WHEN TWO PRINCIPAL STAR OPERATIONS ARE THE SAME

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Abstract. We study when two fractional ideals of the same integral domain generate the same star operation.

1. Introduction

Throughout the paper, R will denote an integral domain with quotient field K and $\mathcal{F}(R)$ will be the set of *fractional ideals* of R, that is, the set of R-submodules I of K such that $xI \subseteq R$ for some $x \in K \setminus \{0\}$.

A *star operation* on R is a map $\star : \mathcal{F}(R) \longrightarrow \mathcal{F}(R)$ such that, for every $I, J \in \mathcal{F}(R)$ and every $x \in K$:

- $I \subseteq I^\star$;
- if $I \subseteq J$, then $I^\star \subseteq J^\star$;
- $(I^\star)^\star = I^\star$;
- $(xI)^\star = x \cdot I^\star$;
- $R^\star = R$.

The usual examples of star operations are the identity (usually denoted by d), the *v-operation* (or *divisorial closure*) $J \mapsto J^v := (R : (R : J))$, the t- and the w-operation (which are defined from v) and the star operations $I \mapsto \bigcap_{T \in \Delta} IT$, where Δ is a set of overrings of R intersecting to R. While these examples are the easiest to work with, they usually cover only a rather small part of the set of star operations.

A much more general construction is given in [9, Proposition 3.2]: if $(I : I) = R$, then the map $J \mapsto (I : (I : J))$ is a star operation. This construction is much more flexible than the more "classical" ones, and allows to construct a much higher number of star operations (see e.g. [10, Proposition 2.1(1)] or [11, Theorem 2.1] for its use to construct an infinite family of star operations, or [14, 15] for constructions in the case of numerical semigroups). In this paper, we slightly generalize this construction (removing the condition $(I : I) = R$) and study under which conditions two ideals I and J generate the same star operation: in particular, we are interested in understanding when this happens only for isomorphic ideals.

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The structure of the paper is as follows: in Section 3 we give some general properties of principal star operations; in Section 4, we generalize some results of [9] from m-canonical ideals to general ideals; in Section 5 we study the effect of localizations on principal star operations; in Section 6 we study operations generated by ideals whose v-closure is R (and, in particular, what happens when R is a unique factorization domain); in Section 7 we study the Noetherian case, reaching a necessary and sufficient condition for $v(I) = v(J)$ under the assumption $(I : I) = (J : J) = R$.

2. Background

By an *ideal* of R we shall always mean a fractional ideal of R, reserving the term *integral ideal* for those contained in R.

Let $\star$ be a star operation on R. An ideal I of R is *$\star$-closed* if $I = I^\star$; the set of $\star$-closed ideals is denoted by $\mathcal{F}^\star(R)$. When $\star = v$ is the divisorial closure, the elements of $\mathcal{F}^v(R)$ are called *divisorial ideals*.

Let $\mathrm{Star}(R)$ be the set of star operation on R. Then, $\mathrm{Star}(R)$ has a natural order structure, where $\star_1 \leq \star_2$ if and only if $I^{\star_1} \subseteq I^{\star_2}$ for every $I \in \mathcal{F}(R)$, or equivalently if $\mathcal{F}^{\star_1}(R) \supseteq \mathcal{F}^{\star_2}(R)$. Under this order, $\mathrm{Star}(R)$ is a complete lattice whose minimum is the identity and whose maximum is the v-operation.

A star operation is said to be *of finite type* if it is determined by its action on finitely generated ideals, or equivalently if

$$I^\star = \bigcup\{J^\star \mid J \subseteq I \text{ is finitely generated}\}$$

for every $I \in \mathcal{F}(R)$. A star operation is *spectral* if there is a subset $\Delta \subseteq \mathrm{Spec}(D)$ such that

$$I^\star = \bigcap\{IR_P \mid P \in \Delta\}$$

for every $I \in \mathcal{F}(R)$.

If $\star$ is a star operation of R, a prime ideal P is a *$\star$-prime* if it is $\star$-closed; the set of the $\star$-primes, denoted by $\mathrm{Spec}^\star(R)$, is called the *$\star$-spectrum*. A *$\star$-maximal ideal* of R is an ideal maximal among the set of proper ideals of R that are $\star$-closed; their set is denoted by $\mathrm{Max}^\star(R)$. Any $\star$-maximal ideal is prime; however, $\star$-maximal ideals need not to exist. If $\star$ is a star operation of finite type, then every $\star$-closed proper integral ideal is contained in some $\star$-maximal ideal; furthermore, for every $\star$-closed ideal I we have $I = \bigcap\{IR_P \mid P \in \mathrm{Spec}^\star(R)\}$.

3. Principal star operations

Definition 3.1. Let R be an integral domain. For every $I \in \mathcal{F}(R)$, the *star operation generated by* I, denoted by $v(I)$, is the supremum of all the star operations $\star$ on R such that I is $\star$-closed. If $\star = v(I)$ for some ideal I, we say that $\star$ is a *principal* star operation. We denote by $\mathrm{Princ}(R)$ the set of principal star operations of R.

We can give a more explicit representation of $v(I)$.

Proposition 3.2. *For every fractional ideal J, we have*

$$J^{v(I)} = J^v \cap (I : (I : J)) = J^v \cap \bigcap_{\alpha \in (I:J)\setminus\{0\}} \alpha^{-1} I. \qquad (1)$$

Furthermore, if $(I : I) = R$ then $J^{v(I)} = (I : (I : J))$.

Proof. The fact that the two maps $J \mapsto J^v \cap (I : (I : J))$ and $J \mapsto J^v \cap \bigcap_{\alpha\in(I:J)\setminus\{0\}} \alpha^{-1}I$ give star operations and coincide follows in the same way as [9, Lemma 3.1 and Proposition 3.2]. The second representation clearly implies that they close I; furthermore, if I is closed then J^v and each $\alpha^{-1}I$ are closed, and thus the two representations of (1) give exactly $v(I)$.

The "furthermore" statement follows again from [9, Lemma 3.1 and Proposition 3.2]. □

In the paper [9] that introduced the map $J \mapsto (I : (I : J))$ when $(I : I) = R$, an ideal I was said to be *m-canonical* if $J = (I : (I : J))$ for every ideal J. This is equivalent to saying that $(I : I) = R$ and that $v(I)$ is the identity.

The definition of $v(I)$ can be extended to semistar operations, as in [13, Example 1.8(2)]; such construction was called the *divisorial closure with respect to I* in [4]. The terminology "generated" is justified by the following Proposition 3.3.

Proposition 3.3. *Let $\star$ be a star operation on R. Then, $\star = \inf\{v(I) \mid I \in \mathcal{F}^\star(R)\}$.*

Proof. Let $\sharp := \inf\{v(I) \mid I \in \mathcal{F}^\star(R)\}$. By definition, $\star \leq v(I)$ for every $I \in \mathcal{F}^\star(R)$, and thus $\star \leq \sharp$. Conversely, let J be a $\star$-ideal; then, $\sharp \leq v(J)$ and thus J is $\sharp$-closed. It follows that $\star \geq \sharp$, and thus $\star = \sharp$. □

Our main interest in this paper is to understand when two ideals generate the same star operation. The first cases are quite easy.

Lemma 3.4. *Let I be a fractional ideal of R. Then, the following hold.*

(a) *$v(I) = v$ if and only if I is divisorial.*
(b) *If $(I : I) = R$, then $v(I) = d$ if and only if I is m-canonical.*
(c) *For every $a \in K$, $a \neq 0$, we have $v(I) = v(aI)$.*
(d) *If L is an invertible ideal of R, then $v(I) = v(IL)$.*

Proof. The only non-trivial part is the last point. If L is invertible, then

$$I^{v(IL)}L \subseteq (I^{v(IL)}L)^{v(IL)} = (IL)^{v(IL)} = IL$$

and thus $I^{v(IL)} \subseteq IL(R : L) = I$, i.e., I is $v(IL)$-closed; it follows that $v(I) \geq v(IL)$. Symmetrically, we have $v(IL) \geq v(IL(R : L)) = v(I)$, and thus $v(I) = v(IL)$. □

We note that if $J = IL$ for some invertible ideal L, then I and J are locally isomorphic. However, the latter condition is neither necessary nor sufficient for I and J to generate the same star operation, even excluding divisorial ideals. For example, if R is an almost Dedekind domain that is not Dedekind, then all ideals are locally isomorphic but not all are divisorial, and two nondivisorial maximal ideal generate different star operations (if $M \neq N$ are two such ideals, then $(M : N) = M$ and so $N^{v(M)} = N^v \cap (M : (M : N)) = R$). For an example of non-locally isomorphic ideals generating the same star operation see Example 7.11.

The following necessary condition has been proved in [14, Lemma 3.7] when I and J are fractional ideals of a numerical semigroup; the proof of the integral domain case (which was also stated later in the same paper) can be obtained in exactly the same way.

Proposition 3.5. *Let R be an integral domain and I, J be non-divisorial ideals of R. If $v(I) = v(J)$ then*

$$I = I^v \cap \bigcap_{\gamma \in (I:J)(J:I) \setminus \{0\}} (\gamma^{-1} I).$$

4. Local rings

As the construction of the principal star operation $v(I)$ generalize the definition of m-canonical ideal, we expect that I is in some way "m-canonical for $v(I)$". Pursuing this strategy, we obtain the following generalization of [9, Lemma 2.2(e)].

Lemma 4.1. *Let I be an ideal of a domain R such that $(I : I) = R$. Let $\{J_\alpha \mid \alpha \in A\}$ be $v(I)$-ideals such that $\bigcap_{\alpha \in A} J_\alpha \neq (0)$. Then,*

$$\left(I : \bigcap_{\alpha \in A} J_\alpha \right) = \left(\sum_{\alpha \in A} (I : J_\alpha) \right)^{v(I)}.$$

Proof. Let $J := \sum_{\alpha \in A} (I : J_\alpha)$. Since $(I : I) = R$, we have $L^{v(I)} = (I : (I : L))$ for every ideal L; therefore,

$$(I : J) = \left(I : \sum_{\alpha \in A} (I : J_\alpha) \right) = \bigcap_{\alpha \in A} (I : (I : J_\alpha)) = \bigcap_{\alpha \in A} J_\alpha^{v(I)} = \bigcap_{\alpha \in A} J_\alpha$$

and thus

$$J^{v(I)} = (I : (I : J)) = \left(I : \bigcap_{\alpha \in A} J_\alpha \right),$$

as claimed. ☐

The following definition abstracts a property proved, for m-canonical ideals of local domains, in [9, Lemma 4.1].

Definition 4.2. Let $\star$ be a star operation on R. We say that an ideal I of R is *strongly $\star$-irreducible* if $I = I^\star \neq \bigcap\{J \in \mathcal{F}^\star(R) \mid I \subsetneq J\}$.

Lemma 4.3. *Let R be a domain and I be a nondivisorial ideal of R. If I is strongly $v(I)$-irreducible and $v(I) = v(J)$, then $I = uJ$ for some $u \in K$.*

Proof. Suppose $v(I) = v(J)$. Then

$$I = I^{v(J)} = I^v \cap \bigcap_{\alpha \in (J:I)\setminus\{0\}} \alpha^{-1}J.$$

Both I^v and each $\alpha^{-1}J$ is a $v(I)$-ideal: hence, either $I = I^v$ (which is impossible since I is not divisorial) or $I = \alpha^{-1}J$ for some $\alpha \in K$. □

Lemma 4.4. *Suppose (R, M) is a local ring and $R = (I : I)$. If M is $v(I)$-closed, then I is strongly $v(I)$-irreducible.*

Proof. Let $\{J_\alpha\}$ be a family of $v(I)$-ideals such that $I = \bigcap J_\alpha$. Then,

$$R = (I : I) = \left(I : \bigcap_\alpha J_\alpha\right) = \left(\sum_\alpha (I : J_\alpha)\right)^{v(I)}$$

by Lemma 4.1.

Hence $(I : J_\alpha) \subseteq R$ for every α; suppose $I \subsetneq J_\alpha$ for all α. Then, $1 \notin (I : J_\alpha)$ and thus $(I : J_\alpha) \subseteq M$; therefore, $\sum(I : J_\alpha) \subseteq M$ and, since M is $v(I)$-closed, also $(\sum_\alpha (I : J_\alpha))^{v(I)} \subseteq M$, a contradiction. Therefore, we must have $J_\alpha = I$ for some α, and I is strongly $v(I)$-irreducible. □

As a consequence of the previous two lemmas, we have a very general result for local rings.

Proposition 4.5. *Let (R, M) be a local domain and I a nondivisorial ideal of R such that $(I : I) = R$. If $M = M^{v(I)}$ (in particular, if M is divisorial), then $v(I) = v(J)$ for some ideal J if and only if $I = uJ$ for some $u \in K$.*

Proof. By Lemma 4.4, I is strongly $v(I)$-irreducible; by Lemma 4.3 it follows that $I = uJ$. □

Corollary 4.6. *Let (R, M) be a local domain, and I and J two nondivisorial ideals of R. If R is completely integrally closed and M is divisorial, then $v(I) = v(J)$ if and only if $I = uJ$ for some $u \in K$.*

Proof. Since R is completely integrally closed, $(L : L) = R$ for all ideals L; furthermore, since M is divisorial $M^{v(L)} = M$ for every L. The claim follows from Proposition 4.5. □

One problem of the previous results is the hypothesis $(I : I) = R$. In the following proposition we eliminate it at the price of forcing more properties of R.

Proposition 4.7. *Let (R, M) be a local ring, and let $T := (M : M)$. Let I, J be ideals of R, properly contained between R and T, such that $v(I) = v(J)$.*

(a) If $(I : I), (J : J) \subseteq T$, then $(I : I) = (J : J)$.
(b) Suppose also that $(I : I) =: A$ is local with divisorial maximal ideal, and that I and J are not divisorial over A. Then, there is a $u \in K$ such that $I = uJ$.

Proof. If M is principal, $T = R$ and the statement is vacuous. Suppose thus M is not principal: then, we also have $T = (R : M)$. We first claim that $L^v = T$ for every ideal L properly contained between R and T. Indeed, the containment $R \subsetneq L$ implies that $(R : L) \subsetneq R$ and thus, since R is local, $(R : L) \subseteq M$ and $L^v \supseteq T \supsetneq L$; hence, $L^v = T$.

(a) Let $T_1 := (I : I)$ and $T_2 := (J : J)$, and define $\star_i$ as the star operation $L^{\star_i} := L^v \cap LT_i$. Since T contains T_1 and T_2, it is both a T_1- and a T_2-ideal. We claim that $L \neq R$ is $\star_i$-closed if and only if it is a T_i-ideal: the "if" part is obvious, while if $L = L^v \cap LT_i$ then $L^v = T$ is a T_i-ideal and thus L is intersection of two T_i-ideals.

If $v(I) = v(J)$, then I is $\star$-closed if and only if J is $\star$-closed; therefore, since I is $\star_1$-closed and J is $\star_2$-closed, both I and J are T_1 and T_2-ideals. But $(I : I)$ (respectively, $(J : J)$) is the maximal overring of R in which I (respectively, J) is an ideal; thus $(I : I) = (J : J)$.

(b) Consider the star operation generated by I on A, i.e., $v_A(I) : L \mapsto (A : (A : L)) \cap (I : (I : L))$ for every $L \in \mathcal{F}(A)$. By the first paragraph of the proof, applied on the A-ideals, we have $(A : (A : L)) = T$ for all ideals L of A properly contained between A and T; in particular, this happen for J (since $R \subset J$ implies $A = AR \subseteq AJ = J$, and $A \neq J$ since J is not divisorial), and thus $J^{v_A(I)} = J^{v(I)} = J$. Symmetrically, $I^{v_A(J)} = I$; hence, $v_A(I) = v_A(J)$. By Proposition 4.5, applied to A, we have $I = uJ$ for some $u \in K$, as claimed. □

Recall that a *pseudo-valuation domain* (PVD) is a local domain (R, M) such that M is the maximal ideal of a valuation overring of R (called the valuation domain *associated* to R) [8].

Corollary 4.8. *Let (R, M) be a pseudo-valuation domain with associated valuation ring V, and suppose that the field extension $R/M \subseteq V/M$ is algebraic. Let I, J be nondivisorial ideals of R. Then, $v(I) = v(J)$ if and only if $I = uJ$ for some $u \in K$.*

Proof. By [12, Proposition 2.2(5)], there are $a, b \in K$ such that $a^{-1}I$ and $b^{-1}J$ are properly contained between R and $V = (M : M)$. Furthermore, since $R/M \subseteq V/M$ is algebraic, every ring between R and V is the pullback of some intermediate field, and in particular it is itself a PVD with maximal ideal M. The claim follows from Proposition 4.7. □

5. Localizations

Let $\star$ be a star operation on R and T a flat overring of R. Then, $\star$ is said to be *extendable* to T if the map

$$\begin{aligned}\star_T\colon \mathcal{F}(T) &\longrightarrow \mathcal{F}(T)\\ IT &\longmapsto I^\star T\end{aligned}$$

is well-defined; when this happens, $\star_T$ is called the *extension* of $\star$ to T and is a star operation on T [16, Definition 3.1]. In general, not all star operations are extendable, although finite-type operations are (see [10, Proposition 2.4] and [16, Proposition 3.3(d)]).

We would like to have an equality $v(I)_T = v(IT)$, where the latter is considered as a star operation on T. In general, this is false, both because $v(I)$ may not be extendable and because the extension $v(I)_T$ may not be equal to $v(IT)$: both these cases happen even for valuation domains.

For example, suppose V is a valuation domain with branched maximal ideal. If I is divisorial, then $v(I) = v$; however, if the maximal ideal is not principal, then v is not extendable to V_P for every non-maximal prime P. On the other hand, if the maximal ideal is principal, then the only star operation on V is the identity, and thus $v(I) = d$ for all ideals I: in particular, $v(I)$ is extendable to every localization of V, and its extension is the identity. Suppose $(0) \subset P \subset Q$ are non-maximal prime ideals of V, and suppose QV_Q is not principal in V_Q: then, the v-operation on V is not the identity. However, $P = PV_Q$ is divisorial in V_Q, and thus $v(PV_Q)$ is the v-operation; on the other hand, $v(P)_{V_Q}$ is the identity on V_Q. In particular, $v(PV_Q) \neq v(P)_{V_Q}$.

In the Noetherian case, however, everything works.

Proposition 5.1. *If R is Noetherian, then $v(I)_T = v(IT)$ for every flat overring T of R.*

Proof. By definition, $J^{v(I)} = (R : (R : J)) \cap (I : (I : J))$; multiplication by a flat overring commutes with intersections, and since every ideal is finitely generated, the colon localizes, and thus

$$\begin{aligned}J^{v(I)}T &= (R : (R : J))T \cap (I : (I : J))T =\\ &= (T : (T : JT)) \cap (IT : (IT : JT)) =\\ &= (JT)^{v_T} \cap (IT : (IT : JT)) = (JT)^{v(IT)},\end{aligned}$$

i.e., $v(I)_T = v(IT)$. □

Another case where localization works well is for Jaffard families. If R is an integral domain with quotient field K, a *Jaffard family* of R is a set Θ of flat overrings of R such that [6, Section 6.3.1]:

- Θ is locally finite;
- $I = \prod\{IT \cap R \mid T \in \Theta, IT \neq T\}$ for every integral ideal I;

- $(IT_1 \cap R) + (IT_2 \cap R) = R$ for every integral ideal I and every $T_1 \neq T_2$ in Θ.

Proposition 5.2. *Let R be an integral domain, and let T be an overring of R that belongs to a Jaffard family of R. For every ideal I of R, the star operation $v(I)$ is extendable to T, and $v(I)_T = v(IT)$.*

Proof. Since T belongs to a Jaffard family of R, we have $(J : L)T = (JT : LT)$ for every pair of fractional ideals J, L of R [16, Lemma 5.3]; the claim follows as in the proof of Proposition 5.1. □

Jaffard families can be used to factorize the set of star operations of a domain R into a direct product of sets of star operations [16, Theorem 5.4]; for principal star operations, we have something similar. We define a "direct sum"-like construction of sets of principal ideals as

$$\bigoplus_{T\in\Theta} \mathrm{Princ}(T) := \{(\star^{(T)})_{T\in\Theta} \mid \star^{(T)} \neq v^{(T)} \text{ for only a finite number of } T\}.$$

Proposition 5.3. *Let R be an integral domain and Θ be a Jaffard family on R. Then, the map*

$$\begin{aligned} \Upsilon\colon \mathrm{Princ}(R) &\longrightarrow \bigoplus_{T\in\Theta} \mathrm{Princ}(T) \\ v(I) &\longmapsto (v(IT))_{T\in\Theta} \end{aligned}$$

is a well-defined order-isomorphism.

Proof. The map Υ is just the restriction of the localization map λ_Θ to $\mathrm{Princ}(R)$, which is an isomorphism (see [16, Theorem 5.4]), so we have only to show that it is well-defined and surjective.

By Proposition 5.2, $v(I)_T = v(IT)$ for every $T \in \Theta$; moreover, $IT = T$ for all but a finite number of T (by definition of a Jaffard family), so that $v(IT) = v(T) = v^{(T)}$ for all but a finite number of T. In particular, the image of Υ lies inside the direct sum $\bigoplus_{T\in\Theta} \mathrm{Princ}(T)$.

Suppose, conversely, that $(v(J_T))_{T\in\Theta} \in \bigoplus_{T\in\Theta} \mathrm{Princ}(T)$. We can suppose that $J_T \subseteq T$ for every T, and that $J_T = T$ if $v(J_T) = v^{(T)}$. Define thus $I := \bigcap_{T\in\Theta} J_T$: then, I is nonzero (since $J_T \neq T$ for only a finite number of T) and $IT = J_T$ for every T [16, Lemma 5.2]. Therefore, $v(I)_T = v(IT) = v(J_T)$, and the image of Υ is exactly $\bigoplus_{T\in\Theta} \mathrm{Princ}(T)$. □

Proposition 5.3 can be interpreted as a way to "factorize" principal star operations.

Corollary 5.4. *Let R be an integral domain and Θ be a Jaffard family on R. Let I be an integral ideal of R. Then, there are $T_1, \dots, T_n \in \Theta$ such that $v(I) = v(IT_1 \cap R) \wedge \cdots \wedge v(IT_n \cap R)$.*

Proof. Since $I \subseteq R$, we have $IT = T$ for all but finitely many $T \in \Theta$; let $T_1, \dots, T_n$ be the exceptions. The claim follows from Proposition 5.3. □

Recall that an integral domain is said to be *h-local* if every ideal is contained in a finite number of maximal ideals and every prime ideal is contained in only one maximal ideal.

Corollary 5.5. *Let R be an h-local Prüfer domain, and let $\mathcal{M}$ be the set of nondivisorial maximal ideals of R. Then, there is a bijective correspondence between* $\mathrm{Princ}(R)$ *and the set* $\mathcal{P}_{\mathrm{fin}}(\mathcal{M})$ *of finite subset of $\mathcal{M}$. Furthermore, $\mathcal{M}$ is finite if and only if every star operation is principal.*

Proof. Since R is h-local, $\{R_M \mid M \in \mathrm{Max}(R)\}$ is a Jaffard family of R, and thus by Proposition 5.3 there is a bijective correspondence Υ between $\mathrm{Princ}(R)$ and $\bigoplus_{M\in\mathrm{Max}(R)} \mathrm{Princ}(R_M)$. If $M \notin \mathcal{M}$, then MR_M is principal and thus $\mathrm{Star}(R_M) = \mathrm{Princ}(R_M) = \{d = v\}$; hence, Υ restricts to a bijection Υ' between $\mathrm{Princ}(R)$ and $\bigoplus_{M\in\mathcal{M}} \mathrm{Princ}(R_M)$. Since R_M is a valuation domain, each $\mathrm{Princ}(R_M)$ is composed by two elements (the identity and the v-operation). Thus, we can construct a bijection Υ_1 from the direct sum to $\mathcal{P}_{\mathrm{fin}}(\mathcal{M})$ by associating to $\star := (\star^{(M)})$ the finite set $\Upsilon_1(\star) := \{M \in \mathcal{M} \mid \star^{(M)} \neq v\}$. The composition $\Upsilon_1 \circ \Upsilon'$ is a bijection from $\mathrm{Princ}(R)$ to $\mathcal{P}_{\mathrm{fin}}(\mathcal{M})$.

The last claim follows immediately. □

A factorization property similar to Corollary 5.4 can be proved for ideals having a primary decomposition with no embedded primes.

Proposition 5.6. *Let $Q_1, \ldots, Q_n$ be primary ideals, let $P_i := \mathrm{rad}(Q_i)$ for all i and let $I := Q_1 \cap \cdots \cap Q_n$. If the P_i are pairwise incomparable, then $v(I) = v(Q_1) \wedge \cdots \wedge v(Q_n)$.*

Proof. For every i, the ideal Q_i is $v(Q_i)$-closed, and thus I is $(v(Q_1) \wedge \cdots \wedge v(Q_n))$-closed; hence, $v(I) \geq v(Q_1) \wedge \cdots v(Q_n)$. To prove the converse, we need to show that each Q_i is $v(I)$-closed.

Without loss of generality, let $i = 1$, and define $\widehat{Q} := Q_2 \cap \cdots \cap Q_n$; we claim that $Q_1 = (I :_R \widehat{Q})$. Since $Q_1\widehat{Q} \subseteq Q_1 \cap \widehat{Q} = I$, clearly $Q_1 \subseteq (I :_R \widehat{Q})$. Conversely, let $x \in (I :_R \widehat{Q})$. Since the radicals of the Q_i are pairwise incomparable, $Q_i \nsubseteq P_1$ for every $i > 1$, and so $\widehat{Q} \nsubseteq P_1$; therefore, there is a $q \in \widehat{Q} \setminus P_1$. Then, $xq \in I$, and in particular $xq \in Q_1$. If $x \notin Q_1$, then since Q_1 is primary we would have $q^t \in Q_1$ for some $t \in \mathbb{N}$; however, this would imply $q \in \mathrm{rad}(Q_1) = P_1$, against the choice of q. Thus, $Q_1 \subseteq (I :_R \widehat{Q})$ and so $Q_1 = (I :_R \widehat{Q})$.

By definition, I is $v(I)$-closed; hence, also $(I :_R \widehat{Q})$ is $v(I)$-closed. It follows that Q_1 is $v(I)$-closed, and thus that each Q_i is $v(I)$-closed, i.e., $v(I) \leq v(Q_1) \wedge \cdots \wedge v(Q_n)$. The claim is proved. □

6. v-trivial ideals

In this section, we analyze principal operations generated by v-trivial ideals.

Definition 6.1. An ideal I of a domain R is *v-trivial* if $I^v = R$.

Lemma 6.2. *If I is v-trivial, then $(I : I) = R$.*

Proof. If $I^v = R$, then $(R : I) = R$, and thus $(I : I) \subseteq (R : I) = R$. □

Definition 6.3. A star operation $\star$ is *semifinite* (or *quasi-spectral*) if every $\star$-closed ideal $I \subsetneq R$ is contained in a $\star$-prime ideal.

All finite-type and all spectral operations are semifinite; on the other hand, if V is a valuation domain with maximal ideal that is branched but not finitely generated, the v-operation on V is not semifinite. The class of semifinite operations is closed by taking infima, but not by taking suprema (see [5, Example 4.5]).

Lemma 6.4. *Let R be an integral domain, and let I, J be v-trivial ideals of R.*

(a) If $J \subsetneq I$, then $J^{v(I)} = I$, and in particular $v(I) \neq v(J)$.

Suppose v is semifinite on R.

(b) $I \cap J$ is v-trivial.
(c) $I \subseteq J^{v(I)}$.
(d) If $I \neq J$, then $v(I) \neq v(J)$.

Proof. (a) Since I is v-trivial, by Lemma 6.2 we have $J^{v(I)} = (I : (I : J))$. However, $R \subseteq (I : J) \subseteq (R : J) = R$ (using the v-triviality of J) and thus $J^{v(I)} = (I : R) = I$, as claimed. In particular, $J = J^{v(J)} \neq J^{v(I)}$ and so $v(I) \neq v(J)$.

(b) If $(I \cap J)^v \neq R$, then by semifiniteness there is a prime ideal P such that $I \cap J \subseteq P = P^v$: But this would imply $I \subseteq P$ or $J \subseteq P$, against the hypothesis that I and J are v-trivial.

(c) Since $J \subseteq J^{v(I)}$, it follows that $J^{v(I)}$ is v-trivial, and by the previous point so it $J^{v(I)} \cap I$. If $I \nsubseteq J^{v(I)}$, it would follow that $J^{v(I)} \cap I \subsetneq I$; but $J^{v(I)} \cap I$ is $v(I)$-closed, against (a). Hence $I \subseteq J^{v(I)}$.

(d) If both I and J are $v(I)$-closed, then so is $I \cap J$; by (b), $(I \cap J)^v = R$. The claim follows applyning (a) to $I \cap J$ and I (or J). □

Corollary 6.5. *Let R be a domain such that v is semifinite. Let I, J be ideals of R such that I^v and J^v are invertible; then, $v(I) = v(J)$ if and only if $I = LJ$ for some invertible ideal L.*

Proof. By invertibility, we have

$$R = I^v(R : I^v) = (I^v(R : I^v))^v = (I(R : I^v))^v;$$

since $I \subseteq I(R : I^v) \subseteq R$, the ideal $I(R : I^v)$ is v-trivial. Analogously, $R = (J(R : J^v))^v$ and $J(R : J^v)$ is v-trivial. Hence, by Lemma 6.4(d) $I(R : I^v) = J(R : J^v)$; thus, $I = I^v(R : J^v)J$, and $L := I^v(R : J^v)$ is invertible. □

Corollary 6.6. *Let R be a unique factorization domain. Then:*

(a) for every principal star operation $\star \neq v$ there is a proper ideal I such that $h(I) > 1$ and $\star = v(I)$;
(b) if I, J are fractional ideals of R, $v(I) = v(J)$ if and only if $I = uJ$ for some $u \in K$.

Proof. Let $\star = v(I)$ for some ideal I. By [7, Corollary 44.5], every v-closed ideal of R is principal; hence, let $I^v = pR$. Then, $(p^{-1}I)^v = R$, i.e., $p^{-1}I$ is v-trivial. Analogously, $q^{-1}J$ is v-trivial for some J; thus $v(p^{-1}I) = v(I) = v(J) = v(q^{-1}J)$. Applying Lemma 6.4(d) to $p^{-1}I$ and $q^{-1}J$ we get $p^{-1}I = q^{-1}J$, i.e., $I = (pq^{-1})J$. □

For star operations generated by v-trivial prime ideals, we can also determine the set of closed ideals.

Proposition 6.7. *Let R be a domain such that v is semifinite and such that I^v is invertible for every ideal I, and let $P \in \mathrm{Spec}(R)$. Then $\mathcal{F}^{v(P)}(R) = \mathcal{F}^v(R) \cup \{LP \mid L$ is an invertible ideal$\}$. In particular, $v(P)$ is a maximal element of $\mathrm{Princ}(R) \setminus \{v\}$.*

Proof. Let I be a non-divisorial ideal; multiplying by an invertible ideal L, we can suppose $I^v = R$. If $I \subseteq P$, by Lemma 6.4(a) $I^{v(P)} = P$, and thus $I \neq I^{v(P)}$ unless $I = P$; suppose $I \nsubseteq P$. Then $(P : I) = P$: we have $(P : I) \subseteq (R : I) = R$, and thus if $xI \subseteq P$ then $x \in P$. Therefore, $I^{v(P)} = I^v \cap (P : (P : I)) = R \cap (P : P) = R \neq I$.

For the "in particular" claim, note that if $v(I) \geq v(P)$ then I should be $\star$-closed: by the previous part of the proof, this means that either I is divisorial (and so $v(I) = v$) or $I = LP$ for some invertible L (and thus $v(I) = v(P)$ by Lemma 3.4(d). □

Corollary 6.8. *Let R be a unique factorization domain, and let $P \in \mathrm{Spec}(R)$. Then, $\mathcal{F}^{v(P)}(R) = \mathcal{F}^v(R) \cup \{aP \mid a \in K\}$.*

We have seen in Proposition 3.3 that all star operation can be "generated" by principal star operations; we can use v-trivial ideals to show that in many cases we need infinitely many of them.

Proposition 6.9. *Let R be a domain such that v is semifinite, and let $I_1, \ldots, I_n$ be v-trivial ideals; let $\star := v(I_1) \wedge \cdots \wedge v(I_n)$. Then, the ideal $I_1 \cap \cdots \cap I_n$ is the minimal v-trivial ideal that is $\star$-closed.*

Proof. Let $J := I_1 \cap \cdots \cap I_n$. By Lemma 6.4(b), J is v-trivial. Clearly J is $\star$-closed. Suppose L is v-trivial; then, applying Lemma 6.4(c),

$$L^\star = L^{v(I_1)\wedge\cdots\wedge v(I_n)} \supseteq I_1 \cap \cdots I_n = J.$$

Therefore, J is the minimum among v-trivial $\star$-closed ideals. □

Corollary 6.10. *Let R be a unique factorization domain, and let $\star \in \mathrm{Star}(R)$ be such that $\star \neq v$. If $\bigcap\{J \in \mathcal{F}^\star(R) \mid J^v = R\} = (0)$, then $\star$ is not the infimum of a finite family of principal star operations.*

Proof. Since R is a UFD, the v-operation is semifinite, and every principal star operation can be generated by a v-trivial ideal. If $\star$ were to be finitely generated, say $\star = v(I_1) \wedge \cdots \wedge v(I_n)$, then $J := I_1 \cap \cdots \cap I_n$ would be the minimal v-trivial $\star$-closed ideal; however, by hypothesis, there must be a v-trivial $\star$-closed ideal J' not containing J, and thus $\star$ cannot be finitely generated. □

Proposition 6.11. *Let R be a domain, and let Δ be a set of overrings whose intersection is R. Let $\star$ be the star operation $I \mapsto \bigcap\{IT \mid T \in \Delta\}$. Suppose that:*

1. *v is semifinite;*
2. *every v-trivial ideal contains a finitely generated v-trivial ideal;*
3. *there is a v-trivial $\star$-closed ideal.*

Then, $\star$ is not the infimum of a finite family of principal star operations.

Proof. By substituting an overring $T \in \Delta$ with $\{T_M \mid M \in \mathrm{Max}(T)\}$, we can suppose without loss of generality that each member of Δ is local.

If $\star$ were finitely generated, by Proposition 6.9 there would be a minimal v-trivial $\star$-closed ideal, say J. By hypothesis, there is finitely generated v-trivial ideal $I \subseteq J$; since $I^\star = J$, by [1, Theorem 2], we have $IT = JT$ for every $T \in \Delta$.

Since $I^\star \neq R$, there must be an $S \in \Delta$ such that $IS \neq S$; by Nakayama's lemma, $I^2S = (IS)^2 \subsetneq IS$, and so $(I^2)^\star \subseteq I^2S \cap R \subsetneq I$. In particular, $(I^2)^\star$ is a v-trivial $\star$-closed ideal, against the definition of I. Thus, $\star$ is not finitely generated. □

The first two hypothesis hold, for example, for unique factorization domains of dimension $d > 1$; the third one holds, for example, in the following cases:

- $\star$ is a spectral star operation of finite type different from the w-operation (see [17, 2]);
- if R is integrally closed and (at least) one maximal ideal is not divisorial, the b-operation/integral closure;
- if R is a UFD, all star operations coming from overrings, except the v-operation.

7. Noetherian domains

In this section, we study in more detail the case of Noetherian domains; in particular, we shall give in Theorem 7.9 a necessary and sufficient condition on when $v(I) = v(J)$, under the assumption that $(I : I) = R = (J : J)$. We first state a case that is already settled, even without this hypothesis.

Proposition 7.1. [14, Proposition 5.4] *Let (R, M) be a local Noetherian integral domain of dimension 1 such that its integral closure V is* *a discrete valuation domain that is finite over* R*; suppose also that the induced map of residue fields* $R/M \subseteq V/M_V$ *is an isomorphism. Then,* $v(I) = v(J)$ *if and only if* $I = uJ$ *for some* $u \in K$*,* $u \neq 0$*.*

We denote by $\mathrm{Ass}(I)$ the set of associated primes of I.

Proposition 7.2. *Let* R *be a domain and* I *an ideal of* R*. Then,* $\mathrm{Spec}^{v(I)}(R) \supseteq \mathrm{Spec}^v(R) \cup \mathrm{Ass}(I)$*, and if* R *is Noetherian the two sets are equal.*

Proof. If $P \in \mathrm{Ass}(I)$, then $P = (I :_R x) = x^{-1}I \cap R$ for some $x \in R$, and thus it is $v(I)$-closed; if $P \in \mathrm{Spec}^v(R)$ then $P = P^v$ and thus $P = P^{v(I)}$.

Conversely, suppose R is Noetherian and $P = P^{v(I)}$. Then $P = P^v \cap (I : (I : P)) = P^v \cap (I : J)$, where $J = (I : P)$; let $J = j_1R + \cdots + j_nR$. We have

$$\begin{aligned} P &= P^v \cap (I : J) = P^v \cap R \cap (I : J) = P^v \cap (I :_R J) = \\ &= P^v \cap (I :_R j_1R + \cdots + j_nR) = P^v \cap \textstyle\bigcap_{i=1}^n (I :_R j_iR), \end{aligned}$$

and, since P is prime, this implies that $P^v = P$ or $(I :_R j_iR) = P$ for some i. In the latter case, since $j_i \in K$, $j_i = a/b$ for some $a, b \in R$; hence $(I :_R j_iR) = (I : ab^{-1}R) \cap R = (bI :_R aR)$, and thus P is associated to bI. There is an exact sequence

$$0 \longrightarrow \frac{bR}{bI} \longrightarrow \frac{R}{bI} \longrightarrow \frac{R}{bR} \longrightarrow 0$$

and, since R is a domain, $bR/bI \simeq R/I$ and thus $\mathrm{Ass}(bI) \subseteq \mathrm{Ass}(I) \cup \mathrm{Ass}(bR)$ [3, Chapter IV, Proposition 3]; therefore, P is associated to I or it is divisorial (since an associated prime of a divisorial ideal – in this case, bR – is divisorial). □

Remark 7.3. Note that, if $P^v = R$, then $(I : P) \subseteq (R : P) = R$, and thus $j_i \in R$; in this case, $b = 1$ and the last part of the proof can be greatly simplified.

The following is a slight improvement of Proposition 6.7. We denote by $X^1(R)$ the set of height-1 prime ideals of R.

Corollary 7.4. *Let* R *be an integrally closed Noetherian domain. Then, the maximal elements of* $\mathrm{Princ}(R) \setminus \{v\}$ *are the* $v(P)$*, as* P *ranges in* $\mathrm{Spec}(R) \setminus X^1(R)$*.*

Proof. Since R is integrally closed, the divisorial prime ideals of R are the height 1 primes. In particular, if P is a prime ideal of height > 1, then $v(P)$ is maximal by Proposition 6.7.

Conversely, suppose $v(I)$ is maximal in $\mathrm{Princ}(R) \setminus \{v\}$. If all associated primes of I have height 1, then $I = \bigcap_{P \in X^1(R)} IR_P$, and so I is divisorial, against $v(I) \neq v$. Hence, there is a $P \in \mathrm{Ass}(I) \setminus X^1(R)$; by Proposition 7.2, $P \in \mathrm{Spec}^{v(I)}(R)$, and thus $v(I) \leq v(P)$. As $v(I)$ is maximal, it follows that $v(I) = v(P)$. The claim is proved. □

Corollary 7.5. *Let R be a Noetherian unique factorization domain. Then, $v(I)$ is a maximal element of $\mathrm{Princ}(R)\setminus\{v\}$ if and only if $I = uP$ for some prime ideal $P \in \mathrm{Spec}(R) \setminus X^1(R)$ and some $u \in K$.*

Proof. It is enough to join Corollary 7.4 (the maximal elements are the $v(P)$) with Corollary 6.6 ($v(I) = v(P)$ if and only if $I = uP$). □

Proposition 7.2 allows to determine, in the Noetherian case, all the spectra of the principal star operations. We need two lemmas.

Lemma 7.6. *Let R be a Noetherian ring and $\Delta \subseteq \mathrm{Spec}(R) \setminus \{(0)\}$ be a finite set. There is an ideal I of R such that $\mathrm{Ass}(I) = \Delta$.*

Proof. We proceed by induction on $n = |\Delta|$. If $n = 1$ and $\Delta = \{P\}$ we can take $I = P$.

Suppose $n > 1$ and let $\Delta = \{P_1, \dots, P_n\}$; without loss of generality we can suppose $P_i \nsubseteq P_j$ for every $i > j$. Let I_0 be an ideal such that $\mathrm{Ass}(I_0) = \{P_1, \dots, P_{n-1}\}$, and let $I_0 = Q_1 \cap \cdots \cap Q_{n-1}$ be a primary decomposition, where $P_i := \mathrm{rad}(Q_i)$. Since the intersection of all P_n-primary ideals is (0), there is a P_n-primary ideal Q_n such that $Q_n \nsubseteq I_0$; let $I := I_0 \cap Q_n$. To show that $\mathrm{Ass}(I) = \Delta$, it is enough to prove that $Q_1 \cap \cdots \cap Q_n$ is an irredundant intersection.

Suppose Q_i is redundant. By construction, $i \neq n$; moreover, if $i = 1$, then $Q_2 \cap \cdots \cap Q_n \subseteq Q_1$ and thus, passing to the radical, $P_2 \cap \cdots \cap P_n \subseteq P_1$, and $P_j \subseteq P_1$ for some $j > 1$, against the hypothesis. Hence suppose $1 < i < n$, and let $L_1 := Q_1 \cap \cdots \cap Q_{i-1}$ and $L_2 := Q_{i+1} \cap \cdots \cap Q_n$. By inductive hypothesis, $Q_1 \cap \cdots \cap Q_i = L_1 \cap Q_i$ is irredundant, and thus $L_1 \nsubseteq Q_i$; let $x \in L_1 \setminus Q_i$. For every $a \in L_2$, we have $xa \in L_1 L_2 \subseteq L_1 \cap L_2 \subseteq Q_i$ (since Q_i is redundant), and thus $L_2 \subseteq (Q_i :_R x)$. However, $\mathrm{rad}((Q :_R x)) \neq R$, and thus $\mathrm{rad}((Q_i :_R x)) = \mathrm{rad}(Q_i) = P_i$; hence, $\mathrm{rad}(L_2) \subseteq \mathrm{rad}(Q_i)$, i.e., $P_{i+1} \cap \cdots \cap P_n \subseteq P_i$. However, this implies that $P_j \subseteq P_i$ for some $j > i$, which still is against the hypothesis. Therefore, no Q_i can be redundant. □

Lemma 7.7. *Let $\star_1, \dots, \star_n \in \mathrm{Star}(R)$, and let $\star := \star_1 \wedge \cdots \wedge \star_n$. Then, $\mathrm{Spec}^\star(R) = \bigcup_i \mathrm{Spec}^{\star_i}(R)$.*

Proof. If $P = P^{\star_i}$ for some i then $P^\star \subseteq P^{\star_i} = P$ and thus $P = P^\star$. Conversely, if $P = P^\star$ then $P = P^{\star_1} \cap \cdots \cap P^{\star_n}$; since P is prime, it follows that $P = P^{\star_i}$ for some i. The claim is proved. □

Proposition 7.8. *Let R be a Noetherian domain, and let $\Delta \subseteq \mathrm{Spec}(R)$. Then, the following are equivalent:*

- *(i)* $\Delta = \mathrm{Spec}^{v(I)}(R)$ *for some ideal I;*
- *(ii)* $\Delta = \mathrm{Spec}^\star(R)$ *for some $\star = v(I_1) \wedge \cdots \wedge v(I_n)$;*
- *(iii)* $\Delta = \mathrm{Spec}^v(R) \cup \Delta'$, *for some finite set Δ'.*

Proof. (i) $\Longrightarrow$ (ii) is obvious. (ii) $\Longrightarrow$ (iii) follows from Lemma 7.7. (iii) $\Longrightarrow$ (i) follows by Lemma 7.6 and Proposition 7.2 (it is enough to take an I such that $\mathrm{Ass}(I) = \Delta'$). □

We now characterize when two nondivisorial ideals with $(I : I) = (J : J) = R$ generate the same star operation.

Theorem 7.9. *Let R be a Noetherian domain, and let I, J be non-divisorial ideals such that $(I : I) = (J : J) = R$. Then, $v(I) = v(J)$ if and only if $\mathrm{Ass}(I) \cup \mathrm{Spec}^v(R) = \mathrm{Ass}(J) \cup \mathrm{Spec}^v(R)$ and, for every $P \in \mathrm{Ass}(I) \cup \mathrm{Spec}^v(R)$, there is an $a_P \in K$ such that $IR_P = a_P JR_P$.*

Proof. Suppose the two conditions hold. By Proposition 7.2, $\mathrm{Ass}(I) \cup \mathrm{Spec}^v(R) = \mathrm{Spec}^{v(I)}(R)$, and thus $\mathrm{Spec}^{v(I)}(R) = \mathrm{Spec}^{v(J)}(R) =: \Delta$. For every ideal L, using Proposition 5.1 we have

$$L^{v(I)} = \bigcap_{P\in\Delta} L^{v(I)}R_P = \bigcap_{P\in\Delta} (LR_P)^{v(I)_{R_P}} = \bigcap_{P\in\Delta} (LR_P)^{v(IR_P)}.$$

Since IR_P and JR_P are isomorphic, $(LR_P)^{v(IR_P)} = (LR_P)^{v(JR_P)}$; it follows that $v(I) = v(J)$.

Conversely, suppose $v(I) = v(J) =: \star$. Then, $\mathrm{Spec}^\star(R)$ is equal to both $\mathrm{Ass}(I) \cup \mathrm{Spec}^v(R)$ and $\mathrm{Ass}(J) \cup \mathrm{Spec}^v(R)$, which thus are equal. Note also that $(I : I) = R$ implies that $R_P = (I : I)R_P = (IR_P : IR_P)$ for every prime ideal P.

Let now $P \in \mathrm{Spec}^\star(R)$. Since $v(I) = v(J)$, clearly $v(I)_{R_P} = v(J)_{R_P}$, which by Proposition 5.1 implies that $v(IR_P) = v(JR_P)$. However, PR_P is $v(IR_P)$-closed because P is $v(I)$-closed; it follows, by Proposition 4.5, that $IR_P = a_P JR_P$ for some $a_P \in K$, as claimed. □

Corollary 7.10. *Let R be an integrally closed Noetherian domain, and let I, J be non-divisorial ideals. Then, $v(I) = v(J)$ if and only if $\mathrm{Ass}(I) \cup X^1(R) = \mathrm{Ass}(J) \cup X^1(R)$ and for every $P \in \mathrm{Ass}(I)$ there is an $a_P \in R_P$ such that $IR_P = a_P JR_P$.*

Proof. Since R is integrally closed and Noetherian, we have $(I : I) = R$ for every ideal I; furthermore, the divisorial primes are the height 1 primes, and for any such P the localizations IR_P and JR_P are isomorphic since R_P is a DVR. The claim now follows from Theorem 7.9. □

Example 7.11. Let R be a Noetherian integrally closed domain, and suppose that R_M is not a UFD for some maximal ideal M. Let P be an height 1 prime contained in M such that PR_M is not principal, and let Q be a prime ideal of height bigger than 1 such that $P + Q = R$ (in particular, $Q \nsubseteq M$). We claim that $v(PQ) = v(Q)$ but PQ and Q are not locally isomorphic.

In fact, since they are coprime, $PQ = P \cap Q$, and thus $\mathrm{Ass}(PQ) = \{P, Q\}$ while $\mathrm{Ass}(Q) = \{Q\}$; moreover, $P \nsubseteq Q$ and thus $PQR_Q = QPR_Q = QR_Q$. Since $P \in X^1(R)$, by Corollary 7.10 it follows that $v(PQ) = v(Q)$. However, $QR_M = R_M$ is principal, while $PQR_M = PR_M$, by hypothesis, is not: therefore, Q and PQ are not locally isomorphic. In particular, there cannot be an invertible ideal L such that $Q = LPQ$, because LR_M would be principal and thus Q and PQ would be locally isomorphic.

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