

# WHEN THE ZARISKI SPACE IS A NOETHERIAN SPACE

ABSTRACT. We characterize when the Zariski space  $\text{Zar}(K|D)$  (where  $D$  is an integral domain,  $K$  is a field containing  $D$  and  $D$  is integrally closed in  $K$ ) and the set  $\text{Zar}_{\min}(L|D)$  of its minimal elements are Noetherian spaces.

## 1. INTRODUCTION

The *Zariski space*  $\text{Zar}(K|D)$  of the valuation ring of a field  $K$  containing a subring  $D$  was introduced by O. Zariski (under the name *abstract Riemann surface*) during its study of resolution of singularities [24, 25]. In particular, he introduced a topology on  $\text{Zar}(K|D)$  (which was later called *Zariski topology*) and proved that it makes  $\text{Zar}(K|D)$  into a compact space [26, Chapter VI, Theorem 40]. Later, the Zariski topology on  $\text{Zar}(K|D)$  was studied more carefully, showing that it is a *spectral space* in the sense of Hochster [14], i.e., that there is a ring  $R$  such that the spectrum of  $R$  (endowed with the Zariski topology) is homeomorphic to  $\text{Zar}(K|D)$  [4, 5, 6]. This topology has also been used to study representations of an integral domain by intersection of valuation rings [16, 17, 18] and, for example, in real and rigid algebraic geometry [15, 21].

In [22], it was shown that in many cases  $\text{Zar}(D)$  is not a Noetherian space, i.e., there are subspaces of  $\text{Zar}(D)$  that are not compact. In particular, it was shown that  $\text{Zar}(D) \setminus \{V\}$  (where  $V$  is a minimal valuation overring of  $D$ ) is often non-compact: for example, this happens when  $\dim(V) > 2 \dim(D)$  [22, Proposition 4.3] or when  $D$  is Noetherian and  $\dim(V) \geq 2$  [22, Corollary 5.2].

In this paper, we study integral domains such that  $\text{Zar}(D)$  is a Noetherian space, and, more generally, we study when the Zariski space  $\text{Zar}(K|D)$  is Noetherian. We show that, if  $D = F$  is a field, then  $\text{Zar}(K|F)$  can be Noetherian only if the transcendence degree of  $K$  over  $F$  is at most 1 and, when  $\text{trdeg}_F K = 1$ , we characterize when this happens in terms of the extensions of the valuation domains of  $F[X]$ , where  $X$  is an element of  $K$  transcendental over  $F$  (Proposition 4.2). In Section 5, we study the case where  $K$  is the quotient field of  $D$ : we first consider the local case, showing that if  $\text{Zar}(D)$  is Noetherian then  $D$

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must be a pseudo-valuation domain (Theorem 5.8) and, subsequently, we globalize this result to the non-local case, showing that  $\text{Zar}(D)$  is Noetherian if and only if so are  $\text{Spec}(D)$  and  $\text{Zar}(D_M)$ , for every maximal ideal  $M$  of  $D$  (Theorem 5.11 and Corollary 5.12). We also prove the analogous results for the set  $\text{Zar}_{\min}(K|D)$  of the minimal elements of  $\text{Zar}(K|D)$ .

## 2. BACKGROUND

**2.1. Overrings and the Zariski space.** Let  $D$  be an integral domain and let  $K$  be a ring containing  $D$ . We define  $\text{Over}(K|D)$  as the set of rings contained between  $D$  and  $K$ . The *Zariski topology* on  $\text{Over}(K|D)$  is the topology having, as a subbasis of closed sets, the sets in the form

$$\mathcal{B}(x_1, \dots, x_n) := \{V \in \text{Over}(K|D) \mid x_1, \dots, x_n \in V\},$$

as  $x_1, \dots, x_n$  range in  $K$ . If  $K$  is the quotient field of  $D$ , an element of  $\text{Over}(K|D)$  is called an *overring* of  $D$ .

If  $K$  is the quotient field of  $D$ , a subset  $X \subseteq \text{Over}(K|D)$  is a *locally finite family* if every  $x \in D$  (or, equivalently, every  $x \in K$ ) is a non-unit in only finitely many  $T \in \text{Over}(K|D)$ .

If  $K$  is a field containing  $D$ , the *Zariski space* of  $D$  in  $K$  is the set of all valuation domains containing  $D$  and whose quotient field is  $K$ ; we denote it by  $\text{Zar}(K|D)$ . The Zariski topology on  $\text{Zar}(K|D)$  is simply the Zariski topology inherited from  $\text{Over}(K|D)$ . If  $K$  is the quotient field of  $D$ , then  $\text{Zar}(K|D)$  will simply be denoted by  $\text{Zar}(D)$ , and its elements are called the *valuation overrings* of  $D$ .

Under the Zariski topology,  $\text{Zar}(K|D)$  is compact [26, Chapter VI, Theorem 40].

We denote by  $\text{Zar}_{\min}(K|D)$  the set of minimal elements of  $\text{Zar}(K|D)$ , with respect to containment. If  $V$  is a valuation domain, we denote by  $\mathfrak{m}_V$  its maximal ideal. Given  $X \subseteq \text{Zar}(D)$ , we define

$$X^\uparrow := \{V \in \text{Zar}(D) \mid V \supseteq W \text{ for some } W \in X\}.$$

Since a family of open sets is a cover of  $X$  if and only if it is a cover of  $X^\uparrow$ , we have that  $X$  is compact if and only if  $X^\uparrow$  is compact.

If  $X$  is a subset of  $\text{Zar}(D)$ , we denote by  $A(X)$  the intersection  $\bigcap \{V \mid V \in X\}$ , called the *holomorphy ring* of  $X$  [20]. Clearly,  $A(X) = A(X^\uparrow)$ .

The *center map* is the application

$$\begin{aligned} \gamma: \text{Zar}(K|D) &\longrightarrow \text{Spec}(D) \\ V &\longmapsto \mathfrak{m}_V \cap D. \end{aligned}$$

If  $\text{Zar}(K|D)$  and  $\text{Spec}(D)$  are endowed with the respective Zariski topologies, the map  $\gamma$  is continuous ([26, Chapter VI, §17, Lemma 1] or [4, Lemma 2.1]), surjective (this follows, for example, from [2, Theorem 5.21] or [11, Theorem 19.6]) and closed [4, Theorem 2.5].

In studying  $\text{Zar}(K|D)$ , it is usually enough to consider the case where  $D$  is integrally closed in  $K$ ; indeed, if  $\overline{D}$  is the integral closure of  $D$  in  $K$ , then  $\text{Zar}(K|D) = \text{Zar}(K|\overline{D})$ .

**2.2. Noetherian spaces.** A topological space  $X$  is *Noetherian* if its open sets satisfy the ascending chain condition, or equivalently if all its subsets are compact. If  $X = \text{Spec}(R)$  is the spectrum of a ring, then  $X$  is a Noetherian space if and only if  $R$  satisfies the ascending chain condition on radical ideals; in particular, the spectrum of a Noetherian ring is always a Noetherian space. If  $\text{Spec}(R)$  is Noetherian, then every ideal of  $R$  has only finitely many minimal primes (see e.g. the proof of [3, Chapter 4, Corollary 3, p.102] or [2, Chapter 6, Exercises 5 and 7]).

Every subspace and every continuous image of a Noetherian space is again Noetherian; in particular, if  $\text{Zar}(D)$  is Noetherian then so are  $\text{Zar}_{\min}(D)$  and  $\text{Spec}(D)$  [22, Proposition 4.1].

**2.3. Kronecker function rings.** Let  $K$  be the quotient field of  $D$ . For every  $V \in \text{Zar}(D)$ , let  $V^b := V[X]_{\mathfrak{m}_V[X]} \subseteq K(X)$ . If  $\Delta \subseteq \text{Zar}(D)$ , the *Kronecker function ring* of  $D$  with respect to  $\Delta$  is

$$\text{Kr}(D, \Delta) := \bigcap \{V^b \mid V \in \Delta\};$$

we denote  $\text{Kr}(D, \text{Zar}(D))$  simply by  $\text{Kr}(D)$ .

The ring  $\text{Kr}(D, \Delta)$  is always a Bézout domain whose quotient field is  $K(X)$ , and, if  $\Delta$  is compact, the intersection map  $W \mapsto W \cap K$  establishes a homeomorphism between  $\text{Zar}(\text{Kr}(D, \Delta))$  and the set  $\Delta^\uparrow$  [4, 5, 6]. Since  $\text{Kr}(D, \Delta)$  is a Prüfer domain, furthermore,  $\text{Zar}(\text{Kr}(D, \Delta))$  is homeomorphic to  $\text{Spec}(\text{Kr}(D, \Delta))$ ; hence,  $\text{Spec}(\text{Kr}(D, \Delta))$  is homeomorphic to  $\Delta^\uparrow$ , and asking if  $\text{Zar}(D)$  is Noetherian is equivalent to asking if  $\text{Spec}(\text{Kr}(D))$  is Noetherian or, equivalently, if  $\text{Kr}(D)$  satisfies the ascending chain condition on radical ideals.

See [11, Chapter 32] or [10] for general properties of Kronecker function rings.

**2.4. Pseudo-valuation domains.** Let  $D$  be an integral domain with quotient field  $K$ . Then,  $D$  is called a *pseudo-valuation domain* (for short, *PVD*) if, for every prime ideal  $P$  of  $D$ , whenever  $xy \in P$  for some  $x, y \in K$ , then at least one between  $x$  and  $y$  is in  $P$ . Equivalently,  $D$  is a pseudo-valuation domain if and only if it is local and its maximal ideal  $M$  is also the maximal ideal of some valuation overring  $V$  of  $D$  (called the valuation domain *associated to*  $D$ ) [12, Corollary 1.3 and Theorem 2.7]. If  $D$  is a valuation domain, then it is also a PVD, and the associated valuation ring is  $D$  itself.

The prototypical examples of a pseudo-valuation domain that is not a valuation domain is the ring  $F + XL[[X]]$ , where  $F \subseteq L$  is a field extension; its associated valuation domain is  $L[[X]]$ .

## 3. EXAMPLES AND REDUCTION

The easiest case for the study of the topology of  $\text{Zar}(D)$  is when  $D$  is a Prüfer domain, i.e., when  $D_M$  is a valuation domain for every maximal ideal  $M$  of  $D$ .

**Proposition 3.1.** *Let  $D$  be a Noetherian domain. Then:*

- (a)  $\text{Zar}(D)$  is a Noetherian space if and only if  $\text{Spec}(D)$  is Noetherian;
- (b)  $\text{Zar}_{\min}(D)$  is Noetherian if and only if  $\text{Max}(D)$  is Noetherian.

*Proof.* Since  $D$  is Prüfer, the center map  $\gamma : \text{Zar}(D) \rightarrow \text{Spec}(D)$  is a homeomorphism [4, Proposition 2.2]. This proves the first claim; the second one follows from the fact that the minimal valuation overrings of  $D$  correspond to the maximal ideals.  $\square$

Another example of a domain that has a Noetherian Zariski space is the pseudo-valuation domain  $D := \mathbb{Q} + Y\mathbb{Q}(X)[[Y]]$ , where  $X, Y$  are indeterminates on  $\mathbb{Q}$ , since in this case  $\text{Zar}(D)$  can be written as the union of the quotient field of  $D$  and two sets homeomorphic to  $\text{Zar}(\mathbb{Q}[X]) \simeq \text{Spec}(\mathbb{Q}[X])$ , which are Noetherian; from this, it is possible to build examples of non-Prüfer domain whose Zariski spectrum is Noetherian, and having arbitrary finite dimension [22, Example 4.7].

More generally, we have the following result, which is probably well-known.

**Lemma 3.2.** *Let  $D$  be an integral domain, and suppose that a prime ideal  $P$  of  $D$  is also the maximal ideal of a valuation overring  $V$  of  $D$ . Then, the quotient map  $\pi : V \rightarrow V/P$  establishes a homeomorphism between  $\{W \in \text{Zar}(D) \mid W \subseteq V\}$  and  $\text{Zar}(V/P|D/P)$ , and between  $\text{Zar}_{\min}(D)$  and  $\text{Zar}_{\min}(V/P|D/P)$ .*

*Proof.* Consider the set  $\text{Over}(V|D)$  and  $\text{Over}(V/P|D/P)$ . Then, the map

$$\begin{aligned} \tilde{\pi} : \text{Over}(V|D) &\longrightarrow \text{Over}(V/P|D/P) \\ A &\longmapsto \pi(A) = A/P \end{aligned}$$

is a bijection, whose inverse is the map sending  $B$  to  $\pi^{-1}(B)$ . Furthermore, it is a homeomorphism: indeed, if  $x \in V/P$  then  $\tilde{\pi}^{-1}(\mathcal{B}(x)) = \mathcal{B}(y)$ , for any  $y \in \pi^{-1}(x)$ , while if  $x \in V$  then  $\tilde{\pi}(\mathcal{B}(x)) = \mathcal{B}(\pi(x))$ .

The condition on  $P$  implies that  $D$  is the pullback of the diagram

$$\begin{array}{ccc} D & \xrightarrow{\pi} & D/P \\ \downarrow & & \downarrow \\ V & \xrightarrow{\pi} & V/P; \end{array}$$

hence, every  $A \in \text{Over}(V|D)$  arises as a pullback. By [8, Theorem 2.4(1)],  $A$  is a valuation domain if and only if  $\pi(A)$  is a valuation

domain and  $V/P$  is the quotient field of  $\pi(A)$ ; hence,  $\tilde{\pi}$  restricts to a bijection between  $\text{Zar}(D) \cap \text{Over}(V|D) = \{W \in \text{Zar}(D) \mid W \subseteq V\}$  and  $\text{Zar}(V/P|D/P)$ . Furthermore, since  $\tilde{\pi}$  is a homeomorphism, so is its restriction. The claim about  $\text{Zar}(D)$  and  $\text{Zar}(V/P|D/P)$  is proved; the claim for the space of minimal elements follows immediately.  $\square$

**Proposition 3.3.** *Let  $D$  be an integral domain, and let  $L$  be a field containing  $D$ . Then, there is a ring  $R$  such that:*

- $\text{Zar}(L|D) \simeq \text{Zar}(R) \setminus \{F\}$ , where  $F$  is the quotient field of  $R$ ;
- $\text{Zar}_{\min}(L|D) \simeq \text{Zar}_{\min}(R)$ .

*Proof.* Let  $X$  be an indeterminate over  $L$ , and define  $R := D + XL[[X]]$ . Then, the prime ideal  $P := XL[[X]]$  of  $R$  is also a prime ideal of the valuation domain  $L[[X]]$ ; by Lemma 3.2, it follows that  $\text{Zar}(L|D) \simeq \Delta := \{W \in \text{Zar}(R) \mid W \subseteq L[[X]]\}$ . Furthermore, every valuation overring  $V$  of  $R$  contains  $XL[[X]]$ , and thus it is either in  $\Delta$  or properly contains  $L[[X]]$ ; however, since  $L[[X]]$  has dimension 1, the latter case is possible only if  $V = L((X))$  is the quotient field of  $R$ . The first claim is proved, and the second follows easily.  $\square$

While Proposition 3.3 shows that (theoretically) we only need to consider spaces of valuation overrings, it is usually easier to not be restricted to this case; the following Proposition 3.4 is an example, as will be the analysis of field extensions done in Section 4.

**Proposition 3.4.** *Let  $D$  be an integral domain that is not a field, let  $K$  be its quotient field and  $L$  a field extension of  $K$ . If  $\text{trdeg}_K L \geq 1$ , then  $\text{Zar}(L|D)$  and  $\text{Zar}_{\min}(L|D)$  are not Noetherian.*

*Proof.* If  $\text{trdeg}_K L \geq 1$ , there is an element  $X \in L \setminus K$  that is not algebraic over  $L$ . If  $\text{Zar}(L|D)$  is Noetherian, so is its subset  $\text{Zar}(L|D[X])$ , and thus also  $\text{Zar}(K(X)|D[X]) = \text{Zar}(D[X])$ , which is the (continuous) image of  $\text{Zar}(L|D[X])$  under the intersection map  $W \mapsto W \cap K(X)$ . However, since  $D$  is not a field,  $\text{Zar}(D[X])$  is not Noetherian by [22, Proposition 5.4]; hence, neither  $\text{Zar}(L|D)$  can be Noetherian.

Consider now  $\text{Zar}_{\min}(L|D)$ : it projects onto  $\text{Zar}_{\min}(K(X)|D)$ , and thus we can suppose that  $L = K(X)$ . Let  $V$  be a minimal valuation overring of  $D$ : then, there is an extension  $W$  of  $V$  to  $L$  such that  $X$  is the generator of the maximal ideal of  $W$ ; furthermore,  $W$  belongs to  $\text{Zar}_{\min}(K(X)|D)$ . In particular,  $\text{Spec}(W) \setminus \text{Max}(W)$  has a maximum, say  $P$ . Let  $\Delta := \text{Zar}(L|D) \setminus \{W\}$ : then,  $\Delta$  can be written as the union of  $\Lambda := (\text{Zar}_{\min}(L|D) \setminus \{W\})^\dagger$  and  $\{W_P\}^\dagger$ . The latter is compact since  $\{W_P\}$  is compact; if  $\text{Zar}_{\min}(L|D) \setminus \{W\}$  were compact, so would be  $\Lambda$ . In this case, also  $\Delta$  would be compact, against the proof of [22, Proposition 5.4]. Hence,  $\Delta$  is not compact, and so  $\text{Zar}_{\min}(L|D)$  is not Noetherian.  $\square$

## 4. FIELD EXTENSIONS

In this section, we consider a field extension  $F \subseteq L$  and analyze when the Zariski space  $\text{Zar}(L|F)$  and its subset  $\text{Zar}_{\min}(L|F)$  are Noetherian. By Proposition 3.3, this is equivalent to studying the Zariski space of the pseudo-valuation domain  $F + XL[[X]]$ .

This problem naturally splits into three cases, according to whether the transcendence degree of  $L$  over  $F$  is 0, 1 or at least 2. The first and the last cases have definite answers, and we collect them in the following proposition. Part (b) is a slight generalization of [22, Corollary 5.5(b)].

**Proposition 4.1.** *Let  $F \subseteq L$  be a field extension.*

- (a) *If  $\text{trdeg}_F L = 0$ , then  $\text{Zar}(L|F) = \{L\} = \text{Zar}_{\min}(L|F)$ , and in particular they are Noetherian.*
- (b) *If  $\text{trdeg}_F L \geq 2$ , then  $\text{Zar}(L|F)$  and  $\text{Zar}_{\min}(L|F)$  are not Noetherian.*

*Proof.* (a) is obvious. For (b), let  $X, Y$  be elements of  $L$  that are algebraically independent. Then, the intersection map  $\text{Zar}_{\min}(L|F) \rightarrow \text{Zar}_{\min}(F(X, Y)|F)$  is surjective, and thus it is enough to prove that  $\text{Zar}_{\min}(F(X, Y)|F)$  is not Noetherian.

Let  $V \in \text{Zar}_{\min}(F(X, Y)|F)$  and, without loss of generality, suppose  $X, Y \in V$ . Let  $\Delta := \text{Zar}_{\min}(F(X, Y)|F) \setminus \{V\}$ . Then,  $\Lambda := \text{Zar}(F(X, Y)|F) \setminus \{V\}$  is the union of  $\Delta^\uparrow$  and a finite set (the valuation domains properly containing  $V$ ). If  $\Delta$  were compact, so would be  $\Lambda$ ; hence, so would be  $\Lambda \cap \text{Zar}(F[X, Y])$  (since both  $\Lambda$  and  $\text{Zar}(F[X, Y])$  would be closed in the inverse topology; see e.g. [6, Remark 2.2 and Proposition 2.6]). However,  $\Lambda \cap \text{Zar}(F[X, Y]) = \text{Zar}(F[X, Y]) \setminus \{V\}$ , which is not compact by the proof of [22, Proposition 5.4]. Hence,  $\Lambda$  is not compact, and thus  $\Delta$  cannot be compact. Hence,  $\text{Zar}_{\min}(F(X, Y)|F)$  is not Noetherian.  $\square$

On the other hand, the case of transcendence degree 1 is more subtle. In [22, Corollary 5.5(a)], it was showed that  $\text{Zar}(L|F)$  is Noetherian if  $L$  is finitely generated over  $F$ ; we now state a characterization.

**Proposition 4.2.** *Let  $F \subseteq L$  be a field extension such that  $\text{trdeg}_F L = 1$ . Then, the following are equivalent:*

- (i)  *$\text{Zar}(L|F)$  is Noetherian;*
- (ii)  *$\text{Zar}_{\min}(L|F)$  is Noetherian;*
- (iii) *for every  $X \in L$  transcendental over  $F$ , every valuation on  $F[X]$  has only finitely many extensions to  $L$ ;*
- (iv) *there is an  $X \in L$ , transcendental over  $F$ , such that every valuation on  $F[X]$  has only finitely many extensions to  $L$ ;*
- (v) *for every  $X \in L$  transcendental over  $F$ , the integral closure of  $F[X]$  in  $L$  has Noetherian spectrum;*
- (vi) *there is an  $X \in L$ , transcendental over  $F$ , such that the integral closure of  $F[X]$  in  $L$  has Noetherian spectrum.*

*Proof.* Every valuation domain of  $L$  containing  $F$  must contain the algebraic closure of  $F$  in  $L$ ; hence, without loss of generality we can suppose that  $F$  is algebraically closed in  $L$ .

(i)  $\implies$  (ii) is obvious; (ii)  $\implies$  (i) follows since (being  $\text{trdeg}_F L = 1$ )  $\text{Zar}(L|F) = \text{Zar}_{\min}(L|F) \cup \{L\}$ .

(i)  $\implies$  (iii). Take  $X \in L \setminus F$ , and suppose there is a valuation  $w$  on  $F[X]$  with infinitely many extensions to  $L$ ; let  $W$  be the valuation domain corresponding to  $w$ . Then, the integral closure  $\overline{W}$  of  $W$  in  $L$  would have infinitely many maximal ideals. Since every maximal ideal of  $\overline{W}$  contains the maximal ideal of  $W$ , the Jacobson radical  $J$  of  $\overline{W}$  contains the maximal ideal of  $W$ , and in particular it is nonzero. It follows that  $J$  has infinitely many minimal primes; hence,  $\text{Max}(\overline{W})$  is not a Noetherian space. However,  $\text{Max}(\overline{W})$  is homeomorphic to a subspace of  $\text{Zar}(L|F)$ , which is Noetherian by hypothesis; this is a contradiction, and so every valuation has only finitely many extensions.

(iii)  $\implies$  (v). Let  $T$  be the integral closure of  $F[X]$ . If  $\text{Spec}(T)$  is not Noetherian, then  $T$  is not locally finite; i.e., there is an  $\alpha \in T$  such that there are infinitely many maximal ideals of  $T$  containing  $\alpha$ . Consider the norm  $N(\alpha)$  of  $\alpha$  over  $F[X]$ , i.e., the product of the algebraic conjugates of  $\alpha$  over  $F[X]$ . Then,  $N(\alpha) \neq 0$ , and it is both an element of  $F[X]$  (being equal to the constant term of the minimal polynomial of  $F[X]$  over  $\alpha$ ) and an element of every maximal ideal containing  $\alpha$  (since all the conjugates are in  $T$ ). Since every maximal ideal of  $F[X]$  is contained in only finitely many maximal ideals of  $T$  (since a maximal ideal of  $F[X]$  correspond to a valuation  $v$  and the maximal ideals of  $T$  containing it to the extensions of  $v$ ), it follows that  $N(\alpha)$  is contained in infinitely many maximal ideals of  $F[X]$ . However, this contradicts the Noetherianity of  $\text{Spec}(F[X])$ ; hence,  $\text{Spec}(T)$  is Noetherian.

Now (iii)  $\implies$  (iv) and (v)  $\implies$  (vi) are obvious, while the proof of (iv)  $\implies$  (vi) is exactly the same as the previous paragraph; hence, we need only to show (vi)  $\implies$  (i); the proof is similar to the one of [22, Corollary 5.5(a)].

Let  $X \in L$ ,  $X$  transcendental over  $F$ , be such that the spectrum of the integral closure  $T$  of  $F[X]$  is Noetherian. Since  $X$  is transcendental over  $F$ , there is an  $F$ -isomorphism  $\phi$  of  $F(X)$  sending  $X$  to  $X^{-1}$ ; moreover, we can extend  $\phi$  to an  $F$ -isomorphism  $\overline{\phi}$  of  $L$ . Since  $\phi(F[X]) = F[X^{-1}]$ , the integral closure  $T$  of  $F[X]$  is sent by  $\overline{\phi}$  to the integral closure  $T'$  of  $F[X^{-1}]$ ; in particular,  $T \simeq T'$ , and  $\text{Spec}(T) \simeq \text{Spec}(T')$ . Thus, also  $\text{Spec}(T')$  is Noetherian, and so is  $\text{Spec}(T) \cup \text{Spec}(T')$ . Furthermore,  $\text{Zar}(T) \simeq \text{Spec}(T) \simeq \text{Spec}(L|F[X])$ , and analogously for  $T'$ ; hence,  $\text{Zar}(T) \cup \text{Zar}(T')$  is Noetherian. But every  $W \in \text{Zar}(L|F)$  contains at least one between  $X$  and  $X^{-1}$ , and thus  $W$  contains  $F[X]$  or  $F[X^{-1}]$ ; i.e.,  $W \in \text{Zar}(T)$  or  $W \in \text{Zar}(T')$ . Hence,  $\text{Zar}(L|F) = \text{Zar}(T) \cup \text{Zar}(T')$  is Noetherian.  $\square$

We remark that there are field extensions that satisfy the conditions of Proposition 4.2 without being finitely generated. For example, if  $L$  is purely inseparable over some  $F(X)$ , then every valuation on  $F[X]$  extends uniquely to  $L$ , and thus condition (iii) of the previous proposition is fulfilled; more generally, each valuation on  $F(X)$  extends in only finitely many ways when the separable degree  $[L : F(X)]_s$  is finite [11, Corollary 20.3]. There are also examples in characteristic 0: for example, [19, Section 12.2] gives examples of non-finitely generated algebraic extension  $F$  of the rational numbers such that every valuation on  $\mathbb{Q}$  has only finitely many extensions to  $F$ . The same construction works also on  $\mathbb{Q}(X)$ , and if  $L$  is such an example then  $\mathbb{Q} \subseteq L$  will satisfy the conditions of Proposition 4.2.

## 5. THE DOMAIN CASE

We now want to study when the space  $\text{Zar}(D)$  is Noetherian, where  $D$  is an integral domain; without loss of generality, we can suppose that  $D$  is integrally closed, since  $\text{Zar}(D) = \text{Zar}(\overline{D})$ . We start by studying intersections of Noetherian families of valuation rings.

Recall that a *treed domain* is an integral domain whose spectrum is a tree (i.e., such that, if  $P$  and  $Q$  are non-comparable prime ideals, then they are coprime). In particular, every Prüfer domain is treed.

**Lemma 5.1.** *Let  $R$  be a treed domain. If  $\text{Max}(R)$  is Noetherian, then every ideal of  $R$  has only finitely many minimal primes.*

Note that we cannot improve this result to  $\text{Spec}(R)$  being Noetherian: for example, the spectrum of a valuation domain with unbranched maximal ideal is not Noetherian, while its maximal spectrum – a singleton – is Noetherian.

*Proof.* Let  $I$  be an ideal of  $R$ , and let  $\{P_\alpha \mid \alpha \in A\}$  be the set of its minimal prime ideals. For every  $\alpha$ , choose a maximal ideal  $M_\alpha$  containing  $P_\alpha$ ; note that  $M_\alpha \neq M_\beta$  if  $\alpha \neq \beta$ , since  $R$  is treed. Let  $\Lambda$  be the set of the  $M_\alpha$ .

Let  $X \subseteq \Lambda$ , and define  $J(X) := \bigcap \{IR_M \mid M \in X\} \cap R$ : we claim that, if  $M \in \Lambda$ , then  $J(X) \subseteq M$  if and only if  $M \in X$ . Indeed, clearly  $J(X)$  is contained in every element of  $X$ . On the other hand, suppose  $N \in \Lambda \setminus X$ . Since  $\text{Max}(R)$  is Noetherian,  $X$  is compact, and thus also  $\{R_M \mid M \in X\}$  is compact; by [7, Corollary 5],

$$J(X)R_N = \left( \bigcap_{M \in X} IR_M \right) R_N \cap R_N = \bigcap_{M \in X} IR_MR_N \cap R_N$$

Since  $M, N \in \Lambda$ , no prime contained in both  $M$  and  $N$  contains  $I$ ; hence,  $IR_MR_N$  contains 1 for each  $M \in X$ . Therefore,  $1 \in J(X)R_N$ , i.e.,  $J(X) \not\subseteq N$ .



Hence, every subset  $X$  of  $\Lambda$  is closed in  $\Lambda$ , since it is equal to the intersection between  $\Lambda$  and the closed set of  $\text{Spec}(R)$  determined by  $J(X)$ . Since  $\Lambda$  is Noetherian, it follows that  $\Lambda$  must be finite; hence, also the set of minimal primes of  $I$  is finite. The claim is proved.  $\square$

**Lemma 5.2.** *Let  $D$  be an integral domain with quotient field  $K$ , and let  $V, W \in \text{Zar}(D)$ . If  $VW = K$ , then  $V^bW^b = K(X)$ .*

*Proof.* Let  $Z := V^bW^b$ . Then, Since  $\text{Zar}(D)$  and  $\text{Zar}(\text{Kr}(D))$  are homeomorphic,  $Z = (Z \cap K)^b$ ; however,  $K \subseteq VW \subseteq V^bW^b$ , and thus  $Z \cap K = K$ . It follows that  $Z = K^b = K(X)$ , as claimed.  $\square$

A consequence of Lemma 5.1 is the following generalization of [16, Theorem 3.4(2)].

**Theorem 5.3.** *Let  $\Delta \subseteq \text{Zar}(D)$  be a Noetherian space, and suppose that  $VW = K$  for every  $V \neq W$  in  $\Delta$ . Then,  $\Delta$  is a locally finite space.*

*Proof.* Let  $\Delta^b := \{V^b \mid V \in \Delta\}$ , and let  $R := \text{Kr}(D, \Delta)$ : then (since, in particular,  $\Delta$  is compact),  $\text{Zar}(R)$  is equal to  $(\Delta^b)^\uparrow$ .

Since  $R$  is a Bézout domain, it follows that  $\text{Spec}(R) \simeq (\Delta^b)^\uparrow$ , while  $\text{Max}(R) \simeq \Delta^b$ ; in particular,  $\text{Max}(R)$  is Noetherian, and thus by Lemma 5.1 every ideal of  $R$  has only finitely many minimal primes. However, since  $V^bW^b = K(X)$  for every  $V \neq W$  in  $\Delta$  (by Lemma 5.2), it follows that every nonzero prime of  $R$  is contained in only one maximal ideal; therefore, every ideal of  $R$  is contained in only finitely many maximal ideals, and thus the family  $\{R_M \mid M \in \text{Max}(R)\}$  is locally finite. This family coincides with  $\Delta^b$ ; since  $\Delta^b$  is locally finite, also  $\Delta$  must be locally finite, as claimed.  $\square$

We say that two valuation domains  $V, W \in \text{Zar}(D) \setminus \{K\}$  are *dependent* if  $VW \neq K$ . Since  $\text{Zar}(D)$  is a tree, being dependent is an equivalence relation on  $\text{Zar}(D) \setminus \{K\}$ ; we call an equivalence class a *dependency class*. If  $\text{Zar}(D)$  is finite-dimensional (i.e., if every valuation overring of  $D$  has finite dimension) then the dependency classes of  $\text{Zar}(D)$  are exactly the sets in the form  $\{W \in \text{Zar}(D) \mid W \subseteq V\}$ , as  $V$  ranges among the one-dimensional valuation overrings of  $D$ .

Under this terminology, the previous theorem implies that, if  $D$  is local and  $\text{Zar}(D)$  is Noetherian, then  $\text{Zar}(D)$  can only have finitely many dependency classes: indeed, otherwise, we could form a Noetherian but not locally finite subset of  $\text{Zar}(D)$  by taking one minimal overring in each dependency class, against the theorem. We actually can say (and will need) something more.

Given a set  $X \subseteq \text{Zar}(D)$ , we define  $\text{comp}(X)$  as the set of all valuation overrings of  $D$  that are comparable with some elements of  $X$ ; i.e.,

$$\text{comp}(X) := \{W \in \text{Zar}(D) \mid \exists V \in X \text{ such that } W \subseteq V \text{ or } V \subseteq W\}.$$

If  $X = \{V\}$  is a singleton, we write  $\text{comp}(V)$  for  $\text{comp}(X)$ . Note that, for every subset  $X$ ,  $\text{comp}(\text{comp}(X)) = \text{Zar}(D)$ , since  $\text{comp}(X)$  contains the quotient field of  $D$ .

The purpose of the following propositions is to show that, if  $D$  is local and  $\text{Zar}(D)$  is Noetherian, then  $\text{Zar}(D)$  can be written as  $\text{comp}(W)$  for some valuation overring  $W \neq K$ . The first step is showing that  $\text{Zar}(D)$  is equal to  $\text{comp}(X)$  for some finite  $X$ .

**Proposition 5.4.** *Let  $D$  be a local integral domain. If  $\text{Zar}_{\min}(D)$  is Noetherian, then there are valuation overrings  $W_1, \dots, W_n$  of  $D$ ,  $W_i \neq K$ , such that  $\text{Zar}(D) = \text{comp}(W_1) \cup \dots \cup \text{comp}(W_n)$ .*

*Proof.* Let  $R := \text{Kr}(D)$  be the Kronecker function ring of  $D$ . Then, the extension  $N := MR$  of the maximal ideal  $M$  of  $D$  is a proper ideal of  $R$ , and the prime ideals containing  $N$  correspond to the valuation overrings of  $R$  where  $N$  survives, i.e., to the valuation overrings of  $D$  centered on  $M$ .

Since  $\text{Zar}_{\min}(D)$  is Noetherian, so is  $\text{Max}(R)$ ; since  $R$  is treed (being a Bézout domain), by Lemma 5.1  $N$  has only finitely many minimal primes. Thus, there are finitely many valuation overrings of  $D$ , say  $W_1, \dots, W_n$ , such that every  $V \in \text{Zar}_{\min}(D)$  is contained in one  $W_i$ . We claim that  $\text{Zar}(D) = \text{comp}(W_1) \cup \dots \cup \text{comp}(W_n)$ . Indeed, let  $V$  be a valuation overring of  $D$ . Since  $\text{Zar}(D)$  is compact,  $V$  contains some minimal valuation overring  $V'$ , and by construction  $V' \in \text{comp}(W_i)$  for some  $i$ ; in particular,  $W_i \supseteq V'$ . The valuation overrings containing  $V'$  (i.e., the valuation overrings of  $V'$ ) are linearly ordered; thus,  $V$  must be comparable with  $W_i$ , i.e.,  $V \in \text{comp}(W_i)$ . The claim is proved.  $\square$

The following result can be seen as a generalization of the classical fact that, if  $X = \{V_1, \dots, V_n\}$  is finite, then  $\text{Zar}(A(X))$  is the union of the various  $\text{Zar}(V_i)$  (since  $A(X)$  will be a Prüfer domain and its localization at the maximal ideals will be a subset of  $X$ ).

**Proposition 5.5.** *Let  $D$  be an integral domain and let  $X \subseteq \text{Zar}(D)$  be a finite set. Then,  $\text{Zar}(A(\text{comp}(X))) = \text{comp}(X)$ .*

*Proof.* Since  $\text{comp}(V) \subseteq \text{comp}(W)$  if  $V \subseteq W$ , we can suppose without loss of generality that the elements of  $X$  are pairwise incomparable. Let  $X = \{V_1, \dots, V_n\}$ ,  $A_i := A(\text{comp}(V_i))$  and let  $A := A(\text{comp}(X)) = A_1 \cap \dots \cap A_n$ . Note that  $D \subseteq A$ , and thus the quotient field of  $A$  coincides with the quotient field of  $D$  and of the  $V_i$ .

If  $V \in \text{comp}(X)$ , then clearly  $A \subseteq V$ ; thus,  $\text{comp}(X) \subseteq \text{Zar}(A)$ .

Conversely, let  $V \in \text{Zar}(A)$ , and let  $\mathfrak{m}_i$  be the maximal ideal of  $V_i$ . Then,  $\mathfrak{m}_i \subseteq W$  for every  $W \in \text{comp}(V_i)$ ; in particular,  $\mathfrak{m}_i \subseteq A_i$ . Therefore,  $P := \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_n \subseteq A$ ; since  $A \subseteq V$ , this implies that  $PV \subseteq V$ .

Suppose  $V \notin \text{comp}(X)$ , and let  $T := V \cap V_1 \cap \dots \cap V_n$ . Since the rings  $V, V_1, \dots, V_n$  are pairwise incomparable,  $T$  is a Bézout domain whose

localizations at the maximal ideals are  $V, V_1, \dots, V_n$ . In particular,  $V$  is flat over  $T$ , and each  $\mathfrak{m}_i$  is a  $T$ -module; hence,

$$PV = \left( \bigcap_{i=1}^n \mathfrak{m}_i \right) V = \bigcap_{i=1}^n \mathfrak{m}_i V.$$

Since  $V$  is not comparable with  $V_i$ , for each  $i$ , the set  $\mathfrak{m}_i$  is not contained in  $V$ ; in particular, the family  $\{\mathfrak{m}_i V \mid i = 1, \dots, n\}$  is a family of  $V$ -modules not contained in  $V$ . Since the  $V$ -submodules of the quotient field  $K$  are linearly ordered, the family has a minimum, and thus  $\bigcap_{i=1}^n \mathfrak{m}_i V$  is not contained in  $V$ . However, this contradicts  $PV \subseteq V$ ; hence,  $V$  must be in  $\text{comp}(X)$ , and  $\text{Zar}(A) = \text{comp}(X)$ .  $\square$

The proof of part (a) of the following proposition closely follows the proof of [13, Proposition 1.19].

**Proposition 5.6.** *Let  $X := \{V_1, \dots, V_n\}$  be a finite family of valuation overrings of the domain  $D$ , and suppose that  $V_i V_j = K$  for every  $i \neq j$ , where  $K$  is the quotient field of  $D$ . Let  $A_i := A(\text{comp}(V_i))$ , and let  $A := A(\text{comp}(X))$ . Then:*

- (a) *each  $A_i$  is a localization of  $A$ ;*
- (b) *for each ideal  $I$  of  $A$ , there is an  $i$  such that  $IA_i \neq A_i$ ;*
- (c) *if  $i \neq j$ , then  $A_i A_j = K$ .*

*Proof.* (a) By induction and symmetry, it is enough to prove that  $B := A_2 \cap \dots \cap A_n$  is a localization of  $A$ . Let  $J$  be the Jacobson radical of  $B$ : then,  $J \neq (0)$ , since it contains the intersection  $\mathfrak{m}_{V_2} \cap \dots \cap \mathfrak{m}_{V_n}$ . Furthermore, if  $W \neq K$  is a valuation overring of  $V_1$ , then  $J \not\subseteq W$ , since otherwise (as in the proof of Proposition 5.5)  $\mathfrak{m}_{V_2} \cap \dots \cap \mathfrak{m}_{V_n}$  would be contained in  $\mathfrak{m}_W \cap (W \cap V_2 \cap \dots \cap V_n)$ , against the fact that  $\{W, V_2, \dots, V_n\}$  are independent valuation overrings.

Hence, for every such  $W$  we can apply [13, Proposition 1.13] to  $D := B \cap W$ , obtaining that  $B$  is a localization of  $D$ , say  $B = S^{-1}D$ , where  $S$  is a multiplicatively closed subset of  $D$ ; in particular, there is a  $s_W \in S \cap \mathfrak{m}_W$ . Each  $s_W$  is in  $B \cap A_1 = A$  (since  $\mathfrak{m}_W$  is contained in every member of  $\text{comp}(V_1)$ ); let  $T$  be the set of all  $s_W$ . Then,

$$T^{-1}A = T^{-1}(B \cap A_2) = T^{-1}B \cap T^{-1}A_1.$$

Each  $s_W$  is a unit of  $B$ , and thus  $T^{-1}B = B$ . On the other hand, no valuation overring  $W \neq K$  of  $V_1$  can be an overring of  $T^{-1}A_1$ , since  $T$  contains  $s_W$ , which is inside the maximal ideal of  $W$ . Since  $\text{Zar}(A_1) = \text{comp}(V_1)$ , it follows that  $T^{-1}A_1 = K$ , and thus  $T^{-1}A = B$ ; in particular,  $B$  is a localization of  $A$ .

(b) Without loss of generality, we can suppose  $I = P$  to be prime. There is a valuation overring  $W$  of  $A$  whose center on  $A$  is  $P$ ; since  $\text{Zar}(A) = \text{comp}(X)$  by Proposition 5.5, there is a  $V_i$  such that  $W \in \text{comp}(V_i)$ . Hence,  $PA_i \neq A_i$ .

(c) By Proposition 5.5,  $\text{Zar}(A_i) \cap \text{Zar}(A_j) = \{K\}$ . It follows that  $K$  is the only common valuation overring of  $A_i A_j$ ; in particular,  $A_i A_j$  must be  $K$ .  $\square$

By [23, Proposition 4.3], Proposition 5.6 can also be rephrased by saying that the set  $\{A_1, \dots, A_n\}$  is a *Jaffard family* of  $A$ , in the sense of [9, Section 6.3].

**Proposition 5.7.** *Let  $D$  be an integrally closed domain; suppose that  $\text{Zar}(D) = \text{comp}(V_1) \cup \dots \cup \text{comp}(V_n)$ , where  $X := \{V_1, \dots, V_n\}$  is a family of incomparable valuation overrings of  $D$  such that  $V_i V_j = K$  if  $i \neq j$ . Then:*

- (a) *the restriction of the center map  $\gamma$  to  $X$  is injective;*
- (b)  $|\text{Max}(D)| \geq |X|$ .

*Proof.* (a) If  $P$  is the image of both  $V_i$  and  $V_j$ , then  $P$  survives in both  $A_i$  and  $A_j$ : however, since  $A_i$  and  $A_j$  are localizations of  $A$  (Proposition 5.6(a)),  $A_P$  would be a common overring of  $A_i$  and  $A_j$ , against the fact that  $A_i A_j = K$  (Proposition 5.6(c)). Therefore, the center map is injective on  $X$ .

(b) Let  $M$  be a maximal ideal: then, there is a unique  $i$  such that  $MA_i \neq A_i$ . In particular,  $M$  can contain only one element of  $\gamma(X)$ , namely  $\gamma(V_i)$ ; thus,  $|\text{Max}(D)| \geq |\gamma(X)| = |X|$ , as claimed.  $\square$

We are ready to prove the pivotal result of the paper.

**Theorem 5.8.** *Let  $D$  be an integrally closed local domain. If  $\text{Zar}_{\min}(D)$  is a Noetherian space, then  $D$  is a pseudo-valuation domain.*

*Proof.* Since  $D$  is local, by Proposition 5.4 there are  $W_1, \dots, W_n$ , not equal to  $K$ , such that  $\text{Zar}(D) = \text{comp}(W_1) \cup \dots \cup \text{comp}(W_n)$ . By eventually passing to bigger valuation domains, we can suppose without loss of generality that  $W_i W_j = K$  if  $i \neq j$ ; since  $D$  is local, by Proposition 5.7(b) we have  $1 \geq n$ , and so  $\text{Zar}(D) = \text{comp}(V)$  for some  $V \neq K$ .

Let  $\Delta$  be the set of  $W \in \text{Zar}(D)$  such that  $\text{comp}(W) = \text{Zar}(D)$ ; then,  $\Delta$  is a chain, and thus it has a minimum in  $\text{Zar}(D)$ , say  $V_0$  (explicitly,  $V_0$  is the intersection of the elements of  $\Delta$ ); furthermore, clearly  $V_0 \in \Delta$ . Since  $V \in \Delta$ , we have  $V_0 \subseteq V$ , and in particular  $V_0 \neq K$ . Let  $M$  be the maximal ideal of  $V_0$ : then,  $M$  is contained in every  $W \in \text{comp}(V_0) = \text{Zar}(D)$ , and thus  $M \subseteq D$ .

Consider now the diagram

$$\begin{array}{ccc} D & \xrightarrow{\pi} & D/M \\ \downarrow & & \downarrow \\ V_0 & \xrightarrow{\pi} & V_0/M. \end{array}$$

Clearly,  $D = \pi^{-1}(D/M)$ ; let  $F_1$  be the quotient field of  $D/M$ . By Lemma 3.2, the set of minimal valuation overrings of  $D$  is homeomorphic to  $\text{Zar}_{\min}(V_0/M|D/M)$ , which thus is Noetherian; by Proposition

3.4, it follows that either  $D/M$  is a field and  $\text{trdeg}_{D/M}(V_0/M) = 1$  (in which case  $D$  is a pseudo-valuation domain with associated valuation domain  $V_0$ ) or  $\text{trdeg}_{F_1}(V_0/M) = 0$ .

In the latter case, we note that  $D/M$  is integrally closed in  $V_0/M$ , since  $D/M$  is the intersection of all the elements of  $\text{Zar}(V_0/M|D/M)$ ; hence,  $V_0/M$  is the quotient field of  $D/M$ . If  $D/M$  is not a field, by the same argument of the first part of the proof it follows that  $\text{Zar}(D/M) = \text{comp}(W_0)$  for some valuation overring  $W_0 \neq F_1$ ; however, this contradicts the choice of  $V_0$ , because  $\pi^{-1}(W_0)$  would be comparable with every element of  $\text{Zar}(D)$ . Hence, it must be  $V_0/M = D/M$ , i.e.,  $V_0 = D$ ; that is,  $D$  is a valuation domain and, in particular, a pseudo-valuation domain.  $\square$

With this result, we can find the possible structures of  $\text{Zar}(D)$  and  $\text{Zar}_{\min}(D)$ , when  $D$  is local and  $\text{Zar}_{\min}(D)$  is Noetherian. Indeed,  $D$  is a pseudo-valuation domain; let  $V$  be its associated valuation overring. Then, we have two cases: either  $D = V$  (i.e.,  $D$  itself is a valuation domain) or  $D \neq V$ .

In the first case,  $\text{Zar}_{\min}(D)$  is a singleton, while  $\text{Zar}(D)$  is homeomorphic to  $\text{Spec}(D)$ ; in particular,  $\text{Zar}(D)$  is linearly ordered, and it is a Noetherian space if and only if  $\text{Spec}(D)$  is Noetherian.

In the second case, we can separate  $\text{Zar}(D)$  into two parts:  $\text{Zar}_{\min}(D)$  and  $\Delta := \text{Zar}(D) \setminus \text{Zar}_{\min}(D)$ . The former must be isomorphic to  $\text{Zar}_{\min}(L|F) = \text{Zar}(L|F) \setminus \{L\}$  (where  $F$  and  $L$  are the residue fields of  $D$  and  $V$ , respectively); on the other hand, the latter is linearly ordered, and is composed by the valuation overrings of  $V$ , so in particular it is homeomorphic to  $\text{Spec}(V)$ , which is (set-theoretically) equal to  $\text{Spec}(D)$ . In other words,  $\text{Zar}(D)$  is composed by a long “stalk” ( $\Delta$ ), under which there is an infinite family of minimal valuation overrings. In particular, we get the following.

**Proposition 5.9.** *Let  $D, V, F, L$  as above. Then:*

- (a)  $\text{Zar}_{\min}(D)$  is Noetherian if and only if  $\text{Zar}(L|F)$  is Noetherian.
- (b)  $\text{Zar}(D)$  is Noetherian if and only if  $\text{Zar}(L|F)$  and  $\text{Spec}(V)$  are Noetherian.

*Proof.* If  $\text{Zar}_{\min}(D)$  is Noetherian, then  $\text{Zar}_{\min}(L|F)$  is Noetherian as well. By Propositions 4.1 and 4.2,  $\text{Zar}(L|F)$  is Noetherian.

If  $\text{Zar}(D)$  is Noetherian, so are  $\text{Spec}(D) = \text{Spec}(V)$  and  $\Delta \simeq \text{Zar}(L|F)$  (in the notation above). Conversely, if  $\text{Zar}(L|F)$  and  $\text{Spec}(V)$  are Noetherian then so are  $\text{Zar}_{\min}(D)$  and  $\Delta$ , and thus also  $\text{Zar}_{\min}(D) \cup \Delta = \text{Zar}(D)$  is Noetherian.  $\square$

Furthermore, we can now apply Propositions 4.1 and 4.2 to characterize when  $\text{Zar}(L|F)$  is Noetherian (see the following Corollary 5.12).

We now study the non-local case.

**Lemma 5.10.** *Let  $D$  be an integral domain such that  $D_M$  is a PVD for every  $M \in \text{Max}(D)$  and, for every  $M$ , let  $V(M)$  be the valuation overring associated to  $D_M$ . Then, the space  $\{V(M) \mid M \in \text{Max}(D)\}$  is homeomorphic to  $\text{Max}(D)$ .*

*Proof.* Let  $\Delta := \{V(M) \mid M \in \text{Max}(D)\}$ . If  $\gamma$  is the center map, then  $\gamma(V(M)) = M$  for every  $M$ ; thus,  $\gamma$  restricts to a bijection between  $\Delta$  and  $\text{Max}(D)$ . Since  $\gamma$  is continuous and closed, it follows that it is a homeomorphism.  $\square$

**Theorem 5.11.** *Let  $D$  be an integrally closed domain. Then:*

- (a)  $\text{Zar}_{\min}(D)$  is Noetherian if and only if  $\text{Max}(D)$  is Noetherian and  $\text{Zar}_{\min}(D_M)$  is Noetherian for every  $M \in \text{Max}(D)$ ;
- (b)  $\text{Zar}(D)$  is Noetherian if and only if  $\text{Spec}(D)$  is Noetherian and  $\text{Zar}(D_M)$  is Noetherian for every  $M \in \text{Max}(D)$ .

*Proof.* (a) If  $\text{Zar}_{\min}(D)$  is Noetherian, then  $\text{Max}(D)$  is Noetherian since it is the image of  $\text{Zar}_{\min}(D)$  under the center map, while each  $\text{Zar}_{\min}(D_M)$  is Noetherian since they are subspaces of  $\text{Zar}_{\min}(D)$ .

Conversely, suppose that  $\text{Max}(D)$  is Noetherian and that  $\text{Zar}(D_M)$  is Noetherian for every  $M \in \text{Max}(D)$ . By the latter property and Theorem 5.8, every  $D_M$  is a PVD; by Lemma 5.10, the space  $\Delta := \{V(M) \mid M \in \text{Max}(D)\}$  (in the notation of the lemma) is homeomorphic to  $\text{Max}(D)$ , and thus Noetherian. Let  $\beta$  be the map sending a  $W \in \text{Zar}_{\min}(D)$  to  $V(\mathfrak{m}_W \cap D)$ .

Let  $X$  be any subset of  $\text{Zar}_{\min}(D)$ , and let  $\Omega$  be an open cover of  $X$ ; without loss of generality, we can suppose  $\Omega = \{\mathcal{B}(f_\alpha) \mid \alpha \in A\}$ , where the  $f_\alpha$  are elements of  $K$ . Then,  $\Omega$  is also a cover of  $X' := \{\beta(V) \mid V \in X\}$ ; since  $X'$  is compact (being a subset of the Noetherian space  $\Delta$ ), there is a finite subfamily of  $\Omega$ , say  $\Omega' := \{\mathcal{B}(f_1), \dots, \mathcal{B}(f_n)\}$ , that covers  $X'$ . For each  $i$ , let  $X_i := \{V \in X \mid f_i \in \beta(V)\}$ ; then,  $X = X_1 \cup \dots \cup X_n$ . We want to find, for each  $i$ , a finite subset  $\Omega_i \subset \Omega$  that is a cover of  $X_i$ .

Fix thus an  $i$ , let  $f := f_i$ , and let  $I := (D :_D f)$  be the conductor ideal. For every  $M \in \text{Max}(D)$ , let  $Z(M) := \gamma^{-1}(M) \cap X_i = \{V \in X_i \mid \mathfrak{m}_V \cap D = M\}$ , where  $\gamma$  is the center map. The union of the  $Z(M)$  is  $X_i$ ; we separate the cases  $I \not\subseteq M$  and  $I \subseteq M$ .

If  $I \not\subseteq M$ , then  $1 \in ID_M = (D_M :_{D_M} f)$ , and thus  $f \in D_M$ ; hence, in this cases  $\mathcal{B}(f)$  contains  $Z(M)$ .

Suppose  $I \subseteq M$ ; clearly, we can suppose  $Z(M) \neq \emptyset$ . We claim that in this case  $M$  is minimal over  $I$ . Indeed, if there is a  $V \in Z(M)$  then  $f \in V$ , and thus  $f \in \beta(V)$ ; therefore,  $f \in D_P$  for every prime ideal  $P \subsetneq M$  (since  $D_P \supsetneq \beta(V)$  for every such  $P$ ), and thus  $I \not\subseteq P$ . Therefore,  $M$  is minimal over  $I$ . By Lemma 5.1,  $I$  has only finitely many minimal primes; hence, there are only finitely many  $M$  such that  $I \subseteq M$  and  $Z(M) \neq \emptyset$ . For each of these  $M$ , the set of valuation domains in  $X$  centered on  $M$  is a subset of  $\text{Zar}_{\min}(D_M)$ , and thus it is

compact; hence, for each of them,  $\Omega$  admits a finite subcover  $\Omega(M)$ . It follows that  $\Omega_i := \{\mathcal{B}(f)\} \cup \bigcup \Omega(M)$  is a finite subset of  $\Omega$  that is a cover of  $X_i$ .

Hence,  $\bigcup_i \Omega_i$  is a finite subset of  $\Omega$  that covers  $X$ ; thus,  $X$  is compact. Since  $X$  was arbitrary,  $\text{Zar}_{\min}(D)$  is Noetherian.

(b) If  $\text{Zar}(D)$  is Noetherian, then  $\text{Spec}(D)$  and every  $\text{Zar}(D_M)$  are Noetherian.

Conversely, suppose that  $\text{Spec}(D)$  is Noetherian and that  $\text{Zar}(D_M)$  is Noetherian for every  $M \in \text{Max}(D)$ . By the previous point,  $\text{Zar}_{\min}(D)$  is Noetherian. Furthermore, if  $P \in \text{Spec}(D) \setminus \text{Max}(D)$  then  $D_P$  is a valuation domain; hence,  $\text{Zar}(D) \setminus \text{Zar}_{\min}(D)$  is homeomorphic to  $\text{Spec}(D) \setminus \text{Max}(D)$ , which is Noetherian by hypothesis. Being the union of two Noetherian subspaces,  $\text{Zar}(D)$  itself is Noetherian.  $\square$

**Corollary 5.12.** *Let  $D$  be an integral domain that is not a field, and let  $L$  be a field containing  $D$ ; suppose that  $D$  is integrally closed in  $L$ . Then,  $\text{Zar}(L|D)$  (respectively  $\text{Zar}_{\min}(L|D)$ ) is Noetherian if and only if the following hold:*

- $L$  is the quotient field of  $D$ ;
- $\text{Spec}(D)$  is Noetherian (resp.,  $\text{Max}(D)$  is Noetherian);
- for every  $M \in \text{Max}(D)$ , the ring  $D_M$  is a pseudo-valuation domain such that  $\text{Zar}(L|F)$  is Noetherian, where  $F$  is the residue field of  $D_M$  and  $L$  is the residue field of the associated valuation overring of  $D_M$ .

*Proof.* Join Proposition 3.4, Theorem 5.11 and Proposition 5.9.  $\square$

For our last result, we recall that the *valuative dimension*  $\dim_v(D)$  of an integral domain  $D$  is the supremum of the dimensions of the valuation overrings of  $D$ ; a domain  $D$  is called a *Jaffard domain* if  $\dim(D) = \dim_v(D) < \infty$ , while it is a *locally Jaffard domain* if  $D_P$  is a Jaffard domain for every  $P \in \text{Spec}(D)$  [1]. Any locally Jaffard domain is Jaffard, while the converse does not hold [1, Example 3.2]. The class of Jaffard domains includes, for example, (finite-dimensional) Noetherian domain, Prüfer domains and universally catenarian domains.

**Proposition 5.13.** *Let  $D$  be an integrally closed integral domain of finite dimension, and suppose that  $\text{Zar}_{\min}(D)$  is a Noetherian space. Then:*

- (a)  $\dim_v(D) \in \{\dim(D), \dim(D) + 1\}$ ;
- (b)  $D$  is locally Jaffard if and only if  $D$  is a Prüfer domain.

*Proof.* (a) Let  $M$  be a maximal ideal of  $D$ . Then,  $\text{Zar}_{\min}(D_M)$  is Noetherian, and thus  $D_M$  is a pseudo-valuation domain; by [1, Proposition 2.9],  $\dim_v(D_M) = \dim(D_M) + \text{trdeg}_F L$ , where  $F$  is the residue field of  $D_M$  and  $L$  is the residue field of the associated valuation ring of  $D_M$ . By Propositions 5.9 and 4.1,  $\text{trdeg}_F L \leq 1$ , and thus

$\dim_v(D_M) \leq \dim(D_M) + 1$ . Hence,  $\dim_v(D) \leq \dim(D) + 1$ ; since  $\dim_v(D) \geq \dim(D)$  always, we have the claim.

(b) If  $D$  is a Prüfer domain then it is locally Jaffard. Conversely, if  $D$  is locally Jaffard, then  $\dim_v(D_P) = \dim(D_P)$  for every prime ideal  $P$  of  $D$ . Take any maximal ideal  $M$ , and let  $F, L$  as above; using  $\dim_v(D_M) = \dim(D_M) + \operatorname{trdeg}_F L$ , it follows that  $\operatorname{trdeg}_F L = 0$ . Since  $D$  (and so  $D_M$ ) is integrally closed, it must be  $F = L$ , i.e.,  $D_M$  itself is a valuation domain. Therefore,  $D$  is a Prüfer domain.  $\square$

Note that there are domains  $D$  that are Jaffard domains and have  $\operatorname{Zar}(D)$  Noetherian, but are not Prüfer domains. Indeed, the construction presented in [1, Example 3.2] gives a ring  $R$  with two maximal ideals,  $M$  and  $N$ , such that  $R_M$  is a two-dimensional valuation ring while  $R_N$  is a one-dimensional pseudo-valuation domain with  $\dim_v(R_N) = 2$ ; in particular, it is a Jaffard domain that is not Prüfer. Choosing  $k = K(Z_1)$  in the construction (or, more generally, choosing  $k$  such that  $K(Z_1, Z_2)$  is finite over  $k$ ), the Zariski space of  $R_N$  is Noetherian (being homeomorphic to  $\operatorname{Zar}(K(Z_1, Z_2)|k)$ , which is Noetherian by Proposition 4.2), and thus  $\operatorname{Zar}(R)$  is Noetherian.

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