Extending valuations to the field of rational functions using pseudo-monotone sequences

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For a valuation domain V of rank one and quotient field K, Ostrowski introduced in 1935 the notion of pseudo-convergent sequence and proved his *Fundalmentalsatz*, which describes all the possible rank one extensions of V to $K(X)$. In this paper, we generalize Ostrowski's result for a general valuation domain V by means of the notion of pseudo-monotone sequence, introduced in 2010 by Chabert, which allows us to build extensions of V to $K(X)$, in the same spirit of Ostrowski. We show that this construction describes, when the V-adic completion $\widehat{K}$ is algebraically closed, all extensions of V to $K(X)$, regardless of the rank of the involved valuations, thus generalizing Ostrowski's Fundamentalsatz.

Keywords: pseudo-convergent sequence, pseudo-limit, pseudo-monotone sequence, monomial valuation, extension of valuations.

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1 Introduction

Throughout the paper, V will denote a valuation domain with quotient field K and maximal ideal M, v will denote its valuation and Γ_v its value group. We also fix an algebraic closure $\overline{K}$ of K. The study of extensions of V is one of the central parts of valuation theory, which naturally splits into the study of algebraic and purely transcendental extensions. The former can be considered a generalization of the fundamental problems of algebraic number theory, and is well-studied through the same concept of

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inertia, decomposition and ramification (in what is known as *ramification theory*). The latter – which is essentially the study of extensions of V to function fields – is less well understood, but plays a main role in several facets and applications of the theory (see [10] and the references therein). The first step of this problem is to classify all the extensions of V to $K(X)$.

In the case V has rank one, there are two classical approaches to this problem: the most famous one, due to MacLane, uses key polynomials and augmented valuation and works for arbitrary fields K, but requires the valuation ring to be discrete [14]; it has been recently generalized by Vaquiè in [22] for general valuation domains. The second approach, due to Ostrowski, "makes no discreteness assumptions" but "requires an elaborate construction to obtain values of $\overline{K}$ from those of K", as MacLane acknowledged in his paper [14, p. 380]. More precisely, Ostrowski showed that, for a given extension W of V to $K(X)$, there exists a *pseudo-convergent sequence* $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ (we refer to §2.3 for the definition) such that the valuation w associated to W is given by the real limit $w(\phi) = \lim_{n\to\infty} v(\phi(s_n))$, for all $\phi \in K(X)$; for its importance, Ostrowski called this result *Fundamentalsatz* [16, §11, IX, p. 378]. To our knowledge, Ostrowski's Fundamentalsatz seems to have been mostly forgotten, even if pseudo-convergent sequences have enjoyed some success: for example, Kaplansky used them to characterize immediate extensions of a valued field and maximal fields in [9], and they are linked to the recently introduced notion of approximation type (see [12]).

In generalizing Ostrowski's Fundamentalsatz, we realized that when dealing with the general case (i.e., when the rank of V or of the extension of V to $K(X)$ is not one), pseudo-convergent sequences are not enough to construct all extensions of V to $K(X)$ (see Example 6.4): for this reason, we use the more general notion of *pseudo-monotone sequences*, introduced by Chabert in 2010 [5] to study the so-called *polynomial closure* in the context of rings of integer-valued polynomials. We recall that, given a subset S of V, the ring of *integer-valued polynomials* over S is classically defined as $\mathrm{Int}(S, V) = \{f \in K[X] \mid f(S) \subseteq V\}$, and the polynomial closure of S is the largest subset $\overline{S} \subseteq V$ such that $\mathrm{Int}(S, V) = \mathrm{Int}(\overline{S}, V)$. Since a polynomial induces a continuous function with respect to the v-adic topology, it follows that the v-adic closure of S is contained in the polynomial closure of S; if V is a DVR with finite residue field, McQuillan proved that the two closures are the same for each subset S of V (see [15, Lemma 2]). On the other hand, if V is discrete or its residue field is infinite, Chabert gave examples of subsets S of V such that the topological closure of S is strictly contained in its polynomial closure ([5, Remark 3.3]); however, he proved that the polynomial closure is the closure operator associated to a topology on K. This result was obtained by describing $\overline{S}$ through the set of pseudo-limits of all pseudo-monotone sequences contained in S.

In this paper, continuing our earlier work in [19], we describe the extensions of V to $K(X)$ by means of pseudo-monotone sequences of K, generalizing a natural construction of Loper and Werner, who were interested in studying when the ring of integer-valued polynomials over a pseudo-convergent sequence is a Prüfer domain [13]. More precisely, we associate to every pseudo-monotone sequence $E = \{s_\nu\}_{\nu\in\Lambda} \subset K$ (see §2.3 for the definition) the valuation domain

$$V_E = \{\phi \in K(X) \mid \phi(s_\nu) \in V, \text{ for all sufficiently large } \nu \in \Lambda\}.$$

We first study the properties of V_E in relation with the properties of E; subsequently, we analyze when and how it is possible to associate to an arbitrary extension a pseudo-monotone sequence. Our main result (Theorem 8.2) proves that every extension of V to $K(X)$ can be realized in this way if and only if the v-adic completion $\widehat{K}$ of K is algebraically closed. In particular, the statement holds if K is algebraically closed, giving a generalization of Ostrowski's result. Moreover, our result shows that, under the same condition, every extension of V to $K(X)$ which is not immediate is a monomial valuation, a natural way of constructing extensions to the field of rational functions (see §2.1).

The structure of the paper is as follows. In Section 2, after settling the notation used throughout the paper, the notion of monomial valuation and divisorial ideal, we give the definition of pseudo-monotone sequence in a general valued field (K, v) (Chabert's original definition was given only for a rank one valuation, but it easily extends to the general case). We then introduce the notions of pseudo-limit, breadth ideal and gauge separately for the three different types of pseudo-monotone sequences: pseudo-convergent sequences (§2.3.1), pseudo-divergent sequences (§2.3.2) and pseudo-stationary sequences (§2.3.3). In the last part of that section, we characterize pseudo-limits and breadth ideals of pseudo-monotone sequences according to their type (Lemmas 2.5 and 2.6).

In Section 3 we show that the sequence of values of the images under a rational function of a pseudo-monotone sequence is definitively monotone (Proposition 3.2); the result is accomplished by introducing the notion of *dominating degree* of a rational function $\phi \in K(X)$ with respect to a pseudo-monotone sequence $E \subset K$ (Definition 3.1), which roughly speaking counts the number of roots of ϕ in $\overline{K}$ which are pseudo-limits of E. Through this result, we show that, for each pseudo-monotone sequence E, the ring V_E is a valuation domain of $K(X)$ extending V (Theorem 3.4). We then describe the main properties of V_E (residue field, value group and associated valuation) in Proposition 3.7, and show that the image of a pseudo-convergent or a pseudo-divergent sequence under a rational function is definitively either pseudo-convergent or pseudo-divergent (Proposition 3.8), improving the analogous result of Ostrowski [16, III, §64, p. 371] on images of pseudo-convergent sequences under polynomial mappings.

Sections 4 and 5 deal with prime ideals and localizations of a valuation domain V_E: more precisely, in the former section we investigate the behaviour of a pseudo-monotone sequence E under a valuation induced by a localization of V, and in the latter section we use those results to study the rank of V_E in relation with the breadth ideal of E.

In Section 6, we associate to each extension W a subset $\mathcal{L}(W)$ of K (which correspond to the notion of pseudo-limit of a pseudo-monotone sequence) and show that if K is algebraically closed then $\mathcal{L}(W)$ (if nonempty) uniquely determines W (Proposition 6.5). In Section 6.1, given a pseudo-convergent sequence $E \subset K$ of algebraic type, we study the dominating degree of the polynomials of minimal degree having at least one root among the pseudo-limits of E, and describe this quantity in terms of the defect of the associated extensions (Proposition 6.9).

In Section 7, we use the results of the previous section to completely describe (for any field K) when two different pseudo-monotone sequences of K give raise to the same associated extension of V to $K(X)$. Subsequently, in Section 8 we give the proof of the aforementioned main Theorem 8.2.

In the final Section 9, we illustrate the different containments which may occur among the valuation domains V_E of $K(X)$. Moreover, Theorem 9.4 provides a modern proof of Ostrowski's Fundamentalsatz.

2 Background and notation

For an extension U or W of V to a field F containing K, we denote the associated valuation with the corresponding lower case letter (i.e., u or w). For a prime ideal P of V, we let V_P be the localization of V at P, which is a valuation domain of K. We recall that an extension $V \subset U$ is *immediate* if U and V have the same value group and same residue field. We denote by $\widehat{K}$ and $\widehat{V}$, respectively, the *completion* of K and V with respect to the topology induced by the valuation v. The elements of $\widehat{K}$ can be constructed as limits of Cauchy sequences $\{a_\nu\}_{\nu\in\Lambda}$, where Λ is a well-ordered set; Λ is not necessarily countable, but can be considered of cardinality equal to the cofinality of the ordered set Γ_v. See for example [7, Section 2.4] for the details of the construction. For a sequence $\{s_\nu\}_{\nu\in\Lambda}$ of elements in K, the set of indexes Λ will always be a well-ordered set without a maximum.

2.1 Monomial valuations

We recall the definition of monomial valuations, a standard way of extending a valuation v of K to $K(X)$.

Definition 2.1. Let Γ be a totally ordered group containing Γ_v, and let $\alpha \in K$ and $\delta \in \Gamma$. The *monomial valuation* $v_{\alpha,\delta}$ is defined for every polynomial $f(X) = a_0 + a_1(X - \alpha) + \ldots + a_n(X-\alpha)^n \in K[X]$ in the following way:

$$v_{\alpha,\delta}(f) = \inf\{v(a_i) + i\delta \mid i = 0, \ldots, n\}.$$

It is well-known that $v_{\alpha,\delta}$ naturally extends to a valuation on $K(X)$ [4, Chapt. VI, §. 10, Lemme 1]. We denote by $V_{\alpha,\delta}$ the associated valuation domain of $K(X)$. For example, the Gaussian extension $v_G = v_{0,0}$ of v, defined as $v_G(\sum_{i\geq 0} a_i X^i) = \inf_i\{v(a_i)\}$, is a monomial valuation. In general, $v_{\alpha,\delta}$ is residually transcendental over v (i.e., the residue field of $V_{\alpha,\delta}$ is transcendental over the residue field of V) if and only if δ is torsion over Γ_v [18, Lemma 3.5]. Furthermore, every residually transcendental extension of V can be written as $W \cap K(X)$, where W is a monomial valuation domain of $\overline{K}(X)$ with respect to an extension w of v to $\overline{K}$ ([2, 3]).

2.2 Divisorial ideals

Let V be a valuation domain with maximal ideal M, and let $\mathcal{F}(V)$ be the set of fractional ideals of V. The *v-operation* (or *divisorial closure*) on V is the map sending each $I \in \mathcal{F}(V)$ to the ideal I^v equal to the intersection of all principal fractional ideals containing it; equivalently, $I^v = (V : (V : I))$, where, for a fractional ideal I of V, we set $(V : I) = \{x \in K \mid xI \subseteq V\}$. If $I = I^v$, we say that I is a *divisorial ideal*.

If the maximal ideal M of V is principal, then each fractional ideal I of V is divisorial; on the other hand, if M is not principal then (see for example [8, Exercise 12, p. 431])

$$I^v = \begin{cases} cV, & \text{if } I = cM \text{ for some } c \in K \\ I, & \text{otherwise.} \end{cases}$$

We say that I is *strictly divisorial* if I is equal to the intersection of all principal fractional ideals *properly* containing it; in particular, each strictly divisorial ideal is divisorial. We now characterize these ideals.

Lemma 2.2. *I is not strictly divisorial if and only if $I = cM$ for some $c \in K$.*

Proof. Suppose first that I is not principal. Then, I is strictly divisorial if and only if it is divisorial; furthermore, I is not divisorial if and only if $I = cM$ for some $c \in K$ and M is not principal, by the above remark. Hence, the claim holds in this case.

Suppose that $I = c'V$ is principal: if also $I = cM$ for some c then cV is the minimal principal ideal properly containing I, and I is not strictly divisorial. Conversely, if I is not strictly divisorial then there is a minimal principal ideal $c'V$ properly containing I; this implies that c'/c is the generator of the maximal ideal of V, and so $I = cM$. $\square$

2.3 Pseudo-monotone sequences

The central concept of the paper is the following, introduced by Chabert in [5] as a generalization of the classical notion of pseudo-convergent sequence given by Ostrowski in [16].

Definition 2.3. Let $E = \{s_\nu\}_{\nu \in \Lambda} \subset K$ be a sequence. We say that the sequence E is:

- *pseudo-convergent* if $v(s_\rho - s_\nu) < v(s_\sigma - s_\rho)$ for all $\nu < \rho < \sigma \in \Lambda$;
- *pseudo-divergent* if $v(s_\rho - s_\nu) > v(s_\sigma - s_\rho)$ for all $\nu < \rho < \sigma \in \Lambda$;
- *pseudo-stationary* if $v(s_\nu - s_\mu) = v(s_{\nu'} - s_{\mu'})$ for all $\nu \neq \mu \in \Lambda$, $\nu' \neq \mu' \in \Lambda$.

If E satisfies any of these definitions, we say that E is a *pseudo-monotone sequence*. We say that E is *strictly pseudo-monotone* if E is either pseudo-convergent or pseudo-divergent. If E and F are two pseudo-monotone sequences that are either both pseudo-convergent, both pseudo-divergent or both pseudo-stationary we say that E and F are *of the same kind*.

We note that Ostrowski's and Chabert's original definitions only required the above condition to be valid only for all ν large enough. Instead, we adopt Kaplansky's convention that the condition is valid for all ν, both since it is not restrictive for our purposes (see Definition 3.3) and in view of the following remark. If $E = \{s_\nu\}_{\nu\in\Lambda}$ is a sequence in K and $E' = \{s_\nu\}_{\nu\geq N}$ is pseudo-monotone for some $N \in \Lambda$, we say that E is *definitively pseudo-monotone* (and analogously for definitively pseudo-convergent, pseudo-divergent and pseudo-stationary).

Remark 2.4. Strictly pseudo-monotone sequences are "rigid", in the sense that, given a set E, there is at most one way to index E to make it pseudo-monotone. Indeed, if the indexing $\{s_\nu\}_{\nu\in\Lambda}$ makes E pseudo-convergent, then the equality $v(s_\nu - s_\mu) = v(s_\nu - s_{\mu'})$ (for $\mu \neq \mu'$) implies that both μ and μ' are greater than ν; thus, the elements of E that appear before s_ν are exactly the t such that $v(s_\nu - t) \neq v(s_\nu - t')$ for all $t \neq t'$, and this condition depends only on the set E. In the same way, if E is pseudo-divergent, then the elements of E appearing after s_ν are the t such that $v(s_\nu - t) \neq v(s_\nu - t')$ for all $t \neq t'$. In particular, if $E = \{s_\nu\}_{\nu\in\Lambda}$ and $F = \{t_\nu\}_{\nu\in\Lambda}$ are two strictly pseudo-monotone sequences that are equal as sets, then $s_\nu = t_\nu$ for every $\nu \in \Lambda$.

On the other hand, pseudo-stationary sequences are "flexible": any permutation of $E = \{s_\nu\}_{\nu\in\Lambda}$ is again pseudo-stationary. For this reason, it may be more apt to call them "pseudo-stationary *sets*", but we will continue to treat them as sequences for analogy with the strictly pseudo-monotone case.

In this paper, we shall treat pseudo-monotone sequences in a general framework in order to build extensions of the valuation domain V to the field of rational functions $K(X)$, and to give theorems valid for all kind of such sequences. However, there are slight differences in how the main concepts concerning pseudo-monotone sequences (for example the breadth ideal, the pseudo-limit and the gauge) are defined in each of the three cases; hence, we shall describe them separately.

2.3.1 Pseudo-convergent sequences

Let $E = \{s_\nu\}_{\nu\in\Lambda}$ be a pseudo-convergent sequence in K. Then, if ν is fixed, the value $v(s_\rho - s_\nu)$, for $\rho > \nu$, does not depend on ρ. We denote by $\delta_\nu \in \Gamma_v$ this value; the sequence $\{\delta_\nu\}_{\nu\in\Lambda}$ (which, by definition, is a strictly increasing sequence in Γ_v) is called the *gauge* of E.

The *breadth ideal* $\mathrm{Br}(E)$ of E is the set

$$\mathrm{Br}(E) := \{x \in K \mid v(x) > \delta_\nu \text{ for all } \nu \in \Lambda\};$$

this set always a fractional ideal of K. If $c_\nu = s_\rho - s_\nu$, for some $\rho > \nu$, then $\mathrm{Br}(E) = \bigcap_{\nu\in\Lambda} c_\nu V$. If $\mathrm{Br}(E)$ is a principal ideal, say generated by an element $c \in K$, then δ_ν converges to an element $\delta \in \Gamma_v$ (and, clearly, $v(c) = \delta$). When this happens, we call δ the *breadth* of E. Note, however, that the breadth of a pseudo-convergent sequence may not always be defined; if V has rank 1 (that is, if Γ_v can be embedded as a totally ordered group into $\mathbb{R}$), then δ_ν always converges to an element $\delta \in \mathbb{R}$, which may not belong to Γ_v. See [19] and [18, Lemma 2.3] for this case.

An element $\alpha \in K$ is a *pseudo-limit* of E if $v(\alpha - s_\nu) = \delta_\nu$ for all $\nu \in \Lambda$ or, equivalently, if $v(\alpha - s_\nu) < v(\alpha - s_\rho)$ for all $\nu < \rho \in \Lambda$. It also suffices that these conditions hold only for $\nu \geq N$, for some $N \in \Lambda$. If the gauge $\{\delta_\nu\}_{\nu\in\Lambda}$ is cofinal in Γ_v (or, equivalently, if E is a Cauchy sequence), then it is well-known that E converges to a unique pseudo-limit α in the completion $\widehat{K}$, which in this case is called simply *limit*.

Following Kaplansky [9], we say that E is of *transcendental type* if $v(f(s_\nu))$ eventually stabilizes for every $f \in K[X]$; on the other hand, if $v(f(s_\nu))$ is eventually increasing for some $f \in K[X]$, we say that E is of *algebraic type*. As we have already remarked in [19], it follows from the work of Kaplansky in [9] that a pseudo-convergent sequence $E \subset K$ is of algebraic type if and only if E admits pseudo-limits in $\overline{K}$, with respect to some extension u of v. Note that any pseudo-convergent sequences satisfies either one of these two conditions, because the image of a pseudo-convergent sequence by a polynomial is a definitively pseudo-convergent sequence (see [16, III, §64, p. 371] or Proposition 3.8 below).

2.3.2 Pseudo-divergent sequences

Let $E = \{s_\nu\}_{\nu\in\Lambda}$ be a pseudo-divergent sequence in K. Symmetrically to case of pseudo-convergent sequences, for a fixed ν, we have that $v(s_\rho - s_\nu)$ is constant for all $\rho < \nu$; if ν is not the minimum of Λ, we denote by $\delta_\nu \in \Gamma_v$ this value. The sequence $\{\delta_\nu\}_{\nu\in\Lambda}$ is a strictly decreasing sequence in Γ_v, called the *gauge* of E.

The *breadth ideal* $\mathrm{Br}(E)$ of E is the set

$$\mathrm{Br}(E) := \{x \in K \mid v(x) > \delta_\nu \text{ for some } \nu \in \Lambda\};$$

this set is a fractional ideal of K if and only if the gauge of E is bounded from below, while otherwise $\mathrm{Br}(E) = K$. In particular, unlike in the pseudo-convergent case, $\mathrm{Br}(E)$ may not be a fractional ideal. If for each non-minimal $\nu \in \Lambda$ we set $c_\nu = s_\rho - s_\nu$, for some $\rho < \nu$, then $\mathrm{Br}(E) = \bigcup_{\nu\in\Lambda} c_\nu V$. Contrary to the case of a pseudo-convergent sequence, it is easily seen that the breadth ideal of a pseudo-divergent sequence is never a principal ideal. However, if $\delta_\nu \searrow \delta$, for some $\delta \in \Gamma_v$, then $\mathrm{Br}(E) = \{x \in K \mid v(x) > c\} = cM$, where $c \in K$ has value δ. As in the case of a pseudo-convergent sequence, when this condition holds we call δ the *breadth* of F.

An element $\alpha \in K$ is a *pseudo-limit* of E if $v(\alpha - s_\nu) = \delta_\nu$ for all (sufficiently large) $\nu \in \Lambda$ or, equivalently, if $v(\alpha - s_\nu) > v(\alpha - s_\rho)$ for all (sufficiently large) $\nu < \rho \in \Lambda$. Every element of E is a pseudo-limit of E: see [18, §2.1.3] and Lemma 2.5 below.

2.3.3 Pseudo-stationary sequences

Let $E = \{s_\nu\}_{\nu\in\Lambda}$ be a pseudo-stationary sequence in K. Note that the residue field of V is necessarily infinite (see [18, §2.1.2]). The element $\delta = v(s_\nu - s_\mu) \in \Gamma_v$, for $\nu \neq \mu$, is called the *breadth* of E. In analogy with pseudo-convergent and pseudo-divergent sequences, we define the *gauge* of E to be the constant sequence $\{\delta_\nu = \delta\}_{\nu\in\Lambda}$.

The *breadth ideal* $\mathrm{Br}(E)$ of E is the set

$$\mathrm{Br}(E) := \{x \in K \mid v(x) \geq \delta\};$$

this set is always a principal fractional ideal of K, generated by any $c \in K$ whose value is δ. In particular, we can take $c = s_{\nu'} - s_\nu$ for any $\nu' \neq \nu$.

An element $\alpha \in K$ is a *pseudo-limit* of E if $v(\alpha - s_\nu) = \delta$ for all sufficiently large $\nu \in \Lambda$ or, equivalently, if $v(\alpha - s_\nu) = \delta$ for all but at most one $\nu \in \Lambda$. As in the pseudo-divergent case, every element of E is a pseudo-limit of E.

2.4 Pseudo-limits and the breadth ideal

In general, if $E \subset K$ is a pseudo-monotone sequence, we denote the set of pseudo-limits of E in K by $\mathcal{L}_E$ and the breadth ideal by $\mathrm{Br}(E)$ (or $\mathcal{L}_E^v$ and $\mathrm{Br}_v(E)$, respectively, if we need to underline the valuation). We will constantly use the following trivial remark: if u is an extension of v to an overfield F of K, then E is a pseudo-monotone sequence in the valued field (F, u); in particular, $\mathcal{L}_F^u$ will denote the set of pseudo-limits of E in the valued field (F, u). We use the notations $\mathcal{L}_E$ and $\mathrm{Br}(E)$ also in the case E is only definitively pseudo-monotone.

The first part of the next result generalizes the classical result of Kaplansky for pseudo-convergent sequences ([9, Lemma 3]) to pseudo-monotone sequences. The proof is actually the same, but for the sake of the reader we give it here.

Lemma 2.5. *Let $E = \{s_\nu\}_{\nu \in \Lambda} \subset K$ be a pseudo-monotone sequence and let $\alpha \in K$ be a pseudo-limit of E. Then the set of pseudo-limits $\mathcal{L}_E$ of E is equal to $\alpha + \mathrm{Br}(E)$. Moreover, $E \cap \mathcal{L}_E = \emptyset$ if E is a pseudo-convergent sequence and $E \subset \mathcal{L}_E$ if E is either pseudo-divergent or pseudo-stationary.*

Proof. Let $\beta = \alpha + x$, for some $x \in \mathrm{Br}(E)$. If E is either pseudo-convergent or pseudo-divergent, then it is easy to see that

$$v(\beta - s_\nu) = v(\alpha - s_\nu + x) = v(\alpha - s_\nu) = \delta_\nu$$

so that β is a pseudo-limit of E. If E is pseudo-stationary then we have $v(\beta - s_\nu) \geq \delta = v(s_\nu - s_\mu) = v(s_\nu - \beta + \beta - s_\mu)$ and therefore for at most one $\nu \in \Lambda$ we may have the strict inequality $v(\beta - s_\nu) > \delta$. So, also in this case β is a pseudo-limit of E.

Conversely, if β is a pseudo-limit of E, then $v(\alpha - \beta) = v(\alpha - s_\nu + s_\nu - \beta) \geq \delta_\nu$, so that $\alpha - \beta \in \mathrm{Br}(E)$, as we wanted to show.

We prove the last claim. If E is a pseudo-convergent sequence then it is clear (both if E is of algebraic type or of transcendental type). If the sequence E is either pseudo-divergent or pseudo-stationary, the claim is proved in [18, §2.1.2 & §2.1.3]. □

In particular, since pseudo-divergent and pseudo-stationary sequences always admit a pseudo-limit in K, in these cases there is no analogue of the notion of pseudo-convergent sequences of transcendental type.

The following result characterizes which fractional ideals of V are breadth ideals for some pseudo-monotone sequence E of K, and which cosets are the set of pseudo-limits for some pseudo-monotone sequence.

Lemma 2.6. *Let I be a fractional ideal of V and let $\alpha \in K$; let $\mathcal{L} = \alpha + I$.*

(a) *$\mathcal{L} = \mathcal{L}_E$ for some pseudo-convergent sequence E if and only if I is strictly divisorial; in particular, if the maximal ideal of V is not principal this happens if and only if I is divisorial.*

(b) *$\mathcal{L} = \mathcal{L}_E$ for some pseudo-divergent sequence if and only if I is not principal.*

(c) *If V/M is infinite, $\mathcal{L} = \mathcal{L}_E$ for some pseudo-stationary sequence if and only if I is principal.*

Proof. It is easily seen that, if $\mathcal{L}_E \neq \emptyset$ for some pseudo-monotone sequence $E = \{s_\nu\}_{\nu\in\Lambda}$, the set $\alpha + \mathcal{L}_E$ is the set of pseudo-limits of $\alpha + E = \{\alpha + s_\nu\}_{\nu\in\Lambda}$; hence, it is enough to prove the claims for $\alpha = 0$. Furthermore, by Lemma 2.5, under this hypothesis we have $\mathcal{L}_E = \mathrm{Br}(E)$, and thus we only need to find which ideals are breadth ideals.

If $I = \mathrm{Br}(E)$ for some pseudo-convergent $E = \{s_\nu\}_{\nu\in\Lambda}$, for each ν let $c_\nu = s_\rho - s_\nu$, for some $\rho > \nu$; then $I = \bigcap_\nu c_\nu V$, and each $c_\nu V$ properly contains I. Therefore I is a strictly divisorial ideal. Conversely, if $I = \bigcap_{a\in A} aV$, where for each $a \in A$ we have $I \subsetneq aV$, we can take a well-ordered subset $\{a_\nu\}_{\nu\in\Lambda}$ such that $I = \bigcap_\nu a_\nu V$ and $a_\rho V \subsetneq a_\nu V$ for all $\rho > \nu$; then, $\{a_\nu\}_{\nu\in\Lambda}$ is a pseudo-convergent sequence having 0 as a pseudo-limit and breadth ideal I. The last remark follows from Lemma 2.2.

Likewise, if $I = \mathrm{Br}(E)$ for some pseudo-divergent $E = \{s_\nu\}_{\nu\in\Lambda}$, for each ν let $c_\nu = s_\rho - s_\nu$, for some $\rho < \nu$; then $I = \bigcup_\nu c_\nu V$, while if I is not principal we can find a well-ordered sequence $E = \{a_\nu\}_{\nu\in\Lambda} \subset V$ which generates I and such that $a_\nu V \subset a_\rho V$ for every $\nu < \rho$, so that E is a pseudo-divergent sequence and I is its breadth ideal.

If $I = \mathrm{Br}(E)$ for some pseudo-stationary sequence $E = \{s_\nu\}_{\nu\in\Lambda}$, then $I = (s_\nu - s_\mu)V$, for any $\nu \neq \mu$; conversely, if $I = cV$ then we can find a well-ordered set $E = \{s_\nu\}_{\nu\in\Lambda}$ of distinct elements of valuation $v(c)$ whose cosets modulo cM are different (because the residue field of V is infinite); then, E is pseudo-stationary with breadth ideal E. □

3 Valuation domains associated to pseudo-monotone sequences

Let $\phi \in K(X)$ be a rational function, and let $S = \{\alpha_1, \ldots, \alpha_k\}$ be a subset of the multiset of critical points of ϕ in $\overline{K}$. The *weighted sum* of S is the sum $\sum_{\alpha_i\in S} \epsilon_i$, where $\epsilon_i = 1$ if α_i is a zero of ϕ and $\epsilon_i = -1$ if α_i is a pole of ϕ. The *S-part* of ϕ is the rational function $\phi_S(X) = \prod_{\alpha_i\in S}(X - \alpha_i)^{\epsilon_i}$, where ϵ_i is as above.

The following definition generalizes [19, Definition 3.4] to pseudo-monotone sequences.

Definition 3.1. Let $E = \{s_\nu\}_{\nu\in\Lambda}$ be a pseudo-monotone sequence in K, let u be an extension of v to $\overline{K}$ and let $\phi \in K(X)$. The *dominating degree* $\mathrm{degdom}_{E,u}(\phi)$ of ϕ with respect to E and u is the weighted sum of the critical points of $\phi(X)$ which are pseudo-limits of E with respect to u.

The next proposition is a generalization to pseudo-monotone sequences of [19, Theorem 3.2]; in particular, it shows that the dominating degree does not depend on the chosen extension of v to $\overline{K}$.

Proposition 3.2. *Let $E = \{s_\nu\}_{\nu\in\Lambda} \subset K$ be a pseudo-monotone sequence of gauge $\{\delta_\nu\}_{\nu\in\Lambda}$, and let $\phi \in K(X)$. Let u be an extension of v to $\overline{K}$ and let $\lambda = \mathrm{degdom}_{E,u}(\phi)$. Then there exist $\gamma \in \Gamma_v$ and $\nu_0 \in \Lambda$ such that for each $\nu \geq \nu_0$ we have*

$$v(\phi(s_\nu)) = \lambda\delta_\nu + \gamma.$$

Furthermore, if $\beta \in \overline{K}$ is a pseudo-limit of E with respect to u, then $\gamma = u\left(\frac{\phi}{\phi_S}(\beta)\right)$, where S is the set of critical points of ϕ which are pseudo-limits of E with respect to u.

Moreover, the dominating degree of ϕ does not depend on u; that is, if u' is another extension of v to $\overline{K}$, then $\mathrm{degdom}_{E,u}(\phi) = \mathrm{degdom}_{E,u'}(\phi)$.

Proof. If E is a pseudo-convergent sequence then the statement is just [19, Proposition 3.5].

If the sequence E is pseudo-divergent, then the proof is essentially the same as when E is pseudo-convergent: let $\beta \in K$ be a pseudo-limit of E and let $\Delta = \Delta_E$ be the least final segment of $\mathbb{Q}\Gamma_v$ containing the gauge of E (if $\mathrm{Br}(E) = K$, just take $\Delta = \Gamma_v$); we take $\tau \in \Gamma_v \cap \Delta$ such that $C = \mathcal{C}_u(\beta, \Delta \cap (-\infty, \tau))$ contains no critical points of ϕ. Then, $s_\nu \in C$ for all large ν; by construction, the weighted sum of the subset S of Ω_ϕ of those elements α such that $u(\alpha - \beta) > \Delta \cap (-\infty, \tau)$ is exactly $\lambda = \mathrm{degdom}_{E,u}(\phi)$. By [19, Theorem 3.2], it follows that for each $\nu \geq \nu_0$ we have

$$v(\phi(s_\nu)) = \lambda v(s_\nu - \beta) + \gamma = \lambda\delta_\nu + \gamma$$

The fact that $\gamma \in \Gamma_v$ and does not depend on the chosen β follows immediately. For the final claim the proof is analogous to [19, Proposition 3.5(c)].

If E is pseudo-stationary, we cannot apply directly [19, Theorem 3.2], but the same general method works: let $\phi \in K(X)$ and write $\phi(X) = c\prod_{i=1}^{n}(X - \alpha_i)^{\epsilon_i}$, where $c \in K$, $\alpha_i \in \overline{K}$ and $\epsilon_i \in \{1, -1\}$. Let u be a fixed extension of v to $\overline{K}$, let $\beta \in K$ be a pseudo-limit of E and let S be the multiset of critical points of ϕ which are pseudo-limits of E with respect to u. If $\alpha \notin S$, then $u(s_\nu - \alpha) = u(\beta - \alpha) < \delta$; on the other hand, if $\alpha \in S$ then there is at most one ν (say ν_0) such that $u(s_{\nu_0} - \alpha) > \delta$, while $u(s_\nu - \alpha) = \delta$ for all $\nu \neq \nu_0$. Hence, for all large ν we have $u(s_\nu - \alpha) = \delta$. Note that, if $\alpha \notin S$, then $u(\beta - \alpha)$ does not depend on the chosen pseudo-limit β of E. In particular, $u(s_\nu - \alpha) \leq \delta$ and equality holds if and only if α is a pseudo-limit of E, in complete analogy with [19, Remark 4.7(a)]. Now, let λ be the weighted sum of S (which is equal to $\mathrm{degdom}_{E,u}(\phi)$) and $\gamma = u\left(\frac{\phi}{\phi_S}(\beta)\right)$: then, for all large ν, s_ν is not a critical point of ϕ and we have

$$v(\phi(s_\nu)) = v(c) + \sum_{\alpha\in S}\epsilon_i u(s_\nu - \alpha) + \sum_{\alpha\notin S}\epsilon_i u(s_\nu - \alpha) = \lambda\delta + \gamma$$

It is clear as before that $\gamma \in \Gamma_v$ and does not depend on the chosen pseudo-limit β of E, by the above remark. To conclude, we only need to prove that the dominating degree of ϕ with respect to a pseudo-stationary sequence E does not depend on the extension of v to $\overline{K}$. Let u, u' be two extensions of v to $\overline{K}$. By Lemma 2.5, it follows that $\mathcal{L}_E = s + cV$, where a pseudo-limit s of E can be chosen in K and $c \in K$ has value δ_E. Now, for the same Lemma we also have that $\mathcal{L}_{E,u} = s + cU$ and $\mathcal{L}_{E,u'} = s + cU'$; in particular, $\mathcal{L}_{E,u}$ and $\mathcal{L}_{E,u'}$ are conjugate under the action of the Galois group of $\overline{K}$ over K. It is then clear that $\Omega_\phi \cap \mathcal{L}_{E,u}$ and $\Omega_\phi \cap \mathcal{L}_{E,u'}$ are conjugate too, so $\mathrm{degdom}_{E,u}(\phi) = \mathrm{degdom}_{E,u'}(\phi)$, as wanted. □

Note that by Proposition 3.2 we may drop the suffix u in the dominating degree of a rational function. Moreover, if $E = \{s_\nu\}_{\nu\in\Lambda}$ is a pseudo-stationary sequence and $\phi \in K(X)$, the values of ϕ on E are definitively constant, namely $v(\phi(s_\nu)) = \lambda\delta + \gamma$, where $\lambda = \mathrm{degdom}_E(\phi)$, $\delta = \delta_E$ and $\gamma \in \Gamma_v$, for all sufficiently large ν.

Definition 3.3. Let $E = \{s_\nu\}_{\nu\in\Lambda} \subset K$ be a pseudo-monotone sequence. We define

$$V_E := \{\phi \in K(X) \mid \phi(s_\nu) \in V, \text{ for all sufficiently large } \nu \in \Lambda\}.$$

Theorem 3.4. *Let $E = \{s_\nu\}_{\nu\in\Lambda} \subset K$ be a pseudo-monotone sequence. Then V_E is a valuation domain with maximal ideal*

$$M_E = \{\phi \in K(X) \mid \phi(s_\nu) \in M, \text{ for all sufficiently large } \nu \in \Lambda\}.$$

Proof. The proof is exactly as the one of [19, Theorem 3.7], but we repeat it here for completeness.

Let $\phi \in K(X)$. By Proposition 3.2, we have $v(\phi(s_\nu)) = \lambda\delta_\nu + \gamma$, for all $\nu \in \Lambda$ sufficiently large, for some $\lambda \in \mathbb{Z}$ and $\gamma \in \Gamma_v$. In particular, the values of ϕ over E are either definitively positive, definitively negative or definitively zero, so either $\phi(s_\nu) \in V$ or $\phi(s_\nu)^{-1} = \phi^{-1}(s_\nu) \in V$ (in both cases for all $\nu \in \Lambda$ sufficiently large), which shows that V_E is a valuation domain.

The claim about the maximal ideal of V_E follows immediately. □

We call V_E the *extension of V associated to the pseudo-monotone sequence E*. Note that, if E is a pseudo-convergent sequence and its gauge is cofinal in Γ_v (or, equivalently, E is a Cauchy sequence), then $V_E = V_\alpha = \{\phi \in K(X) \mid \phi(\alpha) \in \widehat{V}\}$, where α is the (unique) limit of E in the completion $\widehat{K}$. See [17] for a study of these valuation domains.

The main properties of the valuation domain V_E and its associated valuation v_E are summarized in Proposition 3.7 below, which is a generalization of [19, Proposition 3.9]. We need to introduce another definition.

Definition 3.5. Let $E \subset K$ be a pseudo-monotone sequence. We denote by $\mathcal{P}_E$ the set of the irreducible monic polynomials $p \in K[X]$ which have at least one root in $\overline{K}$ which is a pseudo-limit of E (with respect to some extension of v to $\overline{K}$), or, equivalently, such that $\mathrm{degdom}_E(p) > 0$.

We note that $\mathcal{P}_E$ is nonempty if and only if E has a pseudo-limit in $\overline{K}$; that is, $\mathcal{P}_E$ is empty if and only if E is a pseudo-convergent sequence of transcendental type. If E is a pseudo-convergent sequence of algebraic type which is also a Cauchy sequence, then $\mathcal{P}_E$ contains a unique element, namely the minimal polynomial of the (unique) limit of E in $\widehat{K}$ (and by Lemma 2.5 this is the only case in which $\mathcal{P}_E$ has only one element). We shall give a formula for the minimal value of $\mathrm{degdom}_E(p)$ in Proposition 6.9.

Lemma 3.6. *Let E be a strictly pseudo-monotone sequence having a pseudo-limit in $\overline{K}$, and let $p \in K[X]$. Then:*

(a) $v_E(p) \notin \Gamma_v$ if and only if some irreducible factor of p is in $\mathcal{P}_E$;

(b) if $v_E(p) \notin \Gamma_v$, then $v_E(p)$ is not torsion over Γ_v;

(c) if $p_1, p_2 \in \mathcal{P}_E$ are of minimal degree, then $v_E(p_1) = v_E(p_2)$.

Proof. Let $p \in K[X]$. Then, $v_E(p) = v(t)$ if and only if $v(t) = v(p(s_\nu)) = \mathrm{degdom}_E(p)\delta_\nu + \gamma$ for all ν sufficiently large (Proposition 3.2); since E is strictly pseudo-monotone, it follows that $v_E(p) \in \Gamma_v$ if and only if $\mathrm{degdom}_E(p) > 0$. Since $\mathrm{degdom}_E(q_1 \cdots q_n) = \sum_i \mathrm{degdom}_E(q_i)$, (a) follows.

(b) is a consequence of the previous point applied to the powers p^n of p.

Finally, if $p_1, p_2 \in \mathcal{P}_E$ are polynomials of minimal degree, then $p_1 - p_2 = r$ for some $r \in K[X]$ of lower degree, because p_1, p_2 are monic; by minimality, no factor of r belongs to $\mathcal{P}_E$, and so $v_E(r) \in \Gamma_v$. Hence, it must be $v_E(p_1) = v_E(p_2)$, and (c) holds. □

Proposition 3.7. *Let $E = \{s_\nu\}_{\nu \in \Lambda} \subset K$ be a pseudo-monotone sequence.*

(a) If E is either pseudo-convergent of algebraic type or pseudo-divergent, let $\Delta_E = v_E(p)$ for some $p \in \mathcal{P}_E$ of minimal degree. Then $\Gamma_{v_E} = \Delta_E\mathbb{Z} \oplus \Gamma_v$ (as groups) and $V_E/M_E \cong V/M$.

(b) If E is pseudo-convergent of transcendental type, then $V \subset V_E$ is immediate.

(c) If E is pseudo-stationary then $\Gamma_{v_E} = \Gamma_v$ and V_E/M_E is a purely transcendental extension of V/M: more precisely, $V_E/M_E = V/M(t)$, where t is the residue of $\frac{X-\alpha}{c}$ modulo M_E, where $c \in K$ satisfies $v(c) = \delta_E$ and $\alpha \in \mathcal{L}_E$.

(d) If E is not pseudo-convergent of transcendental type, then $\Gamma_{v_E} = \langle \Gamma_v, \Delta_E \rangle$, where $\Delta_E = v_E(p)$ for some polynomial p of minimal degree in $\mathcal{P}_E$. Furthermore, Δ_E does not depend on p and, if E is a pseudo-stationary sequence, $\Delta_E = \delta_E$.

(e) If E has a pseudo-limit $\beta \in K$, then $v_E = v_{\beta, \Delta_E}$.

Proof. (a) In both cases, E has a pseudo-limit in $\overline{K}$ with respect to some extension of v, and so $\mathcal{P}_E \neq \emptyset$. Fix a polynomial $p \in \mathcal{P}_E$ of minimal degree, and let $\Delta_E = v_E(p)$, which does not depend on p and is not torsion over Γ_v by Lemma 3.6. For every $q \in K[X]$, we can write $q = r_0 + r_1 p + r_2 p^2 + \cdots + r_n p^n$, for some (uniquely determined) $r_0, \ldots, r_n \in K[X]$ such that $\deg r_i < \deg p$. Since Δ_E is not torsion over Γ_v and $v_E(r_i) \in \Gamma_v$ for each i by minimality of the degree of p, we have

$$v_E(r_i p^i) = v_E(r_i) + i\Delta_E \neq v_E(r_j) + j\Delta_E = v_E(r_j p^j)$$

for every $i \neq j$; therefore, $v_E(q) = \min\{v_E(r_0), v_E(r_1 p), \ldots, v_E(r_n p^n)\}$, and in particular $v_E(q) \in \Gamma_v \oplus \Delta_E\mathbb{N}$. Hence, $\Gamma_{v_E} = \Gamma_v \oplus \Delta_E\mathbb{Z}$.

We now show that $V_E/M_E = V/M$. If E has a pseudo-limit in K then the result follows in the same way as in [19, Proposition 3.9]. Suppose instead $\mathcal{L}_E = \emptyset$ (in particular, E

must be a pseudo-convergent sequence, by Lemma 2.5), and let ϕ be a unit of V_E. Let u be an extension of v to $\overline{K}$ and let $\alpha \in \mathcal{L}_E^u$. Then, the residue field of U_E is equal to the residue field of U (by the previous case); hence, there is a unit β of U such that $\phi - \beta \in M_{U_E}$. Thus, $\phi(s_\nu) \in \beta + M_U$ for all $\nu \geq N$.

Since ϕ is a unit of V_E, $\phi(s_\nu)$ is a unit of V for all large ν; without loss of generality, for $\nu \geq N$. Let a be such that $\phi(s_N) \in a + M$: then, for every $\nu > N$ we have $\phi(s_\nu) - \phi(s_N) \in M_U \cap V = M$, and thus also $\phi(s_\nu) \in a + M$. Hence, the image of ϕ is in V/M, and so $V/M = V_E/M_E$. The claim is proved.

(b) This follows from Kaplansky's results in [9].

(c) Suppose that E is a pseudo-stationary sequence. It is clear that, without loss of generality, we may suppose that K is algebraically closed. In order to prove the claim, by [16, §11, IV, p. 366] it is sufficient to show that $v_E(X - \alpha - \beta) = \min\{v_E(X - \alpha), v(\beta)\}$ for each $\beta \in K$. By Proposition 3.2, we have $v_E(X - \alpha) = \delta$. If $\delta \neq v(\beta)$ we are done. If $\delta = v(\beta)$ then by Lemma 2.5, $\alpha + \beta$ is a pseudo-limit of E, so again by Proposition 3.2 we have $v_E(X - \alpha - \beta) = \delta$.

(d) For pseudo-convergent sequences of algebraic type or pseudo-divergent sequences the claim follows from the proof of part (a). For a pseudo-stationary sequence E, $\Delta_E = v_E(X - \alpha) = \delta_E$ for all pseudo-limits $\alpha \in \mathcal{L}_E$, and we are done. (e) follows in the same way. $\square$

The next proposition constitutes an important generalization of [9, Lemma 5] and [16, III, §64, p. 371], which says that the image under a polynomial of a pseudo-convergent sequence is a definitively pseudo-convergent sequence.

Proposition 3.8. *Let $E = \{s_\nu\}_{\nu\in\Lambda} \subset K$ be a strictly pseudo-monotone sequence and let $\phi \in K(X)$ be non-constant. Then $\phi(E) = \{\phi(s_\nu)\}_{\nu\in\Lambda}$ is a definitively strictly pseudo-monotone sequence, which is of the same kind of E if* $\mathrm{degdom}_E(\phi) > 0$*, and not of the same kind if* $\mathrm{degdom}_E(\phi) < 0$*; if* $\mathrm{degdom}_E(\phi) \neq 0$*, then $\mathcal{L}_{\phi(E)} = \mathrm{Br}(\phi(E))$. Furthermore, if $\phi(E)$ is definitively pseudo-convergent then $\phi(X)$ is a pseudo-limit of $\phi(E)$ with respect to v_E.*

Proof. Let $\lambda = \mathrm{degdom}_E(\phi)$. Suppose first that $\lambda > 0$ and E is a pseudo-convergent sequence. By Proposition 3.2 we have $v(\phi(s_\nu)) = \lambda\delta_\nu + \gamma < v(\phi(s_\mu)) = \lambda\delta_\mu + \gamma$ for all $\nu < \mu$ sufficiently large (say greater than $\nu_0 \in \Lambda$), which shows that $\phi(E)$ is a definitively pseudo-convergent sequence with gauge $\{\lambda\delta_\nu + \gamma\}_{\nu\geq\nu_0}$. Since $v(\phi(s_\nu))$ increases, 0 is a pseudo-limit of $\phi(E)$, and thus by Lemma 2.5 we have the equality $\mathcal{L}_{\phi(E)} = \mathrm{Br}(\phi(E))$. Since $v(\phi(s_\rho)) > v(\phi(s_\nu))$ if $\rho > \nu$ (sufficiently large), we have $v_E\left(\frac{\phi(X)}{\phi(s_\nu)}\right) > 0$ for all ν sufficiently large; hence, definitively, $v_E(\phi(X) - \phi(s_\nu)) = v_E(\phi(s_\nu))$, and in particular $\{v_E(\phi(X) - \phi(s_\nu))\}_{\nu\in\Lambda}$ is strictly increasing. Hence, $\phi(X)$ is a pseudo-limit of $\phi(E)$.

If $\lambda > 0$ and E is a pseudo-divergent sequence then as above $\phi(E)$ is definitively pseudo-divergent. If $\lambda < 0$, then in the same way we can prove that $\phi(E)$ is strictly pseudo-monotone, not of the same kind of E, and $\phi(X)$ is a pseudo-limit of $\phi(E)$ with respect to v_E.

Suppose now that $\lambda = 0$ and E is a pseudo-convergent sequence. Without loss of generality, we may also suppose that $K = \overline{K}$. Let $\phi(X) = p(X)/q(X)$, where $p, q \in K[X]$. Since K is algebraically closed, we can write $q(X) = q_1(X)q_2(X)$ in such a way that all zeros of q_1 are pseudo-limits of E while no zero of q_2 is a pseudo-limit of E (if E has no pseudo-limits then $q(X) = q_2(X)$ and $q_1(X) = 1$). In particular, $\deg q_1 = \mathrm{degdom}_E(q_1)$. Dividing p by q_1, we have

$$\phi(X) = \frac{p(X)}{q(X)} = \frac{a(X)q_1(X) + b(X)}{q(X)} = \frac{a(X)}{q_2(X)} + \frac{b(X)}{q(X)},$$

where $a, b \in K[X]$ and $\deg b < \deg q_1$. The rational function $\phi_2(X) = \frac{b(X)}{q(X)}$ has dominating degree

$$\mathrm{degdom}_E(\phi_2) = \mathrm{degdom}_E(b) - \mathrm{degdom}_E(q_1) \leq \deg b - \deg q_1 < 0,$$

and thus, by the previous part of the proof, $\{\phi_2(s_\nu)\}_{\nu\in\Lambda}$ is a definitively pseudo-divergent sequence.

Consider now $\phi_1(X) = \frac{a(X)}{q_2(X)}$. If E has a pseudo-limit in $K = \overline{K}$, let $\alpha \in \mathcal{L}_E$. If not, then E is a pseudo-convergent sequence of transcendental type, and we can extend v to a transcendental extension $K(z)$ of K such that z is a pseudo-limit of E ([9, Theorem 2]), and we set $\alpha = z$; with a slight abuse of notation, we still denote by v this extension to $K(z)$. Note that in any case $q_2(\alpha) \neq 0$ since $\mathrm{degdom}_E(q_2) = 0$. Consider the following rational function over $K(\alpha)$:

$$\psi(X) = \phi_1(X) - \phi_1(\alpha) = \frac{a(X)q_2(\alpha) - a(\alpha)q_2(X)}{q_2(\alpha)q_2(X)}.$$

Since $\psi(\alpha) = 0$, the dominating degree of the numerator of ψ is positive; on the other hand, $\mathrm{degdom}_E(q_2(\alpha)q_2) = \mathrm{degdom}_E(q_2) = 0$. Hence, $\mathrm{degdom}_E(\psi) > 0$, and by the previous part of the proof $\{\psi(s_\nu)\}_{\nu\in\Lambda}$ is a definitively pseudo-convergent sequence in $K(\alpha)$. Thus, also $\{\phi_1(s_\nu)\}_{\nu\in\Lambda} = \{\psi(s_\nu) + \phi_1(\alpha)\}_{\nu\in\Lambda}$ is definitively pseudo-convergent in $K(\alpha)$; however, $\phi_1(s_\nu) \in K$ for every ν, and thus $\{\phi_1(s_\nu)\}_{\nu\in\Lambda}$ is a definitively pseudo-convergent sequence in K.

By definition, $\phi(s_\nu) = \phi_1(s_\nu) + \phi_2(s_\nu)$ and, by the previous points, the sequences $\{\phi_1(s_\nu)\}_{\nu\in\Lambda}$ and $\{\phi_2(s_\nu)\}_{\nu\in\Lambda}$ are definitively pseudo-convergent and definitively pseudo-divergent, respectively. In particular, for large ν, $v(\phi_1(s_\rho) - \phi_1(s_\nu))$, $\rho > \nu$, is increasing and $v(\phi_2(s_\rho) - \phi_2(s_\nu))$, $\rho > \nu$, is decreasing; it follows that $v(\phi(s_\rho) - \phi(s_\nu))$, $\rho > \nu$, is definitively equal to one of the two. Hence, $\phi(s_\nu)$ is definitively strictly pseudo-monotone, as claimed.

Suppose in particular that $\phi(E)$ is definitively pseudo-convergent: then,

$$v_E(\phi(X) - \phi(s_\nu)) = v_E((\phi_1(X) - \phi_1(s_\nu)) + (\phi_2(X) - \phi_2(s_\nu))).$$

By the case $\lambda > 0$, we have $v_E((\phi_1(X) - \phi_1(s_\nu)) = v_E(\phi_1(s_\nu))$ for all large ν. On the other hand, since $\phi(E)$ is pseudo-convergent we have $v_E(\phi_1(s_\nu)) < v_E(\phi_2(s_\rho))$ for all large

$\nu < \rho$; in particular, we also have $v_E(\phi_2(X)) \geq v_E(\phi_1(X))$ and so $v_E(\phi_2(X) - \phi_2(s_\nu))$ is bigger than both $v_E(\phi_1(X))$ and $v_E(\phi_1(s_n))$. Hence,

$$v_E(\phi(X) - \phi(s_\nu)) = v_E(\phi_1(X) - \phi_1(s_\nu)) = v_E(\phi_1(s_\nu)),$$

which is definitively strictly increasing. Hence, $\phi(X)$ is a pseudo-limit of $\phi(E)$ with respect to v_E, as claimed.

If E is pseudo-divergent, the same reasoning applies (with the only difference that $\phi_1(E)$ will be pseudo-divergent and $\phi_2(E)$ pseudo-convergent). $\square$

4 Localizations

Let E be a pseudo-convergent sequence with respect to a valuation domain V with quotient field K, and let P be a prime ideal of V. We can also consider E as a sequence in the valued field (K, v_P), where v_P is the valuation associated to V_P. In some cases, E will be a pseudo-convergent sequence also with respect to v_P: this happens, for example, if we take $E = \{\pi^n\}_{n\in\mathbb{N}}$, where $\pi \in P$ is different from zero. However, sometimes E may not remain pseudo-convergent: for example, if $\pi \in V \setminus P$ is not a unit of V, then the sequence $\{\pi^n\}_{n\in\mathbb{N}}$ is pseudo-convergent with respect to V, but it is pseudo-stationary with respect to V_P, since $v_P(\pi^n - \pi^{n+k})$ is equal to 0 for every $k > 0$.

We want to understand when each of these possibilities occurs. We shall use the following notation: if $\theta \in \Gamma_v$, we denote by $\theta^{(P)}$ the corresponding element of Γ_{v_P} (i.e., if $\theta = v(c)$, then $\theta^{(P)} = v_P(c)$).

Lemma 4.1. *Let $E = \{s_\nu\}_{\nu\in\Lambda} \subset K$ be a pseudo-convergent (pseudo-divergent, respectively) sequence and let $P \in \mathrm{Spec}(V)$. Then, exactly one of the following happens:*

- *there is a subsequence $E' \subseteq E$ which is a pseudo-convergent (pseudo-divergent, respectively) sequence with respect to v_P;*
- *E is definitively pseudo-stationary sequence with respect to v_P.*

Proof. We give the proof only for the pseudo-convergent case, since the pseudo-divergent one is essentially analogous.

Let $\{\delta_\nu\}_{\nu\in\Lambda}$ be the gauge of E: then, $\Delta = \{\delta_\nu^{(P)}\}_{\nu\in\Lambda}$ is a non-decreasing sequence in Γ_{v_P}.

Suppose Δ is definitively constant: then, there is an N such that $\delta_\nu^{(P)} = \chi$ for all $\nu \geq N$. Hence, $v_P(s_\nu - s_\mu) = \delta_\nu^{(P)} = \chi$ for all $\mu > \nu \geq N$, and thus $E' = \{s_\nu\}_{\nu\geq N}$ is a pseudo-stationary sequence in V_P. Conversely, if $E' = \{s_\nu\}_{\nu\geq N}$ is pseudo-stationary in V_P for some N, then Δ must become definitively constant.

Suppose that Δ is not definitively constant: then, we can index Δ in a well-ordered way as $\{\chi_\kappa\}_{\kappa\in\Lambda'}$, with $\chi_\kappa < \chi_\mu$ for $\kappa < \mu$. For every κ, let $t_\kappa = s_{\nu_\kappa}$, where $\nu_\kappa = \min\{\nu \mid \delta_\nu = \chi_\kappa\}$. Then, $E' = \{t_\kappa\}_{\kappa\in\Lambda'}$ is a subsequence of E such that

$$v_P(t_\kappa - t_{\kappa+1}) = v_P(s_{\nu_\kappa} - s_{\nu_{\kappa+1}}) = \delta_{\nu_{\kappa+1}}^{(P)} = \chi_{\kappa+1};$$

therefore, E' is pseudo-convergent on V_P. Conversely, if $E' = \{t_\kappa = s_{\nu_\kappa}\}_{\kappa\in\Lambda} \subseteq E$ is pseudo-convergent on V_P, then $v_P(t_\kappa - t_{\kappa+1}) = v_P(s_{\nu_\kappa} - s_{\nu_{\kappa+1}}) = \delta_{\nu_\kappa}$. In particular, $\{\delta_{\nu_\kappa}\}_{\kappa\in\Lambda}$ is strictly increasing, and thus Δ is infinite.

Therefore, these two possibilities are the only ones and each one excludes the other, as claimed. □

Theorem 4.2. *Let $E = \{s_\nu\}_{\nu\in\Lambda} \subset K$ be a pseudo-convergent (pseudo-divergent, respectively) sequence on V and let $P \in \mathrm{Spec}(V)$. The following are equivalent:*

- *(i) there is a subsequence $E' \subseteq E$ which is pseudo-convergent (pseudo-divergent, respectively) on V_P;*
- *(ii) $\mathrm{Br}(E) = \mathrm{Br}_{V_P}(E)$;*
- *(iii) $\mathrm{Br}(E)$ is a strictly divisorial fractional ideal of V_P (is a non-principal V_P-submodule of K, respectively).*

Proof. As in Lemma 4.1, we only give the proof for E pseudo-convergent. Let $\{\delta_\nu\}_{\nu\in\Lambda}$ be the gauge of E.

Suppose that (i) holds, i.e., that there is a subsequence $E' \subseteq K$ that is pseudo-convergent subsequence on V_P. Since E' is pseudo-convergent also on V with $\mathrm{Br}(E') = \mathrm{Br}(E)$, without loss of generality we can suppose $E = E'$.

If $x \in \mathrm{Br}(E)$ then $v(x) > \delta_\nu$ for all ν, and thus $v_P(x) > \delta_\nu^{(P)}$ for all ν, i.e., $x \in \mathrm{Br}_{V_P}(E)$. On the other hand, if $x \in \mathrm{Br}_{V_P}(E)$, then $v_P(x) > \delta_\nu$ for all ν. If $x \notin \mathrm{Br}(E)$, then $v(x) < \delta_\kappa$ for some κ, so that $v_P(x) \leq \delta_\kappa^{(P)}$ and, for $\mu > \kappa$,

$$\delta_\kappa^{(P)} < \delta_\mu^{(P)} \leq v_P(x) \leq \delta_\kappa^P,$$

a contradiction. Hence, $\mathrm{Br}(E) = \mathrm{Br}_{V_P}(E)$ and (ii) holds. Furthermore, since E is pseudo-convergent on V_P (iii) follows from Lemma 2.6.

Suppose now that (i) does not hold. If $\mathrm{Br}(E)$ is not a V_P-ideal then (ii) and (iii) cannot hold. Suppose $\mathrm{Br}(E)$ is a V_P-ideal: we claim that $\mathrm{Br}(E) = cP$ for some $c \in K$. Indeed, by Lemma 4.1 $\delta_\nu^{(P)}$ is definitively constant, say equal to $v_P(c)$, for some $c \in K$. In particular, $\mathrm{Br}_{V_P}(E) = cV_P$. Since E is a pseudo-convergent sequence in V, we have $\delta_\nu < \delta_\mu$ for $\nu < \mu$. Thus, if $v(c_\nu) = \delta_\nu$, we have $c_\nu \notin \mathrm{Br}(E)$ and so $\mathrm{Br}(E) \subsetneq c_\nu V_P = cV_P$ for all large ν. It easily follows that $\mathrm{Br}(E) \subseteq cP$. Conversely, if $d \in cP$ then $v_P(d) > v_P(c) = \delta_\nu^{(P)}$ for all large ν, and thus $v(d) > \delta_\nu$ for all large ν; hence, $d \in \mathrm{Br}(E)$, and $\mathrm{Br}(E) = cP$. In particular, $\mathrm{Br}(E) = cP \neq cV_P = \mathrm{Br}_{V_P}(E)$ and (ii) does not hold; furthermore, there is a smallest V_P-ideal properly containing $\mathrm{Br}(E)$ (namely, cV_P), and so $\mathrm{Br}(E)$ can't be strictly divisorial on V_P. Thus, neither (iii) holds.

It follows that the three conditions are equivalent, as claimed. □

To conclude this section, we analyze the case of pseudo-stationary sequences.

Proposition 4.3. *Let $E = \{s_\nu\}_{\nu\in\Lambda} \subset K$ be a pseudo-stationary sequence on V and let $P \in \mathrm{Spec}(V)$. Then E is pseudo-stationary with respect to V_P and $\mathrm{Br}_{V_P}(E) = \mathrm{Br}(E)V_P$.*

Proof. If $v(s_\nu - s_\mu) = \delta$ for all $\nu \neq \mu$, then $v_P(s_\nu - s_\mu)$ is equal to $\delta^{(P)}$; thus, E is pseudo-stationary. Furthermore, any element c such that $v_P(c) = \delta^{(P)}$ generated $\mathrm{Br}_{V_P}(E)$; in particular, if c satisfies $v(c) = \delta$ (and so generated $\mathrm{Br}(E)$), we have $\mathrm{Br}_{V_P}(E) = cV_P = (cV)V_P = \mathrm{Br}(E)V_P$. □

5 Rank and prime ideals

In this section, we study the relationship between the spectrum of V_E and the spectrum of V. As for every other extension of V, the restriction map

$$\begin{aligned} \eta\colon \mathrm{Spec}(V_E) &\longrightarrow \mathrm{Spec}(V) \\ Q &\longmapsto Q \cap V \end{aligned} \tag{1}$$

is always surjective. The pseudo-stationary case is the easiest one.

Proposition 5.1. *Let $E \subset K$ be a pseudo-monotone sequence which is either pseudo-stationary or pseudo-convergent of transcendental type. Then, the map η defined in* (1) *is injective.*

Proof. In both cases, we have $\Gamma_{v_E} = \Gamma_v$ (by Proposition 3.7(c) in the pseudo-stationary case, by [9, Theorem 2] in the transcendental case). Since the prime ideals of a valuation domain correspond to the convex subgroups of its value group, the claim follows. □

On the other hand, by Proposition 3.7(a), if the sequence E is either pseudo-convergent of algebraic type or pseudo-divergent (thus, it is strictly pseudo-monotone with a pseudo-limit in $\overline{K}$ with respect to some extension of v) then we have $\Gamma_{v_E} \simeq \Gamma_v \oplus \mathbb{Z}$ (as groups): hence, there is at most one prime ideal P of V such that $\eta^{-1}(P)$ is not a singleton, in which case $|\eta^{-1}(P)| = 2$.

To individuate this (potential) prime ideal P, we start by constructing prime ideals of V_E.

Proposition 5.2. *Let $E = \{s_\nu\}_{\nu\in\Lambda} \subset K$ be a pseudo-monotone sequence, and let $P \in \mathrm{Spec}(V)$. Then, the following hold.*

- *(a)* $\mathrm{rad}(PV_E)$ *is the minimal prime ideal of V_E lying over P. If P is not principal in V_P, then* $\mathrm{rad}(PV_E) = PV_E$.
- *(b) The set*
  $$P_E = \{\phi \in V_E \mid \phi(s_\nu) \in P, \textit{ for all large } \nu \in \Lambda\}$$
  is a prime ideal of V_E lying above P. In particular, $\mathrm{rad}(PV_E) \subseteq P_E$.
- *(c) If P is maximal, then* $\mathrm{rad}(PV_E) = P_E$ *if and only if* $|\eta^{-1}(P)| = 1$.

We note that part (a) holds, more generally, for any extension W of V.

Proof. (a) Since V_E is a valuation domain, $\mathrm{rad}(PV_E)$ is prime; furthermore, any prime ideal Q of V_E over P contains the elements of P, so that $PV_E \subseteq Q$ and $\mathrm{rad}(PV_E) \subseteq Q$; hence $\mathrm{rad}(PV_E)$ is minimal.

For the second part, let $\phi \in V_E$ be such that $\phi^n \in PV_E$: then $\phi^n = p\psi$ for some $p \in P$, $\psi \in V_E$ and $n \geq 1$. Since PV_P is not principal, there exists $q \in PV_P = P$ such that $nv_P(q) < v_P(p)$. Hence, $p = q^n t$ for some $t \in V$. Then $\phi^n = q^n t\psi$ implies that $(\phi/q)^n = t\psi \in V_E$, so that $\phi/q \in V_E$ (since V_E is integrally closed) and $\phi \in qV_E \subseteq PV_E$.

(b) Suppose $\phi_1, \phi_2 \in V_E$ satisfy $\phi_1\phi_2 \in P_E$. By Proposition 3.2, the sequences $\{v(\phi_1(s_\nu))\}_{\nu\in\Lambda}$ and $\{v(\phi_2(s_\nu))\}_{\nu\in\Lambda}$ are definitively monotone; in particular, there are integers N_1, N_2, N' such that $v(\phi_1(s_\nu))$ ($v(\phi_2(s_\nu))$, $v((\phi_1\phi_2)(s_\nu))$, respectively) is either in P or out of P for all $\nu \geq N_1$ ($\nu \geq N_2$, $\nu \geq N'$, respectively). Take $N \geq \max\{N_1, N_2, N'\}$; then, either $\phi_1(s_\nu) \in P$ or for all $\nu \geq N$ or $\phi_2(s_\nu) \in P$ for all $\nu \geq N$ (or both). In the former case $\phi_1 \in P_E$, while in the latter $\phi_2 \in P_E$; hence, P_E is a prime ideal.

Since $P_E \cap V = P$, the last statements follow immediately.

(c) If $|\eta^{-1}(P)| = 1$, then $\mathrm{rad}(PV_E) = P_E$ regardless of the fact that P is maximal. If P is maximal, then P_E is the maximal ideal of V (Theorem 3.4); hence if $\mathrm{rad}(PV_E) = P_E$ then $|\eta^{-1}(P)|$ must be 1. □

As we shall see, point (c) does not hold in general, so we need a different characterization; furthermore, the results for pseudo-convergent and pseudo-divergent sequences are slightly different. The case where the characterizations work best is the algebraically closed case; we premit two lemmas.

Lemma 5.3. *Let W be an extension of V (to an arbitrary field F), and let $P \subset V$ be a prime ideal such that PV_P is not principal. Then, there are two prime ideals of W over P if and only if there is a $\phi \in W \setminus K$, not a unit of W, such that $p \in \phi W$ for all $p \in P$ and $\phi \in mW$ for all $m \in V \setminus P$.*

Note that the last two conditions of the statement are equivalent to $\phi W \cap V = P$.

Proof. Suppose that $Q_1 \subset Q_2 \subset W$ are two prime ideals above P, and let $\phi \in Q_2 \setminus Q_1$. If $p \in P \subset Q_1$, then $p \in \phi W$; on the other hand, if $\phi \notin mW$ for some $m \in V \setminus P$ then $mV \subseteq mW \cap V \subseteq \phi W \cap V = P$, a contradiction.

Conversely, suppose such a ϕ exists. By Proposition 5.2(a) (which holds for a general extension of V to $K(X)$, see the remark after the statement), PW is a prime ideal of W. Furthermore, $PW \subseteq \phi W$; if they are equal, there should be a $p \in P$ such that $\phi W \in pW$, i.e., $\phi W = pW$. However, this would imply $PW = pW$ and thus $P = pV$, a contradiction. Hence, $PW \subsetneq \phi W$. Let $Q = \mathrm{rad}(\phi W)$: if $Q \cap V \neq P$ then there is a $m \in V \setminus P$, which is not a unit of V, and an integer $k \geq 1$ such that $m^k \in \phi W$. However, this means that $m^{k+1} \subsetneq \phi W$, against the properties of ϕ. Thus, it must be $Q \cap V = P$, and Q is the second prime ideal above P. □

Lemma 5.4. *Suppose K is algebraically closed, and let E be a strictly pseudo-monotone sequence. Suppose $Q_1 \subset Q_2$ are two prime ideals of V_E that restrict to the same $P \in$* $\mathrm{Spec}(V)$*. Then, for all $\alpha \in \mathcal{L}_E$, there are $d \in K$ and $\epsilon \in \{+1, -1\}$ such that $d(X-\alpha)^\epsilon \in Q_2 \setminus Q_1$.*

Proof. Note that $\mathcal{L}_E \neq \emptyset$, by Proposition 5.1 and since K is algebraically closed.

Let $\phi \in Q_2 \setminus Q_1$, and let S be the multiset of critical points of ϕ that are pseudo-limits of E. If $\phi_1 = \frac{\phi}{\phi_S} \in K(X)$, then $\mathrm{degdom}_E(\phi_1) = 0$, and so there is $\gamma \in \Gamma_v$ such that $v(\phi_1(s_\nu)) = \gamma$ for all large ν. In particular, if $c \in K$ has valuation γ, then $v_E(\phi_1) = v_E(c)$; hence, $\phi V_E = \phi_S \phi_1 V_E = c\phi_S V_E$, and thus $c\phi_S \in Q_2 \setminus Q_1$.

Write now $\phi_S(X) = \prod_{\beta \in S}(X-\beta)^{\epsilon_\beta}$, with $\epsilon_\beta \in \{+1,-1\}$, and let $\lambda = \mathrm{degdom}_E(\phi)$. Then, for all $\beta \in S \cup \{\alpha\}$, there is a $N(\beta)$ such that $v(s_\nu - \beta) = \delta_\nu$ for all $\nu > N(\beta)$; thus, there is an N such that $v(s_\nu - \beta) = v(s_\nu - \alpha)$ for all $\nu > N$. Let $\widetilde{\phi}(X) = (X-\alpha)^\lambda$; then, $v(\phi_S(s_\nu)) = v(\widetilde{\phi}(s_\nu))$ for large ν, and so $\phi_S V_E = \widetilde{\phi} V_E$. Hence $c\widetilde{\phi} \in Q_2 \setminus Q_1$. Let $d \in K$ be such that $d^{|\lambda|} = c$: then,

$$c\widetilde{\phi}(X) = d^{|\lambda|}(X-\alpha)^{\lambda/|\lambda|} = (d(X-\alpha)^\epsilon)^{|\lambda|},$$

where $\epsilon = \lambda/|\lambda|$. Since $c\widetilde{\phi} \in Q_2 \setminus Q_1$, we must have $d(X-\alpha)^\epsilon \in Q_2 \setminus Q_1$, as claimed. $\square$

Theorem 5.5. *Suppose K is algebraically closed, and let E be a pseudo-convergent sequence. Let $P \in \mathrm{Spec}(V)$ and let η be the map defined in* (1)*. Then, the following hold.*

(a) $|\eta^{-1}(P)| = 2$ *if and only if there is a $c \in K$ such that* $\mathrm{Br}(E) = cV_P$ *or* $\mathrm{Br}(E) = cP$.

(b) Suppose P is not maximal. The following are equivalent:

- *(i)* $\mathrm{Br}(E) = cV_P$ *for some $c \in K$;*
- *(ii)* $PV_E \subsetneq P_E$*;*
- *(iii)* $|\eta^{-1}(P)| = 2$ *and there is a subsequence $E' \subseteq E$ that is pseudo-convergent on V_P.*

(c) The following are equivalent:

- *(i)* $\mathrm{Br}(E) = cP$ *for some $c \in K$;*
- *(ii)* $|\eta^{-1}(P)| = 2$ *and* $PV_E = P_E$*;*
- *(iii)* $|\eta^{-1}(P)| = 2$ *and E is definitively pseudo-monotone on V_P.*

Proof. By Proposition 5.1, we can suppose that $\mathcal{L}_E \neq \emptyset$; fix thus a pseudo-limit α of E. Let also $\{\delta_\nu\}_{\nu\in\Lambda}$ be the gauge of $E = \{s_\nu\}_{\nu\in\Lambda}$. Note also that, by Proposition 5.2(a), PV_E is a prime ideal of V_E.

Suppose first that $|\eta^{-1}(P)| = 2$. By Lemma 5.4, there are $c \in K$ and an $\epsilon \in \{+1,-1\}$ such that $c(X-\alpha)^\epsilon \in Q_2 \setminus Q_1$ (where Q_1, Q_2 are the prime ideals of V_E over P). We distinguish the two cases.

Suppose that $\epsilon = -1$, i.e., that $\phi(X) = \frac{c}{X-\alpha} \in Q_2 \setminus Q_1$. If $t \in V \setminus P$, then $tV_E \cap K \supsetneq P$, and thus $\phi V_E \subsetneq tV_E$; hence, $\phi/t \in V_E$ and

$$v\left(\frac{\phi(s_\nu)}{t}\right) = v(c/t) - \delta_\nu > 0$$

for all large ν. This is equivalent to saying that $c/t \in \mathrm{Br}(E)$; hence, $cV_P \subseteq \mathrm{Br}(E)$. Conversely, if $t \notin cV_P$ then $t/c \notin V_P$, and so $c/t \in P$; hence, $c/t \in \phi V_E = \frac{c}{X-\alpha}V_E$ and thus $1/t \in 1/(X-\alpha)V_E$, that is, $\frac{X-\alpha}{t} \in V_E$. Therefore, $\delta_\nu > v(t)$ for all large ν, and so $t \notin \mathrm{Br}(E)$. Therefore, $\mathrm{Br}(E) = cV_P$.

In the same way, suppose that $\epsilon = +1$ and let $\phi(X) = \frac{X-\alpha}{c} \in Q_2 \setminus Q_1$. If $p \in P$, then $p \in \phi V_E$, and so $v(cp) - \delta_\nu > 0$ for all large ν, which implies $cp \in \mathrm{Br}(E)$ and $cP \subseteq \mathrm{Br}(E)$. Conversely, if $t \notin cP$ then $t/c \notin P$, and thus $\phi \in (t/c)V_E$, or equivalently $\delta_\nu - v(c) - (v(t) - v(c)) > 0$ for all large ν, i.e., $\delta_\nu > v(t)$ for all large ν. This implies $t \notin \mathrm{Br}(E)$, and so $\mathrm{Br}(E) = cP$.

Therefore, if $|\eta^{-1}(P)| = 2$ then $\mathrm{Br}(E)$ is equal either to cV_P or to cP, for some $c \in K$. To show the converse, we consider the two cases separately.

Suppose that P is maximal. Then, $\mathrm{Br}(E)$ is strictly divisorial, and thus it can't be equal to cP. If $\mathrm{Br}(E) = cV_P = cV$, let $\phi(X) = \frac{c}{X-\alpha}$. Then, $v(c) > \delta_\nu = v(s_\nu - \alpha)$ for all ν, and so $\phi \in P_E$; on the other hand, if $\phi \in PV_E$ then $\phi \in pV_E$ for some $p \in P$, and thus $c/p \in \mathrm{Br}(E)$, a contradiction. Hence, $|\eta^{-1}(P)| = 2$. When P is not maximal, we distinguish between the case $\mathrm{Br}(E) = cV_P$ (treated as part of (b)) and the case $\mathrm{Br}(E) = cP$ (treated as part of (c)).

(b) Suppose now that P is not maximal.

(i) $\Longrightarrow$ (ii) and (iii). If $\mathrm{Br}(E) = cV_P$, consider $\phi(X) = \frac{c}{X-\alpha}$. We claim that $\phi \in P_E \setminus PV_E$. Consider E as a sequence over V_P: by Lemma 4.1, either E is definitively pseudo-stationary with respect to v_P or there is a subsequence $E' \subseteq E$ that is pseudo-convergent on V_P.

In the former case, the gauge $\delta_\nu^{(P)}$ of E in (K, v_P) is constant for $\nu > N$, and this value must be exactly $v_P(c)$. Consider $t = s_\nu - s_N$. Then, $v_P(t) = v_P(c)$ and thus $t \in cV_P$; however, $v(t) = \delta_\nu < \delta_\mu$ for $\nu < \mu$, and thus $t \notin \mathrm{Br}(E)$. This contradicts $cV_P = \mathrm{Br}(E)$; hence, there is an $E' \subseteq E$ that is pseudo-convergent on V_P, and without loss of generality we can suppose $E' = E$.

Under this condition, the sequence $\{v_P(\phi(s_\nu))\}_{\nu\in\Lambda}$ must be strictly decreasing; since it is nonnegative, it follows that $v_P(\phi(s_\nu)) > 0$ for all ν, i.e., that $\phi(s_\nu) \in P$, or equivalently that $\phi \in P_E$.

If $\phi \in PV_E$, then $\phi \in qV_E$ for some $q \in P$; however, for all $p \in P$,

$$v\left(\frac{p}{\phi(s_\nu)}\right) = v\left(\frac{p}{c}(s_\nu - \alpha)\right) = \delta_\nu - v(c/p) > 0,$$

since $c/p \notin cV_P = \mathrm{Br}(E)$. Therefore, $p/\phi \in V_E$, i.e., $p \in \phi V_E$. Choosing $p = q^{1/2}$ we get $q^{1/2}V_E \subseteq \phi V_E \subseteq qV_E$, a contradiction. Hence, $\phi \notin PV_E$; in particular, $|\eta^{-1}(P)| = 2$.

(ii) $\Longrightarrow$ (i) If $PV_E \subsetneq P_E$, then clearly $|\eta^{-1}(P)| = 2$; by Lemma 5.4, there are $d \in K$, $\alpha \in \mathcal{L}_E$ and $\epsilon \in \{+1, -1\}$ such that $\phi(X) = d(X-\alpha)^\epsilon \in P_E \setminus PV_E$. If $\epsilon = +1$, then $v(\phi(s_\nu))$ is increasing; in particular, we can find $p = \phi(s_\nu) \in P$, and $\phi(s_\mu) \in pV$ for every $\mu > \nu$. Hence, $\phi \in pV_E$, against $\phi \notin PV_E$. Therefore, $\epsilon = -1$; by the first part of the proof, we must have $\mathrm{Br}(E) = cV_P$ for some c.

(iii) $\Longrightarrow$ (i) By the first part of the proof, $\mathrm{Br}(E)$ can be only cV_P or cP; by Theorem 4.2, $\mathrm{Br}(E)$ is a strictly divisorial ideal of V_P. Hence, it must be $\mathrm{Br}(E) = cV_P$ for some $c \in K$.

(c) As before, suppose P is not maximal.
If $\mathrm{Br}(E) = cP$, consider $\phi(X) = \frac{X-\alpha}{c}$. If $p \in P$ then $cp \in \mathrm{Br}(E)$ and thus

$$v\left(\frac{p}{\phi(s_\nu)}\right) = v(pc) - \delta_\nu > 0,$$

so that $p \in \phi V_E$, while if $m \notin P$ then $cm \notin \mathrm{Br}(E)$ and so

$$v\left(\frac{\phi(s_\nu)}{m}\right) = \delta_\nu - v(cm) > 0,$$

i.e., $\phi \in mV_E$. Then, $|\eta^{-1}(P)| = 2$ by Lemma 5.3. Furthermore, if $PV_E \subsetneq P_E$ then we would be in case (b) above; since P is not principal, we must have $PV_E = P_E$, so (i) $\Longrightarrow$ (ii). In the same way, if (ii) holds then by part (b) there can be no pseudo-convergent subsequence $E' \subseteq E$, and thus (iii) holds by Lemma 4.1. Finally, if (iii) holds then $\mathrm{Br}(E)$ is cV_P or cP by the first part of the proof, and again we can't be in the former case by part (b). $\square$

In the general case, the main characterization is similar to the algebraically closed case, but we cannot distinguish the various alternatives.

Proposition 5.6. *Let E be a pseudo-convergent sequence of algebraic type, and let P be an idempotent prime of V. Then, there are two prime ideals of V_E over P if and only if there is a positive integer k and a $c \in K$ such that $\mathrm{Br}(E)^k$ is cV_P or cP.*

Proof. Let U be an extension of V to $\overline{K}$. Then, E is pseudo-convergent also with respect to U.

Suppose that there are two prime ideals of V_E over $P \in \mathrm{Spec}(V)$, and let Q be the prime ideal of U over P. Then, U_E has two prime ideals over Q; hence, by Theorem 5.5 there is a $c \in \overline{K}$ such that $\mathrm{Br}_u(E)$ is equal either to cU_Q or cQ. If $c \in K$, then $\mathrm{Br}(E)$ is equal either to cV_P or cP, respectively. In general, since the value group of U is the divisible hull of the value group of V, there is a positive integer k such that $ku(c) \in \Gamma_v$; in particular, there is a $d \in K$ such that $u(c^k) = u(d)$. Without loss of generality, we may suppose that $k > 1$. Moreover, eventually substituting c with $d^{1/k}$, we can suppose without loss of generality that $c^k \in K$.

Since $\mathrm{Br}(E) = \mathrm{Br}_u(E) \cap K$ and $\mathrm{Br}_u(E)$ is a U_Q-ideal, it follows that $\mathrm{Br}(E)$ is a V_P-ideal. In particular, we have (note that since $k > 1$ then in the following we have a strict inequality)

$$\begin{aligned}\mathrm{Br}(E) = &\left\{x \in K \mid v_P(x) > u_Q(c) = \frac{1}{k}v_P(d)\right\} = \\ =&\{x \in K \mid v_P(x^k) > v_P(d)\};\end{aligned}$$

furthermore, $\mathrm{Br}(E)^k = \{y \in K \mid v_P(y) \geq v_P(x^k) \text{ for some } x \in \mathrm{Br}(E)\}$. We claim that in this case $\mathrm{Br}(E)^k = dP$ (regardless of whether $\mathrm{Br}_u(E) = cU_Q$ or $\mathrm{Br}_u(E) = cQ$). If $y \in \mathrm{Br}(E)^k$, then $v_P(y) \geq v_P(x^k) > v_P(d)$ for some $x \in \mathrm{Br}(E)$, and so $y \in dP$. Conversely, if $v_P(y) > v_P(d)$ then, since $P = PV_P$ is idempotent, there is an $x \in K$ such that $ku_Q(c) = v_P(d) < kv_P(x) \leq v_P(y)$; in particular, $x \in \mathrm{Br}(E)$ and $v_P(y) \geq v_P(x^k)$, and so $y \in \mathrm{Br}(E)^k$. Therefore, $\mathrm{Br}(E)^k = dP$.

Conversely, suppose that $\mathrm{Br}(E)^k$ is equal either to cV_P or to cP for some $P \in \mathrm{Spec}(V)$. In the former case, $\mathrm{Br}(E)$ must be principal in V_P, i.e., $\mathrm{Br}(E) = dV_P$ for some $d \in K$; we claim that $\mathrm{Br}_u(E) = dU_Q$. Suppose not: then, there is a $d' \in \mathrm{Br}_u(E) \setminus dU_Q$, i.e., a $d' \in \mathrm{Br}_u(E)$ satisfying $u_Q(d') < u_Q(d)$. Then, $d/d' \in Q$. Since P is not principal in V_P, there are positive elements of Γ_{v_P} that are arbitrarily close to 0; hence, there is a $t \in K$ such that $0 < v_P(t) < u_Q(d/d')$. Therefore, $u_Q(d') < v_P(t) < v_P(d)$; it would follow that $t \in d'U_Q \cap K \subseteq \mathrm{Br}_u(E) \cap K = \mathrm{Br}(E)$ while $t \notin dV_P = \mathrm{Br}(E)$, a contradiction. Hence, $\mathrm{Br}_u(E) = dU_Q$, and so there are two prime ideals of U_E over Q; it follows that there are two prime ideals of V_E over P.

Suppose $\mathrm{Br}(E)^k = cP$ for some $k > 1$. If $\{u_Q(t) \mid t \in \mathrm{Br}(E)\}$ does not have an infimum, then neither can $\mathrm{Br}(E)^k$, against the hypothesis; hence, there is a $d \in \overline{K}$ such that $u_Q(d) = \inf\{u_Q(t) \mid t \in \mathrm{Br}(E)\}$. This implies that $\mathrm{Br}(E)U$ is equal to dU_Q or to dQ, and so that $\mathrm{Br}_u(E)$ must also be equal to one of them. Hence, there are two primes of U_E over Q, and so two primes of V_E over P, as claimed. □

The case of pseudo-divergent sequences is essentially analogous, and for this reason we only sketch the proof. Note that the conditions equivalent to $\mathrm{Br}(E) = cV_P$ or $\mathrm{Br}(E) = cP$ are switched with respect to the pseudo-convergent case.

Theorem 5.7. *Suppose K is algebraically closed, and let E be a pseudo-divergent sequence. Let $P \in \mathrm{Spec}(V)$ and let η be the map defined in* (1). *Then, the following hold.*

(a) $|\eta^{-1}(P)| = 2$ *if and only if there is a* $c \in K$ *such that* $\mathrm{Br}(E) = cV_P$ *or* $\mathrm{Br}(E) = cP$.

(b) The following are equivalent:
   - *(i)* $\mathrm{Br}(E) = cP$ *for some* $c \in K$*;*
   - *(ii)* $PV_E \subsetneq P_E$*;*
   - *(iii)* $|\eta^{-1}(P)| = 2$ *and there is a subsequence* $E' \subseteq E$ *that is pseudo-divergent on* V_P.

(c) Suppose P is not maximal. The following are equivalent:
   - *(i)* $\mathrm{Br}(E) = cV_P$ *for some* $c \in K$*;*
   - *(ii)* $|\eta^{-1}(P)| = 2$ *and* $PV_E = P_E$*;*
   - *(iii)* $|\eta^{-1}(P)| = 2$ *and E is definitively pseudo-monotone on* V_P.

Proof. Let $\alpha \in \mathcal{L}_E$, and let $\{\delta_\nu\}_{\nu\in\Lambda}$ be the gauge of E.

If $|\eta^{-1}(P)| = 2$, the proof proceed as in Theorem 5.5: we find a $c(X-\alpha)^\epsilon \in Q_2 \setminus Q_1$ (where $c \in K$, $\epsilon \in \{+1,-1\}$ and Q_1, Q_2 are the prime ideals of V_E over P). Hence, if $\epsilon = -1$ then $\mathrm{Br}(E) = cV_P$, while if $\epsilon = +1$ then $\mathrm{Br}(E) = cP$; the only difference in the proof is substituting "for all ν" with "for some ν" when dealing with the membership in $\mathrm{Br}(E)$.

Suppose that P is maximal. Then, $\mathrm{Br}(E)$ is not finitely generated, so it must be equal to cP; in particular $c \notin \mathrm{Br}(E)$, thus $v(c) < \delta_\nu$ for all ν. If $\phi(X) = \frac{X-\alpha}{c}$ then $\phi \in P_E$ while $c/p \notin \mathrm{Br}(E)$ for every $p \in P$ and so $\phi \notin PV_E$. Hence, $|\eta^{-1}(P)| = 2$.

(b) Suppose now that P is not maximal.

(i) $\Longrightarrow$ (ii) and (iii). If $\mathrm{Br}(E) = cP$, consider $\phi(X) = \frac{X-\alpha}{c}$. If E is definitively pseudo-monotone on V_P, then $\delta_\nu^{(P)} = v_P(c)$ for $\nu > N$; if $t = s_\nu - s_N$, then $v_P(t) = v_P(c)$ and so $t \notin \mathrm{Br}(E)$. However, $t \in \mathrm{Br}(E)$ since E is pseudo-divergent, a contradiction.

Hence, there is a pseudo-divergent subsequence $E' \subseteq E$ on V_P; in particular, $\{v_P(\phi(s_\nu))\}_{\nu\in\Lambda}$ must be strictly decreasing and thus positive, i.e., $\phi \in P_E$. If $\phi \in PV_E$, then as in the pseudo-convergent case we would have $q^{1/2}V_E \subseteq \phi V_E \subseteq qV_E$ for some $q \in P$, a contradiction; thus, $\phi \notin PV_E$ and $|\eta^{-1}(P)| = 2$.

(ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i) follows exactly as in the proof of Theorem 5.5(b).

(c) As before, suppose P is not maximal. If $\mathrm{Br}(E) = cV_P$, then we consider $\phi(X) = \frac{c}{X-\alpha}$, which gives $|\eta^{-1}(P)| = 2$ by Lemma 5.3; the rest of the reasoning follows as in the proof of Theorem 5.5(c). ☐

Proposition 5.8. *Let E be a pseudo-divergent sequence, and let P be an idempotent prime of V. Then, there are two prime ideals of V_E over P if and only if there is a positive integer k and a $c \in K$ such that $\mathrm{Br}(E)^k$ is cV_P or cP.*

6 Extensions

We now start the proof of our generalization of Ostrowski's Fundamentalsatz (Theorem 8.2): we want to show that, under some hypothesis, we can obtain every extension W of V to $K(X)$ as a valuation domain V_E associated to a pseudo-monotone sequence E contained in K. In order to accomplish this objective, we want to associate to each such an extension W a subset of K which is the analogous of to the set of pseudo-limits of a pseudo-monotone sequence.

Definition 6.1. Let W be an extension of V to $K(X)$. We define the following subsets of K:

$$\begin{aligned}
\mathcal{L}_1(W) :=& \{\alpha \in K \mid w(X-\alpha) \notin \Gamma_v\};\\
\mathcal{L}_2(W) :=& \{\alpha \in K \mid w(X-\alpha) \in \Gamma_v, \text{ and } w(X-\alpha+c) = w(X-\alpha) \text{ if } w(X-\alpha) = v(c)\};\\
\mathcal{L}(W) :=& \mathcal{L}_1(W) \cup \mathcal{L}_2(W).
\end{aligned}$$

Equivalently, $\alpha \in \mathcal{L}_2(W)$ if $w(X-\alpha) = v(c)$ for some $c \in K$, and the image of $\frac{X-\alpha}{c}$ in the residue field of W does not belong to the residue field of V.

Proposition 6.2. *Let W be an extension of V to $K(X)$.*

(a) Suppose K is algebraically closed. Then $V \subset W$ is immediate if and only if $\mathcal{L}(W) = \emptyset$.

(b) If $\alpha, \beta \in \mathcal{L}(W)$ then $w(X-\alpha) = w(X-\beta)$.

(c) If $\mathcal{L}(W) \neq \emptyset$, then exactly one between $\mathcal{L}_1(W)$ and $\mathcal{L}_2(W)$ is nonempty.

(d) If $\mathcal{L}_1(W) \neq \emptyset$ is nonempty, then it is equal to K or to $\alpha + I$ for some $\alpha \in K$ and some (fractional) ideal I.

(e) If $\mathcal{L}_2(W) \neq \emptyset$ is nonempty, then it is equal to $\alpha + cV$ for some $\alpha, c \in K$.

Proof. (a) Suppose K is algebraically closed. If $V \subset W$ is immediate, then $\Gamma_w = \Gamma_v$ (so $\mathcal{L}_1(W) = \emptyset$); furthermore, since $W/M_W = V/M$, also $\mathcal{L}_2(W) = \emptyset$. Conversely, suppose that $V \subset W$ is not immediate. If $\Gamma_v \neq \Gamma_w$, then $w(p) \notin \Gamma_v$ for some $p \in K[X]$, and thus $w(p') \notin \Gamma_v$ for some irreducible factor p' of p; since K is algebraically closed, $p'(X) = X - \alpha$ and $\alpha \in \mathcal{L}_1(W)$. If $\Gamma_v = \Gamma_w$, then $V/M \subsetneq W/M_W$ and this extension must be transcendental (since K is algebraically closed, so is V/M). By the proof of [2, Proposition 2], we can find $\alpha, c \in K$ such that $w(X-\alpha) = v(c)$ and the image of $\frac{X-\alpha}{c}$ is transcendental over V/M; it follows that $\alpha \in \mathcal{L}_2(W)$, which in particular is nonempty.

(b)-(e) If $\mathcal{L}(W) \neq \emptyset$ and $\mathcal{L}(W) \neq K$, let $\alpha \in \mathcal{L}(W)$. Then, if $\beta \in K$ we have:

$$w(X-\beta) = w(X-\alpha+\alpha-\beta) = \begin{cases} w(X-\alpha), & \text{if } v(\alpha-\beta) > w(X-\alpha) \\ v(\alpha-\beta), & \text{if } v(\alpha-\beta) < w(X-\alpha) \\ w(X-\alpha), & \text{if } v(\alpha-\beta) = w(X-\alpha). \end{cases} \tag{2}$$

In particular, if $\alpha \in \mathcal{L}_1(W)$ then $w(X-\beta)$ is equal either to $w(X-\alpha) \notin \Gamma_v$ or to $v(\alpha-\beta)$; in the former case $\beta \in \mathcal{L}_1(W)$, in the latter $\beta \notin \mathcal{L}_1(W)$. In particular, $\mathcal{L}_1(W) = \alpha + \{x \in K \mid v(x) > w(X-\alpha)\}$, and the latter set is an ideal.

If $\alpha \in \mathcal{L}_2(W)$ and $v(\alpha-\beta) \geq w(X-\alpha)$, then $w(X-\beta) = w(X-\alpha) \in \Gamma_v$, so $\beta \in \mathcal{L}_2(W)$ because $(X-\beta)/c = (X-\alpha)/c + (\beta-\alpha)/c$: over the residue field of W $(X-\alpha)/c$ is not in V/M so it follows that the same holds for $(X-\beta)/c$. In particular, $\mathcal{L}_2(W) = \alpha + \{x \in K \mid v(x) \geq w(X-\alpha)\} = \alpha + cV$, while $\mathcal{L}_1(W) = \emptyset$. Note that this argument shows that only one of the sets $\mathcal{L}_i(W)$, $i = 1, 2$, can be non-empty.

In all cases, $w(X-\alpha) = w(X-\beta)$ for all α, β in $\mathcal{L}(W)$ and $w(X-\beta) \leq w(X-\alpha)$ for all $\beta \in K$. $\square$

Proposition 6.3. *Let $E \subset K$ be a pseudo-monotone sequence.*

(a) If E is a strictly pseudo-monotone sequence, then $\mathcal{L}_1(V_E) = \mathcal{L}_E$.

(b) If E is pseudo-stationary, then $\mathcal{L}_2(V_E) = \mathcal{L}_E$.

In both cases, $\mathcal{L}(V_E)$ is the set of pseudo-limits of E in K.

Proof. Suppose first that $E = \{s_\nu\}_{\nu\in\Lambda}$ is a strictly pseudo-monotone sequence. Let $\alpha \in K$, and suppose $w(X-\alpha) = w(c)$ for some $c \in K$. Then, $(X-\alpha)/c$ is a unit of W, and in particular for large ν both $(s_\nu - \alpha)/c$ and $c/(s_\nu - \alpha)$ belong to V. Therefore, $v(s_\nu - \alpha) = v(c)$ for large ν; hence, $w(X-\alpha) \in \Gamma_v$ if and only if $\alpha \notin \mathcal{L}_E$. Thus, $\mathcal{L}_1(V_E) = \mathcal{L}_E$; furthermore, by Proposition 3.7(a), $V_E/M_E = V/M$, and so $\mathcal{L}_2(V_E) = \emptyset$. Hence, $\mathcal{L}(V_E) = \mathcal{L}_E$.

Suppose now that E is pseudo-stationary: then, by Proposition 3.7(e), $v_E = v_{\alpha,\delta_E}$. By Proposition 6.2(e), $\mathcal{L}(V_E) = \mathcal{L}_2(V_E) = \alpha + cV$, where $c \in K$ has value $v_E(X-\alpha) = \delta_E$. By Lemma 2.5 this is precisely $\mathcal{L}_E$. □

Example 6.4. Proposition 6.3 allows to show that there are extensions of V to $K(X)$ which cannot be realized as V_E, for any pseudo-convergent sequence $E \subset K$. For example, consider the following valuation domain of $K(X)$ introduced in [17]:

$$V_\infty = \{\phi \in K(X) \mid \phi(\infty) \in V\},$$

where $\phi(\infty)$ is defined as $\psi(0)$, where $\psi(X) = \phi(1/X)$. Then, V_∞ is the image of $V_0 = \{\phi \in K(X) \mid \phi(0) \in V\}$ under the K-automorphism Φ of $K(X)$ sending X to $1/X$. The valuation domain V_0 is equal to V_F, where $F = \{t_\nu\}_{\nu\in\Lambda}$ is a Cauchy sequence with limit 0; applying Φ to F, by Proposition 3.8 we obtain a pseudo-divergent sequence $E = \{s_\nu = \Phi(t_\nu) = t_\nu^{-1}\}_{\nu\in\Lambda}$ with breadth ideal $\mathrm{Br}(E) = K$ (as $v(t_\nu)$ is cofinal in Γ_v, $v(s_\nu)$ is coinitial). Thus, $V_\infty = V_E$ has $\mathcal{L}(V_\infty) = \mathcal{L}_E = K$, which is different from $\mathcal{L}(V_G) = \mathcal{L}_G$ for every pseudo-convergent sequence G (by Lemma 2.6). In particular, $V_\infty \neq V_G$. Note also that V_∞ is contained in the DVR $K[1/X]_{(1/X)}$ ([17, Proposition 2.2]).

Note that Proposition 6.2(a) is false without the assumption on K: in fact, if $E \subset K$ is a pseudo-convergent sequence of algebraic type without pseudo-limits in K, then, for some extension u of v to $\overline{K}$, by Proposition 6.3 we have $\mathcal{L}(U_E) = \mathcal{L}_E^u \neq \emptyset$, so by contracting down to K we have $\mathcal{L}(V_E) = \mathcal{L}_E = \emptyset$ while $V \subset V_E$ is not immediate by Proposition 3.7.

Proposition 6.5. *Suppose K is algebraically closed, and let W_1, W_2 be two extensions of V to $K(X)$. If either $\mathcal{L}_1(W_1) = \mathcal{L}_1(W_2) \neq \emptyset$ or $\mathcal{L}_2(W_1) = \mathcal{L}_2(W_2) \neq \emptyset$ then $W_1 = W_2$.*

Proof. Let $\mathcal{L} = \mathcal{L}(W_1) = \mathcal{L}(W_2)$; we shall use w to indicate either w_1 or w_2. Fix also $\alpha \in \mathcal{L}$.

Let $\phi \in K(X)$, and write it as $\phi(X) = c\prod_{\gamma\in\Omega}(X-\gamma)^{\epsilon_\gamma}$, where Ω is the multiset of critical points of ϕ, $c \in K$ and $\epsilon_\gamma \in \{-1,+1\}$.

For every $\gamma \notin \mathcal{L}$, by (2) we have $w(X-\gamma) = v(\alpha-\gamma)$; furthermore, if $\gamma_1, \gamma_2 \in \mathcal{L}$ then $w(X-\gamma_1) = w(X-\gamma_2)$. Hence, $w(\phi) = w(\psi)$, where $\psi(X) = d(X-\alpha)^t$ for some $d \in K$, $t \in \mathbb{Z}$ (More precisely, $d = c\prod_{\gamma\in\Omega\setminus\mathcal{L}}(\alpha-\gamma)^{\epsilon_\gamma}$ and $t = \sum_{\gamma\in\Omega\cap\mathcal{L}}\epsilon_\gamma$.) Note that, in particular, we have both $w_1(\phi) = w_1(\psi)$ and $w_2(\phi) = w_2(\psi)$.

If $t = 0$, then $w(\phi) = v(d)$ and so its sign does not depend on whether $w = w_1$ or $w = w_2$; i.e., $\phi \in W_1$ if and only if $\phi \in W_2$. If $t \neq 0$, then $\psi = (e(X-\alpha)^\epsilon)^{|t|}$, where

$e \in K$ is such that $e^{|t|} = d$ and $\epsilon = t/|t|$; thus, $\psi \in W_i$ if and only if $e(X-\alpha)^\epsilon \in W_i$, for $i = 1, 2$, since a valuation domain is integrally closed.

Suppose now that $\alpha \in \mathcal{L}_1(W)$ and $t > 0$. Then,

$$w(e(X-\alpha)) \geq 0 \iff w(X-\alpha) \geq v(e^{-1}) \iff w(X-\alpha+e^{-1}) = v(e^{-1})$$

(since $w(X-\alpha) \notin \Gamma_v$), i.e., if and only if $\alpha - e^{-1} \notin \mathcal{L}_1(W)$. Since $\mathcal{L}_1(W_1) = \mathcal{L}_1(W_2)$, it follows that $w_1(e(X-\alpha)) \geq 0$ if and only if $w_2(e(X-\alpha)) \geq 0$, i.e., $\phi \in W_1$ if and only if $\phi \in W_2$, as claimed. Analogously, if $t < 0$ then

$$w\left(\frac{e}{X-\alpha}\right) \geq 0 \iff w(X-\alpha) \leq v(e) \iff w(X-\alpha+e) = w(X-\alpha),$$

that is, if and only if $\alpha - e \in \mathcal{L}_1(W)$. As before, this implies that $\phi \in W_1$ if and only if $\phi \in W_2$; hence, $W_1 = W_2$.

Suppose now that $\alpha \in \mathcal{L}_2(W)$. If $t > 0$, then $w(e(X-\alpha)) \geq 0$ if and only if $w(X-\alpha) > w(f)$ for all $f \in K$ such that $v(e^{-1}) > v(f)$; that is, if and only if $w(X-\alpha+f) = v(f)$ for all such f. This happens if and only if $\alpha - f \notin \mathcal{L}$ for all these f; since $v(e^{-1}) > v(f)$ depends only on V, it follows as before that $w_1(e(X-\alpha)) \geq 0$ if and only if $w_2(e(X-\alpha)) \geq 0$, i.e., $\phi \in W_1$ if and only if $\phi \in W_2$, as claimed. If $t < 0$ then, in the same way, $w(e/(X-\alpha)) \geq 0$ if and only if $w(X-\alpha) < v(f)$ for all f such that $v(f) < v(e)$; as above, this implies that $\phi \in W_1$ if and only if $\phi \in W_2$. Hence, $W_1 = W_2$. □

Example 6.6. Note that in Proposition 6.5 we can't drop the hypothesis that K is algebraically closed: for example, take $\alpha \in K$ and let $\delta \in \mathbb{Q}\Gamma_v \setminus \Gamma_v$. Let $E \subset K$ be a pseudo-convergent sequence having a pseudo-limit α and such that $\mathrm{Br}(E) = I = \{x \in K \mid v(x) > \delta\}$; by Proposition 6.3(a), $\mathcal{L}_1(V_E) = \alpha + I \neq \emptyset$. Take now the monomial valuation $w = v_{\alpha,\delta}$: then, $\mathcal{L}_1(W) = \alpha + I = \mathcal{L}_1(V_E)$, but $W \neq V_E$ since the value group of w is contained in the divisible hull of the value group of v, while $\Gamma_{v_E} = \Gamma_v \oplus \Delta_E\mathbb{Z}$ is not (by Proposition 3.7 and Lemma 3.6).

Joining the previous propositions, we can prove that if K is algebraically closed then any extension of V to $K(X)$ is in the form V_E for some pseudo-monotone sequence E; however, we postpone this result to Theorem 8.2 in order to cover a more general case.

Proposition 6.7. *Let $E \subset K$ be a pseudo-monotone sequence, and let U be an extension of V to $\overline{K}$. Then U_E is the unique common extension of U and W to $\overline{K}(X)$. Moreover, if $F \subset K$ is another pseudo-monotone sequence such that E and F are either both pseudo-stationary or both strictly pseudo-monotone, then $V_E = V_F$ if and only if $U_E = U_F$.*

Proof. The first claim can be proved in the same way as [19, Theorem 5.5], but we repeat the proof for clarity. Clearly, U_E extends both U and W. Suppose there is another extension U' of U and W to $\overline{K}(X)$: then, by [4, Chapt. VI, §8, 6., Corollaire 1], there is a $K(X)$-automorphism σ of $\overline{K}(X)$ such that $U_E = \sigma(W')$. Let $\rho = \sigma^{-1}$: then,

$$\begin{aligned}\rho(U_E) =& \{\rho \circ \phi \in \overline{K}(X) \mid \phi(s_\nu) \in U \text{ definitively}\} = \\ & \{\rho \circ \phi \in \overline{K}(X) \mid \sigma \circ \rho(\phi(s_\nu)) \in U \text{ definitively}\}.\end{aligned}$$

Since $s_\nu \in K$ and $\rho|_K$ is the identity, $\rho(\phi(s_\nu)) = (\rho \circ \phi)(s_\nu)$; hence,

$$\begin{aligned}\rho(U_E) =& \{\rho \circ \phi \in \overline{K}(X) \mid \sigma((\rho \circ \phi)(s_\nu)) \in U \text{ definitively}\} = \\ & \{\psi \in \overline{K}(X) \mid \sigma(\psi(s_\nu)) \in U \text{ definitively}\}.\end{aligned}$$

In particular, note that $\rho(U_E) = \rho(U)_E$.

Since both U_E and U' are extensions of U, for any $t \in \overline{K}$ we have that $t \in U$ if and only if $\sigma(t) \in U$; in particular, this happens for $t = \psi(s_\nu)$. It follows that $\rho(U_E) = U' = U_E$, as claimed.

We prove now the last claim. One direction is clear, since $V_E = U_E \cap K$ and $V_F = U_F \cap K$. The other implication follows from the previous claim, since U_E is the unique common extension of V_E and U and U_F is the unique common extension of V_F and U. □

6.1 The minimal dominating degree

Recall that, for a pseudo-monotone sequence E, $\mathcal{P}_E$ is the set of monic irreducible polynomials of $K[X]$ with $\mathrm{degdom}_E(p) > 0$ (Definition 3.5). Suppose E is not a pseudo-convergent sequence of transcendental type (so that $\mathcal{P}_E \neq \emptyset$), and let $q \in K[X]$ be an element of minimal degree of $\mathcal{P}_E$.

When E has a pseudo-limit in K (for example, if E is either pseudo-divergent or pseudo-stationary), then $q(X)$ is linear, and thus $\mathrm{degdom}_E(q) = 1$. On the other hand, if E is pseudo-convergent, then $\mathrm{degdom}_E(q)$ may be strictly greater than one (see Example 6.10); the purpose of this section is to understand this quantity. The next result shows that $\mathrm{degdom}_E(q)$ determines the possible dominating degrees of all rational functions.

Lemma 6.8. *Let $E \subset K$ be a pseudo-convergent sequence of algebraic type, and let $q \in \mathcal{P}_E$ be of minimal degree. For every $\phi \in K(X)$, $\mathrm{degdom}_E(\phi)$ is a multiple of $\mathrm{degdom}_E(q)$.*

Proof. By Proposition 3.7, we have $v_E(\phi) = \lambda\Delta_E + \gamma$ for some unique $\lambda \in \mathbb{Z}$ and some $\gamma \in \Gamma_v$, where $\Delta_E = v_E(q)$. Let $c \in K$ be an element of value γ and let $\psi := \phi/(cq^\lambda)$ (with the convention $q^0 = 1$). Then, $v_E(\psi) = 0$, i.e., ψ is a unit of V_E; it follows that $v(\psi(s_\nu))$ is constantly 0, and thus that $v(\phi(s_\nu)) = \lambda v(q(s_\nu)) + \gamma$ for all large ν. By Proposition 3.2, it follows that $\mathrm{degdom}_E(\phi) = \lambda \mathrm{degdom}_E(q)$. □

In order to calculate $\mathrm{degdom}_E(q)$, we use the defect of a finite extension, which we now briefly recall.

Let $K \subseteq L$ be a finite extension, and let w be an extension of v to L; let k be the residue field of V and k' the residue field of W. The *defect* of the valuation w with respect to v is defined as ([11]):

$$d(L|K, w) := \frac{[L^h : K^h]}{(\Gamma_w : \Gamma_v)[k' : k]},$$

where K^h and L^h are the henselization of K and L, respectively, with respect to an extension of w to the algebraic closure $\overline{K} = \overline{L}$. If $K \subseteq L$ is a normal extension, we have $d(L|K,w) = [L : K]/efg$, where $e = (\Gamma_w : \Gamma_v)$, $f = [k' : k]$ and g is the number of extensions of v to L (for example, see [23, Chapter VI, §12, Corollary of Theorem 25]). If the characteristic of k is 0, then $d(L|K,w)$ is always 1; if $\mathrm{char} k = p > 0$, on the other hand, $d(L|K,w)$ is always a power of p. If $d(L|K,w) = 1$, then w is *defectless* over v.

Proposition 6.9. *Let $E \subset K$ be a pseudo-convergent sequence of algebraic type; let $q \in \mathcal{P}_E$ be of minimal degree and let $\alpha \in \overline{K}$ be a root of q which is a pseudo-limit of E with respect to some extension u of v to $K(\alpha)$. Then*

$$\mathrm{degdom}_E(q) = \frac{d(K(\alpha)|K,u)}{d(K(\alpha)(X)|K(X),u_E)}.$$

In particular, if $K \subset K(\alpha)$ is defectless (and in particular if $\mathrm{char} k = 0$), then every such polynomial has exactly one root that is a pseudo-limit of E.

Proof. By [9, Theorem 3], u is an immediate extension of v, and so $\Gamma_v = \Gamma_u$. By Proposition 3.7, $\Gamma_{u_E} = u_E(X-\alpha)\mathbb{Z}\oplus\Gamma_v$ and $\Gamma_{v_E} = \Delta_E\mathbb{Z}\oplus\Gamma_v$, $\Delta_E = v_E(q)$. In particular, $(\Gamma_{u_E} : \Gamma_{v_E}) = \mathrm{degdom}_E(q)$. Hence, the claim reduces to showing that $(\Gamma_{u_E} : \Gamma_{v_E})$ is equal to the quotient between the defect of u with respect to v and the defect of u_E with respect to v_E.

Let $L = K(\alpha)$. By definition, we have

$$\frac{d(L|K,u)}{d(L(X)|K(X),u_E)} = \frac{[L^h : K^h]}{[L(X)^h : K(X)^h]} \cdot \frac{(\Gamma_{u_E} : \Gamma_{v_E})[k'_\alpha : k_\alpha]}{(\Gamma_u : \Gamma_v)[k' : k]},$$

where k'_α and k_α are the residue field of U_E and V_E, respectively. However, by Proposition 3.7, the residue field of V is equal to the residue field of V_E (and the same holds for U and U_E) and thus $[k'_\alpha : k_\alpha] = [k' : k] = 1$; moreover, $(\Gamma_u : \Gamma_v) = 1$ as u is immediate over v. It follows that

$$\frac{d(L|K,u)}{d(L(X)|K(X),u_E)} = \frac{[L^h : K^h]}{[L(X)^h : K(X)^h]} \cdot (\Gamma_{u_E} : \Gamma_{v_E}),$$

and so we only need to show that $[L^h : K^h] = [L(X)^h : K(X)^h]$.

Since $[L^h : K^h] = [L^h(X) : K^h(X)]$, the previous equality is equivalent to $[L^h(X) : K^h(X)] = [L(X)^h : K(X)^h]$. Furthermore, $K(X)^h = K^h(X)^h$ (and likewise for L), and thus we can suppose that K and L are henselian. In this case, u is the only extension of v to L; hence, by Proposition 6.7, u_E is the only extension of v_E to $L(X)$. Since $L(X)$ is a finite extension of $K(X)$, by [11, (11), p. 326] it follows that $[L(X) : K(X)] = [L(X)^h : K(X)^h]$. The claim is proved. □

Example 6.10.

1. Suppose that E is a Cauchy sequence with limit $\alpha \in \overline{K}$. Then, $\mathcal{P}_E$ contains a unique polynomial q of minimal degree (namely, the minimal polynomial of α) and $\mathrm{degdom}_E(q)$ is equal to the inseparable degree of α over K. In particular, $\mathrm{degdom}_E(q) > 1$ if α is not separable over K.

2. Suppose that K is an henselian field with an immediate extension $K \subset K(\theta)$ which is both immediate and Galois; let u be the extension of v to $K(\theta)$. Since the extension is immediate, we can find a pseudo-convergent sequence $E \subset K$ without pseudo-limits in K but such that θ is a pseudo-limit of E with respect to u. For any conjugate θ' of θ over K, there is a K-automorphism σ such that $\theta' = \sigma(\theta)$; then, θ' will be a pseudo-limit of E with respect to $\sigma \circ u$, which is equal to u since K is henselian. Hence, also θ' is a pseudo-limit of E, and thus all conjugates of θ are pseudo-limits of E; that is, $\mathrm{degdom}_E(q)$ is equal to the degree of θ over K. In particular, $\mathrm{degdom}_E(q) > 1$.

   For an explicit (separable) example, let L be a field of characteristic p and let t be an indeterminate. Let $K = \bigcup_{k\geq 1} L((t^{1/p^k}))$: then, K is henselian. Let θ be a root of the Artin-Schreier polynomial $X^p - X - t^{-1}$: then, $K \subset K(\theta)$ is immediate since each extension $L((t^{1/p^k})) \subset L((t^{1/p^k}))(\theta)$ is immediate, and it is Galois since the conjugates of θ are $\theta+1, \theta+2, \ldots, \theta+p-1$.

7 Equivalence of pseudo-monotone sequences

Using the results of the previous sections, we can now tackle the problem of when two pseudo-monotone sequences have the same associated extension of V to $K(X)$.

Proposition 7.1. *Let $E, F \subset K$ be two pseudo-monotone sequences that are either both pseudo-stationary or both strictly pseudo-monotone. Let u be an extension of v to $\overline{K}$. If $\mathcal{L}_E^u \neq \emptyset$, then $V_E = V_F$ if and only if $\mathcal{L}_E^u = \mathcal{L}_F^u$. Furthermore, if $\mathcal{L}_E \neq \emptyset$ then the previous condition is also equivalent to the corresponding one over K.*

Proof. By Proposition 6.7, it is enough to show that $U_E = U_F$ if and only if $\mathcal{L}_E^u = \mathcal{L}_F^u$.

Suppose $\mathcal{L}_E^u \neq \emptyset$. Then $U \subset U_E$ is not immediate by Proposition 3.7, and by Proposition 6.3 $\mathcal{L}_E^u = \mathcal{L}_2(U_E)$ if E is pseudo-stationary and $\mathcal{L}_E^u = \mathcal{L}_1(U_E)$ if E is strictly pseudo-monotone. Hence, if $\mathcal{L}_E^u = \mathcal{L}_F^u$ then also $\mathcal{L}_F^u \neq \emptyset$; if E and F are both pseudo-stationary then $\mathcal{L}_2(U_F) = \mathcal{L}_2(U_E) \neq \emptyset$ and so $U_E = U_F$ by Proposition 6.5, while if E and F are strictly pseudo-monotone the same conclusion holds by the same Proposition. Conversely, if $U_E = U_F$ then $\mathcal{L}_E^u = \mathcal{L}(U_E) = \mathcal{L}(U_F) = \mathcal{L}_F^u$ and so E and F have the same pseudo-limits (in $\overline{K}$).

Suppose now $\mathcal{L}_E \neq \emptyset$. If $\mathcal{L}_E^u = \mathcal{L}_F^u$ then $\mathcal{L}_E = \mathcal{L}_F$. Conversely, if $\mathcal{L}_E = \mathcal{L}_F$, then by Lemma 2.5 $\mathrm{Br}(E) = \mathrm{Br}(F)$. In particular, $\mathrm{Br}_u(E) = \mathrm{Br}_u(F)$ so by the same Lemma $\mathcal{L}_E^u = \mathcal{L}_F^u$. □

Remark 7.2.

1. Note that, under the same assumptions of Proposition 7.1, by Lemma 2.5 E and F have the same set of pseudo-limits (either over K or over $\overline{K}$) if and only if they have the same breadth ideal and they have at least one pseudo-limit in common.

2. It is possible to have $V_E = V_F$ even if E is pseudo-convergent and F is pseudo-divergent: for example, if I is not finitely generated and it is not equal to cM

for any $c \in K$, we can find both a pseudo-convergent sequence E and a pseudo-divergent sequence F such that $I = 0 + I$ is the set of pseudo-limits of E and F (Lemmas 2.5 and 2.6). By Proposition 7.1, $V_E = V_F$.

3. If E, F are pseudo-divergent sequences with $\mathrm{Br}(E) = K = \mathrm{Br}(F)$ (that is, if the gauges of E, F are not bounded from below, see §2.3.2), then $\mathcal{L}_E = K = \mathcal{L}_F$, and so $V_E = V_F$. This extension is exactly the valuation domain V_∞ considered in Example 6.4.

Let E, F be two Cauchy sequences with limits $x_E, x_F \in K$, respectively. By Proposition 7.1, $V_E = V_F$ if and only if $x_E = x_F$; by extending v to the completion $\widehat{K}$, we see that this happen even if the limits are not in K. Thus, the condition $V_E = V_F$ generalizes the notion of equivalence between Cauchy sequences: for this reason, we say that two pseudo-monotone sequences are *equivalent* if $V_E = V_F$. We now want to characterize this notion in a more intrinsic way, but we need to distinguish between the different types. The first result, involving pseudo-convergent sequences, is a generalization of [19, Theorem 5.4].

Proposition 7.3. *Let $E = \{s_\nu\}_{\nu\in\Lambda}, F = \{t_\mu\}_{\mu\in\Lambda} \subset K$ be pseudo-convergent sequences. Then E and F are equivalent if and only if* $\mathrm{Br}(E) = \mathrm{Br}(F)$ *and, for every $\kappa \in \Lambda$, there are $\nu_0, \mu_0 \in \Lambda$ such that, whenever $\nu \geq \nu_0$, $\mu \geq \mu_0$, we have $v(s_\nu - t_\mu) > v(t_\rho - t_\kappa)$, for any $\rho > \kappa$.*

Proof. By Proposition 7.1, without loss of generality we can suppose that K is algebraically closed. Let $\{\delta_\nu\}_{\nu\in\Lambda}, \{\delta'_\nu\}_{\nu\in\Lambda}$ be the gauges of E and F, respectively. We will use the following remark: $\mathrm{Br}(E) \subseteq \mathrm{Br}(F)$ if and only if for each $\mu \in \Lambda$ there exists $\nu \in \Lambda$ such that $\delta'_\mu \leq \delta_\nu$.

We assume first that the conditions of the statement hold. Suppose that E is of algebraic type: then, E has a pseudo-limit $\beta \in K$. Fix $\mu \in \Lambda$. By the above remark, there exists $\nu_0 \in \Lambda$ such that for all $\nu \geq \nu_0$, $\delta_\nu > \delta'_\mu$. There also exist $\iota_0, \mu_0 \in \Lambda$ such that for all $\nu \geq \iota_0$, $\kappa \geq \mu_0$, we have $v(s_\nu - t_\kappa) > \delta'_\mu$. Then, for $\nu \geq \max\{\iota_0, \nu_0\}$ and $\kappa > \max\{\mu, \mu_0\}$ we have

$$v(\beta - t_\mu) = v(\beta - s_\nu + s_\nu - t_\kappa + t_\kappa - t_\mu) = \delta'_\mu$$

so that β is a pseudo-limit of F. Therefore, F is of algebraic type and $\mathcal{L}_E \subseteq \mathcal{L}_F$. The reverse inclusion is proved symmetrically, and $V_E = V_F$ follows from Proposition 7.1.

Suppose now that E is of transcendental type: by the previous part of the proof, also F must be of transcendental type. We can repeat the previous reasoning by using X instead of β (since X is a pseudo-limit of E with respect to v_E: see [19, Theorem 3.7] or Theorem 3.4); this proves that X is a pseudo-limit of F with respect to v_F. The fact that $V_E = V_F$ now follows from [9, Theorem 2].

Assume now that $V_E = V_F$. Suppose first that E is of algebraic type: then, $\mathcal{L}_E \neq \emptyset$, and by Proposition 7.1 we must have $\mathcal{L}_F = \mathcal{L}_E$, and thus F is also of algebraic type. In particular, $\mathrm{Br}(E) = \mathrm{Br}(F)$. Let $\alpha \in \mathcal{L}_E = \mathcal{L}_F$. Then,

$$v(s_\nu - t_\mu) = v(s_\nu - \alpha + \alpha - t_\mu) \geq \min\{\delta_\nu, \delta'_\mu\}.$$

By the remark, for every κ there is an ι_0 such that $\delta_{\iota_0} > \delta'_\kappa$; choosing $\mu_0 > \kappa$ we have that E and F satisfy the conditions of the statement.

Suppose now that E is of transcendental type; as before, this implies that also F is of transcendental type. Without loss of generality we may suppose that $\mathrm{Br}(F) \subseteq \mathrm{Br}(E)$. If this containment is strict, then there exists a $c \in \mathrm{Br}(E) \setminus \mathrm{Br}(F)$. Then, $\frac{c}{X-\alpha}$ is in V_E for each $\alpha \in K$ (because X is a pseudo-limit of E with respect to v_E and E has no pseudo-limits in K). On the other hand, for every ν we have $\frac{c}{X-t_\nu} \notin V_F$, a contradiction. Therefore $\mathrm{Br}(E) = \mathrm{Br}(F)$. We know that X is a pseudo-limit of F with respect to v_F, so that $\{v_F(X - t_\mu)\}_{\mu\in\Lambda}$ is a (definitively) strictly increasing sequence. In particular, since $V_E = V_F$ implies that $\lambda \circ v_E = v_F$ for some isomorphism of totally ordered groups $\lambda : \Gamma_{v_E} \to \Gamma_{v_F}$, it follows that $\{v_E(X - t_\mu)\}_{\mu\in\Lambda}$ is a (definitively) strictly increasing sequence, so that X is a pseudo-limit of F with respect to v_E. Thus $v_E(X - t_\mu) = \delta'_\mu$, for each $\mu \in \Lambda$ (sufficiently large). The proof now proceeds as above, replacing a pseudo-limit α of E and F by X (which is a pseudo-limit of E and F with respect to v_E). Hence, the conditions of the statement holds. □

The cases of pseudo-divergent and pseudo-stationary sequences are very similar, with the further simplification that in these cases we do not need to consider sequences of transcendental type (which do not exist).

Proposition 7.4. *Let $E = \{s_\nu\}_{\nu\in\Lambda}, F = \{t_\mu\}_{\mu\in\Lambda} \subset K$ be pseudo-divergent sequences. Then E and F are equivalent if and only if $\mathrm{Br}(E) = \mathrm{Br}(F)$ and there exist $\nu_0, \mu_0 \in \Lambda$ such that for all $\nu \geq \nu_0, \mu \geq \mu_0$ there exists $\kappa \in \Lambda$ such that $v(s_\nu - t_\mu) \geq v(t_\rho - t_\kappa)$, for any $\rho < \kappa$.*

Note that the above condition amounts to say that $s_\nu - t_\mu$ is definitively in the breadth ideal $\mathrm{Br}(E) = \mathrm{Br}(F)$.

The following is the analogous result for pseudo-stationary sequences.

Proposition 7.5. *Let $E = \{s_\nu\}_{\nu\in\Lambda}, F = \{t_\mu\}_{\mu\in\Lambda} \subset K$ be pseudo-stationary sequences with breadth δ_E and δ_F, respectively. Then E and F are equivalent if and only if $\delta_E = \delta_F = \delta$ and $v(s_\nu - t_\mu) \geq \delta$ for all $\nu, \mu \in \Lambda$.*

Proof. The conditions of the statement say (using Lemma 2.5) that $E \subset \mathcal{L}_F$ and $F \subset \mathcal{L}_E$. By the same Lemma, this is equivalent to $\mathcal{L}_E = \mathcal{L}_F$, which is equivalent to $V_E = V_F$ by Proposition 7.1. □

8 A generalized Fundamentalsatz

In general, not all the extensions of V to $K(X)$ can be realized via a pseudo-monotone sequence contained in K. For example, let V be the ring of p-adic integers $\mathbb{Z}_p$, for some prime $p \in \mathbb{Z}$. It is not difficult to see that for $\alpha \in \overline{\mathbb{Q}_p} \setminus \mathbb{Q}_p$, the valuation domain $V_{p,\alpha} = \{\phi \in \mathbb{Q}_p(X) \mid v_p(\phi(\alpha)) \geq 0\}$ of $\mathbb{Q}_p(X)$ is not of the form V_E, for any pseudo-monotone sequence $E \subset \mathbb{Q}_p$ (for example, by Proposition 3.7 and [17, Theorem 3.2], see also the proof of Theorem 8.2).

In this section, we show when all extensions of V to $K(X)$ are induced by pseudo-monotone sequences in K. We premit a lemma which allows us to reduce to the algebraically closed case.

Lemma 8.1. *Let L be an extension of K and U a valuation domain of L lying over V such that $\Gamma_u = \Gamma_v$. Let $F \subset L$ be a pseudo-monotone sequence with respect to u having a pseudo-limit $\beta \in K$. Then:*

(a) if F is strictly pseudo-monotone, there is a sequence $E \subset K$ of the same kind of F that is equivalent to F (with respect to u);

(b) if F is pseudo-stationary and the residue field of V is infinite, there is a pseudo-stationary sequence $E \subset K$ that is equivalent to F (with respect to u).

Proof. Let $F = \{t_\nu\}_{\nu\in\Lambda}$.

(a) For every ν, there is a $c_\nu \in K$ such that $u(t_\nu - \beta) = u(c_\nu)$; let $s_\nu = c_\nu + \beta$ and let $E = \{s_\nu\}_{\nu\in\Lambda}$. Then, $E \subset K$ (since $\beta \in K$) and $u(s_\mu - s_\nu) = u(t_\mu - t_\nu)$ for every $\mu > \nu$, so E is pseudo-monotone of the same kind of F and the gauges of E and F coincide, so $\mathrm{Br}_u(E) = \mathrm{Br}_u(F)$. By Proposition 7.1, E and F are equivalent.

(b) Since $u(t_\nu - \beta) = \delta \in \Gamma_v$ and the residue field of V is infinite, we can find an infinite set $\{c_\nu\}_{\nu\in\Lambda} \subset V$ such that $u(c_\nu - \beta) = \delta$ and such that $u(c_\nu - c_\mu) = \delta$ for every $\nu \neq \mu$. Setting $s_\nu = c_\nu + \beta$, as in the previous case we can take $E = \{s_\nu\}_{\nu\in\Lambda}$, and E and F are equivalent by Proposition 7.1. □

Theorem 8.2. *Let V be a valuation domain with quotient field K. Then, every extension W of V to $K(X)$ is of the form $W = V_E$ for some pseudo-monotone sequence $E \subset K$ if and only if $\widehat{K}$ is algebraically closed. In this case, we have*

(a) if $V \subset W$ is immediate, then E is necessarily a pseudo-convergent sequence of transcendental type.

(b) if $V \subset W$ is not immediate, then:

(b1) if $\mathcal{L}(W) = \emptyset$, then E is a pseudo-convergent Cauchy sequence of algebraic type whose limit is in $\widehat{K} \setminus K$;

(b2) if $\mathcal{L}_1(W) = \alpha + I \neq \emptyset$ and I is a divisorial fractional ideal, then E can be taken to be pseudo-convergent of algebraic type;

(b3) if $\mathcal{L}_1(W) = \alpha + I \neq \emptyset$ and I is not a principal ideal, then E can be taken to be pseudo-divergent.

(b4) if $\mathcal{L}_2(W) = \alpha + I \neq \emptyset$, then E is necessarily a pseudo-stationary sequence.

Note that, since every nondivisorial ideal is nonprincipal, cases (b2) and (b3) cover all possibilities. Furthermore, these two cases are not mutually exclusive: see Remark 7.2.

Proof. Throughout the proof we will use the fact that $\widehat{K}$ is algebraically closed if and only if $\overline{K}$ embeds in $\widehat{K}$ (which in turn follows from the fact that the completion of an algebraically closed field is algebraically closed [21, §15.3, Theorem 1]). Loosely speaking, this condition holds if and only if K is dense in its algebraic closure $\overline{K}$.

Suppose that $\widehat{K}$ is not algebraically closed. Then by above there exists $\alpha \in \overline{K}$ such that $K(\alpha)$ cannot be embedded into $\widehat{K}$, that is, α is not the limit of any Cauchy sequence in K. Let U be an extension of V to $\overline{K}$ and let $F \subset \overline{K}$ be a pseudo-convergent Cauchy sequence with limit α. Let $W = U_F \cap K$: we claim that $W \neq V_E$ for any pseudo-monotone sequence E. Indeed, if $W = V_E$ for some pseudo-monotone sequence $E \subset K$, by Proposition 6.7 U_E is the only common extension of U and V_E to $\overline{K}(X)$, so that $U_E = U_F$. By Proposition 7.1, we must have $\mathcal{L}_E^u = \mathcal{L}_F^u = \{\alpha\}$ and $\mathrm{Br}_u(E) = \mathrm{Br}_u(F) = (0)$ and thus $\mathcal{L}_E = \mathcal{L}_E^u \cap K = \emptyset$; hence, $E \subset K$ should be a pseudo-convergent Cauchy sequence with limit α (Lemma 2.5). However, this is impossible by the choice of α, and so $W \neq V_E$ for any pseudo-monotone sequence E.

Suppose now that $\widehat{K}$ is algebraically closed, and let $\mathcal{W}$ be a common extension of $\widehat{V}$ and W to $\widehat{K}(X)$.

If $\widehat{V} \subset \mathcal{W}$ is immediate, then also $V \subset W$ is immediate (since $V \subset \widehat{V}$ is); by Kaplansky's Theorem [9, Theorem 2], there is a pseudo-convergent sequence $E \subset K$ such that $W = V_E$.

Suppose $\widehat{V} \subset \mathcal{W}$ is not immediate. By Proposition 6.2(a), $\mathcal{L}(\mathcal{W}) \subseteq \widehat{K}$ is nonempty, say equal to $\alpha + J$ for some $\alpha \in \widehat{K}$ and some J that is either a fractional ideal of $\widehat{V}$ or the whole $\widehat{K}$.

If $J = (0)$ let $E \subset K$ be a pseudo-convergent Cauchy sequence having limit α: then, $\mathcal{L}(\widehat{V}_E) = \mathcal{L}_1(\widehat{V}_E) = \{\alpha\} = \mathcal{L}_1(\mathcal{W}) = \mathcal{L}(\mathcal{W})$, and by Proposition 6.5 it follows that $\mathcal{W} = \widehat{V}_E$. Hence, $W = \mathcal{W} \cap K = \widehat{V}_E \cap K = V_E$. In particular, if $\alpha \in K$ then $\mathcal{L}(W) = \{\alpha\}$, while if $\alpha \in \widehat{K} \setminus K$ then $\mathcal{L}(W) = \emptyset$; furthermore, by Proposition 3.7 if $V \subset W$ is not immediate then E must be a sequence of algebraic type.

Suppose now that $J \neq (0)$. Then, the open set $\alpha + J$ must contain an element β of K, and in particular $\alpha + J = \beta + J$. Using Lemma 2.6, we construct a pseudo-monotone sequence $F \subset \widehat{K}$ with breadth ideal J and with β as pseudo-limit, with the following properties:

- if $\mathcal{L}_1(\mathcal{W}) \neq \emptyset$ and J is a strictly divisorial fractional ideal, we take F to be a pseudo-convergent sequence;
- if $\mathcal{L}_1(\mathcal{W}) \neq \emptyset$ and J is a nondivisorial fractional ideal, we take F to be a pseudo-divergent sequence (note that, in this case, $J = c\widehat{M}$ is not principal);
- if $\mathcal{L}_1(\mathcal{W}) = \widehat{K}$, we take F to be a pseudo-divergent sequence whose gauge is coinitial in Γ_v;
- if $\mathcal{L}_2(\mathcal{W}) \neq \emptyset$, we take F to be a pseudo-stationary sequence.

Note that the first case falls in case (b2), the second and the third in (b3) and the fourth in (b4).

In all cases, $\mathcal{W} = \widehat{V}_F$ by Proposition 6.5 (in the first three cases using $\mathcal{L}_1$ and in the last one using $\mathcal{L}_2$). Since $V \subset \widehat{V}$ is immediate, we can apply Lemma 8.1 to find a pseudo-monotone sequence $E \subset K$ that is equivalent to F; hence, $V_E = \widehat{V}_F \cap K = \mathcal{W} \cap K = W$. The theorem is now proved. $\square$

Remark 8.3. In particular, by Proposition 3.7 and the main Theorem 8.2, if $\widehat{K}$ is algebraically closed, then every extension of V to $K(X)$ which is not immediate is a monomial valuation. This result was already known to hold but only with the stronger assumption that K is algebraically closed, see [1, pp. 286-289].

We remark that a more direct approach to the proof of Theorem 8.2 can be given by considering the set $M_w = \{w(X-a) \mid a \in K\}$. If M_w has no maximum, then, exactly as in the original proof of Ostrowski, we can extract from M_w a cofinal sequence which determines a pseudo-convergent sequence E in K of transcendental type such that $W = V_E$. If instead M_w has a maximum $\Delta_w = w(X-a_0)$, then, following again Ostrowski's proof, one can show that W is a monomial valuation of the form V_{a_0,Δ_w}: according to whether Δ_w is or not in Γ_v (and, in the latter case, on the properties of the cut induced by Δ_w on Γ_v), we can find a pseudo-monotone sequence $E \subset K$ with a_0 as pseudo-limit and such that $W = V_E$. This approach can be connected to the one given above by noting that $M_w \cap \Gamma_v = v(J)$ (where J is the ideal defined in the proof of Theorem 8.2), and that if Δ_w exists then we have $\{a \in K \mid w(X-a) = \Delta_w\} = \mathcal{L}(W)$.

When M_w has a maximum and V has rank 1, Ostrowski proved in his Fundamentalsatz [16, p. 379] that the rank one valuation associated to W can be realized through a pseudo-convergent sequence $F = \{s_\nu\}_{\nu\in\Lambda} \subset K$ by means of the map defined as $w_F(\phi) = \lim_{\nu\to\infty} v(\phi(s_\nu))$, for each $\phi \in K(X)$ (where the limit is taken in $\mathbb{R}$). If $W = V_E$, where $E \subset L$ is a pseudo-stationary sequence, as in Theorem 8.2(b4), then E and F have the same set of pseudo-limits, and in particular they have the same breadth. Furthermore, by Proposition 9.1 below, in this case we have $V_F \subset W = V_E$. See also [19] for other results regarding the valuation w_F introduced by Ostrowski.

An immediate corollary of Theorem 8.2 is that, for any field K, if U is an extension of V to $\overline{K}$, then every extension W of V to $K(X)$ can be written as the contraction of U_E to $K(X)$, namely $U \cap K(X)$, where $E \subset \overline{K}$ is a pseudo-monotone sequence with respect to U; furthermore, in view of the examples above, we cannot always choose E to be contained in K.

Remark 8.4. The hypothesis that $\widehat{K}$ is algebraically closed is weaker than the hypothesis that K is algebraically closed; we give a few examples.

1. Let $K = \overline{\mathbb{Q}} \cap \mathbb{R}$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$, and let V be an extension to K of $\mathbb{Z}_{(5\mathbb{Z})}$. Then, i belongs to the completion $\widehat{K}$, since the polynomial X^2+1 has a root in $\mathbb{Z}/5\mathbb{Z}$; therefore, $K(i) = \overline{\mathbb{Q}}$ can be embedded into $\widehat{K}$, so $\widehat{K}$ is algebraically closed while K is not.

2. If K is separably closed, then $\widehat{K}$ is algebraically closed (it is enough to adapt the proof of [20, Chapter 2, (N)] to the general case).

3. If V has rank 1, then $\widehat{K}$ is algebraically closed if and only if the residue field k of V is algebraically closed and the value group Γ_v is divisible. Indeed, these two

conditions are necessary, since completion preserves value group and residue field. Conversely, suppose that the two conditions hold. When V has rank 1 then $\widehat{K}$ is henselian, and thus $\widehat{v}$ has a unique extension to the algebraic closure of $\widehat{K}$. By [20, Chapter 2, (N)], $\widehat{K}$ is algebraically closed. This happens, in particular, if K is separably closed ([7, Theorem 3.2.11]).

9 Geometrical interpretation

Throughout this section, we suppose that the maximal ideal M of V is not finitely generated, and that its residue field k is infinite. We also fix a $c \in K$ and an $\alpha \in K$. For any $\delta \in \Gamma_v$, we denote $B(\alpha, \delta) = \{x \in K \mid v(\alpha - x) \geq \delta\}$ and $\mathring{B}(\alpha, \delta) = \{x \in K \mid v(\alpha - x) > \delta\}$ the closed and open ball (respectively) of center α and radius δ.

By Lemma 2.6, we can find both a pseudo-convergent sequence E and a pseudo-stationary sequence F such that $\mathcal{L}_E = \mathcal{L}_F = \alpha + cV = B(\alpha, \delta)$, where $\delta = v(c)$; furthermore, again by Lemma 2.6, for every $z \in \alpha + cV$ we can find a pseudo-divergent sequence D_z such that $\mathcal{L}_{D_z} = z + cM = \mathring{B}(z, \delta)$. Note that by Lemma 2.5 $D_z \subset \mathring{B}(z, \delta)$. In geometrical terms, E and F are associated to the closed ball $B(\alpha, \delta)$, while each D_z is associated to the open ball $\mathring{B}(z, \delta)$, which is contained in $B(\alpha, \delta)$ and has the same radius. In the next proposition we show the containments among the valuation domains associated to these sequences.

Proposition 9.1. *Preserve the notation above. Then, V_F properly contains both V_E and V_{D_z}, for every $z \in B(\alpha, \delta)$. Furthermore, $V_E \neq V_{D_z}$ for every such a z and $V_{D_z} = V_{D_w}$ if and only if $z - w \in cM$.*

Proof. Let U be an extension of V to $\overline{K}$ and let $z \in B(\alpha, \delta)$: then

$$\mathcal{L}^u_{D_z} = z + cM_U \subsetneq \mathcal{L}^u_E = \mathcal{L}^u_F = \alpha + cU$$

Let $\phi \in K(X)$ and, for any sequence G, let λ_G be the dominating degree of ϕ with respect to G.

Since $\mathcal{L}^u_E = \mathcal{L}^u_F$, we have $\lambda_E = \lambda_F$; if $E = \{s_\nu\}_{\nu\in\Lambda}$ and $\{\delta_\nu\}_{\nu\in\Lambda}$ is the gauge of E, by Proposition 3.2 for large ν we have $v(\phi(s_\nu)) = \lambda_E\delta_\nu + \gamma$, where $\gamma = u\left(\frac{\phi}{\phi_S}(\alpha)\right)$. If $\phi \in V_E$, then $v(\phi(s_\nu)) \geq 0$ for all ν sufficiently large; since $\delta_\nu \nearrow \delta$, it follows that $\lambda_E\delta + \gamma \geq 0$. However, if $F = \{t_\nu\}_{\nu\in\Lambda}$, then applying again Proposition 3.2 we have $v(\phi(t_\nu)) = \lambda_F\delta + \gamma = \lambda_E\delta + \gamma$, where γ is the same as the previous case; it follows that $v(\phi(t_\nu)) \geq 0$ for large ν, i.e., $\phi \in V_F$.

Fix now $z \in B(\alpha, \delta)$ and let $D_z = \{r_\nu\}_{\nu\in\Lambda}$. Let $\{\delta'_\nu\}_{\nu\in\Lambda}$ be the gauge of D_z; by mimicking the proof of Proposition 3.2, we have

$$\phi(X) = d \prod_{\alpha\in\mathcal{L}^u_{D_z}\cap S} (X-\alpha)^{\epsilon_\alpha} \prod_{\beta\in(\mathcal{L}^u_F\setminus\mathcal{L}^u_{D_z})\cap S} (X-\beta)^{\epsilon_\beta} \prod_{\gamma\notin\mathcal{L}^u_F\cap S} (X-\gamma)^{\epsilon_\gamma},$$

for some $d \in K$, where S is the multiset of critical points of ϕ. Hence, for large ν, $v(\phi(r_\nu)) = \lambda_{D_z}\delta'_\nu + (\lambda_F - \lambda_{D_z})\delta + \gamma$. As in the previous case, if $\phi \in V_{D_z}$ then $v(\phi(r_\nu)) \geq 0$ for large ν, and so $0 \leq \lambda_F\delta + \gamma = v(\phi(t_\nu))$, i.e., $\phi \in V_F$.

The last two claims follow from Lemma 2.5 and Proposition 7.1 by comparing the set of the pseudo-limits of the sequences involved. $\square$

Consider now the quotient map $\pi : V_F \longrightarrow V_F/M_F$. By Proposition 3.7(c), $V_F/M_F \simeq k(t)$, where t is the image of $\frac{X-\alpha}{c}$. Let W be either V_E or V_{D_z} for some $z \in B(\alpha,\delta)$: then, $M_F \subset W$, and thus we can consider the quotient $\pi(W) = W/M_F$, obtaining the following commutative diagram:

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d, "\pi"'] & W \arrow[r, hook] \arrow[d, "\pi"'] & V_F \arrow[d, "\pi"'] \\ V/M = k \arrow[r, hook] & W/M_F \arrow[r, hook] & k(t). \end{tikzcd} \tag{3}$$

In particular, W/M_F is a (proper) valuation domain of $k(t)$ containing k: hence, by [6, Chapter 1, §3], W/M_F must be equal either to $k[t]_{(f(t))}$, for some irreducible polynomial $f \in k[t]$, or to $k[1/t]_{(1/t)}$. In particular, π induces a one-to-one correspondence between the valuation domains of $k(t)$ containing k and the valuation domains of $K(X)$ contained in V_F. The strictly pseudo-monotone sequences we considered above are exactly the linear case, as we show next.

Proposition 9.2. *Preserve the notation above. Then:*

(a) $\pi(V_E) = k[1/t]_{(1/t)}$*;*

(b) $\pi(V_{D_z}) = k[t]_{(t-\theta(z))}$*, where* $\theta(z) = \pi\left(\frac{z-\alpha}{c}\right)$*;*

(c) $\pi^{-1}(k[t]_{(t-x)}) = V_{D_{\alpha+yc}}$*, where y is an element of V satisfying* $\pi(y) = x$*.*

Proof. Let $\phi(X) = \frac{X-\alpha}{c}$: then, as in the previous discussion, $t = \pi(\phi)$. The ring $k[1/t]_{(1/t)}$ is the only valuation domain of $k(t)$ containing k such that $1/t$ belongs to the maximal ideal: hence, in order to show that $\pi(V_E) = k[1/t]_{(1/t)}$ we only need to show that $1/\phi \in M_E$. This follows immediately from the fact that $v(\phi(s_\nu)) = \delta_\nu - \delta < 0$, where $E = \{s_\nu\}_{\nu\in\Lambda}$ and $\{\delta_\nu\}_{\nu\in\Lambda}$ is the gauge of E.

Analogously, in order to show that $\pi(V_{D_z}) = k[t]_{(t-\theta(z))}$, we need to show that $t-\theta(z)$ is in the maximal ideal of $\pi(V_{D_z})$ or, equivalently, that

$$\phi(X) - \frac{z-\alpha}{c} = \frac{X-z}{c} \in M_{D_z}.$$

This is an immediate consequence of the definition of z and c, and the claim is proved.

The last point follows by the fact that $\theta(\alpha+yc) = \pi\left(\frac{\alpha+yc-\alpha}{c}\right) = \pi(y) = x$. $\square$

If k is algebraically closed (in particular, if K is algebraically closed), then all irreducible polynomials of $k[t]$ are linear; thus, Proposition 9.2 describe all subextensions of V_F. When k is not algebraically closed, on the other hand, it follows that some of the extensions of k to $k(t)$ cannot be obtained by pseudo-divergent sequences contained in K; however, we can construct them by using pseudo-divergent sequences in $\overline{K}$ with respect to a fixed extension of V.

Given an extension u of v to $\overline{K}$ we denote by $\mathcal{D}(U)$ the *decomposition group* of U in $\mathrm{Gal}(\overline{K}/K)$, that is, $\mathcal{D}(U) = \{\sigma \in \mathrm{Gal}(\overline{K}/K) \mid \sigma(U) = U\}$.

Proposition 9.3. *Let W be an extension of V to $K(X)$ which is properly contained in V_F, and suppose that $\pi(W) = k[t]_{(f(t))}$ for some nonlinear irreducible $f \in k[t]$. Let u be an extension of v to $\overline{K}$.*

(a) There exists $z \in \mathcal{L}_F^u$ such that $W = U_{D_z} \cap K(X)$, where $D_z \subset \mathring{B}_u(z, \delta) \subset \overline{K}$ is pseudo-divergent.

Let $\overline{\pi} : U_F \to \overline{k}(t)$ be the canonical residue map.

(b) $\overline{\theta}(z) = \overline{\pi}\left(\frac{z-\alpha}{c}\right)$ is a zero of $f(t)$.

(c) Let $z, w \in \mathcal{L}_F^u$. Then the following are equivalent:

(i) $U_{D_z} \cap K(X) = U_{D_w} \cap K(X)$;

(ii) $\overline{\theta}(z)$ and $\overline{\theta}(w)$ are conjugate over k;

(iii) $\rho(z) - w \in cM_U$ for some $\rho \in \mathcal{D}(U)$.

In particular, the number of extensions of W to $\overline{K}(X)$ is equal to the number of distinct roots of f in $\overline{k}$.

Proof. Let $\mathcal{W}$ be an extension of W to $\overline{K}(X)$ and let $U = \mathcal{W} \cap \overline{K}$; then, U is an extension of V. The diagram (3) lifts to

$$\begin{array}{ccccc} U & \hookrightarrow & \mathcal{W} & \hookrightarrow & U_F \\ \downarrow{\scriptstyle\overline{\pi}} & & \downarrow{\scriptstyle\overline{\pi}} & & \downarrow{\scriptstyle\overline{\pi}} \\ U/M_U = \overline{k} & \hookrightarrow & \mathcal{W}/M_U & \hookrightarrow & \overline{k}(t). \end{array} \tag{4}$$

By Proposition 9.2, $\mathcal{W}$ is equal to U_{D_z}, for some $z \in \mathcal{L}_F^u = \alpha + cU$, and thus $W = U_{D_z} \cap K(X)$, as wanted.

(b) If U_{D_z} is an extension of W to $\overline{K}(X)$, then $\overline{\pi}(U_{D_z}) = \overline{k}[t]_{(t-\overline{\theta}(z))}$ is an extension of $\pi(W) = k[t]_{(f(t))}$ to $\overline{k}(t)$. It is straightforward to see that this implies that $t - \overline{\theta}(z)$ is a factor of $f(t)$ in $\overline{k}[t]$, i.e., that $\overline{\theta}(z)$ is a zero of $f(t)$.

(c) The equivalence of (i) and (ii) follows from the previous point.

(ii) $\iff$ (iii) There is a surjective map from the decomposition group $\mathcal{D}(U)$ of U to the Galois group $\mathrm{Gal}(\overline{k}/k)$, where $\rho \in \mathcal{D}(U)$ goes to the map $\overline{\rho}$ sending $x \in k$ to $\pi(\rho(y))$, where y satisfies $\pi(y) = x$ [4, Chapitre V, §2.2, Proposition 6(ii)]. Hence, if $\overline{\theta}(z)$ and $\overline{\theta}(w)$ are conjugates there is a $\overline{\rho} \in \mathrm{Gal}(\overline{k}/k)$ such that $\overline{\rho}(\overline{\theta}(z)) = \overline{\theta}(w)$, and we have

$$\begin{aligned} \overline{\rho}\left(\overline{\pi}\left(\frac{z-\alpha}{c}\right)\right) = \overline{\pi}\left(\frac{w-\alpha}{c}\right) &\iff \overline{\pi}\left(\rho\left(\frac{z-\alpha}{c}\right)\right) = \overline{\pi}\left(\frac{w-\alpha}{c}\right) \\ &\iff \rho\left(\frac{z-\alpha}{c}\right) - \frac{w-\alpha}{c} \in M_U. \end{aligned}$$

Since $\alpha, c \in K$, the last condition holds if and only if $\rho(z) - w \in cM_U$. Conversely, if $\rho(z) - w \in cM_U$ then we can follow the same reasoning in the opposite order, and so $\overline{\theta}(z)$ and $\overline{\theta}(w)$ are conjugate over k. $\square$

We conclude by reproving Ostrowski's Fundamentalsatz. Recall that, if V has rank 1, we can always consider v as a (not necessarily surjective) map from $K \setminus \{0\}$ to $\mathbb{R}$.

Theorem 9.4. *Suppose that v is a valuation of rank 1 and K is algebraically closed. Let w be an extension of v to $K(X)$ of rank 1. Then the following hold:*

(a) there is a pseudo-convergent sequence $E = \{s_\nu\}_{\nu\in\Lambda} \subset K$ such that

$$w(\phi) = \lim_{\nu\to\infty} v(\phi(s_\nu))$$

for every nonzero $\phi \in K(X)$;

(b) if $V \subset W$ is not immediate, there is also a pseudo-divergent sequence $F = \{t_\nu\}_{\nu\in\Lambda}$ such that

$$w(\phi) = \lim_{\nu\to\infty} v(\phi(t_\nu))$$

for every nonzero $\phi \in K(X)$.

Proof. If $V \subset W$ is immediate, then by [9, Theorems 1 and 3] $W = V_E$ for some pseudo-convergent sequence E of transcendental type and $w(\phi) = v_E(\phi) = v(\phi(s_\nu))$ for $\nu \geq N(\phi)$.

Suppose now that $V \subset W$ is not immediate. By Theorem 8.2, there is a pseudo-monotone sequence $G \subset K$ such that $W = V_G$. We distinguish two cases.

Suppose first that G is pseudo-stationary. Then, $\mathrm{Br}(G) = cV$, and $v_G(\phi) = \lambda_\phi\delta + \gamma$, where $\lambda_\phi = \mathrm{degdom}_G(\phi)$, $\delta = v(c)$ and $\gamma = v\left(\frac{\phi}{\phi_S}(\beta)\right)$ for some pseudo-limit β of G in K. Let $E = \{s_\nu\}_{\nu\in\Lambda} \subset K$ be a pseudo-convergent sequence such that $\mathcal{L}_E = \beta + cV = \mathcal{L}_G$ (Lemma 2.6); then, $\mathrm{degdom}_E(\phi) = \mathrm{degdom}_G(\phi)$ and the gauge $\{\delta_\nu\}_{\nu\in\Lambda}$ of E tends to δ, the gauge of G. By Proposition 3.2,

$$\lim_{\nu\to\infty} v(\phi(s_\nu)) = \lim_{\nu\to\infty}(\lambda_\phi\delta_\nu + \gamma) = \lambda_\phi\left(\lim_{\nu\to\infty}\delta_\nu\right) + \gamma = \lambda_\phi\delta + \gamma = v_G(\phi) = w(\phi),$$

and the claim is proved. In the same way, we can find a pseudo-divergent sequence $F = \{t_\nu\}_{\nu\in\Lambda} \subset K$ such that $\mathcal{L}_F = \beta + cM$; as in the proof of Proposition 9.1, setting $\{\delta'_\nu\}_{\nu\in\Lambda}$ to be the gauge of F, we have (for large ν)

$$v(\phi(t_\nu)) = \lambda'\delta'_\nu + (\lambda_\phi - \lambda')\delta + \gamma,$$

where $\lambda' = \mathrm{degdom}_F(\phi)$. Hence, $v(\phi(t_\nu)) \to \lambda_\phi\delta + \gamma = w(\phi)$, as claimed.

Suppose now that G is strictly pseudo-monotone. Since w and v have the same rank by hypothesis, by Theorems 5.5 and 5.7 $\mathrm{Br}(G)$ is equal neither to cV nor to cM, for all $c \in K$. In particular, $\mathrm{Br}(G)$ is both strictly divisorial and nonprincipal; by Lemma 2.6 we can find a pseudo-convergent sequence E and a pseudo-divergent sequence F in K such that $\mathcal{L}_E = \mathcal{L}_F = \mathcal{L}_G = \beta + \mathrm{Br}(G)$ (where β is any pseudo-limit of G; note that one between E and F could be taken equal to G). In particular, $\mathrm{Br}(E) = \mathrm{Br}(F) = \mathrm{Br}(G)$ and so $\delta_E = \delta_F = \delta$.

Since $W = V_E$ has rank 1, by [19, Theorem 4.9(c)] the valuation relative to V_E is exactly the one mapping $\phi \in K(X)$ to

$$\lambda\delta + \gamma = \lim_{\nu\to\infty}(\lambda\delta_\nu + \gamma)$$

where $\lambda = \mathrm{degdom}_E(\phi) = \mathrm{degdom}_F(\phi)$ and $\gamma = v\left(\frac{\phi}{\phi_S}(\beta)\right)$. Since δ is also the limit of δ'_ν, the claim is proved. □

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