THE ZARISKI TOPOLOGY ON SETS OF SEMISTAR OPERATIONS WITHOUT FINITE-TYPE ASSUMPTIONS

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Abstract. We study, from a topological point of view, spaces of semistar operations without the hypothesis that they are of finite type. In particular, we analyze when the space of all semistar operations, the space of spectral operations, and the set of stable operations are spectral spaces.

1. Introduction

Semistar operations were introduced by Okabe and Matsuda in [18] as a generalization of the previous concept of star operations, whose origin traces back to Krull [16]. In [12] and [10], the set $\mathrm{SStar}(D)$ of semistar operations on an integral domain D was endowed with a Zariski-like topology, and thus considered from a topological point of view. In particular, it was shown that some distinguished subsets of $\mathrm{SStar}(D)$ are *spectral spaces* [13], that is, they are homeomorphic to the prime spectrum of a commutative ring: more precisely, this was shown for the set of finite-type operations [12, Theorem 2.13], of finite-type spectral operations [10, Theorem 4.6] and of finite-type eab operations [10, Theorem 5.11(3)] (the definitions will be recalled in Section 2). These proofs were carried out by using an ultrafilter-theoretic criterion for spectral spaces (introduced in [8]) and the possibility of writing the supremum of a family of finite-type operations in a relatively explicit way (proved in [2, p.1628] in the setting of star operations).

In this paper, we continue this study outside the finite-type case, showing that several natural-looking spaces need not to be spectral, and trying to characterize when they are spectral. More in detail, we study the set of all semistar operations in Section 4, the set of stable operations in Section 5, and the set of spectral operations in Section 6. While we use an ultrafilter-based proof in Theorem 3.2 (and subsequently use this result in Theorems 4.10 and 5.5), we shall often use a more bare-handed approach, based on the compactness of the

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constructible topology [13, Proposition 4] (on which also the ultrafilter criterion is based), which is tested through the finite intersection property of the closed sets.

2. Preliminaries

2.1. **Semistar operations.** In the paper, we shall always use D to denote an integral domain, and K to denote its quotient field. We will denote by $\mathbf{F}(D)$ the set of D-submodules of K, and by $\mathcal{F}(D)$ the set of fractional ideals of D. For basic properties of semistar operations the reader may consult [18].

A *semistar operation* on D is a map $\star : \mathbf{F}(D) \longrightarrow \mathbf{F}(D)$, $I \mapsto I^\star$, such that, for every $I, J \in \mathbf{F}(D)$ and every $z \in K$, the following properties hold:

- $I \subseteq I^\star$ ($\star$ is *extensive*);
- $(I^\star)^\star = I^\star$ ($\star$ is *idempotent*);
- if $I \subseteq J$, then $I^\star \subseteq J^\star$ ($\star$ is *order-preserving*);
- $z \cdot I^\star = (zI)^\star$.

The set of all semistar operations on D is denoted by $\mathrm{SStar}(D)$; it is a complete lattice, where $\inf \Delta$ is the semistar operation such that

$$I^{\inf \Delta} = \bigcap_{\star \in \Delta} I^\star \quad \text{for every } I \in \mathbf{F}(D),$$

and $\sup \Delta$ is the semistar operation $\sharp$ such that $I = I^\sharp$ if and only if $I = I^\star$ for every $\star \in \Delta$.

Given a semistar operation $\star$, its *quasi-spectrum* (denoted by $\mathrm{QSpec}^\star(D)$) is the set of prime ideals P of D such that $P = P^\star \cap D$.

There are several distinguished subsets of semistar operations.

A semistar operation $\star$ is said to be *of finite type* if, for every $I \in \mathbf{F}(D)$,

$$I^\star = \bigcup \{J^\star \mid J \subseteq I, J \text{ is finitely generated}\};$$

the set of finite-type semistar operations is denoted by $\mathrm{SStar}_f(D)$.

A semistar operation $\star$ is *stable* if, for every $I, J \in \mathbf{F}(D)$, we have

$$(I \cap J)^\star = I^\star \cap J^\star;$$

that is, if $\star$ distributes over (finite) intersections. The set of stable operations is denoted by $\mathrm{SStar}_{st}(D)$.

If $\Delta \subseteq \mathrm{Spec}(D)$, the semistar operation s_Δ is defined by

$$s_\Delta : I \mapsto \bigcap \{ID_P \mid P \in \Delta\}.$$

A semistar operation that can be written as s_Δ, for some Δ, is called *spectral*; any spectral operation is stable, and any finite-type stable operation is spectral [1, Corollary 4.2]. We denote by $\mathrm{SStar}_{sp}(D)$ and $\mathrm{SStar}_{f,sp}(D)$, respectively, the sets of spectral operations and of spectral operations of finite type.

A semistar operation $\star$ is *eab* if, for every $G, H \in \mathbf{F}(D)$ and every finitely-generated $F \in \mathbf{F}(D)$, the inclusion $(FG)^\star \subseteq (FH)^\star$ implies that $G^\star \subseteq H^\star$. We denote by $\mathrm{SStar}_{eab}(D)$ and $\mathrm{SStar}_{f,eab}(D)$, respectively, the sets of eab operations and of eab finite-type operations.

In [12], the set $\mathrm{SStar}(D)$ was endowed with a topology, called the *Zariski topology*, that has as a subbasis the family of the sets in the form

$$V_I := \{\star \in \mathrm{SStar}(D) \mid 1 \in I^\star\},$$

as I ranges in $\mathbf{F}(D)$. In the subspace topology, the spaces $\mathrm{SStar}_f(D)$ [12, Theorem 2.13], $\mathrm{SStar}_{f,sp}(D)$ [10, Theorem 4.6] and $\mathrm{SStar}_{f,eab}(D)$ [10, Theorem 5.11(3)] are spectral spaces (see below for the definition).

2.2. **Spectral spaces.** A topological space X is said to be a *spectral space* if it is homeomorphic to the prime spectrum of a (commutative, unitary) ring. Spectral spaces can be characterized in a purely topological way (see [13] or [4]).

Given a topological space X, the *constructible topology* (or *patch topology*) is the coarsest topology on the set X such that the open and compact subsets of X are both open and closed. When X is a spectral space, the set X endowed with the constructible topology (denoted by X^{cons}) is itself a spectral space, which is also Hausdorff [13, Theorem 1]; furthermore, any subset of X which is closed in X^{cons} is a spectral space when endowed with the original topology [5, 1.9.5(vi-vii)].

The constructible topology allows to characterize when X (with the original topology) is a spectral space; the following is a very slight generalization of Proposition 4 and of the Corollary to Proposition 7 of [13].

Lemma 2.1. *Let X be a T_0 topological space with a subbasis $\mathcal{U}$ of open and compact subsets. Then, X is a spectral space if and only if X^{cons} is compact.*

Proof. If X is a spectral space, then any finite intersection of open and compact subsets is again open and compact; hence, the set $\mathcal{B}$ of the finite intersections of elements of $\mathcal{U}$ is a basis of open and compact subsets closed by finite intersections. By [13, Proposition 4] (or [13, Corollary to Proposition 7]), X^{cons} is compact.

Conversely, suppose X^{cons} is compact. Any $U \in \mathcal{U}$ is, by definition, closed in X^{cons}; hence, every finite intersection B of elements of $\mathcal{U}$ is closed in X^{cons} and, being X^{cons} compact, also B is compact. Moreover, B is open in X (being a finite intersection of open sets); hence, the family $\mathcal{B}$ of finite intersections of elements of $\mathcal{U}$ is a basis of open and compact subsets that is closed by finite intersections. Furthermore, if $x, y \in X$ then (without loss of generality) there is a $B \in \mathcal{B}$ such that $x \in B$ while $y \notin B$. Since B is open and closed in X^{cons}, it follows that X^{cons} is Hausdorff. Again by [13, Proposition 4], X is a spectral space. $\square$

A related criterion was proved in [8, Corollary 3.3]: a topological space X is spectral if and only if it is T_0 and there is a subbasis $\mathcal{S}$ such that, for every ultrafilter $\mathscr{U}$ on X, the set

$$X_{\mathcal{S}}(\mathscr{U}) := \{x \in X \mid \forall S \in \mathcal{S}, x \in S \iff S \in \mathscr{U}\}$$

is nonempty.

The *inverse topology* on X is the coarsest topology such that the open and compact subsets of X are closed. When X is a spectral space, the space X^{inv} (i.e., X endowed with the inverse topology) is again spectral, and a subset of X is closed in the inverse topology if and only if it is compact and closed by generizations (where $Y \subseteq X$ is *closed by generizations* if, whenever $y \in Y$ and $y \in \mathrm{Cl}(z)$ for some $z \in X$, then also $z \in Y$).

Let $\boldsymbol{\mathcal{X}}(X)$ be the set of the nonempty subsets of X that are closed in the inverse topology. In [11] and [9, Section 4], $\boldsymbol{\mathcal{X}}(X)$ was endowed with a natural topology, defined by taking, as a subbasis of open sets, the sets of the form

$$\boldsymbol{\mathcal{U}}(\Omega) := \{Y \in \boldsymbol{\mathcal{X}}(X) \mid Y \subseteq \Omega\},$$

as Ω ranges among the compact open subsets of X. This topology coincides with the upper Vietoris topology [11, Proposition 3.1], and in particular it makes $\boldsymbol{\mathcal{X}}(X)$ a spectral space [11, Theorem 3.4(1)].

2.3. **Overrings.** Let D be an integral domain with quotient field K. An *overring* of D is a ring comprised between D and K; the set $\mathrm{Over}(D)$ can be endowed with a topology (called the *Zariski topology*) whose subbasic open sets are the sets of the form

$$\mathcal{B}_F := \{T \in \mathrm{Over}(D) \mid F \subseteq T\},$$

as F ranges among the finite subsets of K. A distinguished subset of $\mathrm{Over}(D)$ is $\mathrm{Zar}(D)$, the set of all overrings of D that are valuation domains. The space $\mathrm{Over}(D)$ can be topologically embedded into $\mathrm{SStar}_f(D)$ by the map that sends every $T \in \mathrm{Over}(D)$ to the semistar operation $\wedge_{\{T\}} : I \mapsto IT$ [12, Proposition 2.5].

Under this topology, both $\mathrm{Over}(D)$ [8, Proposition 3.5] and $\mathrm{Zar}(D)$ [6, 7] are spectral spaces.

3. Sup-normal subsets

Definition 3.1. *Let D be an integral domain, and let $X \subseteq \mathrm{SStar}(D)$. We say that X is* sup-normal *if, for all $\Delta \subseteq X$ and every $I \in \mathbf{F}(D)$, we have*

$$I^{\sup \Delta} = \bigcup \{I^{\star_1 \circ \cdots \circ \star_n} \mid \star_1, \ldots, \star_n \in \Delta\},$$

where $\circ$ denotes the composition of functions.

Note that the definition is inherited by smaller subsets: that is, if $X_1 \subseteq X_2$ and X_2 is sup-normal, so is X_1. Maybe the most important example of sup-normal set is the family of finite-type semistar operations (this is essentially proved in [2, p.1628]).

The following proof is an abstraction of the proof of [12, Theorem 2.13], whose method was also used in proving [10, Theorem 4.6].

Theorem 3.2. *Let D be an integral domain, and let $X \subseteq \mathrm{SStar}(D)$. Suppose there is a family $\mathcal{A} \subseteq \mathbf{F}(D)$ such that:*

(a) $\mathcal{S} := \{V_I \cap X \mid I \in \mathcal{A}\}$ is a subbasis of the Zariski topology on X;

(b) $\inf(V_I \cap X) \in X$ for all $I \in \mathcal{A}$;

(c) X is sup-normal and $\sup \Delta \in X$ for every $\Delta \subseteq X$.

Then X, endowed with the Zariski topology, is a spectral space.

Proof. Clearly X is T_0. Let now $\mathscr{U}$ be an ultrafilter on X. By [8, Corollary 3.3], to show that X is spectral it is enough to show that

$$X_{\mathcal{S}}(\mathscr{U}) := \{x \in X \mid \text{ for all } B \in \mathcal{S}, B \in \mathscr{U} \iff x \in B\}$$

is nonempty. Let

$$\star := \sup\{\inf(B) \mid B \in \mathscr{U} \cap \mathcal{S}\};$$

we claim that $\star \in X_{\mathcal{S}}(\mathscr{U})$.

If $B = V_I \cap X \in \mathscr{U} \cap \mathcal{S}$, then $\star \geq \inf(B)$ and thus $\star \in B$.

Conversely, suppose $\star \in B = V_I \cap X$ and let Δ be the set formed by the $\inf(B)$, as B ranges in $\mathscr{U} \cap \mathcal{S}$. Then, $1 \in I^\star$, and by sup-normality there are $B_1, \ldots, B_n \in \mathscr{U} \cap \mathcal{S}$ such that $1 \in I^{\star_1 \circ \cdots \circ \star_n}$, where $\star_i := \inf(B_i)$. For any $\sharp \in B_1 \cap \cdots \cap B_n$, we have $\sharp \geq \star_i$ for every i, and thus

$$I^{\star_1 \circ \cdots \circ \star_n} \subseteq I^{\sharp \circ \cdots \circ \sharp} = I^\sharp,$$

so that $1 \in I^\sharp$, i.e., $\sharp \in V_I \cap X$. It follows that $B_1 \cap \cdots \cap B_n \subseteq B$; since $\mathscr{U}$ is an ultrafilter, this means that $B \in \mathscr{U}$.

Hence, $\star \in X_{\mathcal{S}}(\mathscr{U})$, as claimed, and X is a spectral space. □

As a corollary, we can obtain again [12, Theorem 2.13] and [10, Theorem 4.6].

Corollary 3.3. *Let X be the set of finite-type semistar operation or the set of finite-type spectral semistar operations on D. Then, X is a spectral space.*

Proof. If $X = \mathrm{SStar}_f(D)$, then it is enough to take $\mathcal{A}$ to be the set of finitely generated D-submodules of K; the fact that $\mathcal{A}$ satisfies the hypotheses of the theorem follows from [12, Remark 2.2(d) and Proposition 2.11(b)] and [2, p.1628].

If $X = \mathrm{SStar}_{f,sp}(D)$, then take $\mathcal{A}$ to be the set of finitely generated ideals of D; the needed properties follows from the references given in the previous case, with the addition of [10, Lemma 4.4(2)]. □

4. The set of all semistar operations

We start with a slight improvement of [12, Propositions 3.4 and 3.5].

Proposition 4.1. *Let D be an integral domain.*

(a) *If $T \in \mathrm{Over}(D)$, then $\mathrm{SStar}(T)$ is homeomorphic to a closed set of $\mathrm{SStar}(D)^{\mathrm{cons}}$.*
(b) *If $\mathrm{SStar}(D)$ is a spectral space, then $\mathrm{SStar}(T)$ is spectral for all $T \in \mathrm{Over}(D)$.*

Proof. Let $\iota : \mathrm{SStar}(T) \hookrightarrow \mathrm{SStar}(D)$ be the map such that, for every $\star \in \mathrm{SStar}(T)$,

$$\begin{aligned}\iota(\star)\colon \mathbf{F}(D) &\longrightarrow \mathbf{F}(D)\\ I &\longmapsto (IT)^\star.\end{aligned}$$

By [12, Proposition 3.5], ι is a topological embedding; thus, $\mathrm{SStar}(T)$ is homeomorphic to $\iota(\mathrm{SStar}(T))$. By [12, Proposition 3.4], moreover, $\iota(\mathrm{SStar}(T)) = \{\star \in \mathrm{SStar}(D) \mid T \subseteq D^\star\}$; hence,

$$\iota(\mathrm{SStar}(T)) = \bigcap\{V_{x^{-1}D} \mid x \in T\}.$$

Each $V_{x^{-1}D}$ is an open and compact set of $\mathrm{SStar}(D)$, and thus is closed in the constructible topology. Therefore, the right hand side is closed in $\mathrm{SStar}(D)^{\mathrm{cons}}$.

The second claim is immediate. $\square$

Proposition 4.2. *Let D be an integral domain. If $\mathrm{SStar}(D)$ is a spectral space, then $\mathrm{SStar}_f(D)$ is closed in $\mathrm{SStar}(D)^{\mathrm{cons}}$.*

Proof. Given a semistar operation $\star$, let $\star_f$ be the map sending an $I \in \mathbf{F}(D)$ to

$$I^{\star_f} := \bigcup\{J^\star \mid J \subseteq I,\ J \text{ is finitely generated}\}.$$

Then, $\star_f$ is a semistar operation of finite type; furthermore, the map

$$\begin{aligned}\Psi_f\colon \mathrm{SStar}(D) &\longrightarrow \mathrm{SStar}(D)\\ \star &\longmapsto \star_f\end{aligned}$$

is continuous and spectral in the Zariski topology, and its image is $\mathrm{SStar}_f(D)$ [12, Proposition 2.4]. Hence, Ψ_f is a closed map in the constructible topology, and so $\mathrm{SStar}_f(D)$ is closed in $\mathrm{SStar}(D)^{\mathrm{cons}}$. $\square$

The main tool of this section is the following theorem, that shows how the spectrality of $\mathrm{SStar}(D)$ implies a very strong condition on the structure of the set of overrings.

Theorem 4.3. *Let D be an integral domain with quotient field K, and suppose that $\mathrm{SStar}(D)$ is a spectral space. Let $T \neq K$ be an overring of D.*

(a) *There are $x_1, \ldots, x_n \in T$ and $y \in D \setminus \{0\}$ such that $yT \subseteq D[x_1, \ldots, x_n]$.*

(b) There is a $b \in D \setminus \{0\}$ such that $T \subseteq D[b^{-1}]$.

Proof. (a) Suppose there is a T that does non verify this property; fix a $z \in K \setminus T$. Let $X := \mathrm{SStar}(D)$, and consider the family

$$\mathcal{G} := \{V_{w^{-1}D} \mid w \in T\} \cup \{V_{z^{-1}T}, X \setminus V_{z^{-1}D}\}.$$

Each element of $\mathcal{G}$ is closed in the constructible topology: indeed, V_I is always open (by definition) and compact (since it has a minimum) and thus V_I and $X \setminus V_I$ are always closed in the constructible topology.

Then, $\bigcap \mathcal{G} = \emptyset$: indeed, suppose $\star \in \bigcap \mathcal{G}$. Then, $1 \in (w^{-1}D)^\star$ for all $w \in T$, and thus $T \subseteq D^\star$; furthermore, since $1 \in (z^{-1}T)^\star$ we have

$$z \in T^\star \subseteq (D^\star)^\star = D^\star.$$

However, this contradicts the fact that $1 \notin (z^{-1}D)^\star$. Hence, $\star$ cannot exist and $\bigcap \mathcal{G} = \emptyset$.

Suppose now that $\mathcal{H} \subseteq \mathcal{G}$ is finite, and let

$$\widetilde{\mathcal{F}}(D) := \bigcup \{\mathcal{F}(D[x_1, \ldots, x_n]) \mid x_1, \ldots, x_n \in T\}.$$

We want to show that $\bigcap \mathcal{H} \neq \emptyset$: therefore, without loss of generality, we can suppose that $\mathcal{H} = \{V_{w_1^{-1}D}, \ldots, V_{w_n^{-1}D}, V_{z^{-1}T}, X \setminus V_{z^{-1}D}\}$ for some $w_1, \ldots, w_n \in T$. Define $\star$ as the map such that

$$I^\star = \begin{cases} ID[w_1, \ldots, w_n] & \text{if } I \in \widetilde{\mathcal{F}}(D) \\ IT[z] & \text{if } I \in \mathbf{F}(D) \setminus \widetilde{\mathcal{F}}(D). \end{cases}$$

We first note that $\star$ is well-defined, since by hypothesis T is not a fractional ideal over $D[w_1, \ldots, w_n]$. Moreover, $\star$ is obviously extensive and $(xI)^\star = x \cdot I^\star$ for every $x \in K$ and every I. If $I \in \widetilde{\mathcal{F}}(D)$ then $I \in \mathcal{F}(D[y_1, \ldots, y_m])$ for some $y_1, \ldots, y_m \in K$, and

$$I^\star \in \mathcal{F}(D[y_1, \ldots, y_m, w_1, \ldots, w_n]) \subseteq \widetilde{\mathcal{F}}(D);$$

hence, $\star$ is idempotent. If now $I \subseteq J$, then either $I, J \in \widetilde{\mathcal{F}}(D)$, $I, J \notin \widetilde{\mathcal{F}}(D)$ or $I \in \widetilde{\mathcal{F}}(D)$ while $J \notin \widetilde{\mathcal{F}}(D)$. In the first two cases, we clearly have $I^\star \subseteq J^\star$; in the last case,

$$I^\star = ID[w_1, \ldots, w_n] \subseteq IT \subseteq IT[z] \subseteq JT[z] = J^\star.$$

Hence, $\star$ is order-preserving. Therefore, $\star$ is a semistar operation; we claim that $\star \in \bigcap \mathcal{H}$.

Since $D^\star = D[w_1, \ldots, w_n]$, we have $\star \in V_{w_i^{-1}D}$ for $i \in \{1, \ldots, n\}$; moreover, $T^\star = T[z]$ and thus $\star \in V_{z^{-1}T}$. On the other hand, $D[w_1, \ldots, w_n] \subseteq T \subsetneq T[z]$, and thus $\star \notin V_{z^{-1}D}$, i.e., $\star \in X \setminus V_{z^{-1}D}$. Therefore, $\star \in \bigcap \mathcal{H}$.

Hence, $\mathcal{G}$ is a family of closed set of X^{cons} that verifies the finite intersection property, but with empty intersection; hence, X^{cons} cannot be compact and X is not spectral.

(b) By the previous point, there are $x_1,\ldots,x_n\in T$, $y\in D\setminus\{0\}$ such that $yT\subseteq D[x_1,\ldots,x_n]$. There are $a_i,b_i\in D\setminus\{0\}$ such that $x_i=a_i/b_i$ for every i; then,

$$\begin{aligned}T\subseteq&y^{-1}D[x_1,\ldots,x_n]\subseteq D[x_1,\ldots,x_n,y^{-1}]=D\left[\frac{a_1}{b_1},\ldots,\frac{a_n}{b_n},\frac{1}{y}\right]\subseteq\\ \subseteq&D[b_1^{-1},\ldots,b_n^{-1},y^{-1}]=D[(b_1\cdots b_ny)^{-1}].\end{aligned}$$

Taking $b:=b_1\cdots b_ny$ we have our claim. □

Our first application involves the complete integral closure. Recall that, if D is a domain and K its quotient field, an element $x\in K$ is *almost integral* over D if there is a $c\in D$, $c\neq 0$ such that $cx^n\in D$ for every $n\in\mathbb{N}$. The set of elements of K that are almost integral over D is a ring, called the *complete integral closure* of D.

Proposition 4.4. *Let D be an integral domain with quotient field K, and let T be its complete integral closure. If $K\neq T$ and T is not a fractional ideal over D, then* $\mathrm{SStar}(D)$ *is not a spectral space.*

Proof. If $\mathrm{SStar}(D)$ were a spectral space, by the previous theorem there would be $x_1,\ldots,x_n\in T$, $y\in D\setminus\{0\}$ such that $yT\subseteq D[x_1,\ldots,x_n]$. By definition, each x_i is almost integral over D; let $c_i\in D\setminus\{0\}$ be such that $c_ix_i^k\in D$ for every $k\in\mathbb{N}$. Then, $cx_i^k\in D$ for every i and k, where $c:=c_1\cdots c_n$; hence, $cD[x_1,\ldots,x_n]\subseteq D$. But this implies that $ycT\subseteq D$, i.e., that T is a fractional ideal over D, against the hypothesis. □

To prove Proposition 4.4, we needed part (a) of Theorem 4.3, and it would not be enough to use part (b). For example, if D is a one-dimensional local ring with complete integral closure T, then T satisfies condition (b) (since $K=D[b^{-1}]$ for any b in the maximal ideal of D); however, if T is different from K and not a fractional ideal of D, then it does not satisfy condition (a). This happens, for example, if D is a Noetherian one-dimensional local domain whose integral closure is not finitely generated over D.

However, (b) is quite useful, as the next result shows.

Proposition 4.5. *Let D be an integral domain, and suppose there is a set Δ of pairwise uncomparable prime ideals of D such that $\bigcap\{P\mid P\in\Delta\}=(0)$. Then,* $\mathrm{SStar}(D)$ *is not a spectral space.*

Proof. Take any $Q\in\Delta$. If $\mathrm{SStar}(D)$ is spectral, there is a $b\in D\setminus\{0\}$ such that $D_Q\subseteq D[b^{-1}]$; in particular, $PD[b^{-1}]=D[b^{-1}]$ for every $P\in\Delta\setminus\{Q\}$, and thus $b\in P$ for every such P. Therefore,

$$bD\cap Q\subseteq\bigcap_{P\in\Delta\setminus\{Q\}}P\cap Q=\bigcap_{P\in\Delta}P=(0).$$

However, neither bD nor Q are equal to (0); since D is an integral domain, this is a contradiction. Hence, $\mathrm{SStar}(D)$ is not spectral. □

Corollary 4.6. *Let D be an integral domain which is not a field and whose Jacobson radical is* (0)*. Then,* $\mathrm{SStar}(D)$ *is not a spectral space.*

Proof. Apply Proposition 4.5 with $\Delta = \mathrm{Max}(D)$. $\square$

4.1. **The Noetherian case.** In the Noetherian context, Proposition 4.5 has a straightforward consequence.

Proposition 4.7. *Let D be a Noetherian or a Krull domain. If* $\dim(D) > 1$*, or if D has infinitely many maximal ideals, then* $\mathrm{SStar}(D)$ *is not a spectral space.*

Proof. Under both hypotheses, we can apply Proposition 4.5 to the set of height-one prime ideals of D. Thus, $\mathrm{SStar}(D)$ cannot be spectral. $\square$

Let now D be a Noetherian domain of dimension 1. Then, the Zariski space $\mathrm{Zar}(D)$ of D consists exactly of the quotient field K and the localizations of the integral closure $\overline{D}$ of D. For any $I \in \mathbf{F}(D)$, define

$$\mathcal{R}(I) := \{V \in \mathrm{Zar}(D) \mid IV \neq K\}.$$

This set can be characterized in a different way.

Lemma 4.8. *Preserve the notation and the hypothesis above. Then, $V \in \mathcal{R}(I)$ if and only if $(I : I) \subseteq V$.*

Proof. If $V \in \mathcal{R}(I)$, then

$$(I : I) \subseteq (IV : IV) = V,$$

with the last equality coming from the fact that IV is a fractional ideal of the one-dimensional valuation domain V.

Conversely, suppose $(I : I) \subseteq V$. Since I is a fractional ideal over $(I : I)$, IV is a fractional ideal over $(I : I)V = V$; in particular, $IV \neq K$. $\square$

Lemma 4.9. *Let D be a one-dimensional Noetherian domain with $|\mathrm{Zar}(D)| < \infty$, let $I \in \mathcal{F}(D)$ and let $T := \bigcap\{V \mid V \in \mathcal{R}(I)\}$. Then:*

(a) T is the integral closure of $(I : I)$;
(b) there is a $x \in K \setminus \{0\}$ such that $xI \subseteq T$;
(c) if $U \in \mathrm{Over}(D)$ is integrally closed and $yI \subseteq U$ for some $y \in K \setminus \{0\}$, then $T \subseteq U$.

Proof. (a) follows directly from Lemma 4.8 (and holds even without the assumption that $|\mathrm{Zar}(D)| < \infty$). To prove (b), let $\mathcal{R}(I) := \{V_1, \dots, V_n\}$; then, for every i there is a $y_i \in D$, $y_i \neq 0$ such that $y_iIV \subseteq V$. Thus, $y_1 \cdots y_nIV_i \subseteq V_i$ for each i, and so

$$y_1 \cdots y_nI \subseteq \bigcap_{i=1}^{n} y_1 \cdots y_nIV_i \subseteq V_1 \cap \cdots \cap V_n = T.$$

Suppose now that $yI \subseteq U$: if $T \nsubseteq U$, then there is a $V \in \mathrm{Zar}(D)$ such that $U \subseteq V$ but $T \nsubseteq V$. Hence, $yIV \subseteq UV = V$, and thus

$IV \neq K$; however, this would mean that $V \in \mathcal{R}(I)$ and thus $T \subseteq V$, a contradiction. (c) is proved. $\square$

We are ready to characterize when $\mathrm{SStar}(D)$ is spectral.

Theorem 4.10. *Let D be a Noetherian integral domain. Then,* $\mathrm{SStar}(D)$ *is a spectral space if and only if D is semilocal,* $\dim(D) = 1$ *and the integral closure of D is finite over D.*

Proof. If $\dim(D) > 1$, or if D is not semilocal, then $\mathrm{SStar}(D)$ is not spectral by Proposition 4.7. Moreover, if D is Noetherian and $\overline{D}$ is its integral closure, then by Theorem 4.3 there are $x_1, \ldots, x_n \in \overline{D}$ and $y \in D \setminus \{0\}$ such that $y\overline{D} \subseteq D[x_1, \ldots, x_n]$. Since each x_i is integral over D, $D[x_1, \ldots, x_n]$ is a finitely-generated module over D; in particular, also $\overline{D}$ must be finitely generated, as claimed.

Conversely, suppose that D verifies the three hypotheses. Then, every overring T of D satisfies the same properties: indeed, every overring of D is Noetherian of dimension 1 [15, Theorem 93], and it is semilocal because its integral closure is an overring of the semilocal Dedekind domain $\overline{D}$. Furthermore, its integral closure is finite over T: indeed, if $\overline{D} = D[x_1, \ldots, x_n]$ then $T[x_1, \ldots, x_n]$ is an overring of T which is integral over T (the x_i are integral over D, so are integral over T) and integrally closed (being also an overring of $\overline{D}$, which is a Dedekind domain); hence, $T[x_1, \ldots, x_n]$ is the integral closure of T and is a finitely generated T-module.

For every $T \in \mathrm{Over}(\overline{D})$, let now

$$\widehat{\mathcal{F}}(T) := \left\{ I \in \mathbf{F}(D) \mid T = \bigcap \{V \mid V \in \mathcal{R}(I)\} \right\}.$$

Clearly, $\mathbf{F}(D)$ is the disjoint union of $\widehat{\mathcal{F}}(T)$, as T ranges in $\mathrm{Over}(\overline{D})$. Furthermore, $\mathcal{R}(I) = \mathcal{R}(xI)$ for every $x \neq 0$, and thus I and xI belong to the same $\widehat{\mathcal{F}}(T)$.

Take now a $\star \in \mathrm{SStar}(D)$, and suppose that $I^\star \in \widehat{\mathcal{F}}(T)$ for some $I \in \mathbf{F}(D)$ and $T \in \mathrm{Over}(\overline{D})$; we claim that $T = T^\star$. Consider the ring $R := (I^\star : I^\star)$: then, R is $\star$-closed, and by Lemma 4.9(a) the integral closure of R is T. By hypothesis, T is finite over R, and thus a fractional ideal of R; since $R = R^\star$, also $T^\star$ is a fractional ideal over R. However, T is maximal among the elements of $\mathrm{Over}(R)$ (or $\mathrm{Over}(D)$) that are also R-fractional ideals; hence, it must be $T^\star = T$. Conversely, if $T = T^\star$ then $\star$ restricts to a map from $\widehat{\mathcal{F}}(T)$ to itself.

Let now Δ be a family of semistar operations on D, and let $\star := \sup \Delta$. Suppose that $x \in I^\star$, and let $T_1, T_2 \in \mathrm{Over}(\overline{D})$ be such that $I \in \widehat{\mathcal{F}}(T_1)$ and $I^\star \in \widehat{\mathcal{F}}(T_2)$. Then, $T_1 \subseteq T_2$: if $xI \subseteq T_1$ and $yI^\star \subseteq T_2$, take a $d \in D$ such that $dxy^{-1} \in D$; then,

$$dxI \subseteq dxI^\star = dxy^{-1}yI^\star \subseteq dxy^{-1}T_2 \subseteq T_2,$$

and thus $T_1 \subseteq T_2$ by Lemma 4.9(c). No ring T such that $T_1 \subseteq T \subsetneq T_2$ can be $\star$-closed: if $xI \subseteq T_1$, then

$$T^\star \supseteq T_1^\star \supseteq (xI)^\star = xI^\star \in \widehat{F}(T_2),$$

and thus $T = T^\star$ would imply $T \in \widehat{F}(T_2)$, a contradiction. Therefore, for each such T there must be a $\star_T \in \Delta$ such that $T \neq T^{\star_T}$. However, there are only a finite number of rings comprised between T_1 and T_2; hence, we can find $\star_1, \dots, \star_m \in \Delta$ such that $J := I^{\star_1 \circ \cdots \circ \star_m} \in \widehat{\mathcal{F}}(T_2)$.

Let now $R := (J : J)^\star$. Then, by the previous reasoning, each $\sharp \in \Delta$ restricts to a map from $\mathcal{F}(R)$ to itself, and each of these maps fixes R; therefore, all the restrictions are star operations on R. Since R is Noetherian, they are all of finite type; by [2, p.1628], we can find $\star_{m+1}, \dots, \star_n \in \Delta$ such that $x \in J^{\star_{m+1} \circ \cdots \circ \star_n}$. This means that

$$x \in (I^{\star_1 \circ \cdots \circ \star_m})^{\star_{m+1} \circ \cdots \circ \star_n} = I^{\star_1 \circ \cdots \circ \star_n}.$$

Hence, $\mathrm{SStar}(D)$ is sup-normal. By Theorem 3.2, applied with $\mathcal{A} = \mathbf{F}(D)$, it follows that $\mathrm{SStar}(D)$ is a spectral space. □

Corollary 4.11. *Let D be a Krull domain. Then, $\mathrm{SStar}(D)$ is a spectral space if and only if D is semilocal and $\dim(D) = 1$.*

Proof. If D is not semilocal, or if $\dim(D) > 1$, then $\mathrm{SStar}(D)$ is not spectral by Lemma 4.8. Conversely, if D is semilocal of dimension 1 then D is Noetherian and we can apply Theorem 4.10. □

4.2. **Prüfer domains.** The case of Prüfer domains is considerably more complex than the case of Noetherian domains. The first result involves quotients, and has a rôle similar to Proposition 4.1.

Proposition 4.12. *Let D be a Prüfer domain.*

- *(a) If $P \in \mathrm{Spec}(D)$, then $\mathrm{SStar}(D/P)$ is homeomorphic of a closed set of $\mathrm{SStar}(D)^{\mathrm{cons}}$.*
- *(b) If $\mathrm{SStar}(D)$ is a spectral space, then $\mathrm{SStar}(D/P)$ is spectral for every $P \in \mathrm{Spec}(D)$.*

Proof. Let $T := \bigcap\{D_Q \mid Q \in V(P)\}$. Since D is Prüfer, T is an overring of D such that every ideal is comparable with P; moreover, $D/P \simeq T/P$. By Proposition 4.1, $\mathrm{SStar}(T)$ is closed in $\mathrm{SStar}(D)^{\mathrm{cons}}$; hence, the claim will follow if we can prove that $\mathrm{SStar}(T/P)$ is homeomorphic to a closed set of $\mathrm{SStar}(T)^{\mathrm{cons}}$.

By [20, Proposition 2.2] (which uses the proofs of [14, Lemmas 2.3 and 2.4]), there is an isomorphism of partially ordered set Ψ between $\mathrm{SStar}(T/P)$ and the set $\Delta := \{\star \in \mathrm{SStar}(T) \mid P = P^\star\}$, defined as follows. Let $\varphi : T \longrightarrow T/P$ be the quotient map, let $\sharp \in \mathrm{SStar}(T/P)$ and $I \in \mathbf{F}(T)$. If $\mathbf{v}_P(I)$ has no infimum in $\mathbf{v}_P(K)$ (where $\mathbf{v}_P$ is the valuation relative to T_P), then $I^{\Psi(\sharp)} := I$; if $\inf \mathbf{v}_P(I)$ is attained at α then

$$I^{\Psi(\sharp)} := \alpha \cdot \varphi^{-1}[(\varphi(\alpha^{-1}I))^\sharp].$$

Conversely, if $\star \in \mathrm{SStar}(T)$ then
$$I^{\Psi^{-1}(\star)} := \varphi(\varphi^{-1}(I)^{\star}) \quad \text{for } I \in \mathbf{F}(T/P).$$

We claim that this bijection is also a homeomorphism. Indeed, suppose $I \in \mathbf{F}(T)$. If $\mathbf{v}_P(I)$ has no infimum, then $\Psi^{-1}(V_I \cap \Delta)$ is either empty (if $1 \notin I$) or the whole $\mathrm{SStar}(T)$ (if $1 \in I$), and in both cases it is an open set. On the other hand, if $\mathbf{v}_P(\alpha) = \inf \mathbf{v}_P(I)$, then $\sharp \in \Psi^{-1}(I)$ if and only if
$$1 \in \alpha \cdot \varphi^{-1}[(\varphi(\alpha^{-1}I))^{\sharp}];$$
applying φ, this is equivalent to
$$1 \in \varphi(\alpha)(\varphi(\alpha^{-1}I))^{\sharp},$$
i.e., $\sharp \in V_J$, where $J := \varphi(\alpha)\varphi(\alpha^{-1}I)$. In particular, it is open, and Ψ is continuous.

Likewise, if $I \in \mathbf{F}(T/P)$, then $\Psi(V_I) = V_{\varphi^{-1}(I)}$, and thus Ψ is also an open map. Hence, Ψ is a homeomorphism.

Now Δ is a closed set of $\mathrm{SStar}(T)$: indeed, it is equal to $\bigcap\{X \setminus V_{w^{-1}P} \mid w \in K \setminus P\}$. In particular, it is a closed set of the constructible topology of $\mathrm{SStar}(T)$ and $\mathrm{SStar}(D)$, as requested.

The second claim is immediate. □

In order to apply Theorem 4.3, we need to look at the prime ideals that survive in a finitely-generated algebra over D. We denote by $\mathcal{D}(I)$ the open set of $\mathrm{Spec}(D)$ associated to I, i.e., $\mathcal{D}(I) := \{P \in \mathrm{Spec}(D) \mid I \nsubseteq P\}$.

Lemma 4.13. *Let D be a Prüfer domain with quotient field K, $P \in \mathrm{Spec}(D)$, $x_1, \ldots, x_n \in K$. Then, $PD[x_1, \ldots, x_n] \neq D[x_1, \ldots, x_n]$ if and only if $P \in \mathcal{D}((D :_D x_1) \cap \cdots \cap (D :_D x_n))$.*

Proof. Let $T := D[x_1, \ldots, x_n]$. We have $PT \neq T$ if and only if $T \subseteq D_P$, i.e., if and only if $x_1, \ldots, x_n \in D_P$. Now $x \in D_P$ if and only if $(D_P :_{D_P} x) = D_P$, i.e., if and only if $(D :_D x)D_P = D_P$, which is equivalent to $(D :_D x) \nsubseteq P$, i.e., to $P \in \mathcal{D}((D :_D x))$. Hence, $PT \neq T$ if and only if $P \in \mathcal{D}((D :_D x_i))$ for every i; the claim follows from the fact that, for any ideal $I_1, \ldots, I_n$ of D, we have $\mathcal{D}(I_1 \cap \cdots \cap I_n) = \mathcal{D}(I_1) \cap \cdots \cap \mathcal{D}(I_n)$. □

Given a prime ideal P, we say that $Q_1, Q_2 \in V(P) \setminus \{P\}$ are *dependent with respect to* P, and we write $Q_1 \sim_P Q_2$, if there is a $P' \in \mathrm{Spec}(D)$ such that $P \subsetneq P' \subseteq Q_1 \cap Q_2$. If D is a Prüfer domain (more generally, if D is a treed domain), then $\sim_P$ is an equivalence relation on $V(P) \setminus \{P\}$. When $P = (0)$, we simply denote $\sim_{(0)}$ with $\sim$.

Lemma 4.14. *Let D be a Prüfer domain, and let K be its quotient field; let V be a valuation overring of D with maximal ideal $\mathfrak{m}$. If $yV \subseteq D$ for some $y \in K \setminus \{0\}$, then $(\mathfrak{m} \cap D) \sim M$ for every maximal ideal M of D.*

Proof. Suppose not: then, there is a maximal ideal M of D such that M and $P := \mathfrak{m} \cap D$ are not dependent, i.e., such that $M \cap P$ does not contain nonzero prime ideals. Hence, the product $D_M V$ is equal to K; however, $yV \subseteq D$ implies $yVD_M \subseteq D_M$, i.e., $yK \subseteq D_M$. Since $y \neq 0$, this is impossible. □

Theorem 4.15. *Let D be a Prüfer domain. If* $\mathrm{SStar}(D)$ *is a spectral space, then for every $P \in \mathrm{Spec}(D)$ the relation $\sim_P$ has only finitely many equivalence classes.*

Proof. If $\mathrm{SStar}(D)$ is spectral, by Proposition 4.12 so is $\mathrm{SStar}(D/P)$, for every $P \in \mathrm{Spec}(D)$; hence, without loss of generality we can suppose $P = (0)$. For $T \in \mathrm{Over}(D)$, let $\mathbf{\Delta}(T)$ be the set of equivalence classes of $\mathrm{Spec}(T) \setminus \{(0)\}$ with respect to $\sim$.

Let Q be a nonzero prime ideal of D. By Theorem 4.3, there are $x_1, \ldots, x_n \in D_Q$, $y \in D \setminus \{0\}$ such that $yD_Q \subseteq D[x_1, \ldots, x_n] =: T$. By construction, $T \subseteq D_Q$: hence, $QT \neq T$, and thus, by Lemma 4.13, Q is contained in the open set $\Omega_Q := \mathcal{D}((D :_D x_1) \cap \cdots \cap (D :_D x_n))$. Moreover, by Lemma 4.14, the ideal $P := QD_Q \cap T$ is dependent (with respect to (0)) from every maximal ideal of T. Hence, $Q \sim M_0$ for every nonzero prime ideal M_0 of D surviving in T. Applying again Lemma 4.13, this means that $\Omega_Q \setminus \{(0)\}$ is contained into the equivalence class of Q modulo $\sim$.

However, the containment $Q \in \Omega_Q$ implies that $\{\Omega_Q \mid Q \in \mathrm{Spec}(D), Q \neq (0)\}$ is an open cover of $\mathrm{Spec}(D)$; by compactness, there is a finite subcover. By the previous reasoning, this implies that $\sim$ admits only finitely many equivalence classes, as claimed. □

Proposition 4.16. *Let D be a finite-dimensional Prüfer domain. Then,* $\mathrm{SStar}(D)$ *is a spectral space if and only if* $\mathrm{Spec}(D)$ *is finite.*

Proof. If $\mathrm{Spec}(D)$ is finite, so is $\mathrm{SStar}(D)$ [14, Theorem 4.5], and since it is a T_0 space it is also a spectral space. Conversely, suppose that $\mathrm{SStar}(D)$ is spectral, and let $\mathrm{Spec}^i(D) := \{P \in \mathrm{Spec}(D) \mid h(P) = i\}$. If $\mathrm{Spec}(D)$ were infinite, there would be a n such that $\mathrm{Spec}^{n-1}(D)$ is finite while $\mathrm{Spec}^n(D)$ is not: in particular, there would be a $P \in \mathrm{Spec}^{n-1}(D)$ which is below infinitely many primes of $\mathrm{Spec}^n(D)$. However, this would imply that $\sim_P$ has infinitely many equivalence classes, contradicting Theorem 4.15. Hence, $\mathrm{Spec}(D)$ is finite. □

When the dimension of D is infinite, things are not so clear. If $D = V$ is a valuation domain, for example, every star operation is stable; in particular, if $\mathrm{Spec}(V)$ is Noetherian then the following Theorem 5.5 implies that $\mathrm{SStar}(V)$ is a spectral space. On the other hand, adding more maximal ideals we can lose the spectrality.

Proposition 4.17. *Let D be a Prüfer domain with quotient field K, and suppose there are a chain Δ of prime ideals and a $N \in$* $\mathrm{Spec}(D)$ *such that:*

- $\bigcap\{P \mid P \in \Delta\} = Q \notin \Delta$;
- $D_P D_N = K$ *for every* $P \in \Delta$;
- $Q \subseteq N$.

Then, $\mathrm{SStar}(D)$ *is not a spectral space.*

Proof. Suppose $\mathrm{SStar}(D)$ is a spectral space. By Proposition 4.12, also $\mathrm{SStar}(D/Q)$ is spectral; since Q contains both N and every $P \in \Delta$, it follows that, without loss of generality, we can suppose $Q = (0)$.

By Theorem 4.3, there is a $b \in D \setminus \{0\}$ such that $D_N \subseteq D[b^{-1}]$; hence, for every $P \in \Delta$,

$$K = D_N D_P \subseteq D[b^{-1}]D_P = D_P[b^{-1}].$$

Hence, $b \in P$ for every $P \in \Delta$; however, since $\bigcap_{P \in \Delta} P = (0)$, this means that $b = 0$, a contradiction. Therefore, $\mathrm{SStar}(D)$ cannot be spectral. $\square$

For example, suppose that D has only two maximal ideals M and N, and suppose that $M \cap N$ does not contain any nonzero prime ideal, and that $\bigcap\{P \in \mathrm{Spec}(D) \mid (0) \neq P \subseteq M\} = (0)$. Then, by the previous proposition, $\mathrm{SStar}(D)$ is not a spectral space. Note that, in this case, $\mathrm{Spec}(D)$ is a a Noetherian space.

5. Stable operations

We start with an analogue of Proposition 4.1.

Proposition 5.1. *Let D be an integral domain.*

(a) $\mathrm{SStar}_{st}(D)$ *is a closed set of* $\mathrm{SStar}(D)^{\mathrm{cons}}$.
(b) If $\mathrm{SStar}(D)$ *is a spectral space, so is* $\mathrm{SStar}_{st}(D)$.

Proof. For every $I, J \in \mathbf{F}(D)$, let $\mathcal{V}(I,J)$ be the union of the four sets

$$\begin{aligned}
\mathcal{V}_1(I,J) &:= V_{I\cap J} \cap V_I \cap V_J,\\
\mathcal{V}_2(I,J) &:= V_I \setminus (V_{I\cap J} \cup V_J),\\
\mathcal{V}_3(I,J) &:= V_J \setminus (V_{I\cap J} \cup V_I) \quad \text{and}\\
\mathcal{V}_4(I,J) &:= X \setminus (V_{I\cap J} \cup V_I \cup V_J).
\end{aligned}$$

We claim that

$$\mathrm{SStar}_{st}(D) = \bigcap_{I,J \in \mathbf{F}(D)} \mathcal{V}(I,J).$$

Indeed, suppose $\star \in \mathrm{SStar}_{st}(D)$, and take $I, J \in \mathbf{F}(D)$. There are four possibilities:

- if $1 \in I^\star$ and $1 \in J^\star$ then $1 \in I^\star \cap J^\star = (I \cap J)^\star$, and thus $\star \in \mathcal{V}_1(I,J)$;
- if $1 \in I^\star$ but $1 \notin J^\star$, then $1 \notin (I \cap J)^\star$ and thus $\star \in \mathcal{V}_2(I,J)$;
- if $1 \notin I^\star$ but $1 \in J^\star$, symmetrically, $\star \in \mathcal{V}_3(I,J)$;
- if $1 \notin I^\star$ and $1 \notin J^\star$ then $\star \in \mathcal{V}_4(I,J)$.

In all cases, $\star \in \mathcal{V}(I, J)$; hence, $\star \in \bigcap \mathcal{V}(I, J)$.

Suppose now $\star$ is in the intersection, and take $I, J \in \mathbf{F}(D)$. We always have $(I \cap J)^\star \subseteq I^\star \cap J^\star$; suppose $x \in I^\star \cap J^\star$. Then, $\star \in V_{x^{-1}I}$ and $\star \in V_{x^{-1}J}$; since $\star \in \mathcal{V}(x^{-1}I, x^{-1}J)$, we must have $\star \in \mathcal{V}_1(x^{-1}I, x^{-1}J)$, and thus $x \in (I \cap J)^\star$. Hence, $(I \cap J)^\star = I^\star \cap J^\star$. Since this holds for every I and J, $\star$ is stable.

Now each $\mathcal{V}_i(I, J)$ is closed in $\mathrm{SStar}(D)^{\mathrm{cons}}$ (since each V_I is open and compact in $\mathrm{SStar}(D)$), and thus $\mathcal{V}(I, J)$ is always closed in the constructible topology. Therefore, their intersection is closed too, and $\mathrm{SStar}_{st}(D)$ is closed in $\mathrm{SStar}(D)^{\mathrm{cons}}$.

The second claim follows easily. $\square$

Contrary to Section 4.1, for stable operations the case of Noetherian domains is essentially trivial.

Proposition 5.2. *Let D be a Noetherian domain. Then,* $\mathrm{SStar}_{st}(D)$ *is a spectral space.*

Proof. Let $\star$ be a stable semistar operation on D. By [10, Proposition 3.4], $\star$ is uniquely determined by $\sharp := \star|_{\mathcal{F}(D)}$; since D is Noetherian, $\sharp$ is of finite type, and thus so is $\star$. Hence, $\star$ is spectral [1, Corollary 4.2], and thus $\mathrm{SStar}_{st}(D) = \mathrm{SStar}_{sp}(D) = \mathrm{SStar}_{f,sp}(D)$. The latter is spectral by [10, Theorem 4.6], and thus $\mathrm{SStar}_{st}(D)$ is spectral. $\square$

In general, it is possible that $\mathrm{SStar}_{st}(D)$ is not a spectral space. For example, let D be an almost Dedekind domain such that all the maximal ideals are principal except for one, say M; suppose also that M is not finitely generated (i.e., that D is not a Dedekind domain). Such a domain exist; see e.g. [17]. For any $I \in \mathbf{F}(D)$, let $U_I := V_I \cap \mathrm{SStar}_{st}(D)$, and let

$$\mathcal{U}_0 := \{U_P \mid P \in \mathrm{Max}(D)\};$$

we claim that $\bigcap \mathcal{U}_0 = \{\wedge_{\{K\}} = s_{\{(0)\}}\}$.

Indeed, suppose that $\star \in \bigcap \mathcal{U}_0$, and let $T := D^\star$. If $P = (p)$ is a principal prime ideal, then

$$1 \in (pD)^\star = pD^\star = pT,$$

and thus $p^{-1} \in T$; hence, the only maximal ideal of D that could survive in T is M, and so $T \in \{D_M, K\}$. However, $\star \in U_M$, and thus $1 \in M^\star \subseteq (MD_M)^\star$; in particular, MD_M is not $\star$-closed. But $MD_M = mD_M$ for some $m \in K$, and thus neither D_M is $\star$-closed; it follows that T must be K, and $\star$ must be $\wedge_{\{K\}}$.

Let J be a nonzero ideal of D contained in infinitely many maximal ideals, and consider $\mathcal{U} := \mathcal{U}_0 \cup \{\mathrm{SStar}_{st}(D) \setminus U_J\}$. Clearly, $\bigcap \mathcal{U} = \emptyset$. On the other hand, if $\mathcal{G} := \{U_{P_1}, \ldots, U_{P_n}, U_M, \mathrm{SStar}_{st}(D) \setminus U_J\}$ is a finite subfamily of $\mathcal{U}$, consider $\Delta := \mathrm{Max}(D) \setminus \{P_1, \ldots, P_n, M\}$. Then, $1 \in P_i^{s_\Delta}$, as well as $1 \in M^{s_\Delta}$; on the other hand, there is at least a minimal prime of J contained in Δ, and thus $1 \notin J^{s_\Delta}$. Hence, $s_\Delta \in \bigcap \mathcal{G}$.

Therefore, $\mathcal{U}$ is a family of closed sets of X^{cons} with the finite intersection property but with empty intersection; hence, X^{cons} is not compact, and X cannot be spectral.

We shall consider one case where we can prove that the space $\mathrm{SStar}_{st}(D)$ is spectral: namely, the case when D is a Prüfer domain where every ideal has only finitely many minimal primes. To this aim, we shall use the following characterization, proved in [19]: under this hypothesis, if $\star$ is a stable semistar operation, then

$$I^\star = \bigcap_{P\in\Delta_1(\star)} ID_P \cap \bigcap_{P\in\Delta_2(\star)} (ID_P)^{v_{D_P}} \tag{1}$$

for every $I \in \mathbf{F}(D)$, where

- $\Delta_1(\star) := \{P \in \mathrm{Spec}(D) \mid 1 \notin P^\star\} = \mathrm{QSpec}^\star(D)$,
- $\Delta_2(\star) := \{P \in \mathrm{Spec}(D) \mid 1 \in P^\star,\ 1 \notin Q^\star$ for some P-primary ideal $Q\}$.

We first need a result of independent interest.

Proposition 5.3. *Let D be a domain. The supremum of a family of stable semistar operations on D is stable.*

Proof. For any semistar operation $\star$, consider the semistar operation $\overline{\star}$ defined by

$$I^{\overline{\star}} := \bigcup\{(I : E) \mid E \subseteq D,\ E^\sharp = D^\sharp\};$$

then, $\overline{\star}$ is the biggest stable semistar operation smaller than $\star$ [3, Theorem 2.14]. In particular, $\star = \overline{\star}$ if and only if $\star$ is stable [3, Theorem 2.6].

Let now $\mathcal{A}$ be a family of stable semistar operations, and let $\sharp$ be its supremum in $\mathrm{SStar}(D)$. If $\sharp$ is not stable, then in particular $\overline{\sharp} < \sharp$, and thus there is a $\star \in \mathcal{A}$ such that $\star \not\leq \overline{\sharp}$. Let now $I \in \mathbf{F}(D)$. Since $\star$ is stable, $\star = \overline{\star}$; thus, for every $x \in I^\star$, there is an $E \subseteq D$ such that $E^\star = D^\star$ and $x \in (I : E)$. But this implies (since $\star \leq \sharp$) that $E^\sharp = D^\sharp$, and thus $x \in I^{\overline{\sharp}}$; it would follow that $I^\star \subseteq I^{\overline{\sharp}}$. Since this holds for every I, we would have $\star \leq \overline{\sharp}$, a contradiction. Hence, $\sharp$ is a stable semistar operation. □

Lemma 5.4. *Let V be a valuation domain, let $\Delta \subseteq \mathrm{SStar}(V)$ and let $\star := \sup\Delta$. Take an $I \in \mathbf{F}(V)$. If $1 \in I^\star$, then $1 \in I^\sharp$ for some $\sharp \in \Delta$.*

Proof. Let P be the minimal prime of I. By the representation (1), $1 \in I^\star$ if and only if $1 \in (IV_P)^\star$: thus, without loss of generality P is the maximal ideal of V and I is P-primary.

Since $1 \in I^\star$, there is a $\sharp \in \Delta$ such that I is not $\sharp$-closed; hence, $I^\sharp$ is a V-module properly containing V. However, if $1 \notin I^\sharp$, then, again by the representation (1), $I \subsetneq P$ and $\sharp = v_V$, which would imply $I^\sharp = I$. This is a contradiction, and $1 \in I^\sharp$. □

Theorem 5.5. *Let D be a Prüfer domain such that every ideal has only finitely many minimal primes. Then,* $\mathrm{SStar}_{st}(D)$ *is a spectral space.*

Proof. The space $\mathrm{SStar}_{st}(D)$ is closed by infima and by suprema (Proposition 5.3); to apply Theorem 3.2 (with $\mathcal{A} = \mathbf{F}(D)$) it is enough to show that $\mathrm{SStar}_{st}(D)$ is sup-normal.

Let thus $\Lambda \subseteq \mathrm{SStar}_{st}(D)$, and let $\star := \sup \Lambda$. Suppose that $x \in I^\star$; by substituting I with $x^{-1}I$, we can suppose $x = 1$, and by substituting I with $I \cap D$ we can suppose $I \subsetneq D$ (this is possible since stable operations, by definition, distribute over finite intersections). By hypothesis, I has only a finite number of minimal primes, $P_1, \ldots, P_k$; let $V_i := D_{P_i}$. Then, $1 \in (IV_i)^\star$; since the restriction of $\star$ to $\mathbf{F}(V_i)$ is the supremum of the set $\{\sharp|_{\mathbf{F}(V_i)} \mid \sharp \in \Lambda\}$, by Lemma 5.4 there is a $\sharp_i \in \Lambda$ such that $1 \in (IV_i)^{\sharp_i}$.

Let now $T_i := \bigcap\{D_P \mid P \in \mathrm{Spec}(D), P \supseteq P_i\}$. Then, every maximal ideal of T_i contains P_i, and thus every prime ideal of T_i is comparable with P_i. Furthermore, $\mathrm{rad}(IT_i) = P_iT_i$. Since $\sharp_i$ does not closes P_iT_i, by the representation (1) the fact that $1 \in (IV_i)^{\sharp_i} = (IT_iV_i)^{\sharp_i}$ implies that $1 \in (IT_i)^{\sharp_i}$.

Since every maximal ideal containing I survives in some T_i, we have $I = IT_1 \cap \cdots \cap IT_n \cap D$; hence,

$$I^{\sharp_1 \circ \cdots \circ \sharp_n} = \bigcap_{i=1}^{n} (IT_i)^{\sharp_1 \circ \cdots \circ \sharp_n} \cap D^{\sharp_1 \circ \cdots \circ \sharp_n} \supseteq \bigcap_{i=1}^{n} (IT_i)^{\sharp_i} \cap D \ni 1.$$

Therefore, $\mathrm{SStar}_{st}(D)$ is sup-normal, and by Theorem 3.2 it is a spectral space. □

In this context, a space closely related to $\mathrm{SStar}_{st}(D)$ is the space

$$\mathrm{SStar}_{sv}(D) := \{\star \in \mathrm{SStar}(D) \mid D^\star \in \mathrm{Zar}(D)\}.$$

The previous theorem immediately yields the following.

Proposition 5.6. *Let D be a Prüfer domain such that every ideal has only finitely many minimal primes. Then,* $\mathrm{SStar}_{sv}(D)$ *is closed in* $\mathrm{SStar}_{st}(D)^{\mathrm{cons}}$*, and in particular it is a spectral space.*

Proof. Clearly, $\mathrm{SStar}_{sv}(D)$ is closed by generizations. Moreover, it is compact, since a family $\mathcal{U}$ of open sets is a cover of $\mathrm{SStar}_{sv}(D)$ if and only if it is a cover of $\{\wedge_{\{V\}} \mid V \in \mathrm{Zar}(D)\}$, and the latter space is compact since it is homeomorphic to $\mathrm{Zar}(D)$. □

More interestingly, $\mathrm{SStar}_{sv}(D)$ can be used to represent $\mathrm{SStar}_{st}(D)$; the following result is a topological version of [19, Proposition 4.10].

Theorem 5.7. *Let D be a Prüfer domain such that every ideal has only finitely many minimal primes. Then,* $\mathrm{SStar}_{st}(D) \simeq \boldsymbol{\mathcal{X}}(\mathrm{SStar}_{sv}(D))$.

Proof. First, note that the construction $\boldsymbol{\mathcal{X}}(\mathrm{SStar}_{sv}(D))$ makes sense since $\mathrm{SStar}_{sv}(D)$ is a spectral space by Proposition 5.6. Consider the map

$$\begin{aligned}\pi\colon \boldsymbol{\mathcal{X}}(\mathrm{SStar}_{sv}(D)) &\longrightarrow \mathrm{SStar}_{st}(D)\\ \Lambda &\longmapsto \inf\Lambda.\end{aligned}$$

We claim that π is a homeomorphism.

To show that it is surjective, consider the set $\mathscr{X}(D)$ formed by all the subspaces of $\mathrm{SStar}_{sv}(D)$ that are closed by generizations. Then, we can factorize π as

$$\boldsymbol{\mathcal{X}}(\mathrm{SStar}_{sv}(D)) \xrightarrow{\ \iota\ } \mathscr{X}(D) \xrightarrow{\ \pi'\ } \mathrm{SStar}_{st}(D),$$

where ι is the natural inclusion map and π' sends Λ to $\inf\Lambda$.

By [19, Proposition 4.10], π' is surjective. Moreover, we claim that, for every $\Lambda\subseteq\mathrm{SStar}_{sv}(D)$, we have $\inf\Lambda=\inf\overline{\Lambda}$, where $\overline{\Lambda}$ is the closure in the inverse topology of Λ. Indeed, obviously $\inf\Lambda\geq\inf\overline{\Lambda}$. On the other hand, if $\star\leq\inf\Lambda$, then $\Lambda\subseteq\{\star\}^{\uparrow}$. But $\{\star\}^{\uparrow}$ is the closure of $\{\star\}$ in the inverse topology of $\mathrm{SStar}_{st}(D)$; in particular, it is closed in the constructible topology, and thus so is $\{\star\}^{\uparrow}\cap\mathrm{SStar}_{sv}(D)$. Hence, $\overline{\Lambda}\subseteq\{\star\}^{\uparrow}\cap\mathrm{SStar}_{sv}(D)$, and $\star\leq\inf\overline{\Lambda}$; hence, $\inf\Lambda\leq\inf\overline{\Lambda}$ and the two infima are equal. Therefore, if $\star=\pi'(\Lambda)$ then $\star=\pi(\overline{\Lambda})$, and thus π is surjective.

We now show that π is injective. Suppose $\inf\Lambda_1=\inf\Lambda_2$ for some distinct $\Lambda_1,\Lambda_2\in\boldsymbol{\mathcal{X}}(\mathrm{SStar}_{sv}(D))$. In this case, $\inf\Lambda_1=\inf(\Lambda_1\cup\Lambda_2)$: hence, without loss of generality, we can suppose $\Lambda_2=\Lambda_1\cup\{\star\}$ for some $\star\in\mathrm{SStar}_{sv}(D)$. Let $V:=D^\star$; identifying $\mathrm{SStar}(V)=\mathrm{SStar}_{sv}(V)$ with its image in $\mathrm{SStar}(D)$ under the topological embedding ι (see the proof of Proposition 4.1), consider $\Lambda_i':=\Lambda_i\cap\mathrm{SStar}(V)$. Since $\mathrm{SStar}(V)$ is closed in the constructible topology of $\mathrm{SStar}(D)$, Λ_1' and Λ_2' are closed in the inverse topology; hence, it is enough to prove that if $\inf\Lambda_1'=\inf\Lambda_2'$ then $\Lambda_1'=\Lambda_2'$.

Suppose $\star\notin\Lambda_1'$: there is a closed set C of the inverse topology of $\mathrm{SStar}(V)$ such that $\Lambda_1'\subseteq C$ but $\star\notin C$. Without loss of generality, $C=V_I\cap\mathrm{SStar}(V)$ for some ideal I of V; hence, $1\in I^\sharp$ for every $\sharp\in\Lambda_1'$ but $1\notin I^\star$. Let Q be a nonmaximal prime ideal of V: then,

$$Q^{\inf\Lambda_1'}=Q^{\inf\Lambda_1'}\cap Q^\star\subseteq Q^\star=Q;$$

hence, $1\notin Q^\sharp$ for some $\sharp\in\Lambda_1'$. In particular, $Q\subsetneq I$, and thus I must be M-primary (where M is the maximal ideal of V).

If $\star=d_V$, then in the same way $M\subsetneq I$, and thus $I=V$; but then, $1\in I^\star$, a contradiction.

If $\star=v_V$, then in the same way I contains every M-primary ideal of V different from M; hence, $M\subseteq I$. However, in this case $1\in M^\star$, again a contradiction.

Therefore, $\inf\Lambda_1'\neq\inf\Lambda_2'$, and so $\inf\Lambda_1\neq\inf\Lambda_2$; hence, $\pi\circ\iota$ is injective, and thus bijective.

To prove that π is continuous, take an $F \in \mathbf{F}(D)$ and let $U_F := V_F \cap \mathrm{SStar}_{st}(D)$. Then,

$$\begin{aligned}(\pi \circ \iota)^{-1}(U_F) &= \{\Delta \subseteq \boldsymbol{\mathcal{X}}(\mathrm{SStar}_{sv}(D)) \mid 1 \in F^{\inf \Delta}\} = \\ &= \{\Delta \subseteq \boldsymbol{\mathcal{X}}(\mathrm{SStar}_{sv}(D)) \mid 1 \in F^{\star}\ \forall \star \in \Delta\} = \\ &= \{\Delta \in \boldsymbol{\mathcal{X}}(\mathrm{SStar}_{sv}(D)) \mid \Delta \subseteq U_F\} = \boldsymbol{\mathcal{U}}(U_F \cap \mathrm{SStar}_{sv}(D))\end{aligned}$$

which is open since $U_F \cap \mathrm{SStar}_{sv}(D)$ is compact. Moreover, it is clear that $\pi(\boldsymbol{\mathcal{U}}(U_F \cap \mathrm{SStar}_{sv}(D))) = U_F$, and thus π is open. Hence, π is a homeomorphism, as claimed. $\square$

Corollary 5.8. *Let D be a Prüfer domain with Noetherian spectrum. Then,* $\mathrm{SStar}_{sv}(D)$ *is a Noetherian space.*

Proof. Keep the notation of the previous proof. By [19, Proposition 4.10], π' is actually bijective; since ι is injective and $\pi = \pi' \circ \iota$ is bijective, it follows that also ι is bijective. But this implies that every subset of $\mathrm{SStar}_{sv}(D)$ closed by generizations is compact; however, this can happen only if $\mathrm{SStar}_{sv}(D)$ is a Noetherian space. The claim is proved. $\square$

It is worthwhile to note that, for an arbitrary domain D, the space $\mathrm{SStar}_{sv}(D)$ is actually very close to $\mathrm{Zar}(D)$; indeed, there is a topological embedding $\mathrm{Zar}(D) \hookrightarrow \mathrm{SStar}_{sv}(D)$ (given by $V \mapsto d_V$) and a continuous surjection $\mathrm{SStar}_{sv}(D) \longrightarrow \mathrm{Zar}(D)$ (given by $\star \mapsto D^\star$), and the latter is (at most) two-to-one. Therefore, it is quite natural to ask the following question: is $\mathrm{SStar}_{sv}(D)$ *always* a spectral space?

6. Spectral operations

The case of the set $\mathrm{SStar}_{sp}(D)$ of the spectral operations of D is slightly different from the cases considered in the previous two sections. One of the reasons is that spectral operations do not mesh well with the constructible topology.

Proposition 6.1. *Let D be an integral domain. Then, the following are equivalent:*

(a) $\mathrm{SStar}_{sp}(D) = \mathrm{SStar}_{f,sp}(D)$*;*
(b) $\mathrm{SStar}_{f,sp}(D)$ *is closed in* $\mathrm{SStar}_{sp}(D)^{\mathrm{cons}}$*;*
(c) $\mathrm{Spec}(D)$ *is Noetherian.*

Furthermore, if $\mathrm{SStar}(D)$ *is a spectral space, the previous properties are equivalent to the following:*

(d) $\mathrm{SStar}_{sp}(D)$ *is closed in* $\mathrm{SStar}(D)^{\mathrm{cons}}$.

Proof. (a) $\Longrightarrow$ (b) is obvious, while (a) $\Longleftrightarrow$ (c) follows from [12, Corollary 4.4].

To prove (b) $\Longrightarrow$ (c), suppose that $\mathrm{Spec}(D)$ is not Noetherian, and let $\Delta := \{P_\alpha\}_{\alpha \in A}$ be an ascending chain of prime ideals of D that does not stabilize. Let $\star := s_\Delta$: then, Δ is not compact, and by [12, Corollary

4.4] $\star$ is not of finite type. If $\mathrm{SStar}_{f,sp}(D)$ is closed in $\mathrm{SStar}_{sp}(D)^{\mathrm{cons}}$, there are ideals $I_1,\dots,I_n$, $J_1,\dots,J_n$ such that

$$\begin{cases}\mathrm{SStar}_{f,sp}(D)\subseteq\bigcup_{i=1}^n V_{I_i}\cap(X\setminus V_{J_i}),\\ \star\notin\bigcup_{i=1}^n V_{I_i}\cap(X\setminus V_{J_i}).\end{cases}$$

The first condition implies that, for every $\alpha\in A$, there is an i such that $I_i\nsubseteq P_\alpha$ and $J_i\subseteq P_\alpha$; since Δ is an ascending chain, this means that there is a k such that $J_k\subseteq P_\alpha$ and $I_k\nsubseteq P_\alpha$ for all large α. However, this means that $1\notin J_k^\star$, while $1\in I_k^\star$; hence, $\star\in V_{I_k}\cap(X\setminus J_k)$. This is a contradiction, and thus $\mathrm{SStar}_{f,sp}(D)$ is not closed in $\mathrm{SStar}_{sp}(D)^{\mathrm{cons}}$

Suppose now $\mathrm{SStar}(D)$ is spectral. Then, $\mathrm{SStar}_{f,sp}(D)$ is closed in $\mathrm{SStar}(D)^{\mathrm{cons}}$, since it is the intersection of $\mathrm{SStar}_f(D)$ (which is closed by Proposition 4.2) and $\mathrm{SStar}_{st}(D)$ (which is closed by Proposition 5.1). In particular, (a) implies (d). Furthermore, if (d) holds then (b) holds, since $\mathrm{SStar}_{f,sp}(D)$ would be closed in the topology induced by $\mathrm{SStar}(D)^{\mathrm{cons}}$ on $\mathrm{SStar}_{sp}(D)$, which is exactly $\mathrm{SStar}_{sp}(D)^{\mathrm{cons}}$. The claim is proved. $\square$

Despite the previous proposition, we can actually fully characterize when the space is spectral. We denote by $\mathrm{Min}(I)$ the set of minimal primes of an (integral) ideal I of D.

Theorem 6.2. *Let D be an integral domain. Then, $\mathrm{SStar}_{sp}(D)$ is a spectral space if and only if every ideal of D has only finitely many minimal primes.*

Proof. Let $X:=\mathrm{SStar}_{sp}(D)$, and let $U_I:=V_I\cap X$ for each ideal I of D.

Suppose that $\mathrm{Min}(I)$ is finite for every ideal I of D. Since $U_I=U_{I\cap D}$ for every $I\in\mathbf{F}(D)$, a subbasis of closed sets of X^{cons} is

$$\mathcal{S}:=\{U_I\mid I\in\mathcal{I}(D)\}\cup\{X\setminus U_H\mid H\in\mathcal{I}(D)\},$$

where $\mathcal{I}(D)$ is the set of integral ideals of D. (Compare [10, Propositions 3.2(1) and 4.3(1)].)

We want to show that any family $\mathcal{G}$ of closed sets of X^{cons} with the finite intersection property has nonempty intersection; by Alexander's Subbasis Theorem (considered on the closed sets) we can suppose that $\mathcal{G}\subseteq\mathcal{S}$. Hence, suppose

$$\mathcal{G}:=\{U_{F_\alpha}\mid\alpha\in A\}\cup\{X\setminus U_{G_\beta}\mid\beta\in B\}$$

has the finite intersection property. Each U_{F_α} has an infimum, say s_{Δ_α}, with $\Delta_\alpha=\Delta_\alpha^\downarrow$. Let $\Delta:=\bigcap\{\Delta_\alpha\mid\alpha\in A\}$. Then, $\star:=s_\Delta\geq s_{\Delta_\alpha}$ for each α, and thus $\star\in\bigcap_\alpha U_{F_\alpha}$. We claim that $\star\notin U_{G_\beta}$ for each $\beta\in B$.

Suppose this is not true, and let $G:=G_{\overline{\beta}}$ be such that $\star\in U_G$; let $\mathrm{Min}(G)=\{P_1,\dots,P_n\}$. Then, $1\in G^\star$, and thus $P_i\notin\Delta$ for each i; in particular, there is a $\Delta_i = \Delta_{\alpha_i}$ such that $P_i \notin \Delta_{\alpha_i}$. Consider the family

$$\mathcal{G}_0 := \{U_{F_{\alpha_1}}, \ldots, U_{F_{\alpha_n}}\} \cup \{X \setminus U_G\}.$$

Being a finite subfamily of $\mathcal{G}$, there is a $\sharp \in \bigcap \mathcal{G}_0$; let $\sharp := s_\Lambda$ with $\Lambda = \Lambda^\downarrow$. Then, $\sharp \geq s_{\Delta_i}$ for each i, and thus $\Lambda \subseteq \Delta_i$ for each i, i.e., $\Lambda \subseteq \bigcap_i \Delta_i$. But this means that $P \notin \Lambda$ for each $P \in \mathrm{Min}(G)$; thus, $1 \in P^\sharp$ for every such P, and $1 \in G^\sharp$, i.e., $\sharp \in U_G$. This is a contradiction, and thus $\star \notin U_{G_\beta}$ for every β.

But this means that $\star \in \bigcap_\beta (X \setminus U_{G_\beta})$; hence, $\star \in \bigcap \mathcal{G}$. It follows that X^{cons} is compact, and thus X is a spectral space.

Conversely, suppose that there is an ideal I of D having infinitely many minimal primes. Let

$$\mathcal{G} := \{U_P \mid P \in \mathrm{Min}(I)\} \cup \{X \setminus U_I\}.$$

If $\star \in \bigcap \mathcal{G}$, then $1 \notin I^\star$, and thus (being $\star$ spectral) $1 \notin P^\star$ for some $P \in \mathrm{Min}(I)$, i.e., $\star \notin U_P$; this contradicts $U_P \in \mathcal{G}$, and thus $\bigcap \mathcal{G}$ must be empty.

Let now $\mathcal{H}$ be a proper subset of $\mathcal{G}$; we claim that $\bigcap \mathcal{H} \neq \emptyset$, and we can suppose that $X \setminus U_I \in \mathcal{H}$. Let $\Delta := \{Q \in \mathrm{Spec}(D) \mid U_Q \in \mathcal{G} \setminus \mathcal{H}\}^\downarrow$; then, the hypotheses imply that Δ is not empty.

Consider $\star := s_\Delta$; then, $1 \notin Q^\star$ for every $Q \in \Delta$ and thus $1 \notin I^\star$, i.e., $\star \in X \setminus U_I$. On the other hand, if $U_Q \in \mathcal{H}$, then $Q \notin \Delta$, and thus $1 \in Q^\star$, i.e., $\star \in U_Q$. It follows that $\star \in \bigcap \mathcal{H}$.

Since $\mathcal{G}$ is infinite, this means that $\mathcal{G}$ is a set with the finite intersection property but with empty intersection. Moreover, $\mathcal{G}$ is a family of closed subsets of X^{cons}; hence, X^{cons} cannot be compact, and X is not a spectral space. □

References

[1] D. D. Anderson. Star-operations induced by overrings. *Comm. Algebra*, 16(12):2535–2553, 1988.

[2] D. D. Anderson and David F. Anderson. Examples of star operations on integral domains. *Comm. Algebra*, 18(5):1621–1643, 1990.

[3] D. D. Anderson and Sylvia J. Cook. Two star-operations and their induced lattices. *Comm. Algebra*, 28(5):2461–2475, 2000.

[4] Guram Bezhanishvili, Nick Bezhanishvili, David Gabelaia, and Alexander Kurz. Bitopological duality for distributive lattices and Heyting algebras. *Math. Structures Comput. Sci.*, 20(3):359–393, 2010.

[5] Jean Dieudonné and Alexander Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I. *Inst. Hautes Études Sci. Publ. Math.*, (20):259, 1964.

[6] David E. Dobbs, Richard Fedder, and Marco Fontana. Abstract Riemann surfaces of integral domains and spectral spaces. *Ann. Mat. Pura Appl. (4)*, 148:101–115, 1987.

[7] David E. Dobbs and Marco Fontana. Kronecker function rings and abstract Riemann surfaces. *J. Algebra*, 99(1):263–274, 1986.

[8] Carmelo A. Finocchiaro. Spectral spaces and ultrafilters. *Comm. Algebra*, 42(4):1496–1508, 2014.

[9] Carmelo A. Finocchiaro, Marco Fontana, and Dario Spirito. New distinguished classes of spectral spaces: a survey. In *Multiplicative Ideal Theory and Factorization Theory: Commutative and Non-Commutative Perspectives*. Springer Verlag, 2016.

[10] Carmelo A. Finocchiaro, Marco Fontana, and Dario Spirito. Spectral spaces of semistar operations. *J. Pure Appl. Algebra*, 220(8):2897–2913, 2016.

[11] Carmelo A. Finocchiaro, Marco Fontana, and Dario Spirito. The upper vietoris topology on the space of inverse-closed subsets of a spectral space and applications. *Rocky Mountain J. Math.*, to appear.

[12] Carmelo A. Finocchiaro and Dario Spirito. Some topological considerations on semistar operations. *J. Algebra*, 409:199–218, 2014.

[13] Melvin Hochster. Prime ideal structure in commutative rings. *Trans. Amer. Math. Soc.*, 142:43–60, 1969.

[14] Evan G. Houston, Abdeslam Mimouni, and Mi Hee Park. Integrally closed domains with only finitely many star operations. *Comm. Algebra*, 42(12):5264–5286, 2014.

[15] Irving Kaplansky. *Commutative Rings*. The University of Chicago Press, Chicago, Ill.-London, revised edition, 1974.

[16] Wolfgang Krull. Beiträge zur Arithmetik kommutativer Integritätsbereiche i-ii. *Math. Z.*, 41(1):545–577; 665–679, 1936.

[17] Alan Loper. Sequence domains and integer-valued polynomials. *J. Pure Appl. Algebra*, 119(2):185–210, 1997.

[18] Akira Okabe and Ryūki Matsuda. Semistar-operations on integral domains. *Math. J. Toyama Univ.*, 17:1–21, 1994.

[19] Dario Spirito. Towards a classification of stable semistar operations on a Prüfer domain. *Comm. Algebra*, 46(4):1831–1842, 2018.

[20] Dario Spirito. The sets of star and semistar operations on semilocal Prüfer domains. *J. Commut. Algebra*, to appear.

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