The Zariski-Riemann space of valuation domains associated to pseudo-convergent sequences

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Let V be a valuation domain with quotient field K. Given a pseudo-convergent sequence E in K, we study two constructions associating to E a valuation domain of $K(X)$ lying over V, especially when V has rank one. The first one has been introduced by Ostrowski, the second one more recently by Loper and Werner. In the first part of the paper we describe the main properties of these valuation domains, and we give a notion of equivalence on the set of pseudo-convergent sequences of K characterizing when the associated valuation domains are equal. In the second part, we analyze the topological properties of the Zariski-Riemann spaces formed by these valuation domains.

Keywords: pseudo-convergent sequence, pseudo-limit, residually transcendental extension, Zariski-Riemann space, constructible topology.

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1 Introduction

Let V be a valuation domain with quotient field K. Determining and describing all the extensions of V to the field $K(X)$ of rational functions is an old and well-studied problem, which plays a vital role in several topics in field theory, commutative algebra and beyond (see for example [13] and the references therein). The problem has been approached in a few different ways, with the main ones being through *key polynomials* (starting from

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the work of MacLane [15] and developed, among many others, by Vaquié [22]), *minimal pairs* (introduced by Alexandru, Popescu and Zaharescu [2, 3]) and *pseudo-convergent sequences*. The latter were introduced by Ostrowski in [17], who used them to describe all rank one extension of the rank one valuation domain V to $K(X)$; subsequently, Kaplansky used this notion in [11] for valuation domains of any rank to characterize immediate extensions of valuation domains and maximally valued fields. More recently, Chabert in [8] generalized Ostrowski's definition by means of *pseudo-monotone sequences* to characterize the polynomial closure of subsets of valued fields of rank one.

In this paper, we study two constructions of extensions of V to $K(X)$ associated to a pseudo-convergent sequence $E \subset K$. The first one, which we denote by V_E, is the same construction introduced by Loper and Werner [14] for certain kinds of pseudo-convergent sequences on rank one valuation domains; we show that it actually defines a valuation domain for every pseudo-convergent sequence and for valuation domains of any rank (Theorem 3.7). The second one (which we denote by W_E) applies only when V has rank one (and E satisfies some conditions), and is defined through its valuation w_E, which was already introduced by Ostrowski [17]; for this reason, we name it the *Ostrowski valuation* associated to E. In Sections 3 and 4, we investigate the structure of these valuation domains, in particular when V has rank one; among other things, we show that $V_E \subseteq W_E$, we characterize when V_E has rank 2 (Theorem 4.9), we find their value group and residue field, and we describe explicitly the valuation v_E associated to V_E as a map from $K(X)$ to $\mathbb{R}^2$ (Theorem 4.10). Many of these results are based on a general theorem (Theorem 3.2) which expresses the valuation of $\phi(t)$, for a rational function $\phi(X)$, as a linear function of $v(t-s)$, in an annulus of center s which contains neither poles nor zeros of $\phi(X)$.

In Section 5, following Ostrowski, we investigate the notion of equivalence between two pseudo-convergent sequences, analogous to the concept of equivalence between two Cauchy sequences. We show that two pseudo-convergent sequences are equivalent if and only if their associated valuation rings are equal; moreover, if they are of algebraic type then these conditions are also equivalent to the property of having the same set of pseudo-limits (in the algebraic closure of K and with respect to the same extension of v; see Theorem 5.4). Using these results, we show that the extensions of an Ostrowski valuation w_E to $\overline{K}(X)$ is completely determined by its restriction to $\overline{K}$ (Theorem 5.5).

In Sections 6, 7 and 8 we study the spaces $\mathcal{V}$ and $\mathcal{W}$ formed, respectively, by the rings in the form V_E and by the rings in the form W_E, from a topological point of view; more precisely, we study the Zariski and the constructible topology they inherit from the Zariski-Riemann space $\mathrm{Zar}(K(X)|V)$. In particular, in Section 6 we analyze the difference between these two topologies, showing that they coincide on $\mathcal{W}$ (Proposition 6.2), while they coincide on $\mathcal{V}$ if and only if the residue field of V is finite (Proposition 6.8); we also show how V_E can be seen as a limit of valuation domains defined from the members of E, mirroring the fact that (classes of) Cauchy sequences can be associated to their limit points (Proposition 6.6). In Section 7, we show that $\mathcal{V}$, endowed with the Zariski topology, is a regular space.

In Section 8, we study two partitions of $\mathcal{V}$. Generalizing the study of valuation domains associated to an element of the completion of K tackled in [19], we show that the set of rings in the form V_E, as E ranges among the pseudo-convergent sequences of fixed breadth, can be seen as a complete ultrametric space under a very natural distance function (Theorem 8.7), although these distances cannot be unified into a metric encompassing all of $\mathcal{V}$ (Proposition 8.11). Subsequently, we study the set of rings in the form V_E having a fixed element of $\overline{K}$ as pseudo-limit; we represent it through a variant of the upper limit topology (Theorem 8.15), and we show that it is metrizable if and only if the value group of V is countable (Proposition 8.18). In particular, this shows that, when the value group of V is not countable, then the space $\mathrm{Zar}(K(X)|V)$, endowed with the constructible topology, is not metrizable (Corollary 8.19).

2 Background and notation

Throughout the article, V is a valuation domain; we denote by K its quotient field, by M its maximal ideal and by v the valuation associated to V. Its value group is denoted by Γ_v. We denote by $\widehat{K}$ and $\widehat{V}$ the completion of K and V, respectively, with respect to the topology induced by the valuation v. We denote by $\overline{K}$ a fixed algebraic closure of K. If u is an extension of v to $\overline{K}$, then the value group of u is the divisible hull of Γ_v, i.e., $\mathbb{Q}\Gamma_v := \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_v$.

The basic object of study of this paper are pseudo-convergent sequences, introduced by Ostrowski in [17] and used by Kaplansky in [11] to describe immediate extensions of valued fields. Related are the concepts *pseudo-stationary* and *pseudo-divergent* sequences [8], but we will not use them.

Definition 2.1. Let $E = \{s_n\}_{n\in\mathbb{N}}$ be a sequence in K. We say that E is a *pseudo-convergent* sequence if $v(s_{n+1} - s_n) < v(s_{n+2} - s_{n+1})$ for all $n \in \mathbb{N}$.

In particular, if $E = \{s_n\}_{n\in\mathbb{N}}$ is a pseudo-convergent sequence and $n \geq 1$, then $v(s_{n+k} - s_n) = v(s_{n+1} - s_n)$ for all $k \geq 1$. We shall usually denote this quantity by δ_n; following [23, p. 327] we call the sequence $\{\delta_n\}_{n\in\mathbb{N}}$ the *gauge* of E.

Definition 2.2. The *breadth ideal* of E is

$$\mathrm{Br}(E) = \{b \in K \mid v(b) > v(s_{n+1} - s_n), \forall n \in \mathbb{N}\}.$$

In general, $\mathrm{Br}(E)$ is a fractional ideal of V and may not be contained in V.

The following definition has been introduced in [11], even though already in [17, p. 375] an equivalent concept appears (see [17, X, p. 381] for the equivalence).

Definition 2.3. An element $\alpha \in K$ is a *pseudo-limit* of E if $v(\alpha - s_n) < v(\alpha - s_{n+1})$ for all $n \in \mathbb{N}$, or, equivalently, if $v(\alpha - s_n) = \delta_n$ for all $n \in \mathbb{N}$. We denote the set of pseudo-limits of E by $\mathcal{L}_E$, or $\mathcal{L}_E^v$ if we need to underline the valuation.

If $\mathrm{Br}(E)$ is the zero ideal then E is a Cauchy sequence in K and converges to a element of $\widehat{K}$, which is the unique pseudo-limit of E. In general, Kaplansky proved the following result.

Lemma 2.4. *[11, Lemma 3] Let $E \subset K$ be a pseudo-convergent sequence. If $\alpha \in K$ is a pseudo-limit of E, then the set of pseudo-limits of E in K is equal to $\alpha + \mathrm{Br}(E)$.*

If w is an extension of v to a field L containing K, and E is a sequence in K, then E is pseudo-convergent under w if and only if E is a pseudo-convergent under v. Moreover, every pseudo-limit of E under v in K is also a pseudo-limit under w.

Suppose now that V has rank one; then we consider Γ_v and $\mathbb{Q}\Gamma_v$ as totally ordered subgroups of $\mathbb{R}$. The valuation v induces an ultrametric distance d on K, defined by

$$d(x, y) := e^{-v(x-y)}.$$

In this metric, V is the closed ball of center 0 and radius 1. Given $s \in K$ and $\gamma \in \Gamma_v$, we denote the ball of center s and radius $r = e^{-\gamma}$ by:

$$B(s, r) = \{x \in K \mid d(x, s) \leq r\} = \{x \in K \mid v(x - y) \geq \gamma\}.$$

A ball in $\overline{K}$ with respect to an extension u of v is denoted by $B_u(s, r)$.

If $E \subset K$ is a pseudo-convergent sequence, then the gauge $\{\delta_n\}_{n\in\mathbb{N}}$ of E is a strictly increasing sequence of real numbers, and so the following definition makes sense.

Definition 2.5. The *breadth* of a pseudo-convergent sequence $E = \{s_n\}_{n\in\mathbb{N}}$ is the limit

$$\delta_E := \lim_{n\to\infty} v(s_{n+1} - s_n) = \lim_{n\to\infty} \delta_n.$$

The breadth δ is an element of $\mathbb{R} \cup \{\infty\}$, and it may not lie in Γ_v. We can use the breadth to characterize the breadth ideal: indeed, $\mathrm{Br}(E) = \{b \in K \mid v(b) \geq \delta_E\}$, or equivalently $\delta_E = \inf\{v(b) \mid b \in \mathrm{Br}(E)\}$. If $\delta = +\infty$, then $\mathrm{Br}(E)$ is just the zero ideal and E is a Cauchy sequence in K. If V is a discrete valuation ring, then every pseudo-convergent sequence is actually a Cauchy sequence. Lemma 2.4 can also be phrased in a geometric way: if $\alpha \in \mathcal{L}_E$, then $\mathcal{L}_E$ is the closed ball of center α and radius $e^{-\delta_E}$, i.e., $\mathcal{L}_E = B(\alpha, e^{-\delta_E})$.

The following concepts have been given by Kaplansky in [11] in order to study the different kinds of immediate extensions of a valued field K. Recall that if L is a field extension of K, a valuation domain W of L *lies over* V if $W \cap K = V$. In this case, the residue field of W is naturally an extension of the residue field of V and similarly the value group of W is an extension of the value group of V. We say that W is immediate over V if both the residue fields and the value groups are the same.

Definition 2.6. Let E be a pseudo-convergent sequence. We say that E is of *transcendental type* if $v(f(s_n))$ eventually stabilizes for every $f \in K[X]$; on the other hand, if $v(f(s_n))$ is eventually increasing for some $f \in K[X]$, we say that E is of *algebraic type.*

The main difference between these two kind of sequences is the nature of the pseudo-limits: if E is of algebraic type, then E has pseudo-limits in $\overline{K}$ (for some extension u of v), while if E is of transcendental type then E admits a pseudo-limit only in a transcendental extension [11, Theorems 2 and 3].

If $L = K(X)$ and W lies over V, then W is said to be a *residually transcendental extension* of V (or simply residually transcendental if V is understood) if the residue field of W is a transcendental extension of the residue field of V [2].

Definition 2.7. Let Γ be a totally ordered group containing Γ_v, and take $\alpha \in K$ and $\delta \in \Gamma$. The *monomial valuation* $v_{\alpha,\delta}$ is defined in the following way: if $f(X) \in K[X]$ is a polynomial, write $f(X) = a_0 + a_1(X-\alpha) + \ldots + a_n(X-\alpha)^n$; then,

$$v_{\alpha,\delta}(f) := \inf\{v(a_i) + i\delta \mid i = 0, \ldots, n\}.$$

It is well-known that $v_{\alpha,\delta}$ naturally extends to a valuation on $K(X)$ [5, Chapt. VI, §. 10, Lemme 1], and $v_{\alpha,\delta}$ is residually transcendental over v if and only if δ has finite order over Γ_v [18, Lemma 3.5]. Furthermore, every residually transcendental extension of V can be written as $W \cap K(X)$, where W is a valuation domain of $\overline{K}(X)$ associated to a monomial valuation [1, 2].

Let D be an integral domain and L be a field containing D (not necessarily the quotient field of D). The *Zariski space of D in L*, denoted by $\mathrm{Zar}(L|D)$, is the set of valuation domains of L containing D endowed with the so-called *Zariski topology*, i.e., with the topology generated by the subbasic open sets

$$B(\phi_1, \ldots, \phi_k) := \{V \in \mathrm{Zar}(L|D) \mid \phi_1, \ldots, \phi_k \in V\},$$

as $\phi_1, \ldots, \phi_k$ range in L. Under this topology, $\mathrm{Zar}(L|D)$ is a compact space [24, Chapter VI, Theorem 40] that is almost never Hausdorff nor T_1 (indeed, $\mathrm{Zar}(L|D)$ is a T_1 space if and only if D is a field and L is an algebraic extension of D).

The *constructible topology* (also called *patch topology*) on $\mathrm{Zar}(L|D)$ is the coarsest topology such that the subsets $B(\phi_1, \ldots, \phi_k)$ are both open and closed; we denote this space by $\mathrm{Zar}(L|D)^{\mathrm{cons}}$. Clearly, the constructible topology is finer than the Zariski topology; however, $\mathrm{Zar}(L|D)^{\mathrm{cons}}$ is still compact, and furthermore it is always Hausdorff [10, Theorem 1].

3 A valuation domain associated to a pseudo-convergent sequence

The following valuation domain associated to a pseudo-convergent sequence has been introduced by Loper and Werner in [14] in the case of a valuation domain V of K of rank one. We generalize it for valuation domains of arbitrary rank.

Definition 3.1. Let $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ be a pseudo-convergent sequence. Let

$$V_E = \{\phi \in K(X) \mid \phi(s_n) \in V, \text{ for all but finitely many } n \in \mathbb{N}\}. \tag{1}$$

The aim of this section is to prove that V_E is a valuation domain of $K(X)$ for every pseudo-convergent sequence E. When the rank of V is one and E is of transcendental type or has zero breadth ideal, this result was already obtained, respectively, in Proposition 5.5 and Theorem 5.8 of [14]. More generally, for any valuation domain, when E is of transcendental type V_E coincides with the valuation domain of $K(X)$ defined by Kaplansky in [11, Theorem 2], which is an immediate extension of V. Since in this case for each $\phi \in K(X)$ we have that $v(\phi(s_n))$ is definitively constant, the value of ϕ with respect to the above valuation is equal to that constant value that $\phi(X)$ assumes over E. Following this example, our method is heavily based on understanding the values of $\phi(X)$ along a pseudo-convergent sequence.

For the next result, which is not a priori related to pseudo-convergent sequences, we introduce some notation. For any rational function $\phi \in K(X)$, the *multiset of critical points* of $\phi(X)$ is the multiset Ω_ϕ of zeroes and poles of ϕ in $\overline{K}$ (each of them counted with multiplicity). Let $S = \{\alpha_1, \dots, \alpha_k\}$ be a sub-multiset of Ω_ϕ. By the *weigthed sum* of S we mean the sum $\sum_{\alpha_i \in S} \epsilon_i$, where ϵ_i is equal either to 1 or to -1, according to whether α_i is a zero or a pole of ϕ, respectively. By the *S-part* of ϕ we mean the rational function $\phi_S(X) = \prod_{\alpha_i \in S}(X - \alpha_i)^{\epsilon_i}$, where $\epsilon_i \in \{\pm 1\}$ is as above. Note that $\phi_{\Omega_\phi}(X)$ is equal to $\phi(X)$ up to a constant.

Given a convex subset Δ of Γ_v, $\beta \in \overline{K}$ and an extension u of v to $\overline{K}$, we set

$$\mathcal{C}_u(\beta, \Delta) = \{s \in \overline{K} \mid u(s - \beta) \in \Delta\} \tag{2}$$

and, if $\gamma \in \mathbb{Q}\Gamma_v$, we write $\gamma < \Delta$ ($\gamma > \Delta$, respectively) if $\gamma < \delta$ ($\gamma > \delta$, respectively) for every $\delta \in \Delta$.

Theorem 3.2. *Let $\phi \in K(X)$ and let $s \in K$; let v be a valuation on K and let u be an extension of v to $\overline{K}$. Let Δ be a convex subset of $\mathbb{Q}\Gamma_v$ such that $C = \mathcal{C}_u(s, \Delta)$ does not contain any critical point of ϕ. Let $\lambda \in \mathbb{Z}$ be equal to the weigthed sum of the multiset S of critical points α of ϕ which satisfy $u(\alpha - s) > \Delta$ and let $\gamma = u\left(\frac{\phi}{\phi_S}(s)\right)$. Then, for all $t \in C \cap K$, we have*

$$v(\phi(t)) = \lambda u(t - s) + \gamma. \tag{3}$$

Proof. Over $\overline{K}$, we can write $\phi(X)$ as a product $c\prod_{i=1}^n (X - \alpha_i)^{\epsilon_i}$, where the α_i are the critical points of ϕ, $\epsilon_i \in \{-1, 1\}$ and $c \in K$. Let $C := \mathcal{C}_u(s, \Delta) \subset \overline{K}$ and let $t \in K \cap C$. If $u(\alpha_i - s) < \Delta$ then $u(t - \alpha_i) = u(s - \alpha_i)$, while if $u(\alpha_i - s) > \Delta$ then $u(t - \alpha_i) = u(s - t)$ (note that, by assumption, there are no other possibilities for the critical points of $\phi(X)$). Therefore, we have

$$\begin{aligned} v(\phi(t)) &= v(c) + \sum_{i:u(\alpha_i - s) < \Delta} \epsilon_i u(t - \alpha_i) + \sum_{i:u(\alpha_i - s) > \Delta} \epsilon_i u(t - \alpha_i) = \\ &= v(c) + \sum_{i:u(\alpha_i - s) < \Delta} \epsilon_i u(\alpha_i - s) + \sum_{i:u(\alpha_i - s) > \Delta} \epsilon_i u(t - s) = \gamma + \lambda v(t - s) \end{aligned}$$

where $\lambda = \sum_{i:u(\alpha_i - s) > \Delta} \epsilon_i$ and $\gamma = v(c) + \sum_{i:u(\alpha_i - s) < \Delta} \epsilon_i u(s - \alpha_i) = u\left(\frac{\phi}{\phi_S}(s)\right)$, with S being the multiset of critical points α_i of ϕ which satisfy $u(\alpha_i - s) > \Delta$. In particular, $\lambda \in \mathbb{Z}$ and $\gamma \in \mathbb{Q}\Gamma_v$ do not depend on t. The claim is proved. $\square$

Remark 3.3. Let $\alpha_1, \dots, \alpha_n$ be the zeros and the poles of ϕ, and let $\rho_i := u(s - \alpha_i)$; without loss of generality, suppose $\rho_1 < \dots < \rho_n$. Then, the sets $\Delta_i := (\rho_i, \rho_{i+1})$, for $i = 0, \dots, n$ (with the convention $\rho_0 := -\infty$ and $\rho_{n+1} := +\infty$) are the maximal convex sets on which Theorem 3.2 can be applied: that is, they satisfy (by definition) the hypothesis of the theorem, and if $\Delta_i \subsetneq \Delta$ for some other convex set Δ then the theorem cannot be applied to Δ.

In order to apply Theorem 3.2 to pseudo-convergent sequences, we need the following definition.

Definition 3.4. Let $E := \{s_n\}_{n\in\mathbb{N}}$ be a pseudo-convergent sequence in K, let u be an extension of v to $\overline{K}$ and let $\phi \in K(X)$. The *dominating degree* $\mathrm{degdom}_{E,u}(\phi)$ of ϕ with respect to E and u is the weighted sum of the critical points of $\phi(X)$ which are pseudo-limits of E with respect to u.

Note that, by definition, if E is a pseudo-convergent sequence of transcendental type, then $\mathrm{degdom}_{E,u}(\phi) = 0$ for every $\phi \in K(X)$.

The following result shows that the values of a rational function over a pseudo-convergent sequence $E \subset K$ form a sequence that is definitively monotone, either strictly increasing, strictly decreasing or stationary, according to whether the dominating degree of ϕ with respect to E is positive, negative or equal to zero, respectively.

Proposition 3.5. *Let $E := \{s_n\}_{n\in\mathbb{N}} \subset K$ be a pseudo-convergent sequence with gauge $\{\delta_n\}_{n\in\mathbb{N}}$, and let u be an extension of v to $\overline{K}$. Let $\phi \in K(X)$.*

(a) If $\lambda := \mathrm{degdom}_{E,u}\,\phi$, then there is $\gamma \in \Gamma_v$ such that, for all sufficiently large n, we have

$$v(\phi(s_n)) = \lambda\delta_n + \gamma.$$

(b) If $\beta \in \overline{K}$ is a pseudo-limit of E with respect to u, then $\gamma = u\left(\frac{\phi}{\phi_S}(\beta)\right)$, where S is the set of critical points of $\phi(X)$ which are pseudo-limits of E.

(c) The dominating degree of ϕ does not depend on u; that is, if u' is another extension of v to $\overline{K}$, then $\mathrm{degdom}_{E,u}\,\phi = \mathrm{degdom}_{E,u'}\,\phi$.

Proof. If E is of transcendental type, then $\lambda = 0$ and all claims follow from the very definition.

Suppose that E is of algebraic type and $\beta \in \mathcal{L}_E^u$; we will prove (a) and (b) together. Let $\Delta = \Delta_E$ be the least initial segment of $\mathbb{Q}\Gamma_v$ containing the gauge of E. There exists $\tau \in \Gamma_v \cap \Delta$ such that $C = \mathcal{C}_u(\beta, \Delta\cap(\tau, +\infty))$ contains no critical points of ϕ. Let λ be the weighted sum of the subset S of Ω_ϕ of those elements α such that $u(\alpha-\beta) > \Delta\cap(\tau,+\infty)$ (or, equivalently, $u(\alpha-\beta) > \Delta$) and $\gamma = u\left(\frac{\phi}{\phi_S}(\beta)\right)$. For all n sufficiently large $s_n \in C$: by Theorem 3.2, it follows that for each such n we have

$$v(\phi(s_n)) = \lambda u(\beta - s_n) + \gamma = \lambda\delta_n + \gamma.$$

Note that $\gamma \in \Gamma_v$ and, by Lemma 2.4, S is the set of critical points of $\phi(X)$ which are pseudo-limits of E, so λ is the dominating degree of ϕ with respect to E.

For (c), we note that $v(\phi(s_n))$ does not depend on the extension u; hence, if $\lambda = \mathrm{degdom}_{E,u}\,\phi$, $\lambda' = \mathrm{degdom}_{E,u'}\,\phi$, $\gamma = u\left(\frac{\phi}{\phi_S}(\beta)\right)$, $\gamma' = u'\left(\frac{\phi}{\phi_S}(\beta)\right)$, we have

$$v(\phi(s_n)) = \lambda\delta_n + \gamma = \lambda'\delta_n + \gamma'$$

for all large n. However, this clearly implies $\lambda = \lambda'$, as claimed. $\square$

In view of point (c) of the previous proposition, we denote the dominating degree of ϕ with respect to E simply as $\operatorname{degdom}_E \phi$.

The term *dominating degree* comes from the following property.

Proposition 3.6. *Let $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ be a pseudo-convergent sequence with pseudo-limit $\beta \in K$, and let $f(X) := \sum_{i=0,\dots,d} a_i(X-\beta)^i \in K[X]$. Then,* $\operatorname{degdom}_E f$ *is the non-negative integer k such that $v(f(s_n)) = v(a_k(s_n-\beta)^k)$ for all large n.*

Proof. Clearly, if $v(f(s_n)) = v(a_k(s_n-\beta)^k)$ for all large n then $v(f(s_n)) = k\delta_n + v(a_k)$ and so $k = \operatorname{degdom}_E f$.

Conversely, suppose $k = \operatorname{degdom}_E f$. Then, by definition, $v(f(s_n)) = k\delta_n + \gamma$ for some $\gamma \in \Gamma_v$ (for all large n), where $\{\delta_n\}_{n\in\mathbb{N}}$ is the gauge of E. We consider the following linear functions from Γ_v to Γ_v:

$$\lambda_i(\eta) := i\eta + v(a_i),\ i \in \{0,\dots,d\}.$$

Let Δ be the least initial segment of Γ_v containing the gauge of E: then, since the λ_i are linear, there is a $\tau \in \Delta$ and an $r \in \{0,\dots,d\}$ such that $\lambda_r(\eta) < \lambda_i(\eta)$ for all $\eta \in \Delta \cap (\tau,+\infty)$. Therefore, whenever $\delta_n \in \Delta\cap(\tau,+\infty)$ we must have

$$v(f(s_n)) = v\left(\sum_i a_i(s_n-\beta)^i\right) = \inf_i\{v(a_i(s_n-\beta)^i)\} = r\delta_n + v(a_r).$$

In particular, it must be $r = k$, as claimed. ☐

Theorem 3.7. *Let $E = \{s_n\}_{n\in\mathbb{N}}$ be a pseudo-convergent sequence. Then $V_E \subset K(X)$ is a valuation domain lying over V with maximal ideal equal to $M_{V_E} = \{\phi \in K(X) \mid v(\phi(s_n)) \in M,$ for all but finitely many $n \in \mathbb{N}\}$. Moreover, X is a pseudo-limit of E with respect to the valuation v_E associated to V_E.*

Proof. Clearly, V_E is a ring and $V_E \cap K = V$.

If E is of transcendental type then V_E is exactly the valuation domain of the immediate extension of the valuation v to $K(X)$ induced by E as in [11, Theorem 2]. We have that X is a pseudo-limit of E by [11, Theorem 2].

Suppose now that E is of algebraic type, and let $\phi \in K(X)$. By Proposition 3.5, $v(\phi(s_n))$ is a linear function of δ_n; hence, it is either definitively positive, definitively zero or definitively negative. Since $v(\phi^{-1}(s_n)) = -v(\phi(s_n))$ (provided that $\phi(s_n) \neq 0$, which happens only finitely many times), we have that $\phi \in V_E$, in the first and second case, while in the third case $\phi^{-1} \in V_E$. Hence, V_E is a valuation domain, and the claim about the maximal ideal follows easily.

Finally, we show that X is a pseudo-limit of E with respect to v_E. Fix $n \in \mathbb{N}$, and let $\phi(X) := \frac{X-s_{n+1}}{X-s_n}$. Then, for $m > n+1$ we have $v(\phi(s_m)) = \delta_{n+1} - \delta_n > 0$, and thus $v_E(X - s_{n+1}) > v_E(X - s_n)$. It follows that X is a pseudo-limit of E, as claimed. ☐

Remark 3.8. In case E is a pseudo-convergent sequence with zero breadth ideal, and $\alpha \in \widehat{K}$ is the (unique) limit of E, then (since rational functions are continuous in the topology induced by v) $V_E = W_\alpha = \{\phi \in K(X) \mid v(\phi(\alpha)) \geq 0\}$. Moreover, α is algebraic (transcendental, respectively) over K if and only if E is of algebraic (transcendental, respectively) type. These kind of valuations domains have been characterized in [19, Proposition 2.2]. We will deal with the case of non-zero breadth ideal in Theorem 4.9.

We conclude this section by describing the valuation v_E, its residue field and its value group when E is a pseudo-convergent sequence of algebraic type.

Proposition 3.9. *Let $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ be a pseudo-convergent sequence and suppose that $\beta \in K$ is a pseudo-limit of E; let $\alpha \in K$. Then, the following hold.*

(a) $v_E(X-\alpha) \leq v_E(X-\beta)$ and equality holds if and only if $\alpha \in \mathcal{L}_E$.

Let $\Delta_E := v_E(X-\beta) \in \Gamma_{v_E}$ (which by above does not depend on the choice of the pseudo-limit β of E).

(b) Δ_E is not a torsion element in Γ_{v_E}/Γ_v (i.e., if $k \in \mathbb{N}$ is such that $k \cdot \Delta_E \in \Gamma_v$, then $k = 0$). In particular, $v_E = v_{\beta,\Delta_E}$.

(c) $\Gamma_{v_E} = \mathbb{Z}\Delta_E \oplus \Gamma_v$ (as groups).

(d) $V_E/M_E \cong V/M$.

Proof. The condition $v_E(X-\alpha) \leq v_E(X-\beta)$ is equivalent to $\phi(X) = \frac{X-\beta}{X-\alpha} \in V_E \Leftrightarrow \phi(s_n) \in V$, for almost all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we have:

$$v(\phi(s_n)) = v(s_n-\beta) - v(s_n-\alpha)$$

Now, we write $v(s_n-\alpha) = v(s_n-\beta+\beta-\alpha)$. Note that $\beta-\alpha \in \mathrm{Br}(E) \Leftrightarrow \alpha \in \mathcal{L}_E$ (Lemma 2.4). If these conditions hold, then $v(\beta-\alpha) > v(s_{n+1}-s_n) = v(s_n-\beta)$ for each $n \in \mathbb{N}$ and therefore $v(s_n-\alpha) = v(s_n-\beta)$. Note that in this case $\phi \in V_E^*$ and so, in particular, $\Delta_E = v_E(X-\beta)$ does not depend on the pseudo-limit β of E we have chosen (in K). If instead $\alpha \notin \mathcal{L}$ then there exists $N \in \mathbb{N}$ such that $v(\beta-\alpha) < v(s_{n+1}-s_n) = v(\beta-s_n)$ for all $n \geq N$. Hence, $v(s_n-\alpha) = v(\beta-\alpha) < v(\beta-s_n)$ for all $n \geq N$, that is, $\phi \in M_E \subset V_E$.

We prove now the other three claims. Suppose there exists $k \in \mathbb{N}$ such that $k\cdot\Delta_E \in \Gamma_v$, that is, $k \cdot v_E(X-\beta) = v_E((X-\beta)^k) = v(a)$, for some $a \in K$. This implies that $\frac{(X-\beta)^k}{a} \in V_E^*$, which is a contradiction, since $k \cdot v(s_n-\beta) - v(a)$ is strictly increasing.

Since $\Delta_E = v_E(X-\beta) \in \Gamma_{v_E}$ is not torsion over Γ_v, by [5, Chapt. VI, §10, Proposition 1] (see also [4, p. 289]) we have that for each $f \in K[X]$, $f(X) = a_0 + a_1(X-\beta) + \ldots + a_n(X-\beta)^n$,

$$v_E(f(X)) = \inf\{v(a_i) + i\Delta_E \mid i = 0,\ldots,n\}$$

(where the inf is in Γ_{v_E}). In fact, we have $v_E(a_i(X-\beta)^i) \neq v_E(a_j(X-\beta)^j)$, for all $i \neq j \in \{0,\ldots,n\}$, otherwise $(i-j)\Delta_E = v(a_j) - v(a_i)$ and Δ_E would be torsion over Γ_v. This implies that $v_E = v_{\beta,\Delta_E}$. Moreover, by the same reference, $\Gamma_{v_E} = \mathbb{Z}\Delta_E \oplus \Gamma_v$ and the residue field of V_E is isomorphic to the residue field of V. □

In the general case, where E is algebraic but has no pseudo-limits in K, we only need to pass to an extension of V.

Corollary 3.10. *Let $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ be a pseudo-convergent sequence of algebraic type, and let U be an extension of V to $\overline{K}(X)$. Let β be a pseudo-limit of E with respect to the valuation u of U, and let $\Delta := u_E(X-\beta)$. Then, v_E is equal to the restriction to $K(X)$ of $u_E = u_{\beta,\Delta}$.*

Proof. Note that $U_E = \{\psi \in K(\beta)(X) \mid \psi(s_n) \in U,$ for all but finitely many $n \in \mathbb{N}\}$. By Proposition 3.9, $u_E = u_{\beta,\Delta}$, where $\Delta = u_E(X-\beta)$. Since $U_E \cap K(X) = V_E$, the claim follows immediately. □

4 The rank of V_E and the Ostrowski valuation w_E

From now on, we assume that V has rank one.

In order to determine the rank of V_E, we need to introduce another kind of valuation on $K(X)$ which lies over V; also these valuations arise from pseudo-convergent sequences and have been first introduced and studied by Ostrowski in [17, 65. p. 374].

Definition 4.1. Let $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ be a pseudo-convergent sequence. We define w_E as the map

$$\begin{aligned} w_E \colon K(X) &\longrightarrow \mathbb{R}\cup\{\pm\infty\} \\ \phi &\longmapsto \lim_{n\to\infty} v(\phi(s_n)) \end{aligned}$$

Note that, for large n, s_n is neither a zero nor a pole of ϕ, so $v(\phi(s_n))$ is defined for all large n.

One of the main accomplishments of Ostrowski (and also the motivation for the introduction of the notion of pseudo-convergent sequence) in his work [17] is the *Fundamentalsatz*, which we now recall.

Theorem 4.2. *[17, 66. IX, p. 378] Let K be an algebraically closed field and let v be a rank one valuation on K. If w is a rank one valuation of $K(X)$ extending v, then there is a pseudo-convergent sequence $E = \{s_n\}_{n\in\mathbb{N}} \subset \overline{K}$ such that $w = w_E$.*

When K is not algebraically closed, this means that the rank one valuations of $K(X)$ extending v can be realized as the contraction to $K(X)$ of the valuations w_E on $\overline{K}(X)$ for some pseudo-convergent sequence $E \subset \overline{K}$ and some extension of v to $\overline{K}$.

For the sake of completeness, in the next two propositions we prove the basic properties of the function w_E.

Proposition 4.3. *Let $E = \{s_n\}_{n\in\mathbb{N}}$ be a pseudo-convergent sequence that is either of transcendental type or of algebraic type and non-zero breadth ideal. Then the map $w_E : K(X) \to \mathbb{R}\cup\{\infty\}$ extends v and is a valuation of rank one on $K(X)$. Furthermore, the valuation ring W_E relative to w_E contains V_E.*

Proof. Suppose first that E is of transcendental type. Then for each $\phi \in K(X)$, $v(\phi(s_n))$ is definitively constant, and furthermore $w_E(\phi) = \infty$ if and only if $\phi = 0$.

Suppose now that E is of algebraic type and the breadth ideal is non-zero. Then also in this case w_E is well-defined, since by Proposition 3.5 for every $\phi \in K(X)$ there is a

$k \in \mathbb{Z}$ and $\gamma \in \Gamma_v$ such that $v(\phi(s_n)) = k\delta_n + \gamma$, and $\delta_n \to \delta$ as $n \to \infty$. Moreover, since $\delta < \infty$, and since ϕ has only finitely many zeros and points where it is not defined, we have $w_E(\phi) = \infty$ if and only if $\phi = 0$.

In either case, if $\phi = a \in K$ is a constant, then $w_E(\phi) = v(a)$; thus, w_E extends v.

If now $\phi_1, \phi_2 \in K(X)$ then

$$v((\phi_1 + \phi_2)(s_n)) = v(\phi_1(s_n) + \phi_2(s_n)) \geq \min\{v(\phi_1(s_n)), v(\phi_2(s_n))\};$$

hence, $w_E(\phi_1 + \phi_2) \geq \min\{w_E(\phi_1), w_E(\phi_2)\}$. In the same way, $w_E(\phi_1\phi_2) = w_E(\phi_1) + w_E(\phi_2)$. Hence, w_E is a valuation.

If now $\phi \in V_E$, then $\phi(s_n) \in V$ for large n, or equivalently $v(\phi(s_n)) \geq 0$ for large n. In particular, $\lim v(\phi(s_n)) \geq 0$, i.e., $w_E(\phi) \geq 0$. Therefore, $\phi \in W_E$. □

If E is of algebraic type and its breadth ideal is zero, on the other hand, w_E is not a valuation: this is due to the fact that $w_E(\phi)$ tends to ∞ when the pseudo-limit of E (in $\overline{K}$) is a zero of ϕ. It is, however, very close to a valuation: recall that a *pseudo-valuation* of a field K is a map v from K to $\Gamma_v \cup \{\infty\}$, where Γ_v is a totally ordered abelian group, which satisfies the same axioms of a valuation except that we are not assuming that $v(x) = \infty \Rightarrow x = 0$. The set $\{x \in K \mid v(x) = \infty\}$ is a prime ideal of the valuation domain V of v, called the *socle* of v.

Proposition 4.4. *Let $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ be a pseudo-convergent sequence of algebraic type and zero breadth ideal. If $q \in K[X]$ is the minimal polynomial of the limit of E in $\widehat{K}$, then the map $w_E : K[X]_{(q)} \to \mathbb{R} \cup \{\infty\}$ extends v and is a pseudo-valuation with socle $q(X)K[X]_{(q)}$. We have that the valuation ring of w_E, that is, $\{\phi \in K[X]_{(q)} \mid w_E(\phi) \geq 0\}$, is equal to V_E.*

Definition 4.5. Let $E \subset K$ be a pseudo-convergent sequence which is either of transcendental type or of algebraic type and non-zero breadth ideal. We call the associated rank one valuation $w_E : K(X) \to \mathbb{R} \cup \{\infty\}$ the *Ostrowski valuation* associated to E, and the corresponding valuation domain W_E the *Ostrowski valuation domain* associated to E.

The following corollary follows at once from Proposition 3.5.

Corollary 4.6. *Let $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ be a pseudo-convergent sequence of algebraic type with breadth δ, and let $\phi \in K(X)$. If $\lambda = \mathrm{degdom}_E(\phi)$ and $\gamma \in \Gamma_v$ is as in Proposition 3.5, then we have*

$$w_E(\phi) = \lambda\delta + \gamma \tag{4}$$

In particular, $\Gamma_{w_E} = \mathbb{Z}\delta + \Gamma_v$.

Remark 4.7. (a) Let u be an extension of v to $\overline{K}$. It follows at once from Corollary 4.6 that if E is a pseudo-convergent sequence of algebraic type with breadth δ and $\beta \in \mathcal{L}_E^u$, then, for each $s \in K$ we have:

$$w_E(X - s) = \lim_{n\to\infty} v(s_n - s) = \begin{cases} \delta, & \text{if } s \in \mathcal{L}_E \\ u(s - \beta) < \delta, & \text{if } s \notin \mathcal{L}_E \end{cases} \tag{5}$$

Note that, in case $s \notin \mathcal{L}_E$ and $\beta' \in \mathcal{L}_E^u$, we have $w_E(X - s) = u(s - \beta) = u(s - \beta')$, thus this value is independent of the chosen pseudo-limit of E. Similarly, if E is of transcendental type, then $w_E(X - s) < \delta$ for any $s \in K$.

(b) Under the assumption of Corollary 4.6, let u be a fixed extension of v to $\overline{K}$ and let w_E be extended to $\overline{K}(X)$ along u (i.e., $w_E(\psi) = \lim u(\psi(s_n))$, for any $\psi \in \overline{K}(X)$). Let S be the multiset of critical points of ϕ which are pseudo-limits of E. Then by (4) and (5) we have

$$w_E(\phi) = w_E(\phi_S) w_E\left(\frac{\phi}{\phi_S}\right) = \lambda\delta + \gamma$$

where $\lambda\delta = w_E(\phi_S)$ and $w_E\left(\frac{\phi}{\phi_S}\right) = u\left(\frac{\phi}{\phi_S}(\beta)\right) = \gamma$, since $w_E(X - \alpha) = v(\beta - \alpha)$ for every $\alpha \notin S$ by the previous remark (we observe that the last value is independent of the chosen $\beta \in \mathcal{L}_E^u$). We stress the strong analogy between this expression of w_E and the valuation associated to a valuation domain of the form W_α, for $\alpha \in \widehat{K}$ which is algebraic over K. See [6, p. 126] and [19, Remark 2.3].

The next lemma is taken from [17]; we repeat it here for the convenience of the reader.

Lemma 4.8. *[17, VII, p. 377] Let $E = \{s_n\}_{n \in \mathbb{N}}$ be a pseudo-convergent sequence of algebraic type and let $\alpha \in \mathcal{L}_E^u$, for some extension u of v to $\overline{K}$. Then $w_E = (u_{\alpha,\delta_E})_{|K(X)} = v_{\alpha,\delta_E}$.*

Proof. We can reduce to proving the statement when K is algebraically closed, so in particular $u = v$. We have to show that $w_E = v_{\alpha,\delta}$, where $\delta = \delta_E$ and $\alpha \in K$ is a pseudo-limit of E. Let $\beta \in K$. To this end, by [17, IV, p. 366], it is sufficient to show that $w_E(X - \alpha + \beta) = \min\{w_E(X - \alpha), v(\beta)\} = \{\delta, v(\beta)\}$. If $\delta \neq v(\beta)$ then this is clear, so suppose that $\delta = v(\beta)$. We have:

$$w_E(X - \alpha + \beta) = \lim_{n \to \infty} v(s_n - \alpha + \beta) = \lim_{n \to \infty} v(s_n - \alpha) = \delta$$

so that also in this case we have the claimed equality. $\square$

In general, since V has rank 1 and $K(X)$ has transcendence degree 1 over K, the rank of V_E can be 1 or 2. We now show in which case each possibility occurs.

Theorem 4.9. *Let $E \subset K$ be a pseudo-convergent sequence.*

(a) If E is of transcendental type, then $V_E = W_E$ has rank 1.

(b) If E is of algebraic type and its breadth is infinite, then V_E has rank 2; furthermore, if q is the minimal polynomial of the pseudo-limit of E, then the one-dimensional overring of E is $K[X]_{(q)}$.

(c) Suppose that E is of algebraic type with breadth $\delta \in \mathbb{R}$. The following conditions are equivalent:

(i) δ is not torsion over Γ_v;

(ii) W_E is not residually transcendental over V;

(iii) V_E has rank one;

(iv) $V_E = W_E$;

(v) $V_E \cap K[X] = W_E \cap K[X]$.

Proof. (a) follows directly from [11, Theorem 2] and the proof of Theorem 3.7, while (b) is a direct consequence of Remark 3.8 (and Proposition 4.4).

(c) Let $E = \{s_n\}_{n\in\mathbb{N}}$ be of algebraic type with finite breadth, and let $\{\delta_n\}_{n\in\mathbb{N}}$ be the gauge of E. Since $V_E \subseteq W_E$ and W_E has rank 1, conditions (iii) and (iv) are clearly equivalent. Since by Lemma 4.8 $w_E = v_{\alpha,\delta}$, by [18, Lemma 3.5] we have that (i) is equivalent to (ii). Clearly, (iv) implies (v).

(v) $\Longrightarrow$ (iv) Let $\phi \in W_E$, i.e., $w_E(\phi) \geq 0$. Clearly, if $w_E(\phi) > 0$ then $\phi \in V_E$, so suppose $w_E(\phi) = 0$. By Proposition 3.5 and Corollary 4.6, there exist $\lambda \in \mathbb{Z}$ and $\gamma \in \Gamma_v$ such that $v(\phi(s_n)) = \lambda\delta_n + \gamma$ for all large n; its limit $\lambda\delta + \gamma$ is equal to $w_E(\phi)$, and thus it is 0. If $\lambda \leq 0$, then $v(\phi(s_n))$ is definitively positive, and $\phi \in V_E$.

Suppose $\lambda > 0$ and let $p \in K[X]$ be the minimal polynomial of some pseudo-limit β of E (with respect to some extension u of v in $\overline{K}$). By Corollary 4.6, there are $\lambda_p \in \mathbb{Z}$, $\gamma_p \in \Gamma_v$ such that $v(p(s_n)) = \lambda_p\delta_n + \gamma_p$ for all large n; furthermore, $\lambda_p > 0$ since p is a polynomial and one of the roots of $p(X)$ is a pseudo-limit of E. Let $c \in K$ be an element of value $\lambda_p\gamma - \lambda\gamma_p$ (which exists since $\lambda, \lambda_p, \gamma, \gamma_p \in \Gamma_v$) and consider $\psi(X) = cp(X)^\lambda \in K[X]$. Then, for all $n \in \mathbb{N}$ sufficiently large we have

$$v(\psi(s_n)) = \lambda_p\gamma - \lambda\gamma_p + \lambda(\lambda_p\delta_n + \gamma_p) = \lambda_p(\gamma + \lambda\delta_n) = \lambda_p v(\phi(s_n)).$$

This quantity has limit 0 as $n \to \infty$ and is strictly increasing, because $\lambda\lambda_p > 0$; hence $v(\psi(s_n)) < 0$ for all $n \in \mathbb{N}$ sufficiently large. Therefore, $\psi \in W_E \setminus V_E$. However, this contradicts the hypothesis because $\psi(X)$ is a polynomial; hence, the claim is proved.

We show now that (i) $\iff$ (iv), and we claim that it is sufficient to prove the equivalence under the further assumption that E has a pseudo-limit β in K. Suppose that (i) is equivalent to (iv) under this assumption and let $\beta \in \mathcal{L}_E^{v'}$ where v' is an extension of v to $K(\beta)$. Let $V'_E \subseteq W'_E$ be the valuation domains of $K(\beta)(X)$ associated to E with respect to the valuation v'. If δ is not torsion over Γ_v then $V'_E = W'_E$ and contracting down to $K(X)$ we get $V_E = W_E$. Conversely, if the rank of V_E is one (thus, $V_E = W_E$) then also the rank of V'_E is one (because $K(X) \subseteq K(\beta)(X)$ is an algebraic extension) and so δ is not torsion over Γ_v.

Suppose thus that $\beta \in K$ is a pseudo-limit of E. By (5) we have $w_E(X - \beta) = \delta$. If δ is torsion over Γ_v, then $k\delta \in \Gamma_v$ for some $k \in \mathbb{N}$, i.e., there is $c \in K$ such that $w_E((X - \beta)^k) = v(c)$; let $\phi(X) := \frac{(X-\beta)^k}{c}$. Then, $w_E(\phi) = 0$ and thus $\phi \in W_E$, while

$$v\left(\frac{(s_n - \beta)^k}{c}\right) = k\delta_n - v(c) < 0, \tag{6}$$

and thus $\phi(s_n) \notin V$ for every n, which implies $\phi \notin V_E$. Hence, $V_E \neq W_E$.

Conversely, suppose that δ is not torsion over Γ_v, and let $\phi \in W_E$. If $w_E(\phi) > 0$ then ϕ belongs to the maximal ideal of W_E, which is contained in V_E. Suppose $w_E(\phi) = 0$, and let k be the dominating degree of E. By definition we have $w_E(\phi) = k\delta + \gamma$ for some $\gamma \in \Gamma_v$ (see also Corollary 4.6 and (4)); since this quantity is 0 and δ is not torsion, we must have $k = 0$, and so also $\gamma = 0$. But this means that $v(\phi(s_n)) = 0$ for large n; in particular, $\phi(s_n) \in V$ for large n. Thus $\phi \in V_E$ and $V_E = W_E$. □

When the rank of V_E is 1, then its valuation v_E is exactly the Ostrowski valuation v_E; on the other hand, if E is algebraic with infinite breadth, then V_E has been characterized in Remark 3.8 and its valuation is described in [19, Remark 2.3]. When V_E has rank 2 and E has finite breadth, a description of v_E has been obtained in Proposition 3.9; however, we want to embed Γ_{v_E} as a totally ordered subgroup in $\mathbb{R}^2$, endowed with the lexicographic order (this can be done by Hahn's theorem [20, Théorème 2, p. 22]).

Theorem 4.10. *Let $E = \{s_n\}_{n\in\mathbb{N}}$ be a pseudo-convergent sequence with non-zero breadth ideal such that V_E has rank 2. Then, the map*

$$\begin{aligned} v_E\colon K(X) \setminus \{0\} &\longrightarrow \mathbb{R}^2 \\ \phi &\longmapsto (w_E(\phi), -\operatorname{degdom}_E(\phi)) \end{aligned}$$

is a valuation on $K(X)$ whose valuation ring is V_E.

Proof. By Theorem 4.9, E is of algebraic type and its breadth δ is torsion over Γ_v. Let $\{\delta_n\}_{n\in\mathbb{N}}$ be the gauge of E. Since w_E is a valuation, we have $w_E(\phi_1\phi_2) = w_E(\phi_1) + w_E(\phi_2)$ for every $\phi_1, \phi_2 \in K(X)$; the same formula holds for the dominating degree, since the multiset of zeros of $\phi_1\phi_2$ is exactly the union of the multisets of zeros of ϕ_1 and ϕ_2. Hence, $v_E(\phi_1\phi_2) = v_E(\phi_1) + v_E(\phi_2)$.

We now want to show that $v_E(\phi_1+\phi_2) \geq \min\{v_E(\phi_1), v_E(\phi_2)\}$. Let $\lambda_1 := \operatorname{degdom}_E(\phi_1)$, $\lambda_2 := \operatorname{degdom}_E(\phi_2)$, $\lambda := \operatorname{degdom}_E(\phi_1+\phi_2)$. By Proposition 3.5, there are $\gamma_1, \gamma_2, \gamma \in \Gamma_v$ such that $v(\phi_i(s_n)) = \lambda_i\delta_n + \gamma_i$, $i = 1, 2$ and $v((\phi_1 + \phi_2)(s_n)) = \lambda\delta_n + \gamma$ for all large n. Furthermore, by Corollary 4.6, $w_E(\phi_i) = \lambda_i\delta + \gamma_i$, $i = 1, 2$ and $w_E(\phi_1 + \phi_2) = \lambda\delta + \gamma$.

We distinguish four cases.

If $w_E(\phi_1) \neq w_E(\phi_2)$, then without loss of generality $w_E(\phi_1) < w_E(\phi_2)$. Hence, we have $v(\phi_1(s_n)) < v(\phi_2(s_n))$ for all large n. Thus,

$$v((\phi_1 + \phi_2)(s_n)) = v(\phi_1(s_n)) = \lambda_1\delta_n + \gamma_1.$$

Hence, $\lambda_1\delta_n + \gamma_1 = \lambda\delta_n + \gamma$ infinitely many times. Thus, it must be $\lambda_1 = \lambda$, and so $v_E(\phi_1 + \phi_2) = v_E(\phi_1) = \min\{v_E(\phi_1), v_E(\phi_2)\}$.

If $w_E(\phi_1) = w_E(\phi_2) < w_E(\phi_1 + \phi_2)$, then $v_E(\phi_1 + \phi_2)$ is bigger than both $v_E(\phi_1)$ and $v_E(\phi_2)$, and we are done.

Suppose that $w_E(\phi_1) = w_E(\phi_2) = w_E(\phi_1 + \phi_2)$ and that $\lambda_1 \neq \lambda_2$; without loss of generality, $\lambda_1 > \lambda_2$ (i.e., $v_E(\phi_1) < v_E(\phi_2)$). Then, $\lambda_1\delta_n + \gamma_1 < \lambda_2\delta_n + \gamma_2$ for all large n. Therefore, as in the first case, $\lambda_1\delta_n + \gamma_1 = \lambda\delta_n + \gamma$ for all large n, and so $v_E(\phi_1 + \phi_2) = v_E(\phi_1)$.

Suppose now that $w_E(\phi_1) = w_E(\phi_2) = w_E(\phi_1 + \phi_2)$ and that $\lambda_1 = \lambda_2 =: \lambda'$. Since the sequences $v(\phi_1(s_n)) = \lambda'\delta_n + \gamma_1$ and $v(\phi_2(s_n)) = \lambda'\delta_n + \gamma_2$ have the same limit, they must be definitively equal, and so $\gamma_1 = \gamma_2 =: \gamma'$. Since v is a valuation, $v((\phi_1 + \phi_2)(s_n)) = \lambda\delta_n + \gamma \geq \lambda'\delta_n + \gamma'$. Since $w_E(\phi_1 + \phi_2) = w_E(\phi_1)$, furthermore, the limits $\lambda\delta + \gamma$ and $\lambda'\delta + \gamma'$ are equal; it follows that $\lambda \leq \lambda'$. Hence,

$$v_E(\phi_1 + \phi_2) = (w_E(\phi_1), -\lambda) \geq (w_E(\phi_1), -\lambda') = v_E(\phi_1) = v_E(\phi_2).$$

Therefore, v_E is a valuation.

The fact that v_E extends v follows from the fact that w_E extends v.

Let V' be the valuation ring associated to v_E. Suppose $\phi \in V_E$. If $w_E(\phi) > 0$ then $v_E(\phi) > 0$. If $w_E(\phi) = 0$ then $v(\phi(s_n)) = \lambda\delta_n + \gamma$ must tend to 0 from above, and thus $\lambda \leq 0$, i.e., $v_E(\phi) \geq 0$. Thus, $V_E \subseteq V'$. Conversely, if $v_E(\phi) \geq 0$ then either $w_E(\phi) > 0$ (and so $\phi \in M_{W_E} \subset V_E$) or $w_E(\phi) = 0$ and $\lambda \leq 0$; in the latter case, $\lambda\delta_n + \gamma \geq 0$, and so $v(\phi(s_n)) \geq 0$. Therefore, $V' \subseteq V_E$, and so $V' = V_E$, as claimed. □

Remark 4.11. Let E be a pseudo-convergent sequence of algebraic type with breadth δ which is torsion over Γ_v, and let $\Delta' := (\delta, -1)$. Take $\phi \in K(X)$ and let $\lambda := \mathrm{degdom}_E(\phi)$. By Theorem 4.10, we have

$$v_E(\phi) = (w_E(\phi), -\lambda);$$

by Corollary 4.6, moreover, $w_E(\phi) = \lambda\delta + \gamma_x$ for some $\gamma_x \in \Gamma_v$. It follows that

$$v_E(\phi) = \lambda\Delta' + \gamma,$$

where $\gamma := (\gamma_x, 0) \in \Gamma_{v_E}$. If, furthermore, E has a pseudo-limit $\beta \in K$, then $\Delta' = \Delta = v_E(X - \beta)$.

5 Equivalence of pseudo-convergent sequences

Classically, two Cauchy sequences $E, F \subset K$ are equivalent if the distance induced by the valuation v between their corresponding terms goes to zero. If α and β are the limits in $\widehat{K}$ of E and F, respectively, it is known that E and F are equivalent if and only if the valuation domains $V_E = W_\alpha, V_F = W_\beta$ (see Remark 3.8) are the same; in particular, E and F determine the same extension of the valuation v to $K(X)$. Ostrowski investigated in [17, p. 387] the similar problem for the valuation domains of the form W_E, for E a pseudo-convergent sequence in K, which led him to give the notion of equivalent pseudo-convergent sequences. In this section, we consider a definition of equivalence for pseudo-convergent sequence as it appears in [12, Section 3.2], even though we correct a mistake there. See also Example 6.7 for a topological interpretation in term of limits.

Definition 5.1. Let $E = \{s_n\}_{n\in\mathbb{N}}$ and $F = \{t_n\}_{n\in\mathbb{N}}$ be two pseudo-convergent sequences in K. We say that E and F are *equivalent* if the breadths δ_E and δ_F are equal and, for every $k \in \mathbb{N}$, there are $i_0, j_0 \in \mathbb{N}$ such that, whenever $i \geq i_0$, $j \geq j_0$, we have

$$v(s_i - t_j) > v(t_{k+1} - t_k).$$

Note that the previous definition boils down to the classical notion of equivalence if E and F are Cauchy sequences.

Remark 5.2. The previous definition was also considered in [12, Section 3.2] without the hypothesis $\delta_E = \delta_F$. However, without this condition the definition is not symmetric: for example, let $F := \{t_n\}_{n\in\mathbb{N}}$ be a sequence in V with $v(t_n) = \delta_n$, where $\{\delta_n\}_{n\in\mathbb{N}}$ is a positive increasing sequence, and let $E := \{s_n := t_n^2\}_{n\in\mathbb{N}}$. Then, for every k and every $i, j \geq k+1$ we have

$$v(s_i - t_j) = \delta_j > \delta_k = v(t_{k+1} - t_k);$$

on the other hand, if $\delta_k > \frac{1}{2}\delta$, then there are no i, j such that

$$v(s_i - t_j) > 2\delta_k = v(s_{k+1} - s_k);$$

hence, E and F are equivalent according to [12], but F and E are not.

On the other hand, suppose that E and F are two pseudo-convergent sequence of K which are equivalent according to Definition 5.1. Then, for every k there is a k' such that $v(s_{k+1}-s_k) < v(t_{k'+1}-t_{k'})$. If now i_0 and j_0 are such that $v(s_i-t_j) > v(t_{k'+1}-t_{k'})$ for all $i \geq i_0$, $j \geq j_0$, then clearly $v(s_i - t_j) > v(s_{k+1} - s_k)$, so F and E are equivalent.

We need first the following preliminary lemma.

Lemma 5.3. *Let $E, F \subset K$ two equivalent pseudo-convergent sequences. Then either E and F are both of transcendental type, or E and F are both of algebraic type. In the latter case, $\mathcal{L}_E^u = \mathcal{L}_F^u$ for every extension u of v to $\overline{K}$.*

Proof. Let $E = \{s_n\}_{n\in\mathbb{N}}$ and $F = \{t_n\}_{n\in\mathbb{N}}$; let $\{\delta_n\}_{n\in\mathbb{N}}, \{\delta'_n\}_{n\in\mathbb{N}}$ be the gauges of E and F, respectively, and δ the breadth of E and F. It is sufficient to prove that if either one of the two pseudo-convergent sequences, say E, is of algebraic type, then also the other is of algebraic type.

Suppose first that K is algebraically closed and let β be a pseudo-limit of E. Fix $k \in \mathbb{N}$. Then there exist $i_0, j_0 \in \mathbb{N}$ such that, for all $m \geq i_0, n \geq j_0$, $v(s_m - t_n) > \delta_k$. For such n and m, suppose also that $m \geq k$. Then $v(t_n - \beta) = v(t_n - s_m + s_m - \beta) \geq \delta_k$. Therefore, $w_F(X - \beta) = \lim_{n\to\infty} v(t_n - \beta) \geq \delta$. If $w_F(X - \beta) > \delta$, then there is a n_0 such that, for all $n \geq n_0$, $v(t_n - \beta) > \delta$; since $v(s_m - \beta) = \delta_m < \delta$, this means that, for every m sufficiently large, $v(t_n - s_m) = v(s_m - \beta)$. This would imply that t_n is a pseudo-limit of E for all $n \geq n_0$, and thus that, in particular, $v(t_{n+1} - t_n) \geq \delta$, which is a contradiction since $v(t_{n+1} - t_n) = \delta'_n \nearrow \delta$. Hence, $\delta_n \leq v(t_n - \beta) < \delta$ for all large n and $w_F(X - \beta) = \delta$; this shows that $v(t_n - \beta)$ is definitively strictly increasing, that is, β is a pseudo-limit of F and thus also F is of algebraic type. Moreover, since $\beta \in \mathcal{L}_E$ was arbitrarily chosen, we also have $\mathcal{L}_E \subseteq \mathcal{L}_F$, which shows that these sets are equal since they are closed balls of the same radius (Lemma 2.4).

If now K is not algebraically closed, let u be any extension of v to $\overline{K}$. Then, E and F are equivalent with respect to u; applying the previous part of the proof, we have that F is of algebraic type and $\mathcal{L}_E^u = \mathcal{L}_F^u$, as claimed. $\square$

Theorem 5.4. *Let $E, F \subset K$ be two pseudo-convergent sequences that are of transcendental type or of algebraic type with nonzero breadth ideal. Then, the following are equivalent:*

(i) E and F are equivalent;
(ii) $V_E = V_F$;
(iii) $W_E = W_F$;
(iv) $w_E = w_F$.

Furthermore, if E and F are of algebraic type, the previous conditions are equivalent to the following:

(v) $\mathcal{L}^u_E = \mathcal{L}^u_F$ for all extensions u of v to $\overline{K}$;

(vi) $\mathcal{L}^u_E = \mathcal{L}^u_F$ for an extension u of v to $\overline{K}$.

Proof. As usual, we set $E = \{s_n\}_{n\in\mathbb{N}}$ and $F = \{t_n\}_{n\in\mathbb{N}}$; let $\{\delta_n\}_{n\in\mathbb{N}}, \{\delta'_n\}_{n\in\mathbb{N}}$ be the gauges of E and F, respectively, and δ, δ' the breadths of E, F, respectively. Recall that, by Proposition 4.3, if E and F are of transcendental type, then $V_E = W_E$ and $V_F = W_F$.

The structure of the proof is as follows:

- we first prove (ii) $\Longrightarrow$ (iii) $\Longleftrightarrow$ (iv) $\Longrightarrow$ (i) in both the algebraic and the transcendental case;
- then we prove (i) $\Longrightarrow$ (iv) and (iii) $\Longrightarrow$ (ii) in the transcendental case;
- finally, we prove (i) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) $\Longrightarrow$ (ii) in the algebraic case.

(ii) $\Longrightarrow$ (iii) and (iv) $\Longrightarrow$ (iii) are obvious.

(iii) $\Longrightarrow$ (iv) Suppose there is a $\phi \in K(X)$ such that $w_E(\phi) \neq w_F(\phi)$; without loss of generality, $w_E(\phi) > w_F(\phi)$. We claim that there is a $c \in K$ such that $w_E(\phi) \geq v(c) > w_F(\phi)$. This is obvious if Γ is dense in $\mathbb{R}$; otherwise, Γ must be isomorphic to $\mathbb{Z}$, and V is a discrete valuation ring. In this case, the breadth of E and F must be infinite, and thus (by hypothesis) E and F must be transcendental. However, by [11, Theorem 2], it follows that $V_E = W_E$ is an immediate extension of V; in particular, the value group of W_E coincide with Γ, and thus we can take a $c \in K$ such that $v(c) = w_E(\phi)$. The existence of c implies that $\frac{\phi}{c} \in W_E$ while $\frac{\phi}{c} \notin W_F$, contradicting $W_E = W_F$. Hence, (iv) holds.

(iv) $\Longrightarrow$ (i) By definition, for every k and every $l \geq 0$,

$$\delta'_k = v(t_{k+l} - t_k) = \lim_{n\to\infty} v(t_n - t_k) = w_F(X - t_k) = w_E(X - t_k) = \lim_{n\to\infty} v(s_n - t_k). \quad (7)$$

If $\delta_E < \delta_F$, then $\delta'_k > \delta_E$ for large k; thus,

$$v(s_n - t_{k+1}) = v(s_n - s_{n+1} + s_{n+1} - t_{k+1}) = \delta_n$$

and thus $\delta_n = \delta'_{k+1}$, a contradiction; hence $\delta_E \geq \delta_F$. By symmetry, we have also $\delta_F \geq \delta_E$, and thus $\delta_E = \delta_F = \delta$.

Fix now k, take n_0' such that $\delta_n > \delta_k'$ for every $n \geq n_0'$; since $\delta_m' > \delta_k'$ if $m > k$, there are $n_0 > n_0'$ and $m_0 > k$ such that $v(s_{n_0} - t_{m_0}) > \delta_k'$. For all $n \geq n_0$, $m \geq m_0$, we have

$$v(s_n - t_m) = v(s_n - s_{n_0} + s_{n_0} - t_{m_0} + t_{m_0} - t_m).$$

The three quantities $v(s_n - s_{n_0})$, $v(s_{n_0} - t_{m_0})$ and $v(t_{m_0} - t_m)$ are all bigger than δ_k'; hence, so is $v(s_n - t_m)$. Since k was arbitrary, E and F are equivalent.

Suppose now that E is of transcendental type. If (iii) holds, then by the previous part of the proof also (i) holds; thus, by Lemma 5.3 both E and F are of transcendental type, and (ii) follows from Theorem 4.9(a).

If (i) holds, then again F is of transcendental type, and the fact that (iv) holds is exactly [12, Satz 3.10] (though note the slight difference in the definition – see Remark 5.2); we give here a proof for the sake of the reader. Without loss of generality, suppose that K is algebraically closed. In order to show that $w_E(\phi) = w_F(\phi)$ for all $\phi \in K(X)$, it is sufficient to show that $w_E(X - \alpha) = w_F(X - \alpha)$ for every $\alpha \in K$. We have

$$w_E(X - \alpha) = \lim_{n\to\infty} v(s_n - \alpha) = v(s_n - \alpha), \ \ \forall n \geq n_1$$
$$w_F(X - \alpha) = \lim_{n\to\infty} v(t_n - \alpha) = v(t_n - \alpha), \ \ \forall n \geq m_1$$

for some $n_1, m_1 \in \mathbb{N}$, since both quantities are definitively constant. We also have that $w_E(X-\alpha)$ and $w_F(X-\alpha)$ are both strictly less δ, since $\alpha \in K$ cannot be a pseudo-limit of E and F, respectively. Hence, there exists $k_1 \in \mathbb{N}$ such that for all $k > k_1$ we have $\delta_k > w_E(X - \alpha)$ and $\delta_k' > w_F(X - \alpha)$. Let $k > \max\{k_1, n_1, m_1\}$. There exists $k_2 \geq k$ such that $\delta_k' < \delta_{k_2}$. Also, there exist $i_0, j_0 \in \mathbb{N}$ such that for each $i \geq i_0$ and $j \geq j_0$ we have $v(s_i - t_j) > \delta_k'$. We have

$$v(s_{k_2} - \alpha) = v(s_{k_2} - s_m + s_m - t_m + t_m - \alpha)$$

Choose $m > \max\{k_2, i_0, j_0\}$. Then $v(s_{k_2} - s_m)$ and $v(s_m - t_m)$ are both strictly bigger than $\delta_k' > v(t_m - \alpha)$. Hence, $w_E(X - \alpha) = v(s_{k_2} - \alpha) = v(t_m - \alpha) = w_F(X - \alpha)$, and the claim is proved.

Suppose now that E is of algebraic type. If (i) holds, then by Lemma 5.3 also F is of algebraic type, and E and F have the same pseudo-limits with respect to any extension u of v to $\overline{K}$; hence, (i) $\Longrightarrow$ (v). Furthermore, (v) $\Longrightarrow$ (vi) is obvious.

We now show that (vi) implies (ii). Let $\phi \in V_E$. By Proposition 3.5, we have $v(\phi(s_n)) = \lambda\delta_n + \gamma$, where $\lambda := \operatorname{degdom}_E \phi$ and $\gamma := u\left(\frac{\phi}{\phi_S}(\beta)\right) \in \Gamma_v$, where β is a pseudo-limit of E (and ϕ_S is defined as in the proposition). Similarly, $v(\phi(t_n)) = \lambda'\delta_n' + \gamma'$; however, since $\mathcal{L}_E^v = \mathcal{L}_F^u$, it follows that $\lambda = \lambda'$ and $\gamma = \gamma'$. Furthermore, by Lemma 2.4, $\delta_E = \delta_F$, and thus $v(\phi(s_n))$ and $v(\phi(t_n))$ have the same limit L as $n \to \infty$. Since $\phi \in V_E$, we have $v(\phi(s_n)) \geq 0$ for large n, and so $L \geq 0$. If $L > 0$, then also $v(\phi(t_n)) > 0$ for large n; this implies that $\phi \in V_F$. If $L = 0$, then $\lambda \leq 0$; in particular, it must be $v(\phi(t_n)) \geq 0$ for large n. Again, it follows that $\phi \in V_F$; therefore, $V_E \subseteq V_F$. Symmetrically, $V_F \subseteq V_E$, and thus $V_E = V_F$, as claimed. $\square$

5.1 Extension of an Ostrowski valuation

Let w_E be an Ostrowski valuation on $K(X)$, where $E \subset K$ is a pseudo-convergent sequence, and let u be an extension of v to $\overline{K}$. The *extension of w_E to $\overline{K}(X)$ along u* is the valuation $\overline{w}_E$ defined by

$$\overline{w}_E(\psi) := \lim_{n\to\infty} u(\psi(s_n))$$

for every $\psi \in \overline{K}(X)$. Clearly, $\overline{w}_E$ extends w_E and has rank 1. A consequence of Theorem 5.4 is that, if E and F are two pseudo-convergent sequences in K, the equality $w_E = w_F$ implies $\overline{w}_E = \overline{w}_F$, since these equalities are both equivalent to the fact that E and F are equivalent pseudo-convergent sequences (which does not depend on the field containing E and F).

We show now that any extension of an Ostrowski valuation on $K(X)$ to $\overline{K}(X)$ is of this kind.

Theorem 5.5. *Let $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ be a pseudo-convergent sequence such that the associated map w_E is a valuation. If $\overline{w}$ is an extension of w_E to $\overline{K}(X)$ and u is the restriction of $\overline{w}$ to $\overline{K}$, then $\overline{w}$ is equivalent to the extension $\overline{w}_E$ of w_E to $\overline{K}(X)$ along u (or, equivalently, the valuation domain of $\overline{w}$ is equal to $\overline{W}_E$).*

Proof. The valuation $\overline{w}_E$ restricts of w_E on $K(X)$ and to u on $\overline{K}$. Suppose there is another valuation w' on $\overline{K}(X)$ with these properties: then, by [5, Chapt. VI, §8, 6., Corollaire 1], there is a $K(X)$-automorphism σ of $\overline{K}(X)$ such that $w' \circ \sigma$ is equivalent to $\overline{w}_E$, that is, $\rho(\overline{W}_E) = W'$, where $\rho = \sigma^{-1}$ and W' is the valuation ring of w'.

Now

$$\begin{aligned}\rho(\overline{W}_E) =&\{\rho\circ\phi \in \overline{K}(X) \mid \lim_n u(\phi(s_n)) \geq 0\} = \\ &\{\rho\circ\phi \in \overline{K}(X) \mid \lim_n u\circ\sigma\circ\rho(\phi(s_n)) \geq 0\}.\end{aligned}$$

Since $s_n \in K$ and $\rho|_K$ is the identity, $\rho(\phi(s_n)) = (\rho\circ\phi)(s_n)$; hence,

$$\begin{aligned}\rho(\overline{W}_E) =&\{\rho\circ\phi \in \overline{K}(X) \mid \lim_n (u\circ\sigma)((\rho\circ\phi)(s_n)) \geq 0\} = \\ &\{\psi \in \overline{K}(X) \mid \lim_n (u\circ\sigma)(\psi(s_n)) \geq 0\}.\end{aligned}$$

Since both $\overline{W}_E$ and $W' = \rho(\overline{W}_E)$ are extensions of U, the valuation domain of u, we have $u(t) = (u\circ\sigma)(t)$ for every $t \in \overline{K}$; in particular, this happens for $t = \psi(s_n)$. It follows that $\rho(\overline{W}_E) = W' = \overline{W}_E$, as claimed. □

Remark 5.6. We note that it is possible for two valuations w_1, w_2 of $\overline{K}(X)$ to be different even if their restriction to $K(X)$ and $\overline{K}$ are equal. For example, let v be a valuation on K, and let w be an extension of v to $K(X)$. If K is complete under the topology induced by v, then there exists a unique extension of v to $\overline{K}$; on the other hand, w can have more than one extension to $\overline{K}(X)$.

For an explicit example, suppose that K is complete under v, let $\overline{v}$ be the unique extension of v to $\overline{K}$ and let $\overline{V}$ be the valuation domain of $\overline{K}$ associated to $\overline{v}$. Let

$\alpha, \beta \in \overline{K}$ be two distinct elements which are conjugate over K, and let w be the valuation associated to the valuation domain

$$W := \{f \in K(X) \mid f(\alpha) \in \overline{V}\} = \{f \in K(X) \mid f(\beta) \in \overline{V}\};$$

note that the second equality follows from the fact that α and β are conjugate over K (see also [19, Theorem 3.2], where such valuation domains are studied; note that they belong to the same class of the valuation domains considered in Remark 3.8). Then, W extends to the following valuation rings of $\overline{K}(X)$:

$$\overline{W}_\alpha := \{f \in \overline{K}(X) \mid f(\alpha) \in \overline{V}\}$$

and

$$\overline{W}_\beta := \{f \in \overline{K}(X) \mid f(\beta) \in \overline{V}\}.$$

However, $\overline{W}_\alpha \neq \overline{W}_\beta$: for example, if $t \in \overline{K}$ satisfies $\overline{v}(t) > \overline{v}(\beta - \alpha)$, then $f(X) := \frac{1}{t}(X - \alpha)$ belongs to $\overline{W}_\alpha$ but not to $\overline{W}_\beta$ (again, the same conclusion follows from the aforementioned result [19, Theorem 3.2]).

By means of Theorem 5.5, in the next result without loss of generality we assume that the extension of w_E to $\overline{K}(X)$ is equal to $\overline{w}_E$ (for some extension u of v to $\overline{K}$; clearly, $u = (\overline{w}_E)_{|\overline{K}}$).

The followins is a variant of Theorem 3.2.

Proposition 5.7. *Let $\phi \in K(X)$ and let $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ be a pseudo-convergent sequence. Let $\overline{w}_E$ be an extension of w_E to $\overline{K}(X)$, and let $\theta_1, \theta_2 \in \mathbb{R}$ be such that $C := \{t \in \overline{K} \mid \theta_1 < \overline{w}_E(X - t) < \theta_2\}$ does not contain any critical point of ϕ. Then, there are $\lambda \in \mathbb{Z}$, $\gamma \in \mathbb{Q}\Gamma_v$ such that*

$$v(\phi(t)) = \lambda w_E(X - t) + \gamma$$

for every $t \in K \cap C$. More precisely, if S is the multiset of critical points α of ϕ such that $\overline{w}_E(X - \alpha) \geq \theta_2$, then λ is the weighted sum of S and $\gamma = \overline{w}_E\left(\frac{\phi}{\phi_S}\right)$.

Proof. Let $\phi(X) = c\prod_{\alpha\in S}(X - \alpha)^{\epsilon_\alpha}\prod_{\beta\in S'}(X - \beta)^{\epsilon_\beta}$, where $S' := \Omega_\phi \setminus S$. Let $t \in K \cap C$ and let u be the restriction of $\overline{w}_E$ to $\overline{K}$. As in the proof of Theorem 3.2, writing $u(t - \alpha) = \overline{w}_E(t - \alpha) = \overline{w}_E(t - X + X - \alpha)$ we see that $u(t - \alpha) = w_E(X - t)$ if $\overline{w}_E(X - \alpha) \geq \theta_2$, while $u(t - \alpha) = \overline{w}_E(X - \alpha)$ if $\overline{w}_E(X - \alpha) \leq \theta_1$ (note that by assumption on C there is no critical point α of ϕ such that $\theta_1 < \overline{w}_E(X - \alpha) < \theta_2$). Hence,

$$v(\phi(t)) = v(c) + \sum_{\alpha\in S}\epsilon_\alpha w_E(X - t) + \sum_{\beta\in S'}\epsilon_\beta \overline{w}_E(X - \beta) = \lambda w_E(X - t) + \gamma,$$

as claimed. $\square$

Given a pseudo-convergent sequence $E \subset K$, an extension $\overline{w}_E$ of w_E and a rational function $\phi \in K(X)$, we define

$$\delta_{\phi,E} := \max\{\overline{w}_E(X-\alpha) \mid \alpha \in \Omega_\phi\}$$

(which we simply write δ_ϕ if E is understood from the context). By Remark 4.7(a), $\delta_\phi \leq \delta$, and $\delta_\phi < \delta$ if no critical point of ϕ is a pseudo-limit of E; in particular, this happens if E is of transcendental type.

Corollary 5.8. *Let $E \subset K$ be a pseudo-convergent sequence and let $\phi \in K(X)$, and suppose that none of the critical points of ϕ is a pseudo-limit of E (with respect to $u = (\overline{w}_E)_{|\overline{K}}$). Then:*

- *(a) if $\delta_\phi < w_E(X-t) \leq \delta_E$, then $v(\phi(t)) = w_E(\phi)$;*
- *(b) if E is of algebraic type and $\alpha \in \mathcal{L}_E^u$, then $w_E(\phi) = u(\phi(\alpha))$.*

Proof. Let $E = \{s_n\}_{n\in\mathbb{N}}$. Since no critical point β of ϕ satisfies $\overline{w}_E(X-\beta) \geq \delta_\phi$, by Proposition 5.7 we have $v(\phi(t)) = \overline{w}_E(\phi) = w_E(\phi)$ for every $t \in C := \{s \in \overline{K} \mid \delta_\phi < \overline{w}_E(X-s) \leq \delta_E\}$, so claim (a) is proved. Claim (b) follows by Proposition 3.5(b). □

In the following sections, we shall be interested in criteria to compare the membership of a rational function to two valuation rings V_E and V_F. To this end, the next result uses the function $\overline{w}_E$ and the pseudo-limits of F.

Proposition 5.9. *Let $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ be a pseudo-convergent sequence, and let $\phi \in K(X)$; let u be an extension of v to $\overline{K}$ and let $\overline{w}_E$ be the extension of w_E to $\overline{K}(X)$ along u. There is a $\delta' < \delta_E$ such that, given $C := \{s \in \overline{K} \mid \delta' < \overline{w}_E(X-s) < \delta_E\}$, whenever F is a pseudo-convergent sequence such that $\delta_F \geq \delta_E$ and $\mathcal{L}_F \cap C \neq \emptyset$, we have $\phi \in V_F$ if and only if $\phi \in V_E$.*

Proof. Let $S \subset \overline{K}$ be the multiset of critical points of ϕ, and let $\delta_1 := \sup\{\overline{w}_E(X-\alpha) \mid \alpha \in S,\ \overline{w}_E(X-\alpha) < \delta_E\}$. Then, there are no critical points of ϕ in $C_1 := \{s \in \overline{K} \mid \delta_1 < \overline{w}_E(X-s) < \delta_E\}$; by Proposition 5.7, there are $\lambda \in \mathbb{Z}$, $\gamma \in \mathbb{Q}\Gamma_v$ such that $u(\phi(t)) = \lambda\overline{w}_E(X-t)+\gamma$ for every $t \in C_1$. Since $v(\phi(s_n)) \to w_E(\phi)$ and s_n is definitively in C_1, we can find $\delta' \in [\delta_1, \delta_E)$ such that the quantity $u(\phi(t))$ is either positive, negative or zero for all $t \in C := \{s \in \overline{K} \mid \delta' < \overline{w}_E(X-s) < \delta_E\}$.

Suppose now $F = \{t_m\}_{m\in\mathbb{N}} \subset K$ is a pseudo-convergent sequence with breadth $\delta_F \geq \delta_E$ and such that $\mathcal{L}_F \cap C \neq \emptyset$: then, if $t \in \mathcal{L}_F \cap C$, we have $\overline{w}_E(X-t_m) = \overline{w}_E(X-t+t-t_m) = \overline{w}_E(X-t)$, for all $m \in \mathbb{N}$ sufficiently large, since $\overline{w}_E(X-t) < \delta_E$ and $v(t_m - t) \nearrow \delta_F$ which is greater than or equal to δ (and so, it is definitively greater than $\overline{w}_E(X-t)$). Hence, t_m is definitively in C and thus $v(\phi(t_m))$ is definitively nonnegative if so is $v(\phi(s_n))$ (in which case $\phi \in V_E \cap V_F$), while it is definitively negative if $v(\phi(s_n))$ is definitively negative (and so $\phi \notin V_E$ and $\phi \notin V_F$). The claim is proved. □

We note that, when we are in the hypothesis of Corollary 5.8 (that is, if ϕ has no critical point which is a pseudo-limit of E), the value δ' of the previous proposition can be taken to be equal to $\delta_{\phi,E}$.

6 Spaces of valuation domains associated to pseudo-convergent sequences

We are now interested in studying, from a topological point of view, the sets formed by the valuation rings V_E and W_E induced by the pseudo-convergent sequences E in K. The topologies we are interested in are the Zariski and the constructible topology (see Section 2 for the definitions). Since we are mainly interested in the former, unless stated otherwise, all the spaces are endowed with the Zariski topology.

We set:

$$\mathcal{V} := \{V_E \mid E \subset K \text{ is a pseudo-convergent sequence}\}$$

and

$$\mathcal{W} := \{W_E \mid E \subset K \text{ is a pseudo-convergent sequence and } w_E \text{ is a valuation}\}.$$

By the results of Section 4, the elements of $\mathcal{W}$ are the rings W_E, when $E \subset K$ is a pseudo-convergent sequence which is either of transcendental type or of algebraic type and non-zero breadth ideal.

When V is discrete, we have the following result.

Theorem 6.1. *[19, Theorem 3.4] Let V be a DVR. Then, $\mathcal{V}$ is homeomorphic to $\widehat{K}$.*

The homeomorphism can also be described explicitly: indeed, if V is a DVR then $\mathcal{V}$ contains only the rings of the form W_α (see Remark 3.8) and we just send W_α to α. Furthermore, in this context, $\mathcal{W}$ is a subset of $\mathcal{V}$, and corresponds to the elements of $\widehat{V}$ that are transcendental over V. In view of these facts, we are mainly interested in the case when V is not discrete.

We start by studying $\mathcal{W}$: indeed, the fact that every Ostrowski valutation domain W_E has rank one has strong consequences on the topology of $\mathcal{W}$. Recall that a topological space X is said to be *zero-dimensional* if it is T_1 and each point $x \in X$ has a neighborhood base consisting of open-closed sets, or, equivalently, if, for each $x \in X$ and closed set $C \subset X$, there exists an open-closed set containing x and not meeting C [9, f-6].

Proposition 6.2. *The Zariski and the constructible topology agree on $\mathcal{W}$. In particular, $\mathcal{W}$ is a zero-dimensional space.*

Proof. The intersection of the maximal ideals of the elements of $\mathcal{W}$ contains the maximal ideal M of V, and thus it is nonzero. Since every W_E has rank 1, the claims follow by [16, Proposition 2.4(b)] and the definition of zero-dimensional space. □

Proposition 6.3. *The space $\mathcal{W}$ is not compact.*

Proof. We claim that

$$\bigcap_{\substack{E \subset K \\ E \text{ pseudo-conv.}}} W_E = \mathrm{Int}^R(K, V),$$

where the right-hand side is the ring of integer-valued rational functions on K, that is, $\mathrm{Int}^R(K, V) = \{\phi \in K(X) \mid \phi(K) \subseteq V\}$ (see [7]). Let $\phi \in \mathrm{Int}^R(K, V)$. Then, clearly

$\phi \in V_E \subseteq W_E$ for all pseudo-convergent sequences E, by definition of V_E. Conversely, if $\phi(K) \nsubseteq V$, then there is a $t \in K$ such that $\phi(t) \notin V$; since V is closed in K and a rational function induces a continuous function (from the subset of K on which it is defined to K), there is a ball $B(t, r)$ such that $\phi(s) \notin V$ for all $s \in B(t, r)$. Choose a ball $B(s, r') \subseteq B(t, r)$ such that ϕ has no critical points in $B(s, r')$, and let $E = \{s_n\}_{n \in \mathbb{N}} \subset K$ be a pseudo-convergent sequence such that $\mathcal{L}_E \subset B(s, r')$ (e.g., a Cauchy sequence with limit s). By Theorem 3.2, for large n we have $v(\phi(s_n)) = v(\phi(s))$, and thus $w_E(\phi) = v(\phi(s)) < 0$, that is, $\phi \notin W_E$.

The intersection of the maximal ideals of the Ostrowski valuation overrings is $M \neq (0)$; hence, if $\mathcal{W}$ is compact then by [16, Theorem 5.3] $\mathrm{Int}^R(K, V)$ is a one-dimensional Prüfer domain with quotient field $K(X)$. Suppose this holds. Then, $\mathrm{Int}^R(V) = \mathrm{Int}^R(V, V)$ is an overring of $\mathrm{Int}^R(K, V)$, and so it has dimension 1; however, by [6, Section X.1, p.260], $\dim(\mathrm{Int}^R(V)) \geq \dim(V) + 1 = 2$, a contradiction. Therefore, $\mathcal{W}$ is not compact, as claimed. $\square$

In order to study the space $\mathcal{V}$, we need a criterion to establish when $V_E \in B(\phi)$, or equivalently when $\phi \in V_E$. To this end, we introduce the following notation: if $\beta \in K$, $\gamma_1 \in \Gamma_v$ and $\gamma_2 \in \Gamma_v \cup \{\infty\}$ with $\gamma_1 < \gamma_2$, the *annulus* of center β and radii γ_1 and γ_2 is

$$\mathcal{C}_v(\beta, \gamma_1, \gamma_2) := \{s \in K \mid \gamma_1 < v(\beta - s) < \gamma_2\}.$$

Note that this definition is a special case of the definition given in (2), when V has rank one. When the valuation v is understood from the context, we shall write simply $\mathcal{C}(\beta, \gamma_1, \gamma_2)$ for $\mathcal{C}_v(\beta, \gamma_1, \gamma_2)$.

Proposition 6.4. *Let $E \subset K$ be a pseudo-convergent sequence of algebraic type with breadth δ, let $\beta \in \mathcal{L}_E^u$ and let $\phi \in K(X)$. The following are equivalent:*

(i) $\phi \in V_E$;

(ii) there are $\theta_1 \in \mathbb{Q}\Gamma_v$, $\theta_2 \in \mathbb{Q}\Gamma_v \cup \{\infty\}$ such that $\theta_1 < \delta \leq \theta_2$ and such that $\phi(s) \in V$ for all $s \in \mathcal{C}_u(\beta, \theta_1, \theta_2)$;

(iii) there is $\tau \in \Gamma_v, 0 < \tau < \delta$ such that $\phi(s) \in V$ for all $s \in \mathcal{C}_u(\beta, \tau, \delta)$.

Proof. (i) $\Longrightarrow$ (ii) Let $\zeta_1 < \zeta_2$ be two elements in $\mathbb{Q}\Gamma_v$ such that $\zeta_1 < \delta \leq \zeta_2$ and there is no critical point of ϕ in $C := \mathcal{C}_u(\beta, \zeta_1, \zeta_2)$. By Theorem 3.2, there are $\lambda \in \mathbb{Z}$, $\gamma \in \Gamma_v$ such that $v(\phi(s)) = \lambda u(\beta - s) + \gamma$ for all $s \in C$. Let $I := \{h \in (\zeta_1, \zeta_2) \mid \lambda h + \gamma \geq 0\}$; then, I is an interval with endpoints $\theta_1, \theta_2 \in \mathbb{Q}\Gamma_v$, and $\phi(s) \in V$ for all $s \in C' := \mathcal{C}_u(\beta, \theta_1, \theta_2)$; we need only to show that $\theta_1 < \delta \leq \theta_2$.

Since $\phi \in V_E$, and $s_n \in C$ for large n, we have (definitively) $\lambda\delta_n + \gamma \geq 0$, where $\{\delta_n\}_{n \in \mathbb{N}}$ is the gauge of E; since $\delta_n \nearrow \delta$ and $\zeta_1 < \delta$, it follows that there is an interval $(\tau, \delta) \subseteq I$, and so $\theta_1 < \delta \leq \theta_2$. The claim is proved.

(ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (i) Suppose that there is an annulus $C := \mathcal{C}_u(\beta, \tau, \delta)$ with this property. Since δ is the breadth of E, for large n we have $s_n \in C$; hence, $\phi(s_n) \in V$ and thus $\phi \in V_E$. $\square$

Remark 6.5. The exact same proof of the previous proposition can be used to show a converse: $\phi \notin V_E$ if and only if there is an annulus $\mathcal{C} := \mathcal{C}(\beta, \tau, \delta)$ such that $\phi(t) \notin V$ for all $t \in \mathcal{C}$ (and similarly for the version with θ_1 and θ_2).

As a first step in the study of $\mathcal{V}$, we analyze the convergence of sequences in $\mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$.

Proposition 6.6. *Let $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ be a pseudo-convergent sequence of algebraic type with breadth δ, and, for each $n \in \mathbb{N}$, let $\zeta_n \in [\delta, \infty]$. If, for each $n \in \mathbb{N}$, $E_n \subset K$ is a pseudo-convergent sequence with pseudo-limit s_n and breadth ζ_n, then V_E is a limit of $\{V_{E_n}\}_{n\in\mathbb{N}}$ and $\{W_{E_n}\}_{n\in\mathbb{N}}$ in $\mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$ (and hence also in the Zariski topology).*

Proof. Let $\mathcal{X} := \mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$; we need to show that, if $V_E \in \Omega$ for some open set Ω, then $V_{E_n}, W_{E_n} \in \Omega$ for large n; without loss of generality, we can consider only the cases $\Omega = B(\phi)$ and $\Omega = \mathcal{X} \setminus B(\phi)$. This amounts to prove that $V_E \in B(\phi)$ if and only if $V_{E_n} \in B(\phi)$ (respectively, $W_{E_n} \in B(\phi)$) for all large n.

Suppose first that E has a pseudo-limit $s \in K$. By Proposition 6.4, there is an annulus $\mathcal{C} := \mathcal{C}(s, \tau, \delta)$ such that $\phi(t) \in V$ for all $t \in \mathcal{C}$. There is a N such that $s_n \in \mathcal{C}$ for $n \geq N$; hence, for these n, $\mathcal{L}_{E_n} \cap \mathcal{C} \neq \emptyset$. For all $t \in \mathcal{C}$, we have $w_E(X - t) = u(s - t)$; hence, $\mathcal{C} = \{t \in K \mid \tau < w_E(X - t) < \delta\}$. Therefore, we can apply Proposition 5.9, and so $V_E \in B(\phi)$ if and only if $V_{E_n} \in B(\phi)$ for $n \geq N$. Thus, the sequence V_{E_n} tends to V_E in the constructible topology.

Since $V_{E_n} \subseteq W_{E_n}$, we also have that if $V_E \in B(\phi)$ then $W_{E_n} \in B(\phi)$ for large n. Furthermore, without loss of generality, $\mathcal{C}$ does not contain any critical point of ϕ and $v(\phi(t)) = \lambda v(t - s) + \gamma$, for each $t \in \mathcal{C}$, for some $\lambda \in \mathbb{Z}$ and $\gamma \in \Gamma_v$ by Theorem 3.2; since $v(t_m - s) = v(t - s)$, for all $m \in \mathbb{N}$ sufficiently large, where $t \in \mathcal{L}_F \cap \mathcal{C}$, then $v(\phi(t_m)) = v(\phi(t))$ for all such m, and so $w_F(\phi) = v(\phi(t))$: hence, if $V_E \notin B(\phi)$ then also $W_{E_n} \notin B(\phi)$. It follows that also the sequence W_{E_n} tends to V_E in $\mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$.

Suppose now that E has a limit $\beta \in \overline{K}$ with respect to some extension u of v to $\overline{K}$; let $U \subset \overline{K}$ be the valuation domain of u. By the previous part of the proof, U_E is the limit of the sequence U_{E_n} in $\mathrm{Zar}(\overline{K}(X)|U)^{\mathrm{cons}}$. The restriction map $\pi : \mathrm{Zar}(\overline{K}(X)|U)^{\mathrm{cons}} \longrightarrow \mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$, $W \mapsto W \cap K(X)$, is continuous; hence, $\pi(U_{E_n}) \to \pi(U_E)$. However, $\pi(U_{E_n}) = V_{E_n}$ and $\pi(U_E) = V_E$; the claim is proved. The same reasoning applies to the sequence $\{W_{E_n}\}_{n\in\mathbb{N}}$.

The claim about the Zariski topology follows since the constructible topology is finer than the Zariski topology. □

Example 6.7. Let $E = \{s_n\}_{n\in\mathbb{N}}$ be a pseudo-convergent sequence of algebraic type and, for each $n \in \mathbb{N}$, let $W_{s_n} = \{\phi \in K(X) \mid \phi(s_n) \in V\}$. Then, by the previous lemma, $\{W_{s_n}\}_{n\in\mathbb{N}}$ converges to V_E in the constructible and in the Zariski topology.

Since we are working with the Zariski topology on $\mathcal{V}$ and $\mathcal{W}$, for ease of notation we set

$$B^{\mathcal{V}}(\phi) = \{V_E \in \mathcal{V} \mid V_E \ni \phi\} = B(\phi) \cap \mathcal{V},$$
$$B^{\mathcal{W}}(\phi) = \{W_E \in \mathcal{W} \mid W_E \ni \phi\} = B(\phi) \cap \mathcal{W}.$$

We denote by $\mathcal{V}(\bullet, \delta)$ the set of valuation domains V_E such that E has breadth δ; these sets will be studied more deeply in Section 8.1.

Proposition 6.8. *Let V be a valuation domain of rank 1 which is not discrete. The following are equivalent:*

(i) the residue field of V is finite;

(ii) the Zariski and the constructible topology coincide on $\mathcal{V}$;

(iii) there is a $\delta \in \mathbb{R} \cup \{+\infty\}$ such that the the Zariski and the constructible topology coincide on $\bigcup_{\delta' \leq \delta} \mathcal{V}(\bullet, \delta')$;

(iv) there is a $\delta \in \mathbb{R} \cup \{+\infty\}$ such that the the Zariski and the constructible topology coincide on $\bigcup_{\delta' < \delta} \mathcal{V}(\bullet, \delta')$.

When V is discrete, $\mathcal{V}$ reduces to $\mathcal{V}(\bullet, \infty)$; we shall see in Theorem 8.7 that in this case the Zariski and the constructible topology do coincide always.

Proof. (i) $\Longrightarrow$ (ii) To show that the Zariski and the constructible topology coincide, it is enough to show that $B(\phi)$ is closed in the Zariski topology for every $\phi \in K(X)$. Let thus $E = \{s_n\}_{n \in \mathbb{N}}$ be a pseudo-convergent sequence with breadth δ such that $V_E \notin B(\phi)$; we want to show that there is an open neighborhood of V_E disjoint from $B(\phi)$.

If E is of transcendental type, then $V_E = W_E$; since the Zariski and the constructible topology agree on $\mathcal{W}$ (Proposition 6.2), the set $B^{\mathcal{W}}(\phi)$ is closed in $\mathcal{W}$, and thus there are $\psi_1, \ldots, \psi_k$ such that $W_E \in B^{\mathcal{W}}(\psi_1, \ldots, \psi_k)$ but $B^{\mathcal{W}}(\psi_1, \ldots, \psi_k) \cap B^{\mathcal{W}}(\phi) = \emptyset$. In particular, $V_E \in B^{\mathcal{V}}(\psi_1, \ldots, \psi_k)$; on the other hand, if $V_F \in B^{\mathcal{V}}(\psi_1, \ldots, \psi_k) \cap B^{\mathcal{V}}(\phi)$, then $\psi_1, \ldots, \psi_k, \phi \in V_F \subseteq W_F$, and thus $W_F \in B^{\mathcal{W}}(\psi_1, \ldots, \psi_k) \cap B^{\mathcal{W}}(\phi)$, a contradiction. Hence, $B^{\mathcal{V}}(\psi_1, \ldots, \psi_k)$ and $B^{\mathcal{V}}(\phi)$ are disjoint, and $B^{\mathcal{V}}(\psi_1, \ldots, \psi_k)$ is the required neighborhood.

Suppose E is of algebraic type without pseudo-limits in K; let $\alpha \in \overline{K} \setminus K$ be a pseudo-limit of E with respect to an extension u of v to $\overline{K}$. By Proposition 6.4 and Remark 6.5, there is an annulus $\mathcal{C} = \mathcal{C}_u(\alpha, \theta_1, \theta_2)$ with $\theta_1, \theta_2 \in \mathbb{Q}\Gamma_v$, $\theta_1 < \delta \leq \theta_2$, such that $\phi(t) \notin V$ for all $t \in \mathcal{C}$. Let $s \in \mathcal{C}$; then $\theta_1 < u(\alpha - s) < \delta$, because otherwise s would be a pseudo-limit of E. Let $d \in K$ be such that $\theta_1 < v(d) < u(\alpha - s)$. Then, $V_E \in B\left(\frac{X-s}{d}\right)$, since, for large n, $v(s_n - s) - v(d) = u(s_n - \alpha + \alpha - s) - v(d) = u(\alpha - s) - v(d) \geq 0$. On the other hand, if $t \in K$ is such that $v\left(\frac{t-s}{d}\right) \geq 0$, then $v(t - s) \geq v(d) > \theta_1$, so $u(t - \alpha) = u(t - s + s - \alpha) > \theta_1$. Since $u(t - \alpha) < \delta$ because E has no pseudo-limits in K, it follows that $t \in \mathcal{C}$, so that $\phi(t) \notin V$; in particular, $B\left(\frac{X-s}{d}\right)$ is a neighborhood of V_E disjoint from $B(\phi)$.

Suppose now that E is of algebraic type with a pseudo-limit $s \in K$. If $\delta \notin \mathbb{Q}\Gamma_v$, then $V_E = W_E$ by Theorem 5.4, so the claim follows as in the transcendental case. Suppose $\delta \in \mathbb{Q}\Gamma_v$, and let $k \in \mathbb{N}^+$ be such that $k\delta \in \Gamma_v$. By Proposition 6.4 and Remark 6.5, there is an annulus $\mathcal{C}(s, \tau, \delta)$, with $\tau < \delta$, such that $\phi(t) \notin V$ for all $t \in \mathcal{C}(s, \tau, \delta)$. Let $d \in K$ be an element such that $v(d) \in (\tau, \delta)$; then, $V_E \in B\left(\frac{X-s}{d}\right)$.

Let $u_1, \dots, u_r$ be a complete set of representatives for the residue field of V; suppose that $u_1 \in M$ and $u_i \in V \setminus M$ for $i = 2, \dots, r$. Let $z \in K$ be an element of valuation $k\delta$. Let

$$\psi(X) := \frac{z^r}{((X-s)^k - zu_1)\cdots((X-s)^k - zu_r)};$$

we claim that $\psi(t) \in V$ if and only if $v(t-s) < \delta$.

Indeed, if $v(t-s) < \delta$ then $v((t-s)^k) = kv(t-s) < k\delta \leq v(zu_i)$ for $i = 1, \dots, r$, and thus

$$v(\psi(t)) = rk\delta - rkv(t-s) > 0.$$

If $v(t-s) > \delta$, then $v((t-s)^k - zu_i) = k\delta$ for $i = 2, \dots, r$ and $v((t-s)^k - zu_1) > k\delta$, and thus

$$v(\psi(t)) < rk\delta - rk\delta = 0.$$

If $v(t-s) = \delta$, then $v((t-s)^k) = k\delta = v(z)$; since $u_1, \dots, u_r$ are a complete set of representatives, there is a (unique) $i \in \{2, \dots, r\}$ such that $v((t-s)^k - zu_i) > k\delta$, while $v((t-s)^k - zu_j) = k\delta$ for $j \in \{1, \dots, r\} \setminus \{i\}$. Hence,

$$v(\psi(t)) = rk\delta - (r-1)k\delta - v((t-s)^k - zu_i) = k\delta - v((t-s)^k - zu_i) < 0.$$

In particular, $V_E \in B(\psi)$ by Proposition 6.4; furthermore, if $\psi(t), \frac{t-s}{d} \in V$, then $t \in \mathcal{C}$. Hence, $B(\psi) \cap B\left(\frac{X-s}{d}\right) \cap B(\phi) = \emptyset$, and thus $B(\psi) \cap B\left(\frac{X-s}{d}\right)$ is a neighborhood of V_E disjoint from $B(\phi)$. It follows that $B(\phi)$ is closed, as claimed.

(ii) $\Longrightarrow$ (iii) and (iv) are obvious.

Suppose now that either (iii) or (iv) hold for some δ, and let $\mathcal{X}$ be $\bigcup_{\delta' \leq \delta} \mathcal{V}(\bullet, \delta')$ or $\bigcup_{\delta' < \delta} \mathcal{V}(\bullet, \delta')$, accordingly. Suppose that the residue field of V is infinite. Let $c \in K$ such that $\eta := v(c) < \delta$: we claim that $B(c^{-1}X) \cap \mathcal{X}$ is not closed in $\mathcal{X}$.

Indeed, let $E = \{s_n\}_{n \in \mathbb{N}}$ be a pseudo-convergent sequence with breadth η and having 0 as a pseudo-limit. Then, $V_E \notin B(c^{-1}X)$. Suppose there is a neighborhood of V_E disjoint from $B(c^{-1}X)$: then, there are $\psi_1, \dots, \psi_k$ such that $V_E \in B(\psi_1, \dots, \psi_k)$ and such that $B(\psi_1, \dots, \psi_k) \cap B(c^{-1}X) = \emptyset$.

Fix an extension u of v to $\overline{K}$. Let $\beta_1, \dots, \beta_m$ be the critical points of $\psi_1, \dots, \psi_k$ having valuation η under u (if there are any). Since the residue field of V is infinite, there is a $t \in K$ such that $v(t) = \eta$ and such that $u(t - \beta_i) = \eta$ for all i. We claim that $v(\psi_i(t)) \geq 0$ for all i.

Indeed, fix i, and let $\alpha_1, \dots, \alpha_r$ be the critical points of $\psi := \psi_i$. By construction, we have

$$u(t - \alpha_j) = \begin{cases} u(\alpha_j) & \text{if } u(\alpha_j) < \eta, \\ v(t) = \eta & \text{if } u(\alpha_j) \geq \eta. \end{cases} \tag{8}$$

In particular, a direct calculation gives $v(\psi(t)) = \lambda\eta + \gamma$, where λ is the weighted sum of the critical points of ψ in the closed ball $B(0, e^{-\eta})$ and $\gamma \in \Gamma_v$. By Corollary 4.6, it follows that $v(\psi(t)) = w_E(\psi)$; in particular, $v(\psi(t)) \geq 0$ since $\psi \in V_E$. Therefore, $v(\psi_i(t)) \geq 0$ for all i. Furthermore, we claim that

$$v(\psi_i(t')) = v(\psi_i(t)) \geq 0, \quad \text{for all } t' \text{ such that } v(t - t') > \eta \tag{9}$$

In fact, by (8) we have $\eta \geq u(t-\alpha_i)$, so $u(t'-\alpha_i) = u(t-\alpha_i)$ for all $i = 1, \dots, r$ and the claim follows.

Hence, if $\eta' > \eta$ and $F = \{t_n\}_{n\in\mathbb{N}}$ is a pseudo-convergent sequence of breadth η' and pseudo-limit t, then $V_F \in B(\psi_1, \dots, \psi_k)$ by (9), since $v(t-t_n) > \eta$ for large n. In particular, we must have $v(t) = v(t_n)$ for every n, since $v(t) = \eta < \eta'$ and $v(t-t_n) \nearrow \eta'$, so 0 is not a pseudo-limit of F (thus, $v(t_n)$ is definitively constant). Hence, V_F also belongs to $B(c^{-1}X)$; therefore, if we choose $\eta' \in (\eta, \delta)$, we have $V_F \in B(\psi_1, \dots, \psi_k) \cap B(c^{-1}X) \cap \mathcal{X}$, against our choice of $\psi_1, \dots, \psi_k$. Therefore, $B(c^{-1}X) \cap \mathcal{X}$ is not closed, and the constructible topology does not agree with the Zariski topology. By contradiction, (i) holds. $\square$

The final part of the previous proof relies heavily on the possibility of choosing both a pseudo-limit and the breadth for a pseudo-convergent sequence E. We shall see in Section 8 (Theorem 8.7 and Proposition 8.17) that, if fix a breadth, or if we fix a pseudo-limit, than the Zariski and the constructible topology actually do coincide.

To conclude this section, we study the the function from $\mathcal{W}$ to $\mathcal{V}$ which map each W_E to V_E. We need the following lemma.

Lemma 6.9. *Let $\phi \in K(X)$ and $\delta \in \mathbb{R}$. Let S be the set of valuation domains V_F, with $F = \{t_n\}_{n\in\mathbb{N}}$, such that $v(\phi(t_n)) \nearrow \delta$. Then, S is a finite set.*

Proof. Let $V_F \in S$, $F = \{t_n\}_{n\in\mathbb{N}}$ with breadth δ_F, and fix an extension u of v to $\overline{K}$. By Proposition 3.5 and Corollary 4.6 there are $\lambda \in \mathbb{Z}$, $\gamma \in \Gamma_v$ depending on F and $\phi(X)$ such that

$$\delta = w_F(\phi) = \lambda\delta_F + \gamma. \tag{10}$$

Since $v(\phi(t_n))$ is definitively strictly increasing, F is of algebraic type and by Proposition 3.5 its dominating degree λ is positive, i.e., some zero of ϕ is a pseudo-limit of F with respect to u. Hence, S is the union of $S_\beta := \{V_F \in S \mid \beta \in \mathcal{L}_F^u\} = S \cap \mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$, as β ranges among the zeroes of ϕ. Since ϕ has only finitely many zeroes, it is enough to show that each S_β is finite.

Let A_β be the set of breadths of the pseudo-convergent sequences in S_β; then, the cardinality of A_β is equal to the cardinality of S_β, by Theorem 5.4. Let $\theta_1 < \dots < \theta_a$ be the elements of Γ_v such that there is a critical point β' of ϕ with $v(\beta-\beta') = \theta_i$; let $\theta_0 = -\infty$ and $\theta_{a+1} = +\infty$. We claim that $A_\beta \cap (\theta_i, \theta_{i+1})$ has at most one element, for every $i \in \{0, \dots, a\}$.

Let $V_F \in S_\beta$ be such that $\delta_F \in (\theta_i, \theta_{i+1})$, and let $F = \{t_n\}_{n\in\mathbb{N}}$. Note that for such pseudo-convergent sequences F, the values of λ and γ in (10) do not depend on F (explicitly, λ is the weighted sum of critical points β' of ϕ such that $v(\beta-\beta') \geq \delta_F$, which is equivalent to $v(\beta-\beta') > \theta_{i+1}$, and γ is defined as in Proposition 3.5). In particular, by (10), δ_F is uniquely determined in (θ_i, θ_{i+1}) (recall that if $V_F \in S$ then the dominating degree is nonzero), and since we are dealing with pseudo-convergent sequences F having β as pseudo-limit, by Theorem 5.4 $|A_\beta \cap (\theta_i, \theta_{i+1})| \leq 1$. Therefore,

$$A_\beta \subseteq \{\theta_1 \dots, \theta_a\} \cup \bigcup_{i=0}^{a} (A_\beta \cap (\theta_i, \theta_{i+1}))$$

is finite. Hence, S_β is finite and the claim is proved. $\square$

If V is a DVR, then we have already remarked at the beginning of Section 6 that $\mathcal{W}$ is a subset of $\mathcal{V}$; in particular, it is a topological embedding. If V is non-discrete, we still have an inclusion, which however is not an embedding.

Proposition 6.10. *Let V be a rank one non-discrete valuation domain. Let Ψ be the map*

$$\begin{aligned}\Psi\colon \mathcal{W} &\longrightarrow \mathcal{V}\\ W_E &\longmapsto V_E\end{aligned}$$

Then, Ψ is continuous and injective, but it is not a topological embedding.

Proof. By Theorem 5.4, Ψ is injective. To show that Ψ is continuous, it is enough to show that every $\Psi^{-1}(B^{\mathcal{V}}(\phi))$ is open.

Since $V_E \subseteq W_E$, we have $\Psi^{-1}(B^{\mathcal{V}}(\phi)) = \{W_E \in \mathcal{W} \mid V_E \ni \phi\} \subseteq B^{\mathcal{W}}(\phi)$, and the inclusion can be strict; more precisely,

$$C := B^{\mathcal{W}}(\phi) \setminus \Psi^{-1}(B^{\mathcal{V}}(\phi)) = \{W_E \mid \phi \in W_E \setminus V_E\} = \{W_E \mid \phi \in W_E^* \setminus V_E\}.$$

If $E = \{s_n\}_{n\in\mathbb{N}}$ is such that $\phi \in W_E^*$, then $w_E(\phi) = 0$; furthermore, if $\phi \notin V_E$ then $v(\phi(s_n))$ is definitively negative. Hence, for every $W_E \in C$ we must have $v(\phi(s_n)) \nearrow 0$, and by Lemma 6.9 the set C is finite (and possibly empty); since $\mathcal{W}$ is T_1 (Proposition 6.2), C is closed. Hence,

$$\Psi^{-1}(B^{\mathcal{V}}(\phi)) = B^{\mathcal{W}}(\phi) \cap (\mathcal{W} \setminus C)$$

is open, and so Ψ is continuous.

Let $\mathcal{V}_0$ be the image of Ψ: to show that Ψ is not a topological embedding, it is enough to show that $\Phi := \Psi^{-1} : \mathcal{V}_0 \longrightarrow \mathcal{W}$ is not continuous. Take a pseudo-convergent sequence E of algebraic type with breadth $\delta \in \Gamma_v$, and let $\zeta > \delta$. By Proposition 6.6, if, for each $n \in \mathbb{N}$, E_n is a pseudo-convergent sequence with limit s_n and breadth ζ, then V_E is the limit of V_{E_n} in the Zariski topology; note that both V_E and the V_{E_n} belong to $\mathcal{V}_0$ since they have finite breadth.

Hence, if Φ were continuous then $\Phi(V_{E_n}) = W_{E_n}$ would have limit $\Phi(V_E) = W_E$ in $\mathcal{W}$; since the Zariski and the constructible topology agree on $\mathcal{W}$ (by Proposition 6.2), it would follow that W_{E_n} has limit W_E in $\mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$. However, this contradicts Proposition 6.6, since $\mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$ is Hausdorff and $V_E \neq W_E$ by Theorem 4.9 (and the choice of δ). Hence, Φ is not continuous and Ψ is not a topological embedding. $\square$

7 Separation properties of $\mathcal{V}$

In this section, we analyze the separation properties of $\mathcal{V}$. In particular, we shall prove that $\mathcal{V}$ is a regular space. We recall that a topological space is *regular* if every point is closed and if, whenever C is a closed set and $x \notin C$ then x and C can be separated by open sets.

We say that two subsets C_1, C_2 of a topological space X can be *separated by open-closed sets* (*open sets*, respectively) if there are disjoint open-closed (open, respectively) subsets Ω_1, Ω_2 of X such that $C_i \subseteq \Omega_i$. If $C_1 = \{c_1\}$ is a singleton, we also say that c_1 and C_2 can be separated by open-closed sets (open sets, respectively).

We need two lemmas.

Lemma 7.1. *Let X be a topological space and $C, D \subseteq X$. If $D = D_1 \cup \cdots \cup D_n$ and each D_i can be separated from C by open-closed sets (respectively, open sets), then C and D can be separated by open-closed sets (resp., open sets).*

Proof. For each $i = 1, \ldots, n$, let O_i, Ω_i be disjoint open-closed sets (or open sets) separating C and D_i, i.e., $C \subseteq O_i$, $D_i \subseteq \Omega_i$, $O_i \cap \Omega_i = \emptyset$. The claim follows by taking $O := O_1 \cap \cdots \cap O_n$ and $\Omega := \Omega_1 \cup \cdots \cup \Omega_n$. $\square$

Lemma 7.2. *Let $\gamma \in \mathbb{Q}\Gamma_v$ and $s \in K$. Then, the set*

$$\Omega(s, \gamma) := \{V_E \in \mathcal{V} \mid w_E(X - s) \leq \gamma\}$$

is both open and closed in $\mathcal{V}$.

Proof. If V is discrete, then by Theorem 6.1 $\Omega(s, \gamma)$ is homeomorphic to the closed ball of $\widehat{K}$ having center s and radius $e^{-\gamma}$, and thus it is both open and closed since $\widehat{K}$ is an ultrametric space.

Suppose V is not discrete, and let $\Omega := \Omega(s, \gamma)$. Let $k > 0$ be an integer such that $k\gamma \in \Gamma_v$, and let $c \in K$ be such that $v(c) = k\gamma$. We claim that

$$\Omega = B\left(\frac{c}{(X - s)^k}\right)$$

and that

$$\mathcal{V} \setminus \Omega = \bigcup_{\substack{d \in K \\ v(d) > \gamma}} B\left(\frac{X - s}{d}\right).$$

Clearly, both right hand sides are open in $\mathcal{V}$.

Let $E = \{s_n\}_{n \in \mathbb{N}}$ be a pseudo-convergent sequence. Then $v(s_n - s)$ is either definitively increasing or definitively constant, and its limit is $w_E(X - s)$ (see Remark 4.7(a)); hence, $V_E \in \Omega$ if and only if $v(s_n - s) \leq \gamma$ for large n, while $V_E \notin \Omega$ if and only if $v(s_n - s) > \gamma$ for large n.

If $V_E \in \Omega$ then

$$v\left(\frac{c}{(s_n - s)^k}\right) = v(c) - kv(s_n - s) \geq k\gamma - k\gamma = 0$$

and so $V_E \in B\left(\frac{c}{(X-s)^k}\right)$. In the same way, if $V_E \in B\left(\frac{c}{(X-s)^k}\right)$ then $v(s_n - s) \leq \gamma$ and so $V_E \in \Omega$.

Similarly, if $V_E \notin \Omega$ then $v(s_n - s) \geq \gamma' > \gamma$ for some $\gamma' \in \Gamma_v$; if $v(d) = \gamma'$ then $V_E \in B\left(\frac{X-s}{d}\right)$ and so it is in the union. Conversely, if V_E is in the union then $V_E \in B\left(\frac{X-s}{d}\right)$ for some d, and $v(s_n - s) \geq v(d) > \gamma$ for large n, so that $V_E \notin \Omega$. The claim is proved. $\square$

Theorem 7.3. *$\mathcal{V}$ is a regular topological space.*

Proof. If V is a DVR, the statement follows from the fact that $\mathcal{V} = \mathcal{V}(\bullet, \infty)$ is an ultrametric space by [19, Theorem 3.4] (see also Theorem 8.7). Henceforth, we assume that V is not discrete.

We first note that each point of $\mathcal{V}$ is closed: indeed, the closure of a point Z in $\mathrm{Zar}(K(X)|V)$ is equal to the set of valuation domains contained in Z. However, two different domains V_E and V_F are never comparable: if they were, then $W_E = W_F$, and thus $V_E = V_F$ by Theorem 5.4.

Let $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ be a pseudo-convergent sequence of breadth δ, let $\{\delta_n\}_{n\in\mathbb{N}}$ be the gauge of E and let $C \subset \mathcal{V}$ be a closed set which does not contain V_E. Then there are rational functions $\phi_1, \ldots, \phi_k \in K(X)$ such that $V_E \in B(\phi_1, \ldots, \phi_k)$ while $B(\phi_1, \ldots, \phi_k) \cap C = \emptyset$. We let $\Lambda := \{\beta_1, \ldots, \beta_m\} \subseteq \overline{K}$ be the set of critical points of $\phi_1, \ldots, \phi_k$. Let also u be an extension of v to $\overline{K}$.

We want to separate V_E and C; to apply Lemma 7.1, we need to distinguish several cases.

Case 1. E is of transcendental type.

By [23, Theorem 31.18, p. 328], there is an n such that no $\beta \in \Lambda$ satisfies $u(\beta - s_n) \geq \delta_n$. Hence, there is a $\gamma < \delta_n$, $\gamma \in \mathbb{Q}\Gamma_v$ such that each $\beta \in \Lambda$ satisfies $u(\beta - s_n) < \gamma$. Moreover, up to considering a bigger $n \in \mathbb{N}$, we may also suppose that $\phi_i(s_n) \in V$ for all $i = 1, \ldots, k$. Let $s := s_n$. By Theorem 3.2, we have $v(\phi_i(t)) = v(\phi_i(s)) \geq 0$ for all t such that $v(t - s) \geq \gamma$ and for all $i = 1, \ldots, k$.

We claim that $\Omega(s, \gamma)$ and its complement separate C and V_E; by Lemma 7.2, this will imply that C and V_E are separated by open-closed sets.

Indeed, clearly $w_E(X - s) = \delta_n > \gamma$ and so $V_E \notin \Omega(s, \gamma)$. On the other hand, if $V_F \in C$ and $F = \{t_n\}_{n\in\mathbb{N}}$, then there is an i such that $v(\phi_i(t_n))$ is definitively negative. By the previous paragraph $v(t_n - s) < \gamma$ for all sufficiently large n; hence, $w_F(X - s) = \lim_n v(t_n - s) \leq \gamma$ and $V_F \in \Omega(s, \gamma)$. Thus, $C \subseteq \Omega(s, \gamma)$, as claimed.

Case 2. E is of algebraic type without pseudo-limits in K.

Let $\alpha \in \overline{K} \setminus K$ be a pseudo-limit of E with respect to u. By Lemma 2.4, there is no element t of K such that $u(\alpha - t) \geq \delta$. By Proposition 6.4, there is an annulus $\mathcal{C} := \mathcal{C}_u(\alpha, \tau, \delta)$ such that $\phi_i(t) \in V$ for all $i = 1, \ldots, k$ and all $t \in \mathcal{C}$; let $s \in \mathcal{C}$ and let $\delta' := u(\alpha - s) \in \mathbb{Q}\Gamma_v$. Note that $\tau < \delta' < \delta$. We claim that $\Omega(s, \tau)$ and its complement separate C and V_E.

Indeed, we have

$$w_E(X - s) = \lim_{n\to\infty} v(s_n - s) = \lim_{n\to\infty} u(s_n - \alpha + \alpha - s) = u(\alpha - s) = \delta'$$

since $\delta_n > \delta'$ for large n; hence, $V_E \notin \Omega(s, \tau)$. On the other hand, if $F = \{t_n\}_{n\in\mathbb{N}}$ is a pseudo-convergent sequence such that $V_F \in C \setminus \Omega(s, \tau)$, then $\tau < v(t_n - s)$ for all $n \in \mathbb{N}$ sufficiently large. Therefore, for each such n we have $\delta > u(t_n - \alpha) = u(t_n - s + s - \alpha) > \tau$. By our assumption this would imply that $\phi_i(t_n) \in V$ for $i = 1, \ldots, k$, for all $n \geq N$, which is a contradiction since $C \cap B(\phi_1, \ldots, \phi_k) = \emptyset$. Hence, $C \subseteq \Omega(s, \tau)$, and we have proved that C and V_E can be separated by an open-closed set.

Case 3. E is of algebraic type and there exists a pseudo-limit α of E in K.

We partition C into the following three sets:

$$\begin{aligned} C_1 :=& \{V_F \in C \mid w_F(X-\alpha) < \delta\}, \\ C_2 :=& \{V_F \in C \mid w_F(X-\alpha) > \delta\}, \\ C_3 :=& \{V_F \in C \mid w_F(X-\alpha) = \delta\}. \end{aligned}$$

By Theorem 3.2 (and Remark 3.3), we can find $\zeta_1, \zeta_2 \in \mathbb{Q}\Gamma_v$ such that $\zeta_1 < \delta \leq \zeta_2$ and such that $v(\phi_i(t)) = \lambda_i v(t-\alpha) + \gamma_i$ for every $t \in \mathcal{C}(\alpha, \zeta_1, \zeta_2)$, for some $\lambda_i \in \mathbb{Z}$ and $\gamma_i \in \Gamma_v$. Since $V_E \in B(\phi_1, \dots, \phi_k)$, by Proposition 6.4 we can find $\theta_1, \theta_2 \in \mathbb{Q}\Gamma_v$, with $\delta \in (\theta_1, \theta_2] \subseteq (\zeta_1, \zeta_2]$, such that $\phi_i(t) \in V$ for all $t \in \mathcal{C}(\alpha, \theta_1, \theta_2)$ and all $i = 1, \dots, k$.

Consider $\Omega(\alpha, \theta_1)$. We have $w_E(X-\alpha) = \delta > \theta_1$, and so $V_E \notin \Omega(\alpha, \theta_1)$; on the other hand, if $V_F \in C_1$, with $F = \{t_n\}_{n \in \mathbb{N}}$, then $v(t_n - \alpha) < \theta_1$ for all large n (because $C_1 \subseteq C$ has empty intersection with $B(\phi_1, \dots, \phi_k)$) and thus also $w_F(X-\alpha) \leq \theta_1$; hence, $C_1 \subseteq \Omega(\alpha, \theta_1)$. Thus, $\Omega(\alpha, \theta_1)$ and its complement are open-closed subsets separating C_1 and V_E.

Similarly, $w_E(X-\alpha) = \delta \leq \theta_2$ and thus $V_E \in \Omega(\alpha, \theta_2)$; if $V_F \in C_2$, $F = \{t_n\}_{n \in \mathbb{N}}$, then $v(t_n - \alpha) > \theta_2$ for all large n (because $C_2 \subseteq C$ has empty intersection with $B(\phi_1, \dots, \phi_k)$) and, since $v(t_n - \alpha)$ is either definitively strictly increasing or definitively constant, we have $w_F(X-\alpha) > \theta_2$, i.e., $C_2 \cap \Omega(\alpha, \theta_2) = \emptyset$. Hence, $\Omega(\alpha, \theta_2)$ and its complement separate V_E and C_2. In particular, if $C_3 = \emptyset$ then V_E and C can be separated by open-closed sets.

Suppose $C_3 \neq \emptyset$ and let $V_F \in C_3$, $F = \{t_n\}_{n \in \mathbb{N}}$: then $\delta \in \mathbb{Q}\Gamma_v$, for otherwise $v(t_n - \alpha)$ should increase to δ, and so t_n would enter in any annulus $\mathcal{C}(\alpha, \tau, \delta)$ and by Proposition 3.5 $\phi_i \in V_F$ for $i = 1, \dots, k$, against the fact that $C \cap B(\phi_1, \dots, \phi_k) = \emptyset$. By the same argument, $v(t_n - \alpha)$ is constantly equal to δ (which therefore is in Γ_v). In particular, α is not a pseudo-limit of F so that $\delta_F > v(t_n - \alpha) = \delta = \delta_E$.

Since $C \cap B(\phi_1, \dots, \phi_k) = \emptyset$, for every $V_F \in C$ there is an $i \in \{1, \dots, k\}$ such that $\phi_i(t_n) \notin V$ for all n sufficiently large; for such an i, $w_F(\phi_i) \leq 0$ and if equality holds then $v(\phi_i(t_n)) \nearrow 0$, where $F = \{t_n\}_{n \in \mathbb{N}}$. For each $i = 1, \dots, k$, let

$$\begin{aligned} D_i :=& \{V_F \in C_3 \mid w_F(\phi_i) < 0\} \quad \text{and} \\ H_i :=& \{V_F \in C_3 \mid w_F(\phi_i) = 0,\ \phi_i \notin V_F\}, \end{aligned}$$

so that $C_3 = \bigcup_{i=1,\dots,k}(D_i \cup H_i)$.

We claim that every D_i can be separated from V_E by open sets: indeed, let

$$\Omega_i := \bigcup_{\substack{d \in K \\ v(d) < 0}} B\left(\frac{d}{\phi_i(X)}\right).$$

As in the proof of Lemma 7.2, if $V_F \in D_i$ then there is a $\kappa < 0$ such that $v(\phi_i(t_n)) \leq \tau$ for all large n and thus, taking $d \in K$ such that $0 > v(d) \geq \kappa$,

$$v\left(\frac{d}{\phi_i(t_n)}\right) \geq \kappa - v(\phi_i(t_n)) \geq 0$$

and so $V_F \in \Omega_i$. Moreover, $\Omega_i \cap B(\phi_1, \dots, \phi_k) = \emptyset$, since otherwise there should be a $t \in K$ such that

$$\begin{cases} v(\phi_i(t)) \geq 0 \\ v\left(\frac{d}{\phi_i(t)}\right) \geq 0; \end{cases}$$

for some $d \in K$ such that $v(d) < 0$, but the latter condition implies that $v(\phi_i(t)) \leq v(d) < 0$. Hence, $B(\phi_1, \dots, \phi_k)$ and Ω_i separate V_E and D_i.

Since for every $V_F \in H_i$, with $F = \{t_n\}_{n \in \mathbb{N}}$, we have $v(\phi_i(t_n)) \nearrow 0$, every H_i is finite by Lemma 6.9. Furthermore, some zero $\beta \in \overline{K}$ of ϕ_i is a pseudo-limit of F, with respect to some extension u of v to $\overline{K}$ (see the proof of Lemma 6.9). If n is sufficiently large, then $\delta_E < u(t_n - \beta) < \delta_F$. Let $\gamma \in \Gamma_v$ be such that $\delta_E < \gamma < u(t_n - \beta)$. If we let $t = t_n$, then $w_E(X - t) \leq \delta_E < \gamma < w_F(X - t)$ (see Remark 4.7(a)). Then $V_E \in \Omega(t, \gamma)$ and $V_F \notin \Omega(t, \gamma)$. Hence, V_F can be separated from V_E by the open-closed set $\Omega(t, \gamma)$, and since H_i is finite it can also be separated from V_E by open-closed sets.

To summarize, we have

$$C = C_1 \cup C_2 \cup \bigcup_{i=1}^{k} D_i \cup \bigcup_{i=1}^{k} H_i,$$

and each of the sets on the right hand side can be separated from V_E by open sets; hence, C and V_E can be separated and $\mathcal{V}$ is regular. □

As a consequence, we can show that under some conditions $\mathcal{V}$ is metrizable.

Proposition 7.4. *Let V be a countable valuation domain. Then, $\mathcal{V}$ is metrizable.*

Proof. A basis for $\mathcal{V}$ is $\mathcal{B} := \{B(\phi_1, \dots, \phi_k) \mid \phi_1, \dots \phi_k \in K(X)\}$. Since V is countable, so are K and $K(X)$; hence, the number of finite subsets of $K(X)$ is countable, and thus also $\mathcal{B}$ is countable. Therefore, $\mathcal{V}$ is second-countable; since it is regular (Theorem 7.3), it follows from Urysohn's metrization theorem [9, e-2] that $\mathcal{V}$ is metrizable. □

8 Two partitions

In this section, we shall study in a more explicit way the Zariski and the constructible topology on the set $\mathcal{V}$ of the rings of the form V_E, as E runs over the set of pseudo-convergent sequences in K.

The starting point of this study is a geometric interpretation of Theorem 5.4. Let $\mathcal{V}_{\mathrm{alg}}$ be the set of the valuation domains V_E, where $E \subset K$ is a pseudo-convergent sequence of algebraic type. Fix an extension u of v to the algebraic closure $\overline{K}$ of K. Then, to every valuation ring $V_E \in \mathcal{V}_{\mathrm{alg}}$ is uniquely associated its set of pseudo-limits $\mathcal{L}_E^u \subset \overline{K}$; furthermore, since $\mathcal{L}_E^u = \beta_E + \mathrm{Br}_u(E)$, where $\beta_E \in \overline{K}$ is a pseudo-limit of E with respect to u (see Lemma 2.4), there is an injective map

$$\begin{aligned} \Sigma \colon \mathcal{V}_{\mathrm{alg}} &\longrightarrow \mathrm{CBall}_u(\overline{K}) \\ E &\longmapsto \mathcal{L}_E^u, \end{aligned} \tag{11}$$

where $\mathrm{CBall}_u(\overline{K})$ is the set of closed balls of the ultrametric space $\overline{K}$, endowed with the metric induced by u.

In general, Σ is not surjective; to find its range, we introduce the following definition. For any $\beta \in \overline{K}$, we consider the minimum distance of the elements of K from β, namely:

$$d_u(\beta, K) := \inf\{d_u(\beta, x) = e^{-u(\beta - x)} \mid x \in K\}.$$

Note that $d_u(\beta, K)$ may be 0 even if $\beta \notin K$: this happens if and only if β is in the completion of K under v. If V is a DVR, then the only closed balls of center $\beta \in \overline{K}$ which can arise as the set of pseudo-limits of a pseudo-convergent sequence $E \subset K$, are those of radius 0 and with $\beta \in \widehat{K}$. If V is non-discrete, we have the following result.

Proposition 8.1. *Let V be a non-discrete rank one valuation domain. Let $\beta \in \overline{K}$, $r \in \mathbb{R}^+$ and u an extension of v to $\overline{K}$; let B be the closed ball of center β and radius r with respect to u. Then, $B = \mathcal{L}_E^u$ for some pseudo-convergent sequence $E \subset K$ if and only if $r \geq d_u(\beta, K)$.*

Proof. Suppose $B = \mathcal{L}_E^u$, and let $E := \{s_n\}_{n\in\mathbb{N}}$. Then, $\{d_u(\beta, s_n)\}_{n\in\mathbb{N}}$ is a decreasing sequence of real numbers with limit $e^{-\delta} = r$, where δ is the breadth of E. By definition,

$$d_u(\beta, K) = \inf\{d_u(\beta, s) \mid s \in K\} \leq d_u(\beta, s_n)$$

for every n, and thus $d_u(\beta, K) \leq r$.

Conversely, suppose $r \geq d_u(\beta, K)$. If $r = d(\beta, x)$ for some $x \in K$, take a sequence $Z = \{z_k\}_{k\in\mathbb{N}} \subseteq K$ such that $v(z_k)$ is increasing and has limit $\delta := -\log(r)$. Then, $x + Z := \{x + z_k\}_{k\in\mathbb{N}}$ is a pseudo-convergent sequence whose set of limits (in $(\overline{K}, u)$) is B.

If $r \neq d(\beta, x)$ for every $x \in K$, we can take a sequence $E := \{s_n\}_{n\in\mathbb{N}}$ such that $d_u(\beta, s_n) = r_n$ decreases to r. Then, E is a pseudo-convergent sequence, and $\mathcal{L}_E^u = B$. □

Corollary 8.2. *Suppose V is not discrete. Then the map Σ defined in (11) is surjective if and only if $\widehat{K}$ is algebraically closed.*

Proof. By Proposition 8.1, Σ is surjective if and only if $\widehat{K}$ contains an algebraic closure of K. This happens if and only if $\widehat{K}$ is algebraically closed. □

The set $\mathcal{H}(X)$ of the closed sets of a topological space X is usually called the *hyperspace* of X; several topologies have been put and studied on $\mathcal{H}(X)$, including the *Vietoris topology*, the *Fell topology* and the topology induced by the *Hausdorff metric* (see e.g. [9, b-6] and the references therein). However, none of these seems to be the right topology to put on $\mathrm{CBall}_u(\overline{K})$ in this context; one reason, as we shall see in Section 8.2, is that the topology on $\mathcal{V}_{\mathrm{alg}}$ depends quite subtly from the value group Γ of V.

We approach the study of $\mathcal{V}$ and $\mathcal{V}_{\mathrm{alg}}$ by considering two natural partitions of them: one of $\mathcal{V}$ obtained by fixing the breadth of the corresponding pseudo-convergent sequence, and the other of $\mathcal{V}_{\mathrm{alg}}$ obtained by considering pseudo-convergent sequences with a prescribed pseudo-limit.

8.1 Fixed breadth

It follows from [19, Theorem 3.4] that the set of the valuation domains $W_\alpha = \{\phi \in K(X) \mid v(\phi(\alpha)) \geq 0\}$, as α ranges in $\widehat{K}$, endowed with the Zariski topology, is homeomorphic to the ultrametric space $\widehat{K}$. By Remark 3.8, this is exactly the set of the valuation domains V_E such that the breadth of E is infinite; the purpose of this section is to generalize this result to an arbitrary breadth $\delta \in \mathbb{R}$.

Definition 8.3. Let $\delta \in \mathbb{R} \cup \{+\infty\}$. We denote by $\mathcal{V}(\bullet, \delta)$ the set of valuation domains V_E such that the breadth of E is δ.

Clearly, if V is discrete then $\mathcal{V} = \mathcal{V}(\bullet, \infty)$ and $\mathcal{V}(\bullet, \delta) = \emptyset$ for each $\delta \in \mathbb{R}$.

Different δ's may yield homeomorphic topological spaces $\mathcal{V}(\bullet, \delta)$'s.

Proposition 8.4. *Let $\delta_1, \delta_2 \in \mathbb{R}$ be such that $\delta_1 - \delta_2 \in \Gamma_v$. Then, $\mathcal{V}(\bullet, \delta_1)$ and $\mathcal{V}(\bullet, \delta_2)$ are homeomorphic.*

Proof. Given a pseudo-convergent sequence $E = \{s_n\}_{n \in \mathbb{N}}$ and a $c \in K$, we denote by cE the sequence $\{cs_n\}_{n \in \mathbb{N}}$. Clearly, cE is again pseudo-convergent, it has breadth $\delta_E + v(c)$, and two sequences E and F are equivalent if and only if cE and cF are equivalent.

Let $c \in K$ be such that $v(c) = \delta_1 - \delta_2$. Then, the map

$$\begin{aligned} \Psi_c \colon \mathcal{V}(\bullet, \delta_2) &\longrightarrow \mathcal{V}(\bullet, \delta_1) \\ V_E &\longmapsto V_{cE} \end{aligned}$$

is well-defined and bijective (its inverse is $\Psi_{c^{-1}} : \mathcal{V}(\bullet, \delta_1) \longrightarrow \mathcal{V}(\bullet, \delta_2)$). Furthermore, $\Psi_c^{-1}(B(\phi)) = B(\psi)$, where ψ is defined by $\psi(X) := \phi(cX)$, and in the same way $\Psi_c(B(\phi)) = B(\psi')$, where $\psi'(X) := \phi(c^{-1}X)$. Hence, Ψ is continuous and open, and thus a homeomorphism. $\square$

Let now $\delta \in \mathbb{R} \cup \{\infty\}$ be fixed, and set $r := e^{-\delta}$. Given two pseudo-convergent sequences $E := \{s_n\}_{n \in \mathbb{N}}$ and $F := \{t_n\}_{n \in \mathbb{N}}$, with $V_E, V_F \in \mathcal{V}(\bullet, \delta)$, we set

$$d_\delta(V_E, V_F) := \lim_{n \to \infty} \max\{d(s_n, t_n) - r, 0\}.$$

It is clear that if $r = 0$ (or, equivalently, $\delta = +\infty$) then $d_\delta(V_E, V_F) = d(\alpha, \beta)$, where α and β are the (unique) limits of E and F, respectively; so in this case we get the same distance as in [19]. We shall interpret d_δ in a similar way in Proposition 8.9; we first show that it is actually a distance.

Proposition 8.5. *Preserve the notation above.*

(a) d_δ is well-defined.

(b) d_δ is an ultrametric distance on $\mathcal{V}(\bullet, \delta)$.

Proof. (a) Let $E := \{s_n\}_{n\in\mathbb{N}}$ and $F := \{t_n\}_{n\in\mathbb{N}}$ be two pseudo-convergent sequences. We start by showing that the limit of $a_n := \max\{d(s_n, t_n) - r, 0\}$ exists. If all subsequences of $\{a_n\}_{n\in\mathbb{N}}$ go to zero, we are done. Otherwise, there is a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ with a positive (possibly infinite) limit; in particular, there is a $\overline{\delta} < \delta$ and $k_0 \in \mathbb{N}$ such that $u(s_{n_k} - t_{n_k}) < \overline{\delta}$ for all $k \geq k_0$. Choose $k_1 \in \mathbb{N}$ such that $\overline{\delta} < \min\{\delta_{k_1}, \delta'_{k_1}\}$ (where $\{\delta_n\}_{n\in\mathbb{N}}$ and $\{\delta'_n\}_{n\in\mathbb{N}}$ are the gauges of E and F, respectively). Fix an $m = n_l$ such that $m > k_1$ and $l > k_0$. Then, for all $n > m$, we have

$$u(s_n - t_n) = u(s_n - s_m + s_m - t_m + t_m - t_n) = u(s_m - t_m)$$

since $u(s_n - s_m) = \delta_m > \delta_{k_1} > \overline{\delta} > u(s_{n_l} - t_{n_l}) = u(s_m - t_m)$, and likewise for $u(t_n - t_m)$. Hence, a_n is definitively constant (more precisely, equal to $e^{-u(s_m - t_m)} - e^{-\delta}$); in particular, $\{a_n\}_{n\in\mathbb{N}}$ has a limit.

In order to show that d_δ is well-defined, we need to show that, if $V_E = V_{E'}$, where $E = \{s_n\}_{n\in\mathbb{N}}$ and $E' = \{s'_n\}_{n\in\mathbb{N}}$, then

$$\lim_{n\to\infty} \max\{d(s_n, t_n) - r, 0\} = \lim_{n\to\infty} \max\{d(s'_n, t_n) - r, 0\}.$$

Let l be the limit on the left hand side and l' the limit on the right hand side. By Theorem 5.4, $V_E = V_{E'}$ if and only if E and E' are equivalent.

If F is equivalent to E and E', for every k there are i_0, j_0, i'_0, j'_0 such that $v(s_i - t_j) > \delta_k$, $v(s'_{i'} - t'_{j'}) > \delta'_k$ for $i \geq i_0$, $j \geq j_0$, $i' \geq i'_0$, $j' \geq j'_0$. Hence, both l and l' are equal to 0, and in particular they are equal.

Suppose that F is not equivalent to E and E'. If l is positive, and $\eta := -\log(l)$, then $v(s_n - t_n) = \eta$ for large n, and $\eta < \delta_k$ for some k; since E and E' are equivalent there is a i_0 such that $v(s_i - s'_i) > \delta_k$ for all $i \geq i_0$. Hence, for all large n,

$$v(s'_n - t_n) = v(s'_n - s_n + s_n - t_n) = v(s_n - t_n) = \eta,$$

as claimed. The same reasoning applies if $l' > 0$; furthermore, if $l = 0 = l'$ then clearly $l = l'$. Hence, $l = l'$ always, as claimed.

(b) d_δ is obviously symmetric. Clearly $d_\delta(V_E, V_E) = 0$; if $d_\delta(V_E, V_F) = 0$, for every $r_k = e^{-\delta'_k} < r$ (where $\delta'_k := v(t_{k+1} - t_k)$) there is i_0 such that $d(s_i, t_i) < r_k$ for all $i \geq i_0$. Thus, if $i, j \geq i_0$, then

$$d(s_i, t_j) = \max\{d(s_i, t_i), d(t_i, t_j)\} = r_k.$$

Hence, E and F are equivalent and $V_E = V_F$. The strong triangle inequality follows from the fact that $d(s_n, t_n) \leq \max\{d(s_n, s'_n), d(s'_n, t_n)\}$ for all $s_n, s'_n, t_n \in K$. Therefore, d_δ is an ultrametric distance. □

Let $\mathcal{V}_K(\bullet, \delta)$ be the set of valuation rings V_E such that E is a pseudo-convergent sequence with breadth δ and such that E has a pseudo-limit in K. When $\delta = \infty$, this set corresponds to K under the homeomorphism between $\mathcal{V}(\bullet, \infty)$ and $\widehat{K}$; in particular, $\mathcal{V}(\bullet, \infty)$ is the completion of $\mathcal{V}_K(\bullet, \infty)$ under d_∞. An analogous result holds for $\delta \in \mathbb{R}$.

Proposition 8.6. *$\mathcal{V}(\bullet,\delta)$ is the completion of $\mathcal{V}_K(\bullet,\delta)$ under the metric d_δ. In particular, $\mathcal{V}(\bullet,\delta)$, under d_δ, is a complete metric space.*

Proof. Let $\{\zeta_k\}_{k\in\mathbb{N}}\subset\Gamma$ be an increasing sequence of real numbers with limit δ and, for every k, let z_k be an element of K of valuation ζ_k; let $Z:=\{z_k\}_{k\in\mathbb{N}}$. It is clear that Z is a pseudo-convergent sequence with 0 as a pseudo-limit and having breadth δ. Then, for every $s\in K$, $s+Z:=\{s+z_k\}_{k\in\mathbb{N}}$ is a pseudo-convergent sequence with pseudo-limit s and breadth δ.

Let $E:=\{s_n\}_{n\in\mathbb{N}}$ be a pseudo-convergent sequence with breadth δ, and let $F_n:=s_n+Z$. By above, $V_{F_n}\in\mathcal{V}_K(\bullet,\delta)$, for each $n\in\mathbb{N}$. We claim that $\{V_{F_n}\}_{n\in\mathbb{N}}$ converges to V_E in $\mathcal{V}(\bullet,\delta)$. Indeed, fix $t\in\mathbb{N}$, and take $k>t$ such that $\zeta_k>\delta_t$. Then,

$$u(s_t+z_k-s_k)=u(s_t-s_k+z_k)=\delta_t;$$

hence, $d(V_E,V_{F_n})=e^{-\delta_n}-e^{-\delta}$. In particular, the distance goes to 0 as $n\to\infty$, and thus V_E is the limit of V_{F_n}.

Conversely, let $\{V_{F_n}\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{V}_K(\bullet,\delta)$, and let $s_n\in K$ be a pseudo-limit of F_n. Then, s_n+Z is another pseudo-convergent sequence with limit s_n and breadth δ; by Theorem 5.4 it follows that $V_{F_n}=V_{s_n+Z}$. There is a subsequence of $E:=\{s_n\}_{n\in\mathbb{N}}$ which is pseudo-convergent; indeed, it is enough to take $\{s_{n_k}\}_{k\in\mathbb{N}}$ such that $d(s_{n_k},s_{n_{k+1}})<d(s_{n_{k-1}},s_{n_k})$. Hence, without loss of generality E itself is pseudo-convergent; we claim that V_E is a limit of $\{V_{F_n}\}_{n\in\mathbb{N}}$. Indeed, as above, $u(s_t+z_k-s_k)=\delta_t$ for large k, and thus $d_\delta(V_E,V_{s_n+Z})=e^{-\delta_t}-e^{-\delta}$. Thus, $\{V_{F_n}\}_{n\in\mathbb{N}}$ has a limit, namely V_E. Therefore, $\mathcal{V}(\bullet,\delta)$ is the completion of $\mathcal{V}_K(\bullet,\delta)$. □

We now wish to prove that the topology induced by d_δ is actually the Zariski topology.

Theorem 8.7. *Let $\delta\in\mathbb{R}\cup\{\infty\}$. On $\mathcal{V}(\bullet,\delta)$, the Zariski topology, the constructible topology and the topology induced by d_δ coincide.*

Proof. If $\delta=\infty$, then the Zariski topology and the topology induced by d_δ coincide (see Theorem 6.1).

Suppose now that V is nondiscrete and fix $\delta\in\mathbb{R}$. Let $V_E\in\mathcal{V}(\bullet,\delta)$ and $\rho\in\mathbb{R}$, $\rho>0$: we show that the open ball $\mathcal{B}(V_E,\rho):=\{V_F\in\mathcal{V}(\bullet,\delta)\mid d_\delta(V_E,V_F)<\rho\}$ of the ultrametric topology induced by d_δ is open in the Zariski topology. Since by Proposition 8.6 $\mathcal{V}_K(\bullet,\delta)$ is dense in $\mathcal{V}(\bullet,\delta)$ under the metric d_δ, without loss of generality we may assume that $V_E\in\mathcal{V}_K(\bullet,\delta)$, i.e., E has a pseudo-limit b in K. To ease the notation, we denote by $B(\phi)$ the intersection $B(\phi)\cap\mathcal{V}(\bullet,\delta)$.

Let $\gamma<\delta$ be such that $\rho=e^{-\gamma}-e^{-\delta}$. We claim that

$$\mathcal{B}(V_E,\rho)=\bigcup_{\delta>v(c)>\gamma}B\left(\frac{X-b}{c}\right).$$

Indeed, suppose $V_F\in\mathcal{B}(V_E,\rho)$, where $F=\{t_n\}_{n\in\mathbb{N}}$. If F is equivalent to E then $V_E=V_F$ and $v\left(\frac{t_n-b}{c}\right)=\delta_n-v(c)$; since $\gamma<\delta$ and Γ is dense in $\mathbb{R}$, there is a $c\in K$ such that $\gamma<v(c)<\delta$, and for such a c the limit of $\delta_n-v(c)$ is positive; hence,

V_E belongs to the union. If F is not equivalent to E, then $0 < d_\delta(V_E, V_F) < \rho$, that is, $e^{-\delta} < \lim_n d(s_n, t_n) < e^{-\delta} + \rho$. By the proof of Proposition 8.5(a), $v(s_n - t_n)$ is definitively constant, and thus there is an $\epsilon > 0$ such that $\delta > v(s_n - t_n) \geq \gamma + \epsilon$ for all large n. Let $c \in K$ be of value comprised between γ and $\gamma + \epsilon$ (such a c exists because Γ is dense in $\mathbb{R}$), then:

$$v\left(\frac{t_n - b}{c}\right) = v(t_n - b) - v(c) = v(t_n - s_n + s_n - b) - v(c) \geq \min\{\gamma + \epsilon, \delta_n\} - v(c) > 0$$

since δ_n becomes bigger than $\gamma + \epsilon$. Hence, $\frac{X-b}{c} \in V_F$, or equivalently $V_F \in B\left(\frac{X-b}{c}\right)$.

Conversely, suppose $V_F \neq V_E$ belongs to $B\left(\frac{X-b}{c}\right)$ for some $c \in K$ such that $\gamma < v(c) < \delta$. Since $\mathcal{L}_E \cap \mathcal{L}_F = \emptyset$, b is not a pseudo-limit of F; therefore, $v(t_n - s_n) = v(t_n - b + b - s_n) = v(b - t_n) \geq v(c) > \gamma$ for sufficiently large n. Thus,

$$d_\delta(V_E, V_F) = \lim_n d(s_n, t_n) - e^{-\delta} = \lim_n d(b, t_n) - e^{-\delta} < e^{-\gamma} - e^{-\delta} = \rho,$$

i.e., $V_F \in \mathcal{B}(V_E, \rho)$. Thus, being the union of sets that are open in the Zariski topology, $\mathcal{B}(V_E, \rho)$ is itself open in the Zariski topology. Therefore, the ultrametric topology is finer than the Zariski topology.

Let now δ be arbitrary, $\phi \in K(X)$ be a rational function, and suppose $V_E \in B(\phi)$ for some $V_E \in \mathcal{V}(\bullet, \delta)$. We want to show that for some $\rho > 0$ there is a ball $\mathcal{B}(V_E, \rho) \subseteq B(\phi)$, and thus that $B(\phi)$ is open in the ultrametric topology induced by d_δ. We distinguish two cases.

Suppose that E is of algebraic type, and let $\beta \in \mathcal{L}_E^u$ for some extension u of v to $\overline{K}$. By Proposition 6.4, there is an annulus $C := \mathcal{C}(\beta, \tau, \delta)$ such that $\phi(s) \in V$ for every $s \in C$. Let $\epsilon := e^{-\tau} - e^{-\delta}$. Let $F := \{t_n\}_{n\in\mathbb{N}}$ be a pseudo-convergent sequence with $d_\delta(V_E, V_F) < \epsilon$. Then, for every n such that $e^{-\delta_n} - e^{-\delta} > d_\delta(V_E, V_F)$,

$$d(t_n, \beta) = \max\{d(t_n, s_n), d(s_n, \beta)\} = e^{-\delta_n},$$

and in particular $v(t_n - \beta)$ becomes larger than τ. Hence, t_n is definitively in C and $\phi(t_n) \in V$ for all large n, and thus $\phi \in V_F$; therefore, $\mathcal{B}(V_E, \epsilon) \subseteq B(\phi)$.

Suppose that E is of transcendental type. Let $\phi(X) = c\prod_{i=1}^{A}(X - \alpha_i)^{\epsilon_i}$ over $\overline{K}$; then, there is an N such that $u(s_n - \alpha_i)$ is constant for every i and every $n \geq N$. Let δ' be the maximum of such constants; then, $\delta' < \delta$ (otherwise the α_i where such maximum is attained would be a pseudo-limit of E, against the fact that E is of transcendental type). Let ϵ be such that $e^{-\delta} + \epsilon < e^{-\delta'}$ and let $V_F \in \mathcal{B}(V_E, \epsilon)$, with $F := \{t_n\}_{n\in\mathbb{N}}$. For all i, and all large n,

$$d(t_n, \alpha_i) = \max\{d(t_n, s_n), d(s_n, \alpha_i)\} = d(s_n, \alpha_i),$$

and thus $u(t_n - \alpha_i) = u(s_n - \alpha_i)$. It follows that $v(\phi(t_n)) = v(\phi(s_n))$ for large n; in particular, $v(\phi(t_n))$ is positive, and $\phi \in V_F$. Hence, $\mathcal{B}(V_E, \epsilon) \subseteq B(\phi)$.

Hence, $B(\phi)$ is open under the topology induced by d_δ and therefore the Zariski topology and the topology induced by d_δ on $\mathcal{V}(\bullet, \delta)$ are the same.

In order to prove that these topologies coincide also with the constructible topology, we need only to show that every $B(\phi)$, $\phi \in K(X)$, is closed in the Zariski topology. Let then $V_E \notin B(\phi)$. If E is of transcendental type, exactly as above there exists $\epsilon > 0$ such that for each $V_F \in \mathcal{B}(V_E, \epsilon)$, where $F = \{t_n\}_{n\in\mathbb{N}}$, $v(\phi(t_n)) = v(\phi(s_n))$ for large n; in particular, $v(\phi(t_n))$ is negative, and $\phi \notin V_F$; thus $\mathcal{B}(V_E, \epsilon)$ is disjoint from $B(\phi)$. If E is of algebraic type, then by Remark 6.5, there exists an annulus $\mathcal{C} := \mathcal{C}(\beta, \tau, \delta)$ such that $\phi(s) \notin V$ for every $s \in \mathcal{C}$. As above, for every pseudo-convergent sequence $F = \{t_n\}_{n\in\mathbb{N}}$ with $d_\delta(V_E, V_F) < \epsilon$, with $\epsilon := e^{-\tau} - e^{-\delta}$, we have $t_n \in \mathcal{C}$ for all but finitely many $n \in \mathbb{N}$, so that $\phi(t_n) \notin V$. Again, this shows that $\mathcal{B}(V_E, \epsilon)$ is disjoint from $B(\phi)$. $\square$

Corollary 8.8. *The set $\mathcal{V}_K := \{V_E \in \mathcal{V} \mid \mathcal{L}_E \cap K \neq \emptyset\}$ is dense in $\mathcal{V}$ (both in the Zariski and in the constructible topology).*

Proof. By Proposition 8.6, $\mathcal{V}_K(\bullet, \delta)$ is dense in $\mathcal{V}(\bullet, \delta)$ under the topology generated by d_δ. By Theorem 8.7, $\mathcal{V}_K(\bullet, \delta)$ is dense in $\mathcal{V}(\bullet, \delta)$ in the Zariski and the constructible topology; hence, $\mathcal{V}_K$ (being the union of the various $\mathcal{V}_K(\bullet, \delta)$) is dense in $\mathcal{V}$ in the Zariski and the constructible topology. $\square$

If we restrict to pseudo-convergent sequences of algebraic type, the distance d_δ can be interpreted in a different way.

Proposition 8.9. *Let $E, F \subset K$ be pseudo-convergent sequences of algebraic type with breadth δ, and let u be an extension of v to $\overline{K}$. If $\beta \in \mathcal{L}_E^u$ and $\beta' \in \mathcal{L}_F^u$, then*

$$d_\delta(V_E, V_F) = \max\{d_u(\beta, \beta') - e^{-\delta}, 0\}.$$

Proof. If $d_u(\beta, \beta') \leq e^{-\delta}$, then the pseudo-limits of E and F coincide, and thus $V_E = V_F$ by Theorem 5.4; hence, $d_\delta(V_E, V_F) = 0$. On the other hand, if $d_u(\beta, \beta') > e^{-\delta}$ then $u(\beta - \beta') < \delta$ and thus, for large n,

$$v(s_n - t_n) = u(s_n - \beta + \beta - \beta' + \beta' - t_n) = u(\beta - \beta');$$

hence, $d_\delta(V_E, V_F) = d_u(\beta, \beta') - e^{-\delta}$, as claimed. Note that, in particular, $u(\beta - \beta')$ lies in Γ_v. $\square$

If V is a DVR, then $\mathcal{V} = \mathcal{V}(\bullet, \infty)$, so, in this case, $\mathcal{V}$ is an ultrametric space whose ultrametric distance is d_∞. On the other hand, if V is not discrete, it is not possible to unify the metrics d_δ in a single metric defined on the whole $\mathcal{V}$. We premise a lemma.

Lemma 8.10. *The closure of $\mathcal{V}(\bullet, \delta)$ in $\mathcal{V}$ is equal to $\bigcup_{\delta' \leq \delta} \mathcal{V}(\bullet, \delta')$.*

Proof. If V is discrete, then the statement is a tautology (see the line immediately after Definition 8.3). We assume henceforth that V is not discrete.

Let $E = \{s_n\}_{n\in\mathbb{N}}$ be a pseudo-convergent sequence with breadth $\delta' < \delta$; we want to show that it is in the closure of $\mathcal{V}(\bullet, \delta)$. By Proposition 8.6, $\mathcal{V}(\bullet, \delta')$ is contained in the closure of $\mathcal{V}_{\mathrm{alg}}(\bullet, \delta')$; hence, we can suppose that E is of algebraic type.

For each $n \in \mathbb{N}$, let E_n be a pseudo-convergent sequence with pseudo-limit s_n and breadth δ: since $\delta' < \delta$, by Proposition 6.6 V_E is the limit of V_{E_n} in the Zariski topology, and thus it belongs to the closure of $\mathcal{V}_{\mathrm{alg}}(\bullet, \delta')$, as claimed.

Conversely, suppose $\delta' > \delta$; we claim that if $E = \{s_n\}_{n\in\mathbb{N}}$ is pseudo-convergent sequence with breadth δ' then there is an open set containing V_E and disjoint from $\mathcal{V}(\bullet, \delta)$. Let $\gamma \in \Gamma_v$ such that $\delta' > \gamma > \delta$; then, there is an N such that $v(s_n - s_{n+1}) > \gamma$ for all $n \geq N$. Take $s := s_N$, and consider the open set $\Omega := \mathcal{V} \setminus \Omega(s, \gamma)$ (see Lemma 7.2). Then, $V_E \in \Omega$ since $w_E(X - s) = \delta'_N > \gamma$; on the other hand, if the breadth of $F = \{t_n\}_{n\in\mathbb{N}}$ is δ, then $w_E(X - s) \leq \delta$ by Remark 4.7(a). In particular, $V_F \in \Omega(s, \gamma)$ and so $V_F \notin \Omega$. Hence, V_E is not in the closure of $\mathcal{V}(\bullet, \delta)$. $\square$

Proposition 8.11. *Let V be a rank one non-discrete valuation domain. Suppose $\mathcal{V}$ is metrizable with a metric d. Then, for any $\delta \in \mathbb{R} \cup \{\infty\}$, the restriction of d to $\mathcal{V}(\bullet, \delta)$ is* not *equal to d_δ.*

Proof. If the restriction of d is equal to d_δ, then by Proposition 8.6 $\mathcal{V}(\bullet, \delta)$ would be complete with respect to d. However, this would imply that $\mathcal{V}(\bullet, \delta)$ is closed, against Lemma 8.10. $\square$

8.2 Fixed pseudo-limit

Throughout this section, let u be a fixed extension of v to $\overline{K}$. We wish to study the set of valuation domains V_E such that E has a prescribed pseudo-limit in $\overline{K}$ with respect to u.

Definition 8.12. Let $\beta \in \overline{K}$. We set

$$\mathcal{V}^u_{\mathrm{alg}}(\beta, \bullet) := \{V_E \in \mathcal{V} \mid \beta \in \mathcal{L}^u_E\}$$

To ease the notation, we set $\mathcal{V}^u_{\mathrm{alg}}(\beta, \bullet) = \mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$.

Equivalently, a valuation domain V_E is in $\mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$ if β is a center of $\mathcal{L}^u_E$, i.e., if $\mathcal{L}^u_E = B_u(\beta, e^{-\delta_E})$.

If V is a DVR, then $\mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$ reduces to the single element $W_\beta = \{\phi \in K(X) \mid \phi(\beta) \in V\}$ (see Remark 3.8), which corresponds to any Cauchy sequence $E \subset K$ converging to β.

We start by showing that each $\mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$ is closed in $\mathcal{V}$.

Proposition 8.13. *Let $\beta \in \overline{K}$, and let u be an extension of v to $\overline{K}$. Then, $\mathcal{V}_{\mathrm{alg}}(\beta, \bullet) := \mathcal{V}^u_{\mathrm{alg}}(\beta, \bullet)$ is closed in $\mathcal{V}$.*

Proof. If V is discrete, then $\mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$ has just one element (see the comments above). By Theorem 6.1 each point of $\mathcal{V}$ is closed, so the statement is true in this case. Henceforth, for the rest of the proof we assume that V is non discrete.

Let $V_E \notin \mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$. We distinguish two cases.

Suppose first that $E = \{s_n\}_{n\in\mathbb{N}}$ is of algebraic type, and let $\alpha \in \overline{K}$ be a pseudo-limit of E with respect to u. Since $\beta \notin \mathcal{L}_E \Leftrightarrow u(\alpha - \beta) < \delta_E$ (Lemma 2.4) it follows that there is $m \in \mathbb{N}$ such that $u(\alpha - \beta) < u(\alpha - s_m)$. Let $s = s_m$. Choose a $d \in K$ such that

$$u(\beta - \alpha) = u(\beta - s) < v(d) < u(\alpha - s) < \delta_E,$$

and let $\phi(X) := \frac{X-s}{d}$; we claim that $V_E \in B(\phi)$ but $B(\phi) \cap \mathcal{V}_{\text{alg}}(\beta, \bullet) = \emptyset$.

Indeed,

$$v(\phi(s_n)) = v\left(\frac{s_n - s}{d}\right) = v(s_n - s) - v(d) > 0$$

since $v(s_n - s) = u(s_n - \alpha + \alpha - s) = u(\alpha - s)$ for large n; hence $V_E \in B(\phi)$. On the other hand, if $F = \{t_n\}_{n\in\mathbb{N}}$ has pseudo-limit β, then $v(t_n - s) = u(t_n - \beta + \beta - s) = u(\beta - s)$ for large n and so

$$v(\phi(t_n)) = u(\beta - s) - v(d) < 0,$$

i.e., $V_F \notin B(\phi)$. The claim is proved.

Suppose now that $E = \{s_n\}_{n\in\mathbb{N}}$ is of transcendental type: then, $u(s_n - \beta)$ is definitively constant, say equal to λ. Then, $\lambda < \delta$, for otherwise β would be a pseudo-limit of E; hence, we can take a $d \in K$ such that $\lambda < v(d) < \delta$. Choose an N such that $u(s_N - \beta) = \lambda$ and such that $v(d) < \delta_N$, and define $\phi(X) := \frac{X - s_N}{d}$. Then, $v(\phi(s_n)) = \delta_N - v(d) > 0$ for $n > N$, and thus $V_E \in B(\phi)$. Suppose now $v(\phi(t)) \geq 0$. Then, $v(t - s_N) \geq v(d) > \lambda$; however, $v(t - s_N) = u(t - \beta + \beta - s_N)$, and since $u(\beta - s_N) = \lambda$ we must have $u(t - \beta) = \lambda$. In particular, there is no annulus C of center β such that $\phi(t) \in V$ for all $t \in C$; hence, by Proposition 6.4, $V_F \notin B(\phi)$ for every $V_F \in \mathcal{V}_{\text{alg}}(\beta, \bullet)$, i.e., $\mathcal{V}_{\text{alg}}(\beta, \bullet) \cap B(\phi) = \emptyset$. The claim is proved. $\square$

We now want to characterize the topology of $\mathcal{V}_{\text{alg}}(\beta, \bullet)$. Given $\beta \in \overline{K}$, let

$$\delta(\beta, K) := \sup\{u(\beta - x) \mid x \in K\};$$

then, $\delta(\beta, K)$ is linked to the quantity $d(\beta, K)$ (introduced at the beginning of Section 8) by the equality $d(\beta, K) = e^{-\delta(\beta,K)}$. Hence, by Theorem 5.4 and 8.1, there is a natural bijection

$$\begin{aligned} \Sigma_\beta \colon \mathcal{V}_{\text{alg}}(\beta, \bullet) &\longrightarrow (-\infty, \delta(\beta, K)] \\ V_E &\longmapsto \delta_E. \end{aligned} \tag{12}$$

To describe the needed topology on the interval $(-\infty, \delta(\beta, K)]$, we introduce the following definition.

Definition 8.14. Let $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$, with $a < b$, and let $\Lambda \subseteq \mathbb{R}$. The Λ-*upper limit topology* on $(a, b]$ is the topology generated by the sets $(\alpha, \lambda]$, for $\lambda \in \Lambda \cup \{\infty\}$ and $\alpha \in (a, b]$. We denote this space by $(a, b]_\Lambda$.

The Λ-upper limit topology is a variant of the upper limit topology (see e.g. [21, Counterexample 51]), and in fact coincides with it when $\Lambda = \mathbb{R}$.

Theorem 8.15. *Suppose V is not discrete, and let $\beta \in \overline{K}$ be a fixed element. The map Σ_β defined in* (12) *is a homeomorphism between $\mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$ (endowed with the Zariski topology) and $(-\infty, \delta(\beta, K)]_{\mathbb{Q}\Gamma_v}$.*

Proof. To shorten the notation, let $\mathcal{X} := (-\infty, \delta(\beta, K)]_{\mathbb{Q}\Gamma_v}$.

We start by showing that Σ_β is continuous. Clearly, $\Sigma_\beta^{-1}(\mathcal{X}) = \mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$ is open.

Suppose $\gamma \in \mathbb{Q}\Gamma_v$ satisfies $\gamma < \delta(\beta, K)$. Then, there is a $t \in K$ such that $u(t-\beta) > \gamma$; we claim that

$$\Sigma_\beta^{-1}((-\infty, \gamma]) = \Omega(t, \gamma) \cap \mathcal{V}_{\mathrm{alg}}(\beta, \bullet).$$

(Recall that $\Omega(t, \gamma) = \{V_E \in \mathcal{V} \mid w_E(X - t) \leq \gamma\}$: see Lemma 7.2). Indeed, let $E = \{s_n\}_{n \in \mathbb{N}}$ be a pseudo-convergent sequence having β as a pseudo-limit. If $\delta_E \leq \gamma$, then (since $u(\beta - t) > \gamma$)

$$w_E(X - t) = \lim_{n \to \infty} v(s_n - t) = \lim_{n \to \infty} u(s_n - \beta + \beta - t) = \delta_E$$

and so $V_E \in \Omega(t, \gamma)$. Conversely, if $V_E \in \Omega(t, \gamma) \cap \mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$ then $w_E(X - t) \leq \gamma$, and thus (using again $u(\beta - t) > \gamma$)

$$\delta_E = \lim_{n \to \infty} u(s_n - \beta) = \lim_{n \to \infty} u(s_n - t + t - \beta) = \lim_{n \to \infty} u(s_n - t) = w_E(X - t) \leq \gamma,$$

i.e., $\Sigma_\beta(V_E) \leq \gamma$.

By Lemma 7.2, $\Omega(t, \gamma)$ is open and closed in $\mathcal{V}$; hence, $\Sigma_\beta^{-1}((-\infty, \gamma])$ and $\Sigma_\beta^{-1}((\gamma, \delta(\beta, K)])$ are both open. If now $(a, b]$ is an arbitrary basic open set of $\mathcal{X}$, with $b \in \mathbb{Q}\Gamma$, then

$$\Sigma_\beta^{-1}((a, b]) = \Sigma_\beta^{-1}((-\infty, b]) \cap \left(\bigcup_{\substack{c \in \mathbb{Q}\Gamma_v \\ c > a}} \Sigma_\beta^{-1}((c, \delta(\beta, K)]) \right)$$

is open. Hence, Σ_β is continuous.

Let now ϕ be an arbitrary nonzero rational function over K, and for ease of notation let $B(\phi)$ denote the intersection $B(\phi) \cap \mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$. Suppose $\delta \in \Sigma_\beta(B(\phi))$, and let $E := \{s_n\}_{n \in \mathbb{N}}$ be a pseudo-convergent sequence of breadth δ having β as a pseudo-limit. By Proposition 6.4 there are $\theta_1, \theta_2 \in \mathbb{Q}\Gamma_v$ such that $\theta_2 < \delta \leq \theta_1$ and such that $v(\phi(t)) \geq 0$ for all $t \in \mathcal{C}(\beta, \theta_1, \theta_2)$. In particular, if $V_F \in \mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$, $F = \{t_n\}_{n \in \mathbb{N}}$, is such that $\Sigma_\beta(V_F) \in (\theta_1, \theta_2]$ we have that $t_n \in \mathcal{C}(\beta, \theta_1, \theta_2)$ for each $n \geq N$, for some $N \in \mathbb{N}$, so that $v(\phi(t_n)) \geq 0$ for each $n \geq N$, thus $\phi \in V_F$. Hence, $(\theta_1, \theta_2] \subseteq \Sigma_\beta(B(\phi))$, and thus $(\theta_1, \theta_2]$ is an open neighbourhood of δ in $\Sigma_\beta(B(\phi))$, which thus is open.

Hence, Σ_β is open, and thus Σ_β is a homeomorphism. □

When V is not discrete, we obtain a new proof of the non-compactness of $\mathcal{W}$, independent from Proposition 6.3.

Corollary 8.16. *The spaces $\mathcal{V}$ and $\mathcal{W}$ are not compact.*

Proof. If V is a DVR, then $\mathcal{V}$ is homeomorphic to $\widehat{K}$ (Theorem 6.1). In particular, it is not compact. The space $\mathcal{W}$ is not compact by Proposition 6.3.

Suppose that V is not discrete, and let $\beta \in \overline{K}$ be a fixed element. By Proposition 8.13, $\mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$ is closed in $\mathcal{V}$; hence if $\mathcal{V}$ were compact so would be $\mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$. By Theorem 8.15, it would follow that $\mathcal{X} := (-\infty, \delta(\beta, K)]_{\mathbb{Q}\Gamma_v}$ is compact. However, let $\gamma_1 > \gamma_2 > \cdots$ be a decreasing sequence of elements in $\mathbb{Q}\Gamma_v$, with $\delta(\beta, K) > \gamma_1$. Then, the family $(\gamma_1, \delta(\beta, K)], (\gamma_2, \gamma_1], \ldots, (\gamma_{n+1}, \gamma_n], \ldots$ is an open cover of $\mathcal{X}$ without finite subcovers: hence, $\mathcal{X}$ is not compact, and so neither is $\mathcal{V}$.

Let $\Psi : \mathcal{W} \longrightarrow \mathcal{V}$ be the map defined in Proposition 6.10. Since Ψ is continuous, if $\mathcal{W}$ were compact then so would be its image $\mathcal{V}_0$. Hence, as in the previous part of the proof, also $\mathcal{V}_0 \cap \mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$ would be compact; however, since $\Sigma_\beta(\mathcal{V}_0 \cap \mathcal{V}_{\mathrm{alg}}(\beta, \bullet)) = (-\infty, \delta(\beta, K)] \setminus \{+\infty\}$, we can use the same method as above (eventually substituting $(\gamma_1, +\infty]$ with $(\gamma_1, +\infty)$) to show that this set can't be compact. Hence, $\mathcal{W}$ is not compact, as claimed. □

We note that, when V is a DVR, $\widehat{K}$ (and thus $\mathcal{V}$) is locally compact if and only if the residue field of V is finite [5, Chapt. VI, §5, 1., Proposition 2]. We conjecture that the same happens when V is not discrete.

Proposition 8.17. *Let $\beta \in \overline{K}$, and let u be an extension of v to $\overline{K}$. Then, the Zariski and the constructible topology agree on $\mathcal{V}_{\mathrm{alg}}(\beta, \bullet) := \mathcal{V}^u_{\mathrm{alg}}(\beta, \bullet)$.*

Proof. It is enough to show that $B(\phi) \cap \mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$ is closed for every $\phi \in K(X)$. Suppose $\delta \in C := \Sigma_\beta(\mathcal{V}_{\mathrm{alg}}(\beta, \bullet) \setminus B(\phi))$ and let $V_E \in \mathcal{V}_{\mathrm{alg}}(\beta, \bullet) \setminus B(\phi)$: by Proposition 6.4 and Remark 6.5, there is an annulus $\mathcal{C} := \mathcal{C}(\beta, \theta_1, \theta_2)$ with $\theta_1, \theta_2 \in \mathbb{Q}\Gamma_v$, $\theta_1 < \delta \leq \theta_2$ and such that $\phi(t) \notin V$ for all $t \in \mathcal{C}$. Hence, $(\theta_1, \theta_2]$ is an open neighborhood of δ in $(-\infty, \delta(\beta, K)]_{\mathbb{Q}\Gamma_v}$ contained in C; thus, C is open and $B(\phi) \cap \mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$ is closed, being the complement of the image of C under the homeomorphism Σ_β^{-1} (see Theorem 8.15). □

To conclude, we study the metrizability of $\mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$ and $\mathcal{V}$. It is well-known [21, Counterexample 51(4)] that the upper limit topology is not metrizable, since it is separable but not second countable. Something similar happens for $(a, b]_\Lambda$.

Proposition 8.18. *Let Λ be a subset of $(a, b]$ that is dense in the Euclidean topology. The following are equivalent:*

(i) *Λ is countable;*
(ii) *$(a, b]_\Lambda$ is second-countable;*
(iii) *$(a, b]_\Lambda$ is metrizable;*
(iv) *$(a, b]_\Lambda$ is an ultrametric space.*

Proof. (iii) $\Longrightarrow$(ii) follows from the fact that $(a, b]_\Lambda$ is separable (since, for example, $\mathbb{Q} \cap (a, b]$ is dense in $(a, b]_\Lambda$); (iv) $\Longrightarrow$ (iii) is obvious.

(ii) $\Longrightarrow$ (i) Any basis of $(a, b]_\Lambda$ must contain an open set of the form $(\alpha, \lambda]$, for each $\lambda \in \Lambda$ (and some $\alpha \in (-\infty, \lambda)$). Hence, if $(a, b]_\Lambda$ is second-countable then Λ must be countable.

(i) $\Longrightarrow$ (iv) Suppose that Λ is countable, and fix an enumeration $\{\lambda_1, \lambda_2, \ldots\}$ of Λ. Let $r : \Lambda \longrightarrow \mathbb{R}$ be the map sending λ_i to $1/i$; then, for each $x, y \in (a, b]$ we set

$$d(x,y) := \begin{cases} \max\{r(\lambda) \mid \lambda \in [\min(x,y), \max(x,y)) \cap \Lambda\}, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$$

We claim that d is a metric on $(a, b]$ whose topology is exactly $(a, b]_\Lambda$.

Note first that d is well-defined and nonnegative; it is also clear from the definition (and the fact that Λ is dense in $\mathbb{R}$) that $d(x, y) = 0$ if and only if $x = y$ and that $d(x, y) = d(y, x)$. Let now $x, y, z \in (a, b]$, and suppose without loss of generality that $x \leq y$. If $z \leq x$, then $[z, y) \supseteq [x, y)$, and thus $d(x, y) \leq d(y, z)$; in the same way, if $y \leq z$ then $[x, z) \supseteq [x, y)$ and $d(x, y) \leq d(x, z)$. If $x \leq z \leq y$, then $[x, y) = [x, z) \cup [z, y)$; hence, $d(x, y) = \max\{d(x, z), d(y, z)\}$. In all cases, we have $d(x, y) \leq \max\{d(x, z), d(y, z)\}$, and thus d induces an ultrametric space.

Let now $x \in \Lambda \subseteq (a, b]$ and $\rho \in \mathbb{R}$ be positive; we claim that the open ball $B = B_d(x, \rho) = \{t \in (a, b] \mid d(x, t) < \rho\}$ is equal to $(y, z]$, where

$$y := \max\{\lambda \in \Lambda \cap (-\infty, x) \mid r(\lambda) \geq \rho\},$$
$$z := \min\{\lambda \in \Lambda \cap (x, +\infty) \mid r(\lambda) \geq \rho\}$$

(with the convention $\max \emptyset = a$ and $\min \emptyset = b$). Note that since $\rho > 0$, there are only a finite number of λ with $r(\lambda) \geq \rho$; in particular, $y, z \in \Lambda$ and by definition, $y < x < z$.

Let $t \in (a, b]$. If $t < y$, then $r(\lambda) \geq \rho$ for some $\lambda \in (t, x) \cap \Lambda$, and thus $d(t, x) \geq \rho$, and so $t \notin B$; in the same way, if $y < t < x$, then $r(\lambda) < \rho$ for every $\lambda \in (t, x) \cap \Lambda$, and thus $t \in B$. Symmetrically, if $x < t < z$ then $t \in B$, while if $z < t$ then $t \notin B$. We thus need to analyze the cases $t = y$ and $t = z$.

By definition,

$$d(x, z) = \max\{r(\lambda) \mid \lambda \in [x, z) \cap \Lambda\};$$

since by definition $r(\lambda) < \rho$ for every $\lambda \in [x, z) \cap \Lambda$, we have $d(x, z) < \rho$ and $z \in B_d(x, r)$.

Since $y \in \Lambda$, we have $r(y) \geq \rho$. Thus,

$$d(x, y) = \max\{r(\lambda) \mid \lambda \in [y, x) \cap \Lambda\} \geq r(y) \geq \rho$$

and $y \notin B_d(x, \rho)$. Thus, $B_d(x, \rho) = (y, z]$ as claimed; therefore, $B_d(x, \rho)$ is open in $(a, b]_\Lambda$.

The family of the intervals $(y, z]$, as z ranges in Λ and y in $(a, b]$, is a basis of $(a, b]_\Lambda$; therefore, the topology induced by d on $(a, b]$ is exactly the Λ-upper limit topology. Hence, $(a, b]_\Lambda$ is an ultrametric space, as claimed. $\square$

As a consequence, we obtain that in many cases $\mathcal{V}$ is not metrizable, in contrast with with Proposition 7.4.

Corollary 8.19. *Let V be a valuation ring with uncountable value group. Then, $\mathcal{V}$ and $\mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$ are not metrizable.*

Proof. If $\mathcal{V}$ were metrizable, so would be $\mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$, against Theorem 8.15 and Proposition 8.18 (note that, if the value group of V is uncountable, in particular V is not discrete). Similarly, if $\mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$ were metrizable, so would be $\mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$, endowed with the constructible topology. Since the Zariski and the constructible topology agree on $\mathcal{V}_{\mathrm{alg}}(\beta, \bullet)$ (Proposition 8.17), this is again impossible. $\square$

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