SUPREMA IN SPECTRAL SPACES AND THE CONSTRUCTIBLE CLOSURE

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Abstract. Given an arbitrary spectral space X, we endow it with its specialization order $\leq$ and we study the interplay between suprema of subsets of $(X, \leq)$ and the constructible topology. More precisely, we provide sufficient conditions in order for the supremum of a set $Y \subseteq X$ to exist and belong to the constructible closure of Y. We apply such result to characterize which totally ordered spaces are spectral and to provide density properties of some distinguished spaces of rings and ideals.

1. Introduction

A topological space X is said to be a *spectral space* if it is homeomorphic to the spectrum of a (commutative, unitary) ring, endowed with the Zariski topology; as shown by Hochster [15], being a spectral space is a topological condition, in the sense that it is possible to define spectral spaces exclusively through topological properties, without mentioning any algebraic notion. His proof relies heavily on the passage from the starting topology to a new topology, the *patch* or *constructible* topology (see Section 2.1), which remains spectral but becomes Hausdorff; this topology has recently been interpreted as the topology of ultrafilter limit points with respect to the open and quasi-compact subsets of the original topology (see [9] and [13]). Spectral spaces are related to several other topics, for example Boolean algebras, distributive lattices and domain theory, all of which provide a different context and a different point of view on the underlying topological structure.

The spectrum of a ring carries a natural partial order, the one induced by set inclusion: such order can also be recovered topologically, as it coincides with the specialization order of the Zariski topology. In this paper, we study the interplay between this order and the constructible topology; in particular, we are interested in studying when

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the supremum and the infimum of a subset Y of a spectral space X belong to the constructible closure of Y. Our starting result is Theorem 3.1, which says that if every finite subset of Y has a supremum, then also Y has a supremum and it belongs to the constructible closure of the set of all finite suprema. (The corresponding result for infima, Proposition 3.8(1), is obtained using the *inverse topology*, another spectral topology which inverts the order but preserves the constructible topology.) From a purely topological point of view, we use this result in Section 4 to characterize which totally ordered topological spaces are spectral.

In Section 5, we go back to the algebraic setting, showing the consequences of Theorem 3.1 in the context of spaces of submodules, of overrings and of semistar operations: such spaces have recently been shown to provide several natural examples of spectral spaces when endowed with the Zariski or the hull-kernel topology, and show the deep interplay between algebraic and topological properties. While known results are usually positive, i.e., they concentrate on spaces which are spectral and/or closed in the constructible topology (see for example [22, Example 2.2]), our method allows to find example of subspaces that are dense with respect to the constructible topology and thus, in particular, are not closed: for example, we show that the space of finitely generated submodules of a module M is dense in the constructible topology, and spectral only if M is a Noetherian module (Proposition 5.1 and Corollary 5.2) and we show that the set of Prüfer overrings of an integral domain D is dense in the set of integrally closed overrings of D (Proposition 5.6); we also give Noetherian analogue of the latter result, showing that the set of Dedekind overrings of D is dense in the set of integrally closed overrings, provided that D is Noetherian (Corollary 5.9).

2. Preliminaries

2.1. **The constructible topology on a spectral space.** Let X be a spectral space. The *constructible* (or *patch*) *topology* on X is the coarsest topology for which all open and quasi-compact subspaces of X are clopen sets. In the following we will denote by X^{cons} the space X, with the constructible topology, and, for every $Y \subseteq X$, by $\mathrm{Cl}^{\mathrm{cons}}(Y)$ the closure of Y in X^{cons}. In light of [15, Theorem 1 and Proposition 4], X^{cons} is quasi-compact, Hausdorff and totally disconnected, and thus, a fortiori, a zero-dimensional spectral space.

A mapping $f : X \to Y$ of spectral spaces is called a *spectral map* if, for every open and quasi-compact subspace V of Y, $f^{-1}(V)$ is open and quasi-compact. In particular, any spectral map is continuous; moreover, if X and Y are endowed with the constructible topology, f becomes continuous and closed (see [15, pag. 45]).

Let $x \in X$, let $\emptyset \neq Y \subseteq X$ and let $\mathscr{U}$ be an ultrafilter on Y. We say that x is *an ultrafilter limit point of Y with respect to $\mathscr{U}$* if, for every open and quasi quasi-compact subset S of X, we have $x \in S$ if and only if $S \cap Y \in \mathscr{U}$.

Example 2.1. When $X := \mathrm{Spec}(A)$, for some ring A, and $Y \subseteq \mathrm{Spec}(A)$, it is easily seen that the ultrafilter limit point of Y with respect to an ultrafilter $\mathscr{U}$ on Y is the prime ideal

$$Y_{\mathscr{U}} := \{a \in A \mid V(a) \cap Y \in \mathscr{U}\};$$

see, for instance, [4, Proposition 2.3(1)].

By [9, Proposition 2.13 and Proposition 3.2], the closure of Y in X^{cons} is exactly the set of all ultrafilter limit points of Y with respect to some ultrafilter $\mathscr{U}$ on Y. A *proconstructible* subset of X is a set which is closed with respect to the constructible topology. A subset Y of X is said to be *retrocompact in X* provided that, for every open and quasi-compact subset Ω of X, $Y \cap \Omega$ is quasi-compact. The following well-known fact provides a relation between the notions given above and it will be freely used in what follows.

Proposition 2.2. [15, Pag. 45] *Let X be a spectral space and let $Y \subseteq X$. The following conditions are equivalent.*

(i) *Y is proconstructible.*
(ii) *Y is retrocompact and spectral (with the subspace topology of X).*

Furthermore, if Y is proconstructible (and thus, in particular, spectral), then the subspace topology on Y induced by the constructible topology of X is the constructible topology of Y.

Proof. If Y is proconstructible, the fact that Y is retrocompact and spectral follows from the fact that the constructible topology is Hausdorff. Conversely, if Y is retrocompact and spectral, then the inclusion map $Y \hookrightarrow X$ is a spectral map, and so Y is proconstructible by either [15, p. 45] or [3, 1.9.5(vii)]. $\square$

2.2. **The order induced by a spectral topology.** Let X be any topological space. A natural preorder can be defined on X by setting, for every $x, y \in X$, $x \leq y :\iff y \in \mathrm{Cl}(\{x\})$. In particular, if Ω is an open neighborhood of y and $x \leq y$, then $x \in \Omega$. Since every spectral space is, in particular, a T_0 space, the canonical preorder induced by a spectral topology is in fact a partial order.

Following [15, Proposition 8], given a spectral space X the *inverse topology* on X is the topology on the same base set whose closed sets are the intersections of the open and quasi-compact subspaces of X. The inverse topology is spectral, and the order it induces is exactly the reverse of the order induced by the original spectral topology. In the following we will denote by X^{inv} the space X equipped with the inverse topology and, for every $Y \subseteq X$, $\mathrm{Cl}^{\mathrm{inv}}(Y)$ will denote the closure of Y in X^{inv}. By definition, for every $x \in X$, we have

$$\mathrm{Cl}^{\mathrm{inv}}(\{x\}) = \{y \in X \mid y \leq x\}.$$

For every subset Y of X, define the *closure under generization of* Y to be the set

$$Y^{\mathrm{gen}} := \{x \in X \mid x \leq y, \text{ for some } y \in Y\}.$$

Clearly $Y \subseteq Y^{\mathrm{gen}}$, and when $Y = Y^{\mathrm{gen}}$ then Y is said to be *closed under generizations*. If Y is quasi-compact, then Y^{gen} is proconstructible in X, by [4, Proposition 2.6].

We recall here a well-known fact that will be freely used in the remaining part of the paper.

Lemma 2.3. [24, Remark 2.2(vi)] *Let X be a quasi-compact T_0 space and let $\leq$ be the order induced by the topology. Then for every $x \in X$ there is a maximal point x_0 such that $x \leq x_0$. In particular, X has maximal points.*

In case X is a spectral space and $x \in X$, then the previous lemma (applied to the given spectral topology and to its inverse topology) implies that there are a maximal point y and a minimal point z of X such that $z \leq x \leq y$.

2.3. **Hyperspaces of spectral spaces.** Let X be a spectral space, and set

$$\boldsymbol{\mathcal{X}}(X) := \{H \subseteq X \mid H \neq \emptyset, H \text{ is closed in } X^{\mathrm{inv}}\}.$$

As in [8], we endow $\boldsymbol{\mathcal{X}}(X)$ with the so-called *upper Vietoris topology*, i.e., the topology for which a basis of open sets is given by the sets

$$\mathcal{U}(\Omega) := \{H \in \boldsymbol{\mathcal{X}}(X) \mid H \subseteq \Omega\},$$

where Ω runs among the open and quasi-compact subspaces of X. In [8, Theorem 3.4] it is proven that the canonical map $x \mapsto \mathrm{Cl}^{\mathrm{inv}}(\{x\}) = \{x\}^{\mathrm{gen}}$ is a spectral map and a topological embedding of X into $\boldsymbol{\mathcal{X}}(X)$.

3. Suprema of subsets and the constructible closure

If X is a spectral space and Y is a nonempty subset of X, we shall denote by $\sup(Y)$ the supremum of Y (if it exists) in X, with respect to the order induced by the spectral topology. Furthermore, we define

$$Y_f := \{x \in X \mid x = \sup(F), \text{ for some } \emptyset \neq F \subseteq Y, F \text{ finite}\},$$
$$Y_\infty := \{x \in X \mid x = \sup(Z), \text{ for some } \emptyset \neq Z \subseteq Y\}.$$

We say that Y_f *exists* if $\sup(F)$ exists for every nonempty finite subset $F \subseteq Y$, while we say that Y_∞ *exists* if $\sup(Z)$ exists for every nonempty subset $Z \subseteq Y$.

In this paper, we are mainly interested in studying the relationship between existence of suprema and the constructible topology. The following criterion will be the basis of all our paper.

Theorem 3.1. *Let X be a spectral space and let $Y \subseteq X$. If Y_f exists, then Y_∞ exists and $Y_\infty \subseteq \mathrm{Cl}^{\mathrm{cons}}(Y_f)$.*

Proof. Since $Y \subseteq Y_f$ always (each y is the supremum of the finite set $\{y\}$) we can suppose without loss of generality that $Y = Y_f$.

We first show that, under the hypothesis, $\sup(Y)$ exists and belongs to $\mathrm{Cl}^{\mathrm{cons}}(Y)$. Consider the collection of subsets

$$\mathcal{G} := \{Y \setminus B \mid B \subseteq X \text{ open and quasi-compact and } Y \nsubseteq B\}$$

of Y and observe that $\mathcal{G}$ has the finite intersection property. As a matter of fact, let $B_1, \ldots, B_n$ be open and quasi-compact subspaces of X such that $Y \nsubseteq B_i$, for $1 \leq i \leq n$, and take points $x_1, \ldots, x_n$ such that $x_i \in Y \setminus B_i$. By assumption, the exists the supremum $x^\star := \sup(\{x_i, \ldots, x_n\})$ in X and $x^\star \in Y$. If $x^\star \in B_i$, for some i, the fact that B_i is open and $x_i \leq x^\star$ would imply $x_i \in B_i$, a contradiction. Thus $x^\star \in \bigcap_{i=1}^n (Y \setminus B_i)$, proving that $\mathcal{G}$ has the finite intersection property. It follows that there exists an ultrafilter $\mathscr{U}$ on Y such that $\mathcal{G} \subseteq \mathscr{U}$. Since X is a spectral space, [9, Lemma 2.5 and Corollary 3.3] imply that there exist an ultrafilter limit point $z \in X$ of Y with respect to $\mathscr{U}$ (i.e., by definition, for every open and quasi-compact subset S of X, $z \in S$ if and only if $S \cap Y \in \mathscr{U}$). We claim that z is the supremum of Y in X.

First, suppose that there exist some element $y \in Y$ such that $z \ngeq y$, that is, $z \notin \mathrm{Cl}(\{y\})$. Since the open and quasi-compact subspaces of X form a basis for the given spectral topology of X, there exists an open and quasi-compact subset Ω of X such that $z \in \Omega$ and $y \notin \Omega$. By the definition of ultrafilter limit point we infer that $Y \cap \Omega \in \mathscr{U}$. On the other hand, $y \in Y \setminus \Omega$ and thus $Y \setminus \Omega \in \mathcal{G} \subseteq \mathscr{U}$. It follows that $\emptyset \in \mathscr{U}$, a contradiction. This proves that z is an upper bound for Y in X.

Let now z' be any upper bound for Y in X and assume that $z' \not\geq z$, i.e., assume that $z' \notin \mathrm{Cl}(\{z\})$. There exists an open and quasi-compact subset B of X such that $z' \in B$ and $z \notin B$. The last condition implies, by definition of ultrafilter limit point, that $Y \setminus B \in \mathscr{U}$ and, in particular, $Y \setminus B \neq \emptyset$. Take a point $y_0 \in Y \setminus B$. Since z' is an upper bound of Y in X we have $y_0 \leq z'$ and since B is open and $z' \in B$ we infer $y_0 \in B$, a contradiction. The claim is proved.

Let now Γ be any nonempty subset of Y and consider Γ_f: then, $\Gamma_f \subseteq Y_f = Y$. Furthermore, if $F_1, \ldots, F_n$ are nonempty and finite subsets of Γ, then, by assumption, there is $z := \sup\{\sup(F_i) : 1 \leq i \leq n\}$ and $z = \sup(F_1 \cup \ldots \cup F_n) \in \Gamma_f$. It follows that we can apply the claim to Γ_f, and thus there exists $\sup(\Gamma_f)$ and $\sup(\Gamma_f) \in \mathrm{Cl}^{\mathrm{cons}}(\Gamma) \subseteq \mathrm{Cl}^{\mathrm{cons}}(Y)$. The conclusion follows immediately by noting that $\sup(\Gamma_f) = \sup(\Gamma)$. □

Corollary 3.2. *Let X be a spectral space and let $Y \subseteq X$. If Y_f exists, then Y_f and Y_∞ have the same closure, with respect to the constructible topology.*

Proof. By Theorem 3.1, Y_∞ exists and $Y_\infty \subseteq \mathrm{Cl}^{\mathrm{cons}}(Y_f)$. Since $Y_f \subseteq Y_\infty$, we must have $\mathrm{Cl}^{\mathrm{cons}}(Y_f) \subseteq \mathrm{Cl}^{\mathrm{cons}}(Y_\infty) \subseteq \mathrm{Cl}^{\mathrm{cons}}(Y_f)$. □

In general, Y_f need not to exist: for example, if the topology of X is already Hausdorff (i.e., it coincides with the constructible topology), then no set with two or more elements has maximum. Moreover, even if Y_f exists, its constructible closure may not be limited to Y_∞ (see Example 4.5). We now study one case where the suprema determine the constructible closure.

Definition 3.3. *Let X be a spectral space. We say that X* is locally with maximum *if every point of X admits a local basis of open sets each of which has maximum, with respect to the order induced by the topology.*

While restrictive, this property actually holds for several spectral spaces of algebraic interest, as for example the set of submodules of a module or the set of overrings of an integral domain (see Section 5).

Remark 3.4. Let X be a spectral space.

(1) If U is a subset of X with maximum u_0, then U is quasi-compact (since, if $\mathcal{A}$ is an open cover of U and $A \in \mathcal{A}$ contains u_0, then $A \supseteq U$). If furthermore U is open, since $\mathrm{Cl}^{\mathrm{inv}}(\{u_0\}) = \{x \in X \mid x \leq u_0\}$, we immediately infer $U = \mathrm{Cl}^{\mathrm{inv}}(\{u_0\})$.
(2) Keeping in mind part 1 and the definitions, it is straightforward to infer that the following conditions are equivalent.
   (i) X is locally with maximum.

(ii) Any point of X has a local basis of the form $\mathrm{Cl}^{\mathrm{inv}}(\{x\})$, for suitable elements $x \in X$.
(iii) The open and quasi-compact subspaces of X are precisely finite unions of open sets with maximum.

To use this definition, we need the following connection between closures of different topologies.

Lemma 3.5. *Let X be a spectral space, and let $\mathcal{T}$ be a topology on X such that any open and quasi-compact subset of X is closed, with respect to $\mathcal{T}$. If $Y \subseteq Z$ are nonempty subsets of X, $\mathrm{Cl}^{\mathcal{T}}(Y)$ is the closure of Y with respect to $\mathcal{T}$ and there exists $\sup(Y) \in X$, then there exists $\sup(\mathrm{Cl}^{\mathcal{T}}(Y) \cap Z)$, and we have $\sup(Y) = \sup(\mathrm{Cl}^{\mathcal{T}}(Y) \cap Z)$.*

Proof. Since $Y \subseteq \mathrm{Cl}^{\mathcal{T}}(Y) \cap Z$, it is sufficient to prove that any upper bound $x \in X$ for Y is an upper bound for $\mathrm{Cl}^{\mathcal{T}}(Y) \cap Z$ too. Assume that there exists an element $z \in \mathrm{Cl}^{\mathcal{T}}(Y) \cap Z$ such that $z \nleq x$ (i.e., $x \notin \mathrm{Cl}(\{z\})$). Since X is a spectral space, there exists an open and quasi-compact subset Ω of X such that $x \in \Omega$ and $z \notin \Omega$. Since Ω is closed with respect to $\mathcal{T}$, it follows that Y can't be contained in Ω (otherwise $\mathrm{Cl}^{\mathcal{T}}(Y) \subseteq \Omega$); hence, there is an $y \in Y \cap (X \setminus \Omega)$. Since Ω is open in the starting topology, $\mathrm{Cl}(\{y\}) \subseteq X \setminus \Omega$, and so $x \notin \mathrm{Cl}(\{y\})$, that is, $y \nleq x$, against the fact that x is an upper bound of Y in X. □

Proposition 3.6. *Let X be a spectral space which is locally with maximum and let Y be a proconstructible subset of X such that Y_f exists. Then:*

(1) *Y_∞ exists and $\mathrm{Cl}^{\mathrm{cons}}(Y_f) = Y_\infty$;*
(2) *for every integer $n \geq 1$, the set $Y_n := \{\sup(F) \mid F \subseteq Y, |F| = n\}$ is proconstructible in X;*
(3) *if $Z \subseteq Y$ is dense in Y, then Z_f is dense in Y_∞.*

Proof. (1) Since Y is proconstructible in X, it is spectral, and thus it makes sense to consider the hyperspace
$xcal(Y)$, endowed with the upper Vietoris topology. By Theorem 3.1, the map $\Sigma : \boldsymbol{\mathcal{X}}(Y) \longrightarrow X$ defined by setting $\Sigma(H) := \sup(H)$, for each $H \in \boldsymbol{\mathcal{X}}(Y)$, is well-defined. Since the space is locally with maximum, by [8, Lemma 4.6] Σ is a spectral map; hence, it is continuous when $\boldsymbol{\mathcal{X}}(Y)$ and X are equipped with their constructible topologies, and thus Σ is a closed map (again in the constructible topology). It follows, in particular, that $\Sigma(\boldsymbol{\mathcal{X}}(Y))$ is proconstructible. On the other hand, by Lemma 3.5 (applied by taking as $\mathcal{T}$ the inverse topology on X) we infer that $\Sigma(\boldsymbol{\mathcal{X}}(Y)) = Y_\infty$. Now the conclusion follows Corollary 3.2.

(2) Consider the map

$$\begin{aligned} \Sigma_n\colon Y^n &\longrightarrow \boldsymbol{\mathcal{X}}(X)\\ (y_1,\dots,y_n) &\longmapsto \{y_1,\dots,y_n\}^{\mathrm{gen}}, \end{aligned}$$

where Y^n is endowed with the product topology of the topology induced by X. Since Y is a spectral space (being proconstructible) then so is Y^n, by [15, Theorem 7].

If Ω is open and quasi-compact in X, then

$$\begin{aligned} \Sigma_n^{-1}(\Omega) =& \{(y_1,\dots,y_n)\mid \{y_1,\dots,y_n\}^{\mathrm{gen}}\subseteq\Omega\} =\\ =& \{(y_1,\dots,y_n)\mid y_1,\dots,y_n\in\Omega\} = \Omega^n, \end{aligned}$$

which is quasi-compact as it is the product of quasi-compact spaces. Hence, Σ_n is a spectral map, and in particular $\Sigma_n(Y^n)$ is closed in the constructible topology. By the previous part of the proof, it follows that $\Sigma(\Sigma_n(Y^n)) = Y_n$ is closed, as well.

(3) Note first that Z_f exists since $Z\subseteq Y$ and Y_f exists by hypothesis. If Z is dense in Y, then Z^n is dense in Y^n; setting $\Psi_n := \Sigma_n\circ\Sigma : Y^n \longrightarrow Y_\infty$ (with Σ and Σ_n as in the previous point), we see that $\Psi_n(Z^n) = Z_n$ is dense in $\Psi_n(Y^n) = Y_n$. Therefore, $Z_f = \bigcup_n Z_n$ is dense in $Y_f = \bigcup_n Y_n$; by part (1), it follows that $\mathrm{Cl}^{\mathrm{cons}}(Z_f) = \mathrm{Cl}^{\mathrm{cons}}(Y_f) = Y_\infty$, that is, Z_f is dense in Y_∞. The claim is proved. $\square$

Let X be a set endowed with a spectral topology $\mathcal{T}$. The order induced by the inverse topology $\mathcal{T}^{\mathrm{inv}}$ of $\mathcal{T}$ is the opposite of the order induced by $\mathcal{T}$; hence, the supremum of a subset $Y\subseteq X$ in the inverse topology (if it exists) is exactly the infimum of Y in the original topology. Since the constructible topology of $\mathcal{T}$ and $\mathcal{T}^{\mathrm{inv}}$ coincide, we can reword the previous statements using infima instead of suprema. We denote by $Y_{(f)}$ the set of finite infima of elements of Y, and $Y_{(\infty)}$ the set of all infima of subsets of Y.

Lemma 3.7. *Let X be a spectral space and let $\mathcal{T}$ be a topology on X which is finer than (or equal to) the given spectral topology. If $Y\subseteq Z$ are nonempty subsets of X and there exists $\inf(Y)\in X$, then there exists $\inf(\mathrm{Cl}^{\mathcal{T}}(Y)\cap Z)$, and we have $\inf(Y) = \inf(\mathrm{Cl}^{\mathcal{T}}(Y)\cap Z)$.*

Proof. By Hochster duality, the open and quasi-compact subsets of X^{inv} are precisely the sets of the type $X\setminus\Omega$, where $\Omega\subseteq X$ is open and quasi-compact. By assumption, open and quasi-compact subsets of X^{inv} are closed with respect to $\mathcal{T}$. Then the conclusion follows by applying Lemma 3.5 to the spectral space X^{inv}. $\square$

Proposition 3.8. *Let X be a spectral space and let $Y\subseteq X$ be a subset such that $Y_{(f)}$ exists.*

(1) *$Y_{(\infty)}$ exists, is contained in $\mathrm{Cl}^{\mathrm{cons}}(Y_{(f)})$ and has the same closure (in the constructible topology) of $Y_{(f)}$.*
(2) *If X^{inv} is locally with maximum and Y is proconstructible, then $\mathrm{Cl}^{\mathrm{cons}}(Y_{(f)}) = Y_{(\infty)}$.*

Proof. The first claim is the inverse version of Theorem 3.1, while the second corresponds to Proposition 3.6(1). □

4. Linearly ordered spaces

In this section, we analyze the relationship between spectrality and linear orders.

Proposition 4.1. *Let X be a spectral space and $Y \subseteq X$ be a subspace that is linearly ordered (in the order induced by the topology). The following statements hold.*

(1) *For every nonempty subset $Z \subseteq Y$, $\sup(Z)$ and $\inf(Z)$ exist in X, and they belong to $\mathrm{Cl}^{\mathrm{cons}}(Y)$.*
(2) *$\mathrm{Cl}^{\mathrm{cons}}(Y)$ is totally ordered.*
(3) *$\mathrm{Cl}^{\mathrm{cons}}(Y) = Y_\infty \cup Y_{(\infty)}$.*
(4) *Y is proconstructible if and only if the supremum and the infimum of each nonempty subset of Y belong to Y.*

Proof. (1) Since Y is linearly ordered, $Y = Y_f$, and thus Y_∞ exists and is contained in $\mathrm{Cl}^{\mathrm{cons}}(Y)$ by Theorem 3.1. Analogously, $Y = Y_{(f)}$ and so $Y_{(\infty)} \subseteq \mathrm{Cl}^{\mathrm{cons}}(Y)$ by Proposition 3.8.

(2) Suppose not: then, there are points $x_0, y_0 \in \mathrm{Cl}^{\mathrm{cons}}(Y)$ such that $x_0 \nleq y_0$ and $y_0 \nleq x_0$. This means that there are open and quasi-compact subsets U, V of X such that $x_0 \in U \setminus V$ and $y_0 \in V \setminus U$. By definition, $\Lambda := U \cap (X \setminus V)$ (resp., $\Delta := V \cap (X \setminus U)$) is a clopen neighborhood of x_0 (resp., y_0) in X^{cons}. Since $x_0, y_0 \in \mathrm{Cl}^{\mathrm{cons}}(Y)$, we can pick elements $a \in Y \cap \Lambda, b \in Y \cap \Delta$ and, being Y is totally ordered, we can assume that $a \leq b$. However, $b \in V$ implies that $a \in V$, against the fact that $a \in \Lambda$. Thus, $\mathrm{Cl}^{\mathrm{cons}}(Y)$ is totally ordered.

(3) By part (1), $Y_\infty, Y_{(\infty)}$ exist and $Y' := Y_\infty \cup Y_{(\infty)} \subseteq \mathrm{Cl}^{\mathrm{cons}}(Y)$. Now take any element $z \notin Y'$, and let

$$H := \{y \in Y \mid y < z\}, \quad K := \{y \in Y \mid y > z\}.$$

If $H \neq \emptyset$, since $z \notin Y'$ we have $\sup(H) < z$, and so we can take an open and quasi-compact set Ω such that $\sup(H) \in \Omega$, $z \notin \Omega$; otherwise, if $H = \emptyset$, set $\Omega := \emptyset$. Likewise, if $K \neq \emptyset$ then $z < \inf(K)$ and so there is an open and quasi-compact set Γ such that $z \in \Gamma$ and $\inf(K) \notin \Gamma$; if $K = \emptyset$ take $\Gamma := X$. Consider $U := (X \setminus \Omega) \cap \Gamma$. Since $\sup(H) \in \Omega$, then $H \subseteq \Omega$ and so $H \cap U = \emptyset$; likewise, $\inf(K) \notin \Gamma$ implies $K \subseteq X \setminus \Gamma$, so $K \cap \Gamma = \emptyset$ and $K \cap U = \emptyset$. Therefore, $Y \cap U = \emptyset$; since $z \in U$, it follows that U is an open neighborhood (in the constructible topology) of z disjoint from U, and so $z \notin \mathrm{Cl}^{\mathrm{cons}}(Y)$.

Finally, (4) is an immediate consequence of part (3). □

We now characterize linearly ordered spectral spaces; our characterization can be seen as a topological version of the results in [23], where totally ordered spectral spaces were considered from the point of view of order theory.

If $(\mathcal{P}, \leq)$ is totally ordered and has a minimum, we denote by 0 its minimal element. If $x \in \mathcal{P}$, we denote by $[0, x]$ the initial segment

$$[0, x] := \{y \in \mathcal{P} \mid 0 \leq y \leq x\}.$$

Theorem 4.2. *Let X be a T_0 topological space such that the order induced by the topology is total. Then, X is a spectral space if and only if the following hold:*

1. *every nonempty subspace $Y \subseteq X$ has an infimum and a supremum;*
2. *the set of the initial segments $[0, x]$ which are open is a basis;*
3. *if $[0, x]$ is an open proper subset of X, then x has an immediate successor.*

Proof. Suppose first that X is a spectral space. Then, every subspace has infimum and supremum by Theorem 3.1 and Proposition 3.8(1). Furthermore, the open and quasi-compact subsets of X are the $[0, x]$ whose complements is quasi-compact in the inverse topology; that is, are the $[0, x]$ such that x has immediate successor. The claim is proved.

Conversely, suppose the three conditions hold. Then, X has a minimum 0. Let $\mathcal{B}$ be the basis of quasi-compact open initial segments, and let $\mathscr{U}$ be an ultrafilter on X. Let $B_0 := \bigcap\{B \mid B \in \mathcal{B} \cap \mathscr{U}\}$; then, $0 \in B_0$, so B_0 is nonempty. Define $x := \sup B_0$: we claim that x is the ultrafilter limit point of $\mathscr{U}$ with respect to $\mathcal{B}$. Let thus $B \in \mathcal{B}$.

If $B \in \mathscr{U}$, then $B_0 \subseteq B$. Furthermore, since B is quasi-compact, it has a maximum; hence, $x = \sup(B_0) \leq \sup(B) = \max(B)$. It follows that $x \in B$.

Conversely, suppose $x \in B$. Then, the initial segment $[0, x]$ is contained in B. Suppose that $x' \in B$ for some $x' > x$: by definition, there must be a $B' \in \mathcal{B} \cap \mathscr{U}$ such that $x' \notin B'$. This implies that $B' \subseteq B$, and so $B \in \mathscr{U}$, as claimed. Suppose now that $x' \notin B$ for every $x' > x$: then, B must be equal to $[0, x]$, which thus must be an open set. By hypothesis, there is an immediate successor x' of x. Since $x' \notin B_0$ (otherwise x would not be the supremum), it follows that there is a

$B' \in \mathscr{U}$ not containing x'; hence, B' must be contained in $[0,x]$. It follows that B' must be B, i.e., that $B \in \mathscr{U}$, as claimed.

By [9, Corollary 3.3], X is a spectral space. □

We now see that two known results are consequences of the previous theorem. The first one is [18, Corollary 3.6], of which we give a topological proof (unlike the original, which uses divisibility groups). Conditions (K1) and (K2) are two properties of spectral spaces that were proved (in the algebraic setting) by Kaplansky as, respectively, Theorem 9 and Theorem 11 of [17].

Corollary 4.3. *Let $(X,\leq)$ be a totally ordered set with the following properties:*

- (K1) *every nonempty subset of X has an infimum and a supremum;*
- (K2) *for every $y_1, y_2 \in X$, $y_1 < y_2$, there are $x_1, x_2 \in X$ such that $y_1 \leq x_1 < x_2 \leq y_2$ and such that there are no elements between x_1 and x_2.*

Then, there is a topology $\mathcal{T}$ on X such that $(X,\mathcal{T})$ is a spectral space and such that the order induced by $\mathcal{T}$ is $\leq$.

Proof. Let $\mathcal{B} := \{[0,x] \mid x \text{ has an immediate successor}\}$, and let $\mathcal{T}$ be the topology generated by $\mathcal{B}$. Note that, under set containment, $\mathcal{B}$ is a chain; in particular, $\mathcal{B}$ is a basis of $\mathcal{T}$.

Let $y_1 < y_2$ be in X: by hypothesis, we can find an element x with an immediate successor such that $y_1 \leq x < y_2$. Then, $[0,x] \in \mathcal{B}$ contains y_1 but not y_2; it follows that $(X,\mathcal{T})$ is T_0, and that the order induced by $\mathcal{T}$ is exactly $\leq$. In particular, by (K1), every nonempty subset of X has an infimum and a supremum, with respect to the order induced by $\mathcal{T}$.

Furthermore, if $[0,x]$ is open in $\mathcal{T}$, then $[0,x] = \bigcup_{\alpha\in I}[0,y_\alpha]$ for some family $\{[0,y_\alpha] \mid \alpha \in I\} \subseteq \mathcal{B}$; in particular, x must belong to $[0,y_{\overline{\alpha}}]$ for some $\overline{\alpha} \in I$, and thus $[0,x] = [0,y_{\overline{\alpha}}] \in \mathcal{B}$. Hence, $(X,\mathcal{T})$ satisfies the conditions of Theorem 4.2, and thus $(X,\mathcal{T})$ is a spectral space. □

The second corollary is a different proof of [15, Proposition 13].

Corollary 4.4. *Let $(X,\leq)$ be a linearly ordered set. Then, there is at most one spectral topology inducing the order $\leq$.*

Proof. Suppose there are two distinct spectral topologies, say $\mathcal{T}_1$ and $\mathcal{T}_2$, on X inducing $\leq$. Then, there is a subset $\Omega \subseteq X$ which is open and quasi-compact in $\mathcal{T}_1$ but not in $\mathcal{T}_2$ (or conversely). Since $\mathcal{T}_1$ is linearly ordered, $\Omega = [0,x]$ for some $x \in X$; by Theorem 4.2, x has an immediate successor y. Let now Ω' be an open set of $\mathcal{T}_2$ containing x: then, $[0,x] \subseteq \Omega'$, and $[0,x] \neq \Omega'$ because otherwise $[0,x]$ would be open and quasi-compact in $\mathcal{T}_2$, against the hypothesis. Hence, $y \in \Omega'$; however, this implies that x and y cannot be distinguished by $\mathcal{T}_2$, that is, that $\mathcal{T}_2$ is not T_0. This contradicts the spectrality of $\mathcal{T}_2$. □

If Y is not totally ordered, the characterization given in Proposition 4.1(4) does not hold, as we show in the next example. Recall that an *almost Dedekind domain* is an integral domain D such that $D_{\mathfrak{m}}$ is a discrete valuation ring for every maximal ideal $\mathfrak{m}$ of D.

Example 4.5. Let D be a non-Noetherian almost Dedekind domain and let $\mathfrak{n}$ be a non-finitely generated maximal ideal of D (for explicit examples of almost Dedekind domains which are not Dedekind see, for instance, [21, p. 426] and [19]). Let $X_0 := \mathrm{Spec}(D)$, and define a topological space X in the following way: as a set, X is the disjoint union of X_0 and an element $\infty \notin X_0$, while the open sets of X are X itself and the open sets of X_0. On X_0, the order induced by this topology is the the same order of X_0, while ∞ is bigger than every element of X and so is the unique maximal element of X. In particular, X is T_0 and quasi-compact.

We claim that X is spectral. If $\mathcal{S}_0$ is the basis of X_0 formed by all open and quasi-compact subsets, then $\mathcal{S} := \mathcal{S}_0 \cup \{X\}$ is a basis of X. Let $\mathscr{U}$ be an ultrafilter on X. If $X_0 \notin \mathscr{U}$, then $\{\infty\} \in \mathscr{U}$ (i.e., $\mathscr{U}$ is the principal ultrafilter based on ∞) and thus ∞ is an ultrafilter limit point of $\mathscr{U}$. If $X_0 \in \mathscr{U}$, then $\mathscr{U}_0 := \{U \cap X_0 \mid U \in \mathscr{U}\}$ is an ultrafilter on X_0, and since X_0 is spectral it has an ultrafilter limit point x with respect to $\mathscr{U}_0$, which is also an ultrafilter limit point of X with respect to $\mathscr{U}$. By [9, Corollary 3.3], X is a spectral space.

Consider now $Y := X \setminus \{\mathfrak{n}\}$. Then, every subset H of Y has supremum and infimum in X, and they belong to Y: as a matter of fact, if $|H \cap \mathrm{Max}(D)| \leq 1$ then H is linearly ordered (and finite) while if $|H \cap \mathrm{Max}(D)| \geq 2$ then (0) is the infimum of H and ∞ is its supremum. We claim that $\mathfrak{n} \in \mathrm{Cl}^{\mathrm{cons}}(Y)$; since X_0 is open and quasi-compact in X (and so proconstructible) by Proposition 2.2 its constructible topology is the subspace topology of the constructible topology of X, and thus we need only to show that $\mathfrak{n}$ is in the constructible closure of $Y \cap X_0$ in X_0. If not, then $Y \cap X_0$ would be proconstructible, and thus in particular quasi-compact; hence, $Y \cap X_0$ is the open set induced by a finitely generated ideal I, and so $\mathfrak{n}$ is the radical of a finitely generated ideal. Since D is almost Dedekind, by [14, Theorem 36.4] it would follow that $I = \mathfrak{n}^k$ for some positive integer k; since an almost Dedekind domain is Prüfer, I is invertible, and so $\mathfrak{n}$ would be invertible, a contradiction. Hence, $\mathfrak{n} \in \mathrm{Cl}^{\mathrm{cons}}(Y \cap X_0) \subseteq \mathrm{Cl}^{\mathrm{cons}}(Y)$, as claimed.

5. Applications

5.1. **Spaces of modules.** Let R be any commutative ring, M be a R-module, and let $\mathbf{F}_R(M) := \mathbf{F}(M)$ be the set of R-submodules of M. The *Zariski topology* on $\mathbf{F}(M)$ is the topology generated by the sets

$$\mathcal{B}(x_1, \dots, x_n) := \{N \in \mathbf{F}(M) \mid x_1, \dots, x_n \in N\},$$

as $x_1, \dots, x_n$ range in M. Several spectral spaces naturally appearing in an algebraic setting can be seen as subspaces of $\mathbf{F}(M)$ with the Zariski topology (see for example Section 5.3).

Under this topology, $\mathbf{F}(M)$ itself is spectral: indeed, $\mathbf{F}(M)$ is spectral when it is endowed with the *hull-kernel topology* (that is, the topology generated by the complements of the sets of the type $\mathcal{B}(x_1, \dots, x_n)$, where $x_1, \dots, x_n \in M$) [7, Proposition 2.1], and the Zariski topology is just the inverse topology of the hull-kernel topology.

The order induced by the Zariski topology on $\mathbf{F}(M)$ is the reverse inclusion: i.e., $N \in \mathrm{Cl}(\{N'\})$ if and only if $N \subseteq N'$. In particular, the supremum of a family of submodules is the intersection of the elements of the family, while the infimum is their sum. Furthermore, $\mathbf{F}(M)$ is locally with maximum, since every basic open subset $\mathcal{B}(x_1, \dots, x_n)$ has minimum under inclusion, namely the R-submodule of M generated by $x_1, \dots, x_n$.

Note that the topology induced on $\mathrm{Spec}(R)$ by the Zariski topology on $\mathbf{F}(R)$ is *not* the Zariski topology of the spectrum, but rather its inverse topology.

Proposition 5.1. *Let M be an R-module. Then the set $\mathbf{f}(M)$ of finitely generated submodules of M is dense in $\mathbf{F}(R)$, with respect to the constructible topology.*

Proof. The sum of two finitely generated submodules is finitely generated, and thus $\mathbf{f}(M)$ is closed by finite sums, i.e., by finite infima. By Proposition 3.8(1), the constructible closure of $\mathbf{f}(M)$ contains all sums and thus all submodules of M (since each N is the sum of the principal submodules contained in N). The claim is proved. □

Corollary 5.2. *Let M be an R-module. Then $\mathbf{f}(M)$ is spectral if and only if M is a Noetherian R-module.*

Proof. If M is a Noetherian module, $\mathbf{f}(M) = \mathbf{F}(M)$ is spectral.

Conversely, suppose $\mathbf{f}(M)$ is spectral. The set $\mathbf{f}(M)$ is retrocompact, since the minimum of every subbasic open set $\mathcal{B}(x_1, \dots, x_n)$ belongs to $\mathbf{f}(M)$ and thus every $\mathcal{B}(x_1, \dots, x_n) \cap \mathbf{f}(M)$ is quasi-compact. Therefore, by Proposition 2.2, $\mathbf{f}(M)$ is proconstructible in $\mathbf{F}(M)$; however, by the previous proposition $\mathbf{f}(M)$ is dense in $\mathbf{F}(M)^{\mathrm{cons}}$, and thus we must have

$\mathbf{f}(M) = \mathbf{F}(M)$. By definition, M must be a Noetherian R-module, as claimed. □

5.2. **Spaces of ideals.** Given a ring R, we denote by $\mathcal{I}(R) := \mathbf{F}_R(R)$ the set of ideals of a ring R; we also denote by $\mathcal{I}^\bullet(R)$ the set of proper ideals of R, and endow both with the Zariski topology. Note that $\mathcal{I}^\bullet(R)$ is closed in $\mathcal{I}(R)$ (since it is the complement of the basic open set $\mathcal{B}(1)$), and thus in particular it is proconstructible in $\mathcal{I}(R)$ and a spectral space.

Proposition 5.3. *Let R be a ring, and let $\mathcal{I}_f(R)$ be the set of finitely generated ideals of R. Then, $\mathcal{I}_f(R)$ is dense in $\mathcal{I}(R)$, with respect to the constructible topology, and $\mathcal{I}_f(R)$ is spectral if and only if R is a Noetherian ring.*

Proof. The claims are the translation of Proposition 5.1 and Corollary 5.2 from submodules to ideals. □

Any ring homomorphism $f : R \longrightarrow R'$ induces a map $f^\sharp : \mathcal{I}(R') \longrightarrow \mathcal{I}(R)$, given by $f^\sharp(I) = f^{-1}(I)$, for every $I \in \mathcal{I}(M')$; it is easy to see that $f^\sharp$ is spectral when $\mathcal{I}(R)$ and $\mathcal{I}(R')$ are endowed with the Zariski topology. Therefore, $f^\sharp$ is continuous and closed when $\mathcal{I}(R)$ and $\mathcal{I}(R')$ are endowed with the constructible topology.

We are now interested in studying the set of primary ideals of a ring; we shall show two cases where the behaviour of this set is radically different. Given a ring R, let $\mathcal{P}_R$ be the set of all primary ideals of R, and given a prime P of R let $\mathcal{P}(P) = \mathcal{P}_R(P)$ be the set of P-primary ideals of R. Clearly, $\mathcal{P}(P)$ is always closed by finite intersections.

Proposition 5.4. *Let R be a Noetherian ring, and let $P \in \mathrm{Spec}(R)$. Then, the closure of $\mathcal{P}(P)$ in the constructible topology of $\mathcal{I}^\bullet(R)$ is equal to $f^\sharp(\mathcal{I}^\bullet(R_P))$, where $f : R \to R_P$ is the localization map. In particular, if R is local with maximal ideal $\mathfrak{m}$, then $\mathcal{P}(\mathfrak{m})$ is dense in $\mathcal{I}^\bullet(R)$, with respect to the constructible topology.*

Proof. Suppose first that $(R, \mathfrak{m})$ is local and that $P = \mathfrak{m}$, and let I be a proper ideal of R. Then, R/I is local with maximal ideal $\mathfrak{m}/I$ and, by the Krull Intersection Theorem (see e.g. [1, Theorem 10.17 and Corollary 10.19]), $\bigcap_{n\geq 1}(\mathfrak{m}/I)^n = (0)$; hence, $\bigcap_{n\geq 1}(\mathfrak{m}^n + I) = I$ and thus $I \in \mathcal{P}(\mathfrak{m})_\infty \subseteq \mathrm{Cl}^{\mathrm{cons}}(\mathcal{P}(\mathfrak{m}))$, by Theorem 3.1. Therefore, $\mathcal{P}(\mathfrak{m})$ is dense $\mathcal{I}^\bullet(R)$, i.e., $\mathrm{Cl}^{\mathrm{cons}}(\mathcal{P}(\mathfrak{m})) = \mathcal{I}^\bullet(R) = f^\sharp(\mathcal{I}^\bullet(R))$ (the latter equality coming from the fact that in this case f is the identity).

Now let R be any Noetherian ring, let P be any prime ideal of R and let $f : R \to R_P$ be the localization map. By the first part of the proof, $\mathrm{Cl}^{\mathrm{cons}}(\mathcal{P}(PR_P)) = \mathcal{I}^\bullet(R_P)$. Since $f^\sharp$ is continuous and closed, with respect to the constructible topology, we have

$$f^\sharp(\mathcal{I}^\bullet(R_P)) = \mathrm{Cl}^{\mathrm{cons}}(f^\sharp(\mathcal{P}(PR_P))) = \mathrm{Cl}^{\mathrm{cons}}(\mathcal{P}(P)),$$

as claimed. □

Proposition 5.5. *Let V be a valuation domain. Then, the following hold.*

1. *If P is a branched prime ideal of V, the closure of $\mathcal{P}(P)$ in the constructible topology is equal to $\mathcal{P}(P)$ plus the prime ideal directly below P.*
2. *$\mathcal{P}_V$ is a proconstructible subset of $\mathcal{I}(V)$.*

Proof. (1) By [14, Theorem 17.3(3)], there is a prime ideal $Q \subsetneq P$ of V such that there are no prime ideals properly between P and Q. Then, Q is the intersection of all the P-primary ideals, and thus by Theorem 3.1 is contained in the closure of $\mathcal{P}(P)$ in the constructible topology.

Let $f : V \longrightarrow V_P/QV_P$ be the natural map, and let $f^\sharp$ be the induced map from $\mathcal{I}(V_P/QV_P)$ to $\mathcal{I}(V)$. Then, $f^\sharp(\mathcal{I}^\bullet(V_P/QV_P)) = \mathcal{P}(P) \cup \{Q\}$; since $\mathcal{I}^\bullet(R)$ is proconstructible in $\mathcal{I}(R)$ for every ring R and $f^\sharp$ is closed with respect to the constructible topology, it follows that $\mathcal{P}(P) \cup \{Q\}$ is proconstructible. Considering the previous paragraph, $\mathcal{P}(P) \cup \{Q\}$ must be the closure of $\mathcal{P}(P)$, as claimed.

(2) By Proposition 4.1(4), we need to show that the infimum and the supremum of every nonempty subset $\Delta \subseteq \mathcal{P}_V$ are in $\mathcal{P}_V$.

Let thus $\Delta = \{Q_\alpha\}_{\alpha \in A}$ and, for every α, let P_α be the radical of Q_α; let $\Delta' := \{P_\alpha\}_{\alpha \in A}$. If Δ' has a maximum, say $\overline{P}$, then the supremum of Δ is equal to the supremum of $\overline{\Delta} := \{Q \in \Delta \mid \mathrm{rad}(Q) = \overline{P}\} \subseteq \mathcal{P}(\overline{P})$. This set is proconstructible (if $\overline{P}$ is branched by the previous part of the proof, if $\overline{P}$ is not branched because in that case $\mathcal{P}(\overline{P}) = \{\overline{P}\}$), and thus it has a supremum in $\mathcal{P}(\overline{P}) \subseteq \mathcal{P}_R$; hence, Δ has a supremum.

If Δ' has not a maximum, then for every α there is an α' such that $P_\alpha = \mathrm{rad}(Q_\alpha) \subsetneq \mathrm{rad}(Q_{\alpha'}) = P_{\alpha'}$, and thus $P_\alpha \subseteq Q_{\alpha'}$; hence, the supremum $Q := \bigcup_\alpha Q_\alpha$ of Δ is also equal to the supremum $\bigcup_\alpha P_\alpha$ of Δ'. However, Δ' is closed in the constructible topology, and thus $Q \in \Delta' \subseteq \mathcal{P}_R$. Therefore, Δ has always a supremum.

The claim for the infimum follows in the same way, and thus $\mathcal{P}_V$ is closed by suprema and infima; in particular, it is proconstructible. □

Note that, if $\dim V > 1$, then $\mathcal{P}_V \neq \mathcal{I}^\bullet(V)$: for example, if $\mathfrak{m}$ is the maximal ideal of V and x belongs to a prime ideal P strictly contained in $\mathfrak{m}$, then $x\mathfrak{m}$ is not primary: indeed, if $y \in \mathfrak{m} \setminus P$, then $xy \in x\mathfrak{m}$, while $x \notin x\mathfrak{m}$ and $y^n \notin P$ and thus $y^n \notin x\mathfrak{m}$ for every $n \geq 1$.

5.3. **Overrings of an integral domain.** Let $A \subseteq B$ be a ring extension and let $R(B|A)$ denote the set of all subrings of B containing A as a subring. In the constructible topology of the Zariski topology of $\mathbf{F}_A(B)$, the space $R(B|A)$ is closed; in particular, it is a spectral space [9, Proposition 3.5]. Furthermore, $R(B|A)$ is again locally with maximum, since any set of the type $\mathcal{B}(x_1, \ldots, x_n) \cap R(B|A)$ has a minimum, namely the ring $A[x_1, \ldots, x_n]$.

Let D be an integral domain and K the quotient field of D. We denote by:

- $\mathrm{Over}(D) := R(K|D)$ the set of all *overrings* of D;
- $\mathrm{Zar}(D)$ the set of all valuation overrings of D;
- $I(D)$ the set of integrally closed overrings of D;
- $\mathrm{Pruf}(D)$ the set of Prüfer overrings of D;
- $\mathrm{Pruf}_{\mathrm{sloc}}(D)$ the set of semilocal Prüfer overrings of D.

By [9, Example 2.1(3), Propositions 2.12, 3.2 and 3.6], we see that that $\mathrm{Zar}(D)$ and $I(D)$ are proconstructible in $\mathrm{Over}(D)$; our next results show that this usually does not hold for the last two spaces.

Proposition 5.6. *Let D be an integral domain. Then, the closure of* $\mathrm{Pruf}_{\mathrm{sloc}}(D)$ *in the constructible topology of* $\mathrm{Over}(D)$ *is* $I(D)$.

Proof. Consider the proconstructible subset $Y := \mathrm{Zar}(D)$ of the spectral space $X := \mathrm{Over}(D)$. By [14, Theorem 22.8], $Y_f = \mathrm{Pruf}_{\mathrm{sloc}}(D)$; on the other hand, by [1, Corollary 5.22], $Y_\infty = I(D)$. Since $\mathrm{Over}(D)$ is locally with maximum, the conclusion follows from Proposition 3.6. □

Corollary 5.7. *For an integral domain D, the following conditions are equivalent.*

(i) *The integral closure $\overline{D}$ of D is a Prüfer domain.*
(ii) $\mathrm{Pruf}(D)$ *is quasi-compact as a subspace of* $\mathrm{Over}(D)$.

Proof. (i)$\Longrightarrow$(ii) is trivial since, if $\overline{D}$ is a Prüfer domain, then $\mathrm{Pruf}(D) = I(D)$ by [2, Corollary 4.5].

(ii)$\Longrightarrow$(i). Since the order of the Zariski topology on $\mathrm{Over}(D)$ is the reverse inclusion, $\mathrm{Pruf}(D)$ is closed under generizations, again by [2, Corollary 4.5]. Since by assumption $\mathrm{Pruf}(D)$ is quasi-compact, then it is closed in the inverse topology and thus proconstructible, by [4, Proposition 2.6]. The inclusions $\mathrm{Pruf}_{\mathrm{sloc}}(D) \subseteq \mathrm{Pruf}(D) \subseteq I(D)$ and Proposition 5.6 imply $\mathrm{Pruf}(D) = I(D)$, and in particular $\overline{D} \in \mathrm{Pruf}(D)$, i.e., $\overline{D}$ is a Prüfer domain. □

We now want to prove the following "Noetherian" analogue of the Proposition 5.6. We start by considering the discrete valuation rings; for the notation $\wedge_\Delta$ and the b-operation see the following Section 5.4.

Theorem 5.8. *Let D be a Noetherian domain and let $\Delta(D)$ be the set of all discrete valuation overrings of D. Then $\Delta(D)$ is dense in* $\mathrm{Zar}(D)^{\mathrm{cons}}$.

Proof. By [16, Proposition 6.8], for every finitely generated ideal I of R we have $I^{\wedge_{\Delta(D)}} = I^b$, where $\wedge_{\Delta(D)}$ is the semistar operation induced by $\Delta(D)$ and b is the b-operation (or integral closure) on D. By [6, Lemma 5.8(3)], it follows that $\mathrm{Zar}(D) = \mathrm{Cl}^{\mathrm{inv}}(\Delta(D))$, i.e., that $\Delta(D)$ is dense in $\mathrm{Zar}(D)$ with respect to the inverse topology.

For every finite subset F of the quotient field K of D, let $\mathcal{B}(F) := \mathrm{Zar}(D[F])$ denote (with a small abuse of notation) the generic basic open set of the Zariski topology (induced by that of $\mathbf{F}_D(K)$). Since $\mathcal{B}(F) \cap \mathcal{B}(G) = \mathcal{B}(F \cup G)$, the open and quasi-compact subspaces of $\mathrm{Zar}(D)$ are precisely all finite unions of basic open sets of the type $\mathcal{B}(F)$, where F is a finite subset of K. Since the constructible topology is, by definition, the coarsest topology on $\mathrm{Zar}(D)$ for which open and quasi-compact subspaces of the Zariski topology are clopen, it is easily seen that a basis of $\mathrm{Zar}(R)^{\mathrm{cons}}$ consists of sets of the type

$$\mathcal{B}(F)\cap\left(\mathrm{Zar}(D)\setminus\bigcup_{j=1}^{m}\mathcal{B}(G_j)\right) = \mathrm{Zar}(D[F])\cap\left(\mathrm{Zar}(D[F])\setminus\bigcup_{j=1}^{m}\mathcal{B}(G_j)\right),$$

for some finite subsets $F, G_1, \ldots, G_m$ of K. Let Ω be the previous set. Then, Ω is an open set of the inverse topology of $\mathrm{Zar}(D[F])$; by the first paragraph of the proof, $\emptyset \neq \Delta(D[F]) \cap \Omega \subseteq \Delta(D) \cap \Omega$. Therefore, $\Delta(D)$ intersects all basic open sets of $\mathrm{Zar}(D)^{\mathrm{cons}}$, and thus it is dense in it. The claim is proved. □

Corollary 5.9. *Let D be a Noetherian domain. Then the set of the overrings of D that are Dedekind and semilocal is dense in $I(D)$, with respect to the constructible topology.*

Proof. Let $\Delta := \Delta(D)$ as in Theorem 5.8 and let Λ be the set of the overrings of D that are Dedekind and semilocal. By [20, Theorem 12.2], $\Delta_f = \Lambda$ and, by Proposition 3.6(3) and Theorem 5.8, Λ is dense in $\mathrm{Zar}(D)_\infty = I(D)$, as claimed. □

The following result is a slight generalization of [25, Proposition 7.6], with a different proof. It can be seen as an analogue of Proposition 5.1 for rings instead of modules.

Proposition 5.10. *Let $A \subseteq B$ be a ring extension, and let Y be the set of all A-subalgebras of B that are of finite type over A. Then, Y is dense in $R(B|A)$, with respect to the constructible topology.*

Proof. If $A[x_1, \dots, x_n], A[y_1, \dots, y_m] \in Y$, then their infimum (with respect to the order induced by the Zariski topology in $R(B|A)$) is $A[x_1, \dots, x_n, y_1, \dots, y_m] \in Y$; hence, $Y = Y_{(f)}$. Furthermore, $Y_{(\infty)} = R(B|A)$ since, given any $T \in R(B|A)$, $T = \inf\{A[t] \mid t \in T\}$. Then the conclusion immediately follows from Proposition 3.8(1). □

As a consequence of Corollary 5.9, we can complete [25, Proposition 7.3] by considering the case of principal ideal domains.

Proposition 5.11. *Let D be a Noetherian domain, and let*

$$\Delta := \{T \in \mathrm{Over}(D) \mid T \textit{ is a principal ideal domain}\}.$$

Then the following conditions are equivalent:

(i) *the integral closure of D is a principal ideal domain;*
(ii) *Δ is proconstructible in* $\mathrm{Over}(D)$*;*
(iii) *Δ is quasi-compact.*

Proof. Note first that (ii) and (iii) are equivalent since, in view of [2, Corollary 5.3], Δ is closed under generization (i.e., if $T \subseteq T'$ and $T \in \Delta$, then also $T' \in \Delta$).

If (i) holds, then Δ is equal to $I(D)$, which is proconstructible by Proposition 5.6, and so (ii) holds. Suppose (ii) holds: by Corollary 5.9, the set Λ of overrings of D that are Dedekind and semilocal is dense in $I(D)$; since $\Lambda \subseteq \Delta$ (see, for instance, [14, Corollary 34.7]), also Δ is dense in $I(D)$. Since Δ is proconstructible, we must have $\Delta = I(D)$, and in particular the integral closure $\overline{D}$ of D is in Δ, as claimed. □

An integral domain D is called *rad-colon coherent* if the radical of the conductor $(D :_D x)$ is the radical of a finitely generated ideal for every $x \in K$ (where K is the quotient field of D); likewise, it is called *rad-colon principal* if the radical of each $(D :_D x)$ is the radical of a principal ideal for every $x \in K$ [27].

Proposition 5.12. *Let D be a rad-colon coherent domain. If $\{P_\alpha\}_{\alpha \in A}$ is a chain of prime ideals of D and $P := \bigcup_\alpha P_\alpha$, then $\bigcap_\alpha D_{P_\alpha} = D_P$.*

Proof. Since D is rad-colon coherent, the set X of localizations of D at prime ideals is proconstructible in $\mathrm{Over}(D)$ [27, Theorem 3.2(b)]; in particular, the constructible closure of $\Delta := \{D_{P_\alpha}\}_{\alpha \in A}$ is contained in X. In the Zariski topology, $\sup \Delta$ is exactly the intersection of the elements of Δ; by Theorem 3.1, it follows that $\sup \Delta \in X$. It follows that $\bigcap_\alpha D_{P_\alpha} = D_P$, as claimed. □

Proposition 5.13. *Let D be a rad-colon principal domain. If $\{S_\alpha\}_{\alpha \in A}$ is a descending chain of multiplicatively closed subsets of D, and $S := \bigcap_\alpha S_\alpha$, then $\bigcap_\alpha S_\alpha^{-1} D = S^{-1} D$.*

Proof. The proof is the same as the previous proposition, using [27, Theorem 4.4] to prove that the set of quotient rings of D is proconstructible. ☐

Recall that a prime ideal $\mathfrak{p}$ of a ring R is *branched* if there exists a $\mathfrak{p}$-primary ideal of R distinct from $\mathfrak{p}$. A prime ideal that is not branched is called *unbranched*. We show that the hypothesis in the previous two propositions cannot be dropped in general.

Example 5.14.

(1) Let V be a valuation domain with an unbranched maximal ideal M and such that the residue field is equal to $K(X)$ for some field K and some indeterminate X over K. Let D be the pullback of K in V: that is, $D := \{r \in V \mid \pi(r) \in K\}$, where $\pi : V \longrightarrow V/M$ is the residue map. By [11, Theorem 1.4], the prime spectrum of D is (set-theoretically) equal to the spectrum of V; in particular, M is the maximal ideal of D.

Let $\{P_\alpha\}$ be the chain of non-maximal prime ideals of D: then, for every α, we have $D_{P_\alpha} = V_{P_\alpha}$, and thus $\bigcap_\alpha D_{P_\alpha} = V$. On the other hand, $M = \bigcup_\alpha D_{P_\alpha}$, and $D_M = D$; thus, $\bigcap_\alpha D_{P_\alpha} \neq D_M$. In particular, D is not rad-colon coherent.

(2) Let D be a Dedekind domain with countably many maximal ideals, say $M_0, M_1, \ldots, M_n, \ldots$ and M_∞, and suppose that $M_\infty \subseteq \bigcup_n M_n$: that is, suppose that M_∞ is not the radical of any principal ideal, or equivalently that the class of M_∞ in the Picard group of D is not torsion. In particular, D is not rad-colon principal.

Let $S_n := D \setminus (M_1 \cup \cdots \cup M_n)$: then, $\{S_n\}_{n\in\mathbb{N}}$ is a descending chain of multiplicatively closed subset whose intersection $S = D \setminus \bigcup_n M_n$ is just the set of units of D. Hence, $S_n^{-1}D = D_{M_1} \cap \cdots \cap D_{M_n}$, and thus $T := \bigcap_n S_n^{-1}D = \bigcap_n D_{M_n}$. Since D is a Dedekind domain, the maximal ideals of T are in the form M_nT, for $n \in \mathbb{N}$; in particular, $M_\infty T = T$. On the other hand, $S^{-1}D = D$; in particular, $\bigcap_n S_n^{-1}D \neq S^{-1}D$.

5.4. **Spaces of semistar operations.** Let D be an integral domain with quotient field K. Let $\mathcal{F}_D := \mathbf{F}_D(K)$ denote the set of all D-submodules of K. A *semistar operation* on D is a map $\star : \mathcal{F}_D \longrightarrow \mathcal{F}_D$, $F \mapsto F^\star$ satisfying the following axioms, for any nonzero element $k \in K$ and any $F, G \in \mathcal{F}_D$:

- $F \subseteq F^\star$;
- $F \subseteq G$ implies $F^\star \subseteq G^\star$;
- $(F^\star)^\star = F^\star$;

- $(kF)^\star = kF^\star$.

A semistar operation is said to be *of finite type* if $\star = \star_f$, where $\star_f$ is the semistar operation on D defined by setting

$$F^{\star_f} := \bigcup\{G^\star \mid G \in \mathcal{F}_D, G \subseteq F, G \text{ finitely generated}\}$$

for every $F \in \mathcal{F}_D$. It is straightforward that $\star_f$ is of finite type, for every semistar operation $\star$ on D.

We denote by $\mathrm{SStar}(D)$ and $\mathrm{SStar}_f(D)$, respectively, the set of all semistar operations and the set of all semistar operations of finite type. These sets have a natural partial order $\preceq$, defined by saying that $\star \preceq \star'$ if $F^\star \subseteq F^{\star'}$ for every $F \in \mathcal{F}_D$; under this order, any subset $\mathcal{S} \subseteq \mathrm{SStar}(D)$ admits an infimum, namely the semistar operation $\bigwedge(\mathcal{S})$ defined by setting

$$F^{\bigwedge(\mathcal{S})} := \bigcap_{\star\in\mathcal{S}} F^\star \qquad \text{for any } F \in \mathcal{F}_D.$$

The set $\mathrm{SStar}(D)$ can be endowed with a natural topology, called the *Zariski topology*, generated by the sets

$$V_F := \{\star \in \mathrm{SStar}(D) \mid 1 \in F^\star\},$$

as F ranges in $\mathcal{F}_D$. Under this topology, $\mathrm{SStar}_f(D)$ is always a spectral space [10, Theorem 2.13], while $\mathrm{SStar}(D)$ may not be spectral [26, Section 4]. The order induced by the Zariski topology is the opposite of the order $\preceq$ defined above.

For every semistar operation $\star$, we set $\mathcal{F}_D^\star := \{I \in \mathcal{F}_D \mid I = I^\star\}$.

Proposition 5.15. *Let $\star$ be a semistar operation. Then,* $\mathrm{Cl}^{\mathrm{cons}}(\mathcal{F}_D^\star) = \mathcal{F}_D^{\star_f}$.

Proof. If $\star$ is of finite type, then $\mathcal{F}_D^\star$ is proconstructible by [12, Proposition 2.9 and Corollary 2.10] (note that in their definition a semistar operation is defined only on nonzero submodules). In particular, the constructible closure (in $\mathcal{F}_D$) of every subset X of $\mathcal{F}_D^\star$ is equal to the closure of X in the constructible topology of $\mathcal{F}_D^\star$ (by Proposition 2.2).

Suppose now that $\star$ is any semistar operation. Then, $\mathcal{F}_D^\star \subseteq \mathcal{F}_D^{\star_f}$, and thus $\mathrm{Cl}^{\mathrm{cons}}(\mathcal{F}_D^\star) \subseteq \mathcal{F}_D^{\star_f}$. Let $X := \{F^\star \mid F \in \mathcal{F}_D \text{ is finitely generated}\}$. Then, X is closed by finite infima in $\mathcal{F}_D^{\star_f}$: indeed, if $F_1, \ldots, F_n$ are finitely generated then the infimum of $\{F_1^\star, \ldots, F_n^\star\}$ in $\mathcal{F}_D^{\star_f}$ is

$$(F_1^\star + \cdots + F_n^\star)^\star = (F_1 + \cdots + F_n)^\star \in X$$

as $F_1 + \cdots + F_n$ is finitely generated. By Proposition 3.8(1), the constructible closure of X in $\mathcal{F}_D^{\star_f}$ contains all infima of subsets of X. However, if $I \in \mathcal{F}_D^{\star_f}$ then by definition $I = \bigcup\{F^\star \mid E \subseteq I, F$ is finitely generated}; i.e., $I = \bigcup\{F \in X \mid F \subseteq I\} = \inf\{F \in X \mid F \subseteq I\}$; it follows that X is dense in $\mathcal{F}_D^{\star_f}$, with respect to the constructible topology.

However, $X \subseteq \mathcal{F}_D^{\star}$ since $F^\star = F^{\star_f}$ for every finitely generated F; therefore, $\mathcal{F}_D^{\star_f} = \mathrm{Cl}^{\mathrm{cons}}(X) \subseteq \mathrm{Cl}^{\mathrm{cons}}(\mathcal{F}_D^{\star}) \subseteq \mathrm{Cl}^{\mathrm{cons}}(\mathcal{F}_D^{\star_f})$. The claim is proved. □

For every overring R of D, the map $\star_R$ defined by setting $F^{\star_R} := FR$, for every $F \in \mathcal{F}_D$, is a semistar operation of finite type. if $\Delta \subseteq \mathrm{Over}(D)$, then we set $\wedge_\Delta := \bigwedge(\{\star_R \mid R \in \Delta\})$, and in particular we set $b := \wedge_{\mathrm{Zar}(D)}$.

There is a natural topological embedding $\iota : \mathrm{Over}(D) \longrightarrow \mathrm{SStar}_f(D)$, defined by $\iota(R) := \star_R$. (This is the reason why the topology is called the Zariski topology.) We will use our results to show that this map is, in general, not spectral.

Proposition 5.16. *Let D be an integral domain, and let $\iota : \mathrm{Over}(D) \longrightarrow \mathrm{SStar}_f(D)$ the canonical embedding. If ι is a spectral map (equivalently, if $\iota(\mathrm{Over}(D))$ is closed in the constructible topology of $\mathrm{SStar}_f(D)$) then the integral closure $\overline{D}$ of D is a Prüfer domain.*

Proof. Let Δ be the set of semilocal Prüfer overrings of D. We first prove that if $T \in \Delta$, then $\iota(T) := \star_T$ is the infimum of a finite family of elements of $\iota(\mathrm{Zar}(D))$: indeed, if I is any D-submodule of the quotient field K of D and $V_1, \ldots, V_n$ are the localizations at the maximal ideals of T, then, keeping in mind [14, Theorem 4.10], we have

$$I^{\star_T} = IT = IT\left(\bigcap_{i=1}^{n} V_i\right) = \bigcap_{i=1}^{n} ITV_i = \bigcap_{i=1}^{n} I^{\star_{V_i}},$$

i.e., $\star_T = \inf_{1\leq i\leq n} \star_{V_i}$. On the other hand, if $V_1, \ldots, V_n \in \mathrm{Zar}(D)$, then $T := V_1 \cap \cdots \cap V_n$ is a Prüfer domain whose localizations at the maximal ideals are a subset of $\{V_1, \ldots, V_n\}$, and so $\inf_{1\leq i\leq n} \star_{V_i} = \star_T$. Therefore, the set $\iota(\Delta)$ is closed by finite infima (that is, it is closed by finite suprema, with respect to the Zariski topology).

Suppose $\iota(\mathrm{Over}(D))$ is proconstructible in $\mathrm{SStar}_f(D)$. By Theorem 3.1, it follows that

$$\iota(\Delta)_\infty \subseteq \iota(\mathrm{Over}(D)),$$

and thus, in particular, $b = \wedge_{\mathrm{Zar}(D)} \in \iota(\mathrm{Over}(D))$. Let $R \in \mathrm{Over}(D)$ be such that $b = \iota(R) = \star_R$. Then, $D^b = D^{\star_R} = DR = R$; however, D^b is equal to the integral closure $\overline{D}$ of D [1, Corollary 5.22], and thus $R = \overline{D}$. Hence, for every $\overline{D}$-submodule J of K, we have $J^b = J^{\star_{\overline{D}}} = J\overline{D} = J$. This proves that b is the identity on $\mathcal{F}_{\overline{D}}$, and this happens if and only if $\overline{D}$ is a Prüfer domain, by [14, Theorem 24.7]. The conclusion follows. □

By [10, Proposition 2.7], if $\mathcal{S}$ is a quasi-compact set of semistar operations then $\bigwedge(\mathcal{S})$ is a semistar operation of finite type; the converse holds if $\mathcal{S}$ is a set of closures in the form $\star_R$ if the rings R are either all localizations of D [10, Corollary 4.4] or valuation overrings of D [10, Proposition 4.5]. In [10], it was also conjectured that the converse is valid for every family of semistar operations induced by overrings; while this conjecture has already been disproved in [5, Example 3.6] using numerical semigroup rings, we now show that it can fail on every non-Prüfer domain.

Example 5.17. Let D be an integrally closed domain that is not a Prüfer domain and, as before, let b be the semistar operation $b : I \mapsto \bigcap\{IV \mid V \in \mathrm{Zar}(D)\}$, for every $I \in \mathcal{F}_D$. Note that b is a semistar operation of finite type (for example, since $\mathrm{Zar}(D)$ is quasi-compact). However, if T is a Prüfer overring of D and I is any D-submodule of the quotient field of D, then, again by [14, Theorem 4.10], we have

$$IT = \bigcap_{P \in \mathrm{Spec}(T)} IT_M = \bigcap_{V \in \mathrm{Zar}(T)} IV,$$

and thus $b = \bigwedge(\mathcal{S})$, where $\mathcal{S} := \{\star_R \mid R \in \mathrm{Pruf}(D)\}$. In particular, $\bigwedge(\mathcal{S})$ is of finite type, while $\mathcal{S}$ is not quasi-compact since it is homeomorphic to $\mathrm{Pruf}(D)$ [10, Proposition 2.5] and $\mathrm{Pruf}(D)$ is not quasi-compact by Corollary 5.7.

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