $D[y^{-1}] \in \mathcal{B}(x)$  if and only if  $y \in I$ .

If  $x \in D[y^{-1}]$ , then

$$(1) \quad 1 \in (D[y^{-1}] :_{D[y^{-1}]} x) = (D :_D x)D[y^{-1}],$$

since  $D[y^{-1}]$  is flat over  $D$ .

If now  $P \in \mathcal{V}(I)$  (i.e.,  $I \subseteq P$ ), then in particular  $(D :_D x) \subseteq P$ , and so  $PD[y^{-1}] = D[y^{-1}]$ ; it follows that  $y \in P$ . Since this happens for every  $P \in \mathcal{V}(I)$  and  $I$  is a radical ideal,  $y \in I$ .

Suppose now that  $y \in I$ . Then, every prime ideal containing  $I$  explodes in  $D[y^{-1}]$ , and thus  $ID[y^{-1}] = D[y^{-1}]$ . Therefore, the same happens to  $(D :_D x)$ , and so  $x \in D[y^{-1}]$  (with the same calculation of (1), just backwards).

Let now  $\mathcal{U} := \{\mathcal{B}(z^{-1}) \mid z \in I\}$ .

*Claim 2:*  $\mathcal{U}$  is an open cover of  $\mathcal{B}(x) \cap \text{Over}_{qr}(D)$ .

Let  $T \in \mathcal{B}(x) \cap \text{Over}_{qr}(D)$ : then,  $1 \in (T :_T x) = (D :_D x)T$ , and thus there are  $d_1, \dots, d_n \in (D :_D x)$ ,  $t_1, \dots, t_n \in T$  such that  $1 = d_1t_1 + \dots + d_nt_n$ . For every  $i$ , there is a  $w_i \in D$  such that  $w_i^{-1} \in T$



and  $w_i t_i \in D$ ; let  $w := w_1 \cdots w_n$ . Then,  $w$  is invertible in  $T$ , and thus  $D[w^{-1}] \subseteq T$ , that is,  $T \in \mathcal{B}(w^{-1})$ ; moreover,

$$w = d_1 w t_1 + \cdots + d_n w t_n \in d_1 D + \cdots + d_n D \subseteq (D :_D x) \subseteq I,$$

and so  $\mathcal{B}(w^{-1}) \in \mathcal{U}$ . Therefore,  $\mathcal{U}$  is a cover of  $\mathcal{B}(x) \cap \text{Over}_{\text{qr}}(D)$ .

*Claim 3:* there are no finite subsets of  $\mathcal{U}$  that cover  $\mathcal{B}(x) \cap \text{Over}_{\text{qr}}(D)$ .

Consider a finite subset  $\mathcal{U}_0 := \{\mathcal{B}(z_1^{-1}), \dots, \mathcal{B}(z_n^{-1})\}$  of  $\mathcal{U}$ , for some  $z_1, \dots, z_n \in I$ . In particular,  $\text{rad}(z_i D) \subseteq I$  for every  $I$ ; moreover,  $\text{rad}(z_i D) \neq I$  since  $I$  is not the radical of any principal ideal. It follows that for every  $i$  there is a prime ideal  $P_i$  containing  $z_i$  but not  $I$ . By prime avoidance, there is an  $y \in I \setminus (P_1 \cup \cdots \cup P_n)$ ; in particular,  $D[y^{-1}] \in \mathcal{B}(x) \cap \text{Over}_{\text{qr}}(D)$ .

We claim that  $D[y^{-1}] \notin \mathcal{B}(z_i^{-1})$  for every  $i$ : indeed,  $z_i \in P_i$ , and  $P_i D[y^{-1}] \neq D[y^{-1}]$ . Therefore,  $z_i$  is not invertible in  $D[y^{-1}]$ , and  $z_i^{-1} \notin D[y^{-1}]$ . Hence,  $D[y^{-1}]$  is an element of  $\mathcal{B}(x) \cap \text{Over}_{\text{qr}}(D)$  not contained in any element of  $\mathcal{U}_0$ , which thus is not a cover.

Therefore,  $\mathcal{B}(x) \cap \text{Over}_{\text{qr}}(D)$  is not compact; it follows that  $\text{Over}_{\text{qr}}(D)$  is not proconstructible, as claimed.  $\square$

We remark that the first implication of the previous theorem follows also from [24, Theorem 2.5] and the following Theorem 5.5.

**Corollary 4.5.** *Let  $D$  be a Noetherian domain, and let  $X^1(D)$  be the set of height-1 prime ideals of  $D$ . The following are equivalent:*

- (i)  $\text{Over}_{\text{qr}}(D)$  is proconstructible in  $\text{Over}(D)$ ;
- (ii)  $D = \bigcap \{D_P \mid P \in X^1(D)\}$  and every  $P \in X^1(D)$  is the radical of a principal ideal.

*Proof.* (i  $\implies$  ii) Suppose that  $\text{Over}_{\text{qr}}(D)$  is proconstructible.

Let  $Q$  be a prime  $t$ -ideal, and consider  $A := \bigcap \{D_P \mid P \in \mathcal{D}(Q)\}$ . We claim that  $A \neq D$ : indeed, if  $A = D$ , then the map  $\star : I \mapsto \bigcap \{ID_P \mid P \in \mathcal{D}(Q)\}$  would be a star operation of finite type (since  $\mathcal{D}(Q)$  is compact) such that  $Q^\star = D \not\subseteq Q = Q^t$ , i.e., it would not be smaller than the  $t$ -operation, an absurdity. Hence, there is an  $x \in A \setminus D$ , and  $\text{rad}((D :_D x)) = Q$ . By Theorem 4.4,  $Q = \text{rad}(yD)$  for some  $y \in D$ .

If  $Q$  has not height 1, then this contradicts the Principal Ideal Theorem; thus,  $\text{QSpec}^t(D) = X^1(D)$ , and  $D = \bigcap \{D_P \mid P \in X^1(D)\}$ .

(ii  $\implies$  i) Conversely, suppose that the two conditions hold; the first one implies that  $\text{QSpec}^t(D) = X^1(D)$  (since  $X^1(D)$  is a compact subspace of  $\text{Spec}(D)$ ). For every  $x \in K \setminus D$ ,  $(D :_D x)$  is a proper  $t$ -ideal, and thus its minimal primes are  $t$ -ideals, i.e., have height 1. However,  $(D :_D x)$  has only finitely many minimal primes, say  $P_1, \dots, P_n$ , and by hypothesis  $P_i = \text{rad}(y_i D)$  for some  $y_i \in D$ ; hence,  $\text{rad}((D :_D x))$  is the radical of the principal ideal  $y_1 \cdots y_n D$ . By Theorem 4.4,  $\text{Over}_{\text{qr}}(D)$  is proconstructible.  $\square$

**Corollary 4.6.** *Let  $D$  be a Krull domain, and let  $X^1(D)$  be the set of height-1 prime ideals of  $D$ . Then, the following are equivalent:*

- (i)  $\text{Over}_{\text{qr}}(D)$  is proconstructible in  $\text{Over}(D)$ ;
- (ii) each  $P \in X^1(D)$  is the radical of a principal ideal;
- (iii) the class group of  $D$  is a torsion group.

*Proof.* The equivalence between (i) and (ii) follows as in the previous corollary, noting that  $D = \bigcap \{D_P \mid P \in X^1(D)\}$  holds for every Krull domain; the equivalence of (ii) and (iii) follows from the proof of Theorem 1 of [32].  $\square$

## 5. SUBLOCALIZATIONS

Our first result about  $\text{Over}_{\text{sloc}}(D)$  shows a striking difference between the space of sublocalizations and the spaces we considered in the previous sections.

**Proposition 5.1.** *Let  $D$  be an integral domain. Then,  $\text{Over}_{\text{sloc}}(D)$  is a spectral space if and only if it is proconstructible in  $\text{Over}(D)$ .*

*Proof.* If  $\text{Over}_{\text{sloc}}(D)$  is proconstructible, then it is spectral. On the other hand, for every  $x_1, \dots, x_n \in K$ , the intersection  $\mathcal{B}(x_1, \dots, x_n) \cap \text{Over}_{\text{sloc}}(D)$  is compact, since it has a minimum, namely the intersection of the localizations of  $D$  that contain  $x_1, \dots, x_n$ . Since  $\{\mathcal{B}(x_1, \dots, x_n) \cap \text{Over}_{\text{sloc}}(D) \mid x_1, \dots, x_n \in K\}$  is a subbasis of  $\text{Over}_{\text{sloc}}(D)$ , by Lemma 2.1 if  $\text{Over}_{\text{sloc}}(D)$  is spectral then it is also proconstructible in  $\text{Over}(D)$ .  $\square$

We are now tasked to study the spectrality of  $\text{Over}_{\text{sloc}}(D)$ . To this end, we use spectral semistar operations; more precisely, we use the fact that there is a map

$$\begin{aligned} \pi: \text{SStar}_{\text{sp}}(D) &\longrightarrow \text{Over}_{\text{sloc}}(D) \\ \star &\longmapsto D^\star \end{aligned}$$

that is continuous [12, Proposition 3.2(2)] and surjective (by definition of  $\text{Over}_{\text{sloc}}(D)$ ). We shall use the following topological lemma.

**Lemma 5.2.** *Let  $\phi: X \longrightarrow Y$  be a continuous surjective map between two topological spaces. Suppose that:*

- (a)  $X$  is spectral;
- (b)  $Y$  is  $T_0$ ;
- (c) there is a subbasis  $\mathcal{C}$  of  $Y$  such that, for every  $C \in \mathcal{C}$ ,  $\phi^{-1}(C)$  is compact.

*Then,  $Y$  is a spectral space and  $\phi$  is a spectral map.*

*Proof.* Let  $\Omega := O_1 \cap \dots \cap O_m$  be a finite intersection of elements of  $\mathcal{C}$ . Then,  $\phi^{-1}(\Omega) = \bigcap_i \phi^{-1}(O_i)$  is compact, since  $X$  is spectral and each  $\phi^{-1}(O_i)$  is compact by hypothesis; moreover, since  $\phi$  is surjective, also  $\Omega_i = \phi(\phi^{-1}(\Omega))$  is compact. Therefore, the set  $\mathcal{C}_0$  of finite intersections

of elements of  $\mathcal{C}$  is a basis of compact subsets. If now  $\Omega'$  is any open and compact subset of  $Y$ , then  $\Omega$  is a finite union of elements of  $\mathcal{C}_0$ , and thus  $\phi^{-1}(\Omega')$  is also compact.

The claim now follows from [8, Proposition 9].  $\square$

**Proposition 5.3.** *Let  $D$  be an integral domain. If  $\text{SStar}_{sp}(D)$  is a spectral space, then so is  $\text{Over}_{\text{sloc}}(D)$ .*

*Proof.* Let  $\mathcal{B} := \{\mathcal{B}(x) \cap \text{Over}_{\text{sloc}}(D) \mid x \in K\}$  be the canonical subbasis of  $\text{Over}_{\text{sloc}}(D)$ . Then,

$$\begin{aligned} \pi^{-1}(\mathcal{B}(x) \cap \text{Over}_{\text{sloc}}(D)) &= \{\star \in \text{SStar}_{sp}(D) \mid x \in D^\star\} = \\ &= \{\star \in \text{SStar}_{sp}(D) \mid 1 \in x^{-1}D^\star\} = \\ &= \{\star \in \text{SStar}_{sp}(D) \mid 1 \in (x^{-1}D)^\star\} = \\ &= \{\star \in \text{SStar}_{sp}(D) \mid 1 \in (x^{-1}D \cap D)^\star\} = \\ &= V_{x^{-1}D \cap D} \cap \text{SStar}_{sp}(D) = V_{(D:Dx)} \cap \text{SStar}_{sp}(D). \end{aligned}$$

However,  $V_{(D:Dx)} \cap \text{SStar}_{sp}(D)$  is compact since it has a minimum (explicitly,  $s_{\mathcal{D}((D:Dx))}$ ). Hence, the map  $\pi : \text{SStar}_{sp}(D) \rightarrow \text{Over}_{\text{sloc}}(D)$  satisfies the hypothesis of Lemma 5.2, and thus  $\text{Over}_{\text{sloc}}(D)$  is a spectral space.  $\square$

However,  $\text{SStar}_{sp}(D)$  is not, in general, a spectral space. To avoid this problem, we restrict  $\pi$  to the space  $\text{SStar}_{f,sp}(D)$  (which is always spectral [13, Theorem 4.6]), obtaining the map  $\pi_s : \text{SStar}_{f,sp}(D) \rightarrow \text{Over}_{\text{sloc}}(D)$ ; analogously to the previous proof, we need to show that  $\pi_s$  is surjective and that  $\pi_s^{-1}(\mathcal{B}(x) \cap \text{Over}_{\text{sloc}}(D))$  is compact. We claim that  $D$  being rad-colon coherent is a sufficient condition for this to happen; we need a lemma.

**Lemma 5.4.** *Let  $D$  be an integral domain, and let  $\star$  be a spectral semistar operation on  $D$ .*

- (a) *If  $\mathcal{D}(F \cap D)$  is a compact subset of  $\text{Spec}(D)$  for every finitely generated fractional ideal  $F$  of  $D$ , then  $\star_f = \tilde{\star}$ .*
- (b) *If  $D$  is rad-colon coherent, then  $D^{\star_f} = D^\star$ .*

Note that the equality  $\star_f = \tilde{\star}$  may actually fail; see [2, p.2466].

*Proof.* (a) Since  $\star_f$  and  $\tilde{\star}$  are of finite type, it is enough to show that  $F^{\star_f} = F^{\tilde{\star}}$  if  $F$  is finitely generated. The containment  $F^{\tilde{\star}} \subseteq F^{\star_f}$  always holds; suppose  $x \in F^{\star_f}$ . Then, since  $F^{\star_f} \subseteq F^\star$ , we have  $x \in F^\star$ . Consider  $I := x^{-1}F \cap D$ . Then,  $xI = F \cap xD \subseteq F$ . Moreover,

$$I^\star = (x^{-1}F \cap D)^\star = x^{-1}F^\star \cap D^\star$$

since  $\star$  is spectral, and thus  $1 \in I^\star$ . Since  $x^{-1}F$  is finitely generated, by hypothesis  $\mathcal{D}(I)$  is compact, and thus there is a finitely generated ideal  $J$  of  $D$  such that  $\text{rad}(I) = \text{rad}(J)$ ; passing, if needed, to a power of  $J$ , we can suppose  $J \subseteq I$ , so that  $xJ \subseteq xI \subseteq F$ . For any spectral operation  $\sharp$ ,  $\text{rad}(A) = \text{rad}(B)$  implies that  $1 \in A^\sharp$  if and only if  $1 \in B^\sharp$ ;

therefore,  $1 \in J^*$ , and thus  $x \in (F : J) \subseteq F^{\tilde{*}}$ , and  $x \in F^{\tilde{*}}$ . Hence,  $\star_f = \tilde{\star}$ , as requested.

(b) It is enough to repeat the proof of the previous point by using  $F = D$ , and noting that  $\mathcal{D}(x^{-1}D \cap D)$  is compact since  $D$  is rad-colon coherent.  $\square$

**Theorem 5.5.** *Let  $D$  be an integral domain. If  $D$  is rad-colon coherent, then  $\text{Over}_{\text{sloc}}(D)$  is a spectral space.*

*Proof.* Suppose  $D$  is rad-colon coherent. If  $T \in \text{Over}_{\text{sloc}}(D)$ , then there is a  $\sharp \in \text{SStar}_{sp}(D)$  such that  $T = D^\sharp$ ; since  $D$  is  $D$ -finitely generated, moreover, we have  $D^\sharp = D^{\sharp_f}$ . By Lemma 5.4(b),  $D^{\sharp_f} = D^{\tilde{\sharp}}$ ; but  $\tilde{\sharp} \in \text{SStar}_{f,sp}(D)$ , and thus  $\pi_s$  is surjective.

As in the proof of Proposition 5.3,

$$\pi_s^{-1}(\mathcal{B}(x) \cap \text{Over}_{\text{sloc}}(D)) = V_{(D:Dx)} \cap \text{SStar}_{f,sp}(D),$$

which is compact since it has a minimum  $(s_{\mathcal{D}((D:Dx))})$ . Since  $\text{SStar}_{f,sp}(D)$  is a spectral space [13, Theorem 4.6], by Lemma 5.2  $\text{Over}_{\text{sloc}}(D)$  is spectral.  $\square$

**Corollary 5.6.** *If  $D$  is a domain with Noetherian spectrum (in particular, if  $D$  is Noetherian) then  $\text{Over}_{\text{sloc}}(D)$  is a spectral space.*

Note that it is not hard to see that, if  $\mathcal{D}(J)$  is not compact in  $\text{Spec}(D)$ , then  $V_J \cap \text{SStar}_{f,sp}(D)$  is actually not compact; therefore, the proof of Theorem 5.5 cannot easily be further generalized.

Another natural question is whether  $\pi_s$  is injective; however, this is usually false. For example, if  $\Delta$  is any subset of  $\text{Spec}(D)$  containing the  $t$ -spectrum, then  $\pi_s(s_\Delta) = D$ . Thus,  $\pi_s$  does not give a way to “represent”  $\text{Over}_{\text{sloc}}(D)$  like  $\text{Spec}(D)$  does for  $\text{Loc}(D)$  and  $\mathcal{S}(D)$  for  $\text{Over}_{\text{qr}}(D)$ . To circumvent this problem, we shall use, instead of the whole spectrum, the  $t$ -spectrum; note that  $\text{QSpec}^t(D)$  is a proconstructible subspace of  $\text{Spec}(D)$  [5, Proposition 2.5], so a spectral space, and thus the space  $\mathcal{X}(\text{QSpec}^t(D))$  is defined and spectral.

Consider the map

$$\begin{aligned} \pi_t: \mathcal{X}(\text{QSpec}^t(D)) &\longrightarrow \text{Over}_{\text{sloc}}(D) \\ \Delta &\longmapsto D^{s_\Delta}. \end{aligned}$$

Note that, if  $D$  is rad-colon coherent,  $\pi_t$  is continuous and spectral, since it is the composition of the spectral inclusion  $\mathcal{X}(\text{QSpec}^t(D)) \hookrightarrow \mathcal{X}(D)$  ([15, Proposition 4.1], noting the inclusion  $\text{QSpec}^t(S) \hookrightarrow \text{Spec}(D)$  is spectral since  $\text{QSpec}^t(D)$  is proconstructible), the homeomorphism  $\mathcal{X}(D) \longrightarrow \text{SStar}_{f,sp}(D)$  and the map  $\pi_s: \text{SStar}_{f,sp}(D) \longrightarrow \text{Over}(D)$  (which is spectral by Lemma 5.2 and the proof of Theorem 5.5).

We first show that, using  $\pi_t$ , we do not lose anything.

**Proposition 5.7.** *Let  $D$  be an integral domain. Then:*

- (a) for any  $\Delta, \Lambda \in \mathcal{X}(D)$ , if  $\Delta \cap \text{QSpec}^t(D) = \Lambda \cap \text{QSpec}^t(D)$  then  
 $\pi_s(s_\Delta) = \pi_s(s_\Lambda)$ ;  
 (b)  $\pi_s(\text{SStar}_{f,sp}(D)) = \pi_t(\mathcal{X}(\text{QSpec}^t(D)))$ .

*Proof.* It is enough to show that, for every  $\Delta \in \mathcal{X}(D)$ ,  $\pi_s(\Delta) = \pi_s(\Delta_0)$ , where  $\Delta_0 := \Delta \cap \text{QSpec}^t(D)$ . Let  $T := \pi_s(s_\Delta)$ ; then, since  $\Delta$  is a proconstructible subset of  $\text{Spec}(D)$ , also  $\Delta_0$  is proconstructible. In particular,  $\Delta_0$  is compact and closed by generizations relative to  $\text{QSpec}^t(D)$ , and so it belongs to  $\mathcal{X}(\text{QSpec}^t(D))$ . We claim that  $T = \pi_t(\Delta_0)$ .

Indeed, let  $P \in \Delta$ . Then,  $t_P : ID_P \mapsto I^t D_P$  is a star operation of finite type on  $D_P$  (see [26]), and  $QD_P$  is a maximal  $t_P$ -ideal if and only if  $Q$  is maximal among the  $t$ -prime ideals contained in  $P$ . Hence,  $D_P = \bigcap \{D_Q \mid Q \subseteq P, Q = Q^t\}$ , and

$$T = \bigcap \{D_Q \mid Q = Q^t, Q \subseteq P \text{ for some } P \in \Delta\}.$$

The set of primes on the right hand side is exactly  $\Delta_0$ . Therefore,  $T = \pi_t(\Delta_0) \in \pi_t(\mathcal{X}(\text{QSpec}^t(D)))$ , and (a) is proved.

Moreover, this also shows that  $\pi_s(\text{SStar}_{f,sp}(D)) \subseteq \pi_t(\mathcal{X}(\text{QSpec}^t(D)))$ ; since the other inclusion is obvious, (b) holds.  $\square$

The  $t$ -spectrum is much less redundant than  $\text{Spec}(D)$ : indeed, if  $D = \bigcap \{D_P \mid P \in \Delta\}$  for some compact  $\Delta \subseteq \text{QSpec}^t(D)$ , then  $\Delta$  must contain the  $t$ -maximal ideals, since  $t$  is the biggest star operation of finite type. In general,  $\pi_t$  is not always injective; however, when this happens then  $\pi_t$  is also a homeomorphism, as the next proposition shows.

**Proposition 5.8.** *Let  $D$  be a rad-colon coherent domain. Then, the following are equivalent:*

- (i)  $\pi_t$  is a homeomorphism;
- (ii)  $\pi_t$  is injective;
- (iii) if  $\Delta, \Lambda \in \mathcal{X}(D)$  are such that  $\pi_s(s_\Delta) = \pi_s(s_\Lambda)$ , then  $\Delta \cap \text{QSpec}^t(D) = \Lambda \cap \text{QSpec}^t(D)$ .

*Proof.* The implication (i  $\implies$  ii) is obvious; the equivalence between (ii) and (iii) follows from Proposition 5.7.

Suppose now that  $\pi_t$  is injective; then,  $\pi_t$  is bijective (since it is also surjective by Theorem 5.5, being  $D$  rad-colon coherent), continuous and spectral. Clearly, if  $\Delta \supseteq \Lambda$  then  $\pi_t(\Delta) \subseteq \pi_t(\Lambda)$ . Conversely, suppose  $\pi_t(\Delta) \subseteq \pi_t(\Lambda)$ : then,  $T := \bigcap \{D_P \mid P \in \Delta\} \subseteq \bigcap \{D_Q \mid Q \in \Lambda\}$ , and thus  $T \subseteq D_Q$  for every  $Q \in \Lambda$ . Hence,  $\pi_t(\Delta) = \pi_t(\Delta \cup \Lambda)$ , and by the injectivity of  $\pi_t$  it must be  $\Delta = \Delta \cup \Lambda$ , i.e.,  $\Lambda \subseteq \Delta$ . Therefore,  $\pi_t$  is also an order isomorphism (in the order induced by the respective topologies of  $\mathcal{X}(\text{QSpec}^t(D))$  and  $\text{Over}_{\text{sloc}}(D)$ ); by [25, Proposition 15],  $\pi_t$  is a homeomorphism.  $\square$

A prime ideal  $P$  of  $D$  is *well-behaved* if  $PD_P$  is  $t$ -closed in  $D_P$  [34]; this is equivalent to  $D_P$  being a DW-domain, i.e., to the fact that,

on  $D_P$ , the  $w$ -operation coincides with the identity (this follows from [29, Proposition 2.2]). A domain is called *well-behaved* if every  $t$ -prime ideal is well-behaved; examples of well-behaved domains are Noetherian domains, Krull domains and domains where every  $t$ -prime ideal has height 1.

**Proposition 5.9.** *Let  $D$  be an integral domain. Then,  $D$  is well-behaved if and only if the map  $\pi_t : \mathcal{X}(\text{QSpec}^t(D)) \rightarrow \text{Over}_{\text{sloc}}(D)$  is injective.*

*Proof.* Suppose  $\pi_t$  is injective, and let  $P \in \text{QSpec}^t(D)$  and  $\Delta := \text{QSpec}^t(D_P)$ . Then,  $\Delta$  is compact (being proconstructible in  $\text{Spec}(D_P)$ ), and thus  $\Delta \cap D := \{Q \cap D \mid P \in \Delta\}$  is a compact subspace of  $\text{QSpec}^t(D)$ , since it is the continuous image of  $\Delta$  under the canonical map  $\text{Spec}(D_P) \rightarrow \text{Spec}(D)$ . If  $PD_P \notin \Delta$ , then  $P \notin \Delta \cap D$ ; however,

$$\pi_t(\Delta \cap D) = \bigcap \{D_{Q \cap D} \mid Q \in \Delta\} = \bigcap \{(D_P)_Q \mid Q \in \Delta\} = D_P,$$

with the last equality coming from the properties of the  $t$ -spectrum. If we denote by  $\Lambda_1$  the closure in the inverse topology of  $\text{QSpec}^t(D)$  of  $\Delta \cap D$ , and by  $\Lambda_2$  the closure of  $(\Delta \cap D) \cup \{P\}$ , we have thus  $\pi_t(\Lambda_1) = \pi_t(\Lambda_2)$  while  $\Lambda_1 \neq \Lambda_2$ , against the injectivity of  $\pi_t$ .

On the other hand, suppose  $D$  is well-behaved. Suppose  $\pi_t(\Delta) = \pi_t(\Lambda) =: T$  for some  $\Delta, \Lambda \in \mathcal{X}(\text{QSpec}^t(D))$ ,  $\Delta \neq \Lambda$ , and let  $P \in \Delta \setminus \Lambda$ . By [7, Lemma 2.4], the subspace  $\{D_Q \mid Q \in \Lambda\} \subseteq \text{Over}(D)$  is compact; then,

$$D_P = D_P T = D_P \bigcap_{Q \in \Lambda} D_Q = \bigcap_{Q \in \Lambda} D_P D_Q,$$

with the last equality coming from [17, Corollary 5]. The family  $\{D_P D_Q \mid Q \in \Lambda\}$  is again compact [17, Lemma 4]; thus,  $\star : I \mapsto \bigcap_{Q \in \Lambda} I D_P D_Q$  is a finite-type spectral semistar operation such that  $D^\star = D_P$ , and thus it restricts to a finite-type *star* operation  $\star'$  on  $D_P$ . Since  $PD_P$  is  $t$ -closed, and  $\star'$  is of finite type,  $(PD_P)^{\star'}$  must be equal to  $PD_P$ ; however,

$$P^{\star'} = P^\star = \bigcap_{Q \in \Lambda} PD_Q D_P = \bigcap_{Q \in \Lambda} D_Q D_P = D_P,$$

since  $P \not\subseteq Q$  for every  $Q \in \Lambda$ . This is a contradiction, and  $\pi_t$  is injective.  $\square$

**Remark 5.10.**

- (1) There are examples of integral domains that are not well-behaved (see [34, Section 2] or [1, Example 1.4]), and thus  $\pi_t$  is not always injective.
- (2) It would be tempting to substitute the space  $\mathcal{X}(\text{QSpec}^t(D))$  with  $\mathcal{X}(\Delta)$ , where  $\Delta$  is the set of well-behaved  $t$ -prime ideals of  $D$ . However,  $\Delta$  may not be compact and thus, *a fortiori*, may not be a spectral space. For example, consider a domain  $D$  and

a prime ideal  $Q$  that is a maximal  $t$ -ideal (that is,  $P$  is maximal among the ideals  $I$  such that  $I = I^t$ ) but not well-behaved. (An explicit example is  $E + XE_S[X]$ , where  $E$  is the ring of entire functions,  $X$  is an indeterminate and  $S$  is the set of finite products of elements of the form  $Z - \alpha$ , as  $\alpha$  ranges in  $\mathbb{C}$ ; see [33, Example 2.6, Section 4.1 and Proposition 4.3].) Let  $\Lambda$  be the set of prime ideals that are associated to some principal ideal; then,  $P \in \Lambda$  if and only if  $P$  is minimal over the ideal  $(bD :_D aD)$ , for some  $a, b \in D$ .

Since a principal ideal is  $t$ -closed, so is  $(bD :_D aD) = \frac{b}{a}D \cap D$ ; moreover, a minimal prime over a  $t$ -ideal is again a  $t$ -ideal, and thus  $\Lambda \subseteq \text{QSpec}^t(D)$ . Moreover, if  $P \in \Lambda$  then  $PD_P$  will be associated to a principal ideal of  $D_P$  (if  $P$  is minimal over  $(bD :_D aD)$ , then  $PD_P$  is minimal over  $(bD :_D aD)D_P = (bD_P :_{D_P} aD_P)$ ). Hence, each prime of  $\Lambda$  is well-behaved, and  $\Lambda \subseteq \Delta$ .

By [4], we have  $D = \bigcap \{D_P \mid P \in \Lambda\}$ , and thus also  $D = \bigcap \{D_P \mid P \in \Delta\}$ . If  $\Delta$  were compact, it would define a finite-type star operation  $\star : I \mapsto \bigcap \{ID_P \mid P \in \Delta\}$  such that  $Q^\star = D$ . On the other hand, we should have  $\star \leq t$  and thus  $Q^\star \subseteq Q^t = Q$ , a contradiction. Hence,  $\Delta$  is not compact.

Recall that a domain is  $v$ -coherent if, for any ideal  $I$ ,  $(D : I) = (D : J)$  for some finitely generated ideal  $J$ .

**Corollary 5.11.** *Let  $D$  be a  $v$ -coherent domain. Then,  $\pi_t$  is injective.*

*Proof.* Since  $D$  is  $v$ -coherent,  $(ID_Q)^t = I^t D_Q$  for every ideal  $I$  of  $D$  [26, proof of Proposition 4.6] and every  $Q \in \text{Spec}(D)$ ; thus, if  $P \in \text{QSpec}^t(D)$  then  $(PD_P)^t = P^t D_P = PD_P$ . By Proposition 5.9,  $\pi_t$  is injective.  $\square$

## 6. FLAT OVERRINGS

The space  $\text{Over}_{\text{flat}}(D)$  of flat overrings of  $D$  is much more mysterious than  $\text{Over}_{\text{qr}}(D)$  and  $\text{Over}_{\text{sloc}}(D)$ , and we are not able to characterize when it is spectral or proconstructible. The main theorem of this section is the following partial result.

**Proposition 6.1.** *Let  $D$  be an integral domain. Then,  $\text{Over}_{\text{flat}}(D)$  is a proconstructible subspace of  $\text{Over}(D)$  if and only if  $\text{Over}_{\text{flat}}(D) \cap \mathcal{B}(x_1, \dots, x_n)$  is compact for every  $x_1, \dots, x_n \in K$ .*

*Proof.* If  $\text{Over}_{\text{flat}}(D)$  is proconstructible, the compactness of  $\text{Over}_{\text{flat}}(D) \cap \mathcal{B}(x_1, \dots, x_n)$  follows like in the proof of Proposition 5.1.

Suppose that the compactness property holds, and let  $x_1, \dots, x_n \in K$ . Consider the canonical subbasis  $\mathcal{S} := \{\mathcal{B}(x) \cap X \mid x \in K\}$  of  $X := \text{Over}_{\text{flat}}(D)$ . By [10, Proposition 3.3] and [19, Theorem 8] (or [10, Corollary 2.17]), we need to show that, for every ultrafilter  $\mathcal{U}$  on  $X$ , the ring  $A_{\mathcal{U}} := \{x \in K \mid \mathcal{B}(x) \cap X \in \mathcal{U}\}$  is flat.

Take  $a_1, \dots, a_n \in D$ ,  $x_1, \dots, x_n \in A_{\mathcal{U}}$  such that  $a_1x_1 + \dots + a_nx_n = 0$ . For all  $C \in \text{Over}_{\text{flat}}(D) \cap \mathcal{B}(x_1, \dots, x_n)$ , by the equational characterization of flatness (see e.g. [28, Theorem 7.6] or [9, Corollary 6.5]) there are  $b_{jk}^{(C)} \in D$ ,  $y_k^{(C)} \in C$  such that

$$(2) \quad \begin{cases} 0 = a_1b_{1k}^{(C)} + \dots + a_nb_{nk}^{(C)} & \text{for all } k \\ x_i = b_{i1}^{(C)}y_1^{(C)} + \dots + b_{iN}^{(C)}y_N^{(C)} & \text{for all } i. \end{cases}$$

Let  $\Omega(C) := \mathcal{B}(y_1^{(C)}, \dots, y_n^{(C)})$ . Then, the family of the  $\Omega(C)$  is an open cover of  $\text{Over}_{\text{flat}}(D) \cap \mathcal{B}(x_1, \dots, x_n)$ . Hence, there is a finite subcover  $\{\Omega(C_1), \dots, \Omega(C_n)\}$ ; by the properties of ultrafilters, it follows that  $\Omega(C_j) \in \mathcal{U}$  for some  $j$ . Thus,  $y_i^{(C_j)} \in A_{\mathcal{U}}$  for all  $i$ ; then, (2) holds in  $A_{\mathcal{U}}$ . Hence, applying again the equational criterion,  $A_{\mathcal{U}}$  is flat.  $\square$

**Corollary 6.2.** *Let  $D$  be an integral domain such that  $\text{Over}_{\text{flat}}(D) = \text{Over}_{\text{sloc}}(D)$ . Then,  $\text{Over}_{\text{flat}}(D)$  is a proconstructible subset of  $\text{Over}(D)$ . In particular,  $D$  is rad-colon coherent.*

*Proof.* It is enough to note that  $\text{Over}_{\text{sloc}}(D) \cap \mathcal{B}(x_1, \dots, x_n)$  has always a minimum, and apply Proposition 6.1.  $\square$

**Example 6.3.** The space of flat overrings can be spectral even if it is not proconstructible.

Let  $K$  be a field, and let  $D := K[[X^2, X^3, XY, Y]]$ ; that is,  $D$  is the set of the power series in two variables over  $K$  without the monomial corresponding to  $X$ . Then,  $D$  is a two-dimensional local Noetherian domain; its integral closure is  $A := K[[X, Y]] = D[X]$ , which is also equal to the intersection of the localizations at the height-1 primes of  $D$ . (In particular,  $A$  is a local sublocalization of  $D$  that is not a localization.) By Corollary 5.11, it is easy to see that the sublocalizations of  $D$  are  $D$  itself and the intersections  $T(\Delta) := \bigcap \{D_P \mid P \in \Delta\}$ , as  $\Delta$  ranges among the subsets of  $X^1(D) := \{P \in \text{Spec}(D) \mid P \text{ has height } 1\}$ .

A power series  $\phi := \sum_{i,j \geq 0} a_{ij}X^iY^j$  is invertible in  $A$  if and only if  $a_{00} \neq 0$ ; hence, if  $\phi \in A$  is not invertible then  $\phi^2 \in D$ . Since every height-1 prime ideal of  $A$  is principal (being  $A$  a unique factorization domain) and the canonical map  $\text{Spec}(A) \rightarrow \text{Spec}(D)$  is surjective, every height-1 prime ideal of  $D$  is the radical of a principal ideal (if  $P = Q \cap D$ , for  $Q \in \text{Spec}(A)$ ,  $Q = \phi A$ , then  $P$  is the radical of  $\phi^2 D$ ). Hence,  $T(\Delta)$  is a quotient ring of  $D$  for every  $\Delta \subsetneq X^1(D)$ ; in particular, they are all flat. Hence,  $\text{Over}_{\text{qr}}(D) = \text{Over}_{\text{flat}}(D)$  is spectral; however,  $(D :_D X)$  is equal to the maximal ideal of  $D$ , which cannot be the radical of a principal ideal since it is of height 2. By Theorem 4.4,  $\text{Over}_{\text{qr}}(D)$  (and so  $\text{Over}_{\text{flat}}(D)$ ) is not proconstructible.

The space  $\text{Over}_{\text{flat}}(D)$  is, however, amenable to generalizations. Indeed, if  $R$  is a ring and  $M$  is an  $R$ -module, then the set  $\text{SMod}_R(M)$  of  $R$ -submodules of  $M$  can be endowed with a topology (called the *Zariski*



topology) whose basic open sets are of the form

$$\mathcal{D}(x_1, \dots, x_n) := \{N \in \text{SMod}_R(M) \mid x_1, \dots, x_n \in N\},$$

as  $x_1, \dots, x_n$  vary in  $M$ . Under this topology,  $\text{SMod}_R(M)$  is a spectral space [31, Example 2.2(2)]; moreover, if  $D$  is an integral domain with quotient field  $K$ , then the Zariski topology on  $\text{Over}(D)$  is exactly the restriction of the Zariski topology on  $\text{SMod}_D(K) = \mathbf{F}(D)$ , and  $\text{Over}(D)$  is proconstructible in  $\mathbf{F}(D)$ .

We can consider on  $\text{SMod}_R(M)$  the subspace  $\text{SModFlat}_R(M)$  consisting of all flat  $R$ -submodules of  $M$ . Surprisingly, in many cases spectrality and proconstructibility of  $\text{SModFlat}_R(M)$  are equivalent.

**Proposition 6.4.** *Let  $R$  be a ring and  $M$  be an  $R$ -module; suppose that  $R$  is an integral domain or that  $M$  is torsion-free. Then,  $\text{SModFlat}_R(M)$  is a spectral space if and only if it is proconstructible in  $\text{SMod}_R(M)$ .*

*Proof.* Clearly if  $\text{SModFlat}_R(M)$  is proconstructible in  $\text{SMod}_R(M)$  then it is spectral.

Conversely, suppose that  $Y := \text{SModFlat}_R(M)$  is spectral. By Lemma 2.1,  $Y$  is proconstructible if and only if  $\Omega \cap Y$  is compact for every  $\Omega$  in some subbasis of  $\text{SMod}_R(M)$ ; since  $\mathcal{D}(x_1, \dots, x_n) = \mathcal{D}(x_1) \cap \dots \cap \mathcal{D}(x_n)$  for every  $x_1, \dots, x_n \in M$ , we can consider the subbasis  $\{\mathcal{D}(x) \cap Y \mid x \in M\}$ . By definition,  $\mathcal{D}(x) \cap Y := \{N \in Y \mid x \in N\}$ .

Let  $x \in M$ . If  $x$  has no torsion (so, in particular, if  $M$  is torsion-free), then the principal submodule  $\langle x \rangle$  is isomorphic to  $R$ , which is flat; thus,  $\mathcal{D}(x) \cap Y$  has a minimum, namely  $\langle x \rangle$ , and  $\mathcal{D}(x) \cap Y$  is compact. On the other hand, if  $R$  is an integral domain, then every flat  $R$ -module is torsion-free [3, I.2, Proposition 3]; thus, if  $x$  has torsion then no module containing  $x$  can be flat, and so  $\mathcal{D}(x) \cap Y$  must be empty (and in particular compact).

In all the cases considered, it follows that  $\text{SModFlat}_R(M)$  is proconstructible in  $\text{SMod}_R(M)$ .  $\square$

**Corollary 6.5.** *Let  $D$  be an integral domain with quotient field  $K$ , and suppose that  $D$  is not rad-colon coherent. Then,  $\text{SModFlat}_D(K)$  is not a spectral space.*

*Proof.* The space  $\text{Over}(D)$  is proconstructible in  $\text{SMod}_D(K)$  [31, Example 2.2(5)], and thus  $\text{Over}_{\text{flat}}(D)$  is proconstructible in  $\text{Over}(D)$  if and only if it is proconstructible in  $\text{SMod}_D(K)$ . If  $\text{SModFlat}_D(K)$  were spectral, by Proposition 6.4, it would follow that it is proconstructible in  $\text{SMod}_D(K)$ ; thus, also the intersection  $\text{Over}(D) \cap \text{SMod}_D(K) = \text{Over}_{\text{flat}}(D)$  would be proconstructible in  $\text{SMod}_D(K)$ .

However, if  $D$  is not rad-colon coherent then  $\text{Over}_{\text{flat}}(D)$  is not proconstructible in  $\text{Over}(D)$  (Proposition 3.4); hence,  $\text{SModFlat}_D(K)$  cannot be spectral.  $\square$

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