

TOPOLOGICAL PROPERTIES OF SUBSETS OF THE ZARISKI SPACE

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ABSTRACT. We study the properties of some distinguished subspaces of the Zariski space $\text{Zar}(F|D)$ of a field F over a domain D , in particular the topological properties of subspaces defined through algebraic means. We are mainly interested in two classes of problems: understanding when spaces in the form $\text{Zar}(F|D) \setminus \{V\}$ are compact (which is strongly linked to the problem of determining when $\text{Zar}(F|D)$ is a Noetherian space), and studying spaces of rings defined through pseudo-convergent sequences on an (arbitrary, but fixed) rank one valuation domain.

1. INTRODUCTION AND NOTATION

Let D be an integral domain and K be a field containing D (not necessarily the quotient field of D). In the Thirties, studying the problem of resolution of singularities, Zariski introduced the *Zariski space* of K over D (under the name *generalized Riemann surface*) as the set $\text{Zar}(K|D)$ of all valuation domains of K containing D [22, 23]. He introduced on this set a topology (later called the *Zariski topology*) which is generated by the open sets

$$\mathcal{B}(x_1, \dots, x_n) := \{V \in \text{Zar}(K|D) \mid x_1, \dots, x_n \in V\},$$

as x_1, \dots, x_n range in K , and showed that, under this topology, $\text{Zar}(K|D)$ is a compact space [24, Chapter VI, Theorem 40].

Later, it was shown that $\text{Zar}(K|D)$ is actually a *spectral space* [9], that is, that for every K and D there is a ring R such that $\text{Zar}(K|D) \simeq \text{Spec}(R)$; such an R can also be constructed explicitly as a Bézout domain having quotient field $K(X)$ (called the *Kronecker function ring* of K over D) [4, 5, 6]. As a spectral space, $\text{Zar}(K|D)$ can also be endowed with the *inverse topology* (the topology generated by the complements of the open and compact subspaces of the original topology) and the *constructible* (or *patch*) *topology* (the topology generated by both the open and compact subspaces and their complements). These two

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topologies are both spectral (so, in particular, compact) and, more importantly, $\text{Zar}(K|D)^{\text{cons}}$ (i.e., $\text{Zar}(K|D)$ under the constructible topology) is an Hausdorff space, something that does not happen for the Zariski or the inverse topology unless D is a field and K is an algebraic extension, i.e., unless $\text{Zar}(K|D)$ is just $\{K\}$. A particular importance have the closed set of $\text{Zar}(K|D)^{\text{cons}}$: they are called *proconstruable* subsets, and they are again spectral spaces (in the Zariski topology).

These three topology are closely linked with the algebraic properties of the valuation domains, and in particular there is a connection between the topological properties of $X \subseteq \text{Zar}(K|D)$ and the algebraic properties of the intersection of the elements of X (called the *holomorphy ring* $A(X)$ of X) [12, 13, 14, 15]: for example, if X is a compact subset of one-dimensional valuation domains such that $\bigcap_{V \in X} \mathfrak{m}_V \neq (0)$, then $A(X)$ is a one-dimensional Bézout domain [15, Theorem 5.3]. In particular, for Prüfer domains, the set $\text{Zar}(D)$ (that is, $\text{Zar}(K|D)$ with K being the quotient field of D) is homeomorphic to the spectrum of D (under the Zariski topology). More generally, there is always a map $\gamma : \text{Zar}(K|D) \rightarrow \text{Spec}(D)$, $V \mapsto \mathfrak{m}_V \cap D$, called the *center map*, which is continuous ([24, Chapter VI, §17, Lemma 1] or [4, Lemma 2.1]), surjective (this follows, for example, from [1, Theorem 5.21] or [8, Theorem 19.6]) and closed [4, Theorem 2.5].

The space $\text{Zar}(K|D)$ can also be considered as a subspace of the set $\text{Over}(K|D)$ of the rings comprised between D and K , as a subspace of the set of D -submodules of K or, even more generally, as a subspace of the power set of K ; all these sets become spectral spaces under the natural extension of the Zariski topology [3, 1.9.5(vi-vii)]. It is to be noted that a closer look at Zariski's proof of the compactness of $\text{Zar}(K|D)$ actually shows that $\text{Zar}(K|D)$ is a proconstruable subset of the power set $\mathcal{P}(K)$ [14, discussion after Proposition 2.1].

2. COMPACTNESS

In general, it is hard to find subsets of $\text{Zar}(K|D)$ that are *not* compact. A general algebro-geometric criterion was given in [7, Lemma 5.8(2)] through the theory of *semistar operations*; to be useful, however, it has to be applied together with the theory of the b -operation/integral closure, which can be defined either as the semistar operation induced by the whole $\text{Zar}(D)$ or through integral dependence of ideals [10]. A first consequence is the following.

Theorem 2.1. [19, Proposition 7.1] *Let D be a Noetherian ring with quotient field K , and let Δ be the set of Noetherian valuation overrings of D . Then, Δ is compact if and only if $\dim(D) \leq 1$.*

(Note that, when $\dim(D) \leq 1$, the set Δ is actually just $\text{Zar}(D)$.) If Δ is as in the theorem, then we can write $\Delta = X(D) \cap \text{Zar}(D)$, where

$X(D)$ is the set of Noetherian overrings of D ; in particular, $X(D)$ cannot be proconstructible in the Zariski topology of $\text{Over}(D)$, since this would imply that Δ , as the intersection of two proconstructible subspaces, is itself proconstructible. The same happens for other subsets of Noetherian rings.

Proposition 2.2. [19, Proposition 7.3 and Corollary 7.7] *Let D be a Noetherian domain. Then, the following are equivalent:*

- (i) $\dim(D) = 1$;
- (ii) $X(D)$ is compact;
- (iii) the set $\{T \in \text{Over}(D) \mid T \text{ is a Dedekind domain}\}$ is compact;
- (iv) the set $\{T \in \text{Over}(D) \mid T \text{ is Noetherian of dimension } 1\}$ is compact;

The same holds if “compact” is substituted with “proconstructible”.

Another interesting case is the one in which we delete just one valuation domain.

Theorem 2.3. [19, Theorem 3.6] *Let D be an integral domain and V be a minimal element of $\text{Zar}(D)$. If $\text{Zar}(D) \setminus \{V\}$ is compact, then V is equal to the integral closure of $D[x_1, \dots, x_n]_M$ for some $x_1, \dots, x_n \in K$ and some $M \in \text{Max}(D[x_1, \dots, x_n])$.*

This condition is very strong; for example, it cannot happen in all the following cases:

- D is Noetherian and $\dim(V) \geq 2$;
- $\dim(V) > 2 \dim(D)$ [19, Proposition 4.3];
- D is local and $\bigcap \{P \mid P \in \mathcal{Y}\} = (0)$ for some family \mathcal{Y} of nonzero incomparable prime ideals [19, Theorem 5.1].

A topological space X is *Noetherian* if all its subsets are compact; equivalently, if the open sets of X satisfy the ascending chain condition. For example, the prime spectrum of any Noetherian ring is a Noetherian space [1, Chapter 6, Exercises 5-8]. On the other hand, by either of the previous two cases, $\text{Zar}(D)$ is not a Noetherian space as soon as D is a Noetherian domain of dimension 2 or more. Indeed, the Noetherianity of $\text{Zar}(K|D)$ is an extremely rare phenomenon.

Proposition 2.4. *Let D be an integral domain and let K be a field containing D ; suppose that D is integrally closed in K .*

- (a) [20, Proposition 4.2] *If $D = F$ is a field, then $\text{Zar}(K|F)$ is a Noetherian space if and only if $\text{trdeg}_F K \leq 1$ and, for every $T \in K$ transcendental over F , every valuation on $F[T]$ extends to finitely many valuations of K .*

- (b) [20, Theorem 5.11 and Corollary 5.12] *If D is local and not a field, then $\text{Zar}(D)$ is Noetherian if and only if D is a pseudo-valuation domain,¹ K is the quotient field of D and $\text{Zar}(L|F)$ is Noetherian, where F is the residue field of D and L is the residue field of the associated valuation domain.*
- (c) [20, Theorem 5.11 and Corollary 5.12] *If D is not a field, then $\text{Zar}(K|D)$ is Noetherian if and only if K is the quotient field of D , $\text{Spec}(D)$ is Noetherian and $\text{Zar}(D_M)$ is Noetherian for every $M \in \text{Max}(D)$.*

In particular, these domains have a fairly peculiar Zariski space: in the local case, the non-minimal valuations of D are all comparable, and the valuative dimension of D can be only $\dim(D)$ or $\dim(D) + 1$ [20, Proposition 5.13].

3. PSEUDO-CONVERGENT SEQUENCES

Let now V be a one-dimensional valuation ring with valuation v , value group $\Gamma_v \subseteq \mathbb{R}$ and quotient field K . A *pseudo-convergent sequence* of V is a sequence $E = \{s_n\}_{n \in \mathbb{N}} \subset K$ such that

$$v(s_n - s_{n-1}) < v(s_{n+1} - s_n)$$

for all $n \in \mathbb{N}$, $n \geq 1$. Pseudo-convergent sequences were introduced by Ostrowski to determine all the rank-one extensions of V to $K(X)$ [16, 17], and subsequently used by Kaplansky to investigate maximal valued fields [11]. They can be generalized to *pseudo-monotone sequences* [2, Definition 4.6].

The *gauge* of E is the sequence of the $\delta_n := v(s_{n+1} - s_n)$ [21, p. 327]; it is strictly increasing, and its limit $\delta_E \in \mathbb{R} \cup \{\infty\}$ is called the *breadth* of E . In particular, δ_E is infinite if and only if E is a Cauchy sequence (in the topology induced by the valuation). If V is discrete, every pseudo-convergent sequence has infinite breadth. The ideal $\text{Br}(E) := \{x \in V \mid v(x) \geq \delta_E\}$ is called the *breadth ideal* of E .

Pseudo-convergent sequences can be divided into two classes: E is of *algebraic type* if $v(f(s_n))$ is definitively increasing for some polynomial $f \in K[X]$, while it is of *transcendental type* otherwise [11, Definitions, p. 306]. If $v(\alpha - s_n) < v(\alpha - s_{n+1})$ for all $n \in \mathbb{N}$ (or, equivalently, if $v(\alpha - s_n) = \delta_n$), then α is said to be a *pseudo-limit* of E ; if $\alpha \in \overline{K}$ (the algebraic closure of K), then we can use the same definition once we fix an extension u of v to \overline{K} . In particular, E is of algebraic type if and only if it has a pseudo-limit in \overline{K} [11, Theorems 2 and 3]. Pseudo-limits are not unique, but if α is one of them, then the set $\mathcal{L}(E)$ of the pseudo-limits of E is the coset $\alpha + \text{Br}(E)$ [11, Lemma 3]. The

¹A *pseudo-valuation domain* (PVD) is a local domain (D, \mathfrak{m}) having a valuation overring V whose maximal ideal is \mathfrak{m} ; such V is called the valuation domain associated to D .

name “algebraic” and “transcendental” derive from the fact that, if E is a Cauchy sequence, the limit of E in \widehat{K} is algebraic (resp., transcendental) over K if and only if E is of algebraic (resp., transcendental) type.

To each pseudo-convergent sequence E we associate the map $w_E : K(X) \longrightarrow \mathbb{R} \cup \{\infty\}$ such that [18, Propositions 4.3 and 4.4]

$$w_E(\phi) := \lim_{n \rightarrow \infty} v(\phi(s_n)).$$

Then, w_E is a valuation on $K(X)$ if E is of transcendental type or if E is of algebraic type and $\delta_E < \infty$; if E is of algebraic type and $\delta_E = \infty$, then w_E is only a pseudo-valuation². If w_E is a valuation, the corresponding valuation ring W_E is a one-dimensional extension of V to $K(X)$; if K is algebraically closed, then every rank-one extension of V to $K(X)$ is in this form [16, 17]. We denote the set of all rings in the form W_E as \mathcal{W} : then, the Zariski and the constructible topology agree on \mathcal{W} , and under them \mathcal{W} is a regular zero-dimensional space that is not compact [18, Propositions 6.2 and 6.3].

To every pseudo-convergent sequence E can be associated another valuation domain, defined as

$$V_E := \{\phi \in K(X) \mid \phi(s_n) \in V \text{ for all large } n\}.$$

The ring V_E is always an extension of V to $K(X)$, and it is contained in W_E (if W_E is defined). If E is of transcendental type, then $V_E = W_E$ is an immediate extension of E [18, Theorem 4.9(a)]. On the other hand, if E is of algebraic type, then the value group of V_E is always isomorphic to $\Gamma_v \oplus \mathbb{Z}$, and the rank of V_E depends on the breadth [18, Theorem 4.9(b,c)]:

- if $k\delta \in \Gamma_v$ for some positive $k \in \mathbb{N}$, then V_E has rank 2 and W_E has rank 1;
- if $\delta < \infty$ and $k\delta \notin \Gamma_v$ for all positive $k \in \mathbb{N}$, then $V_E = W_E$ has rank 1;
- if $\delta = \infty$, then V_E has rank 2 and its one-dimensional overring is $K[X]_{(q)}$, where q is the minimal polynomial of the limit of E .

The valuation v_E can also be described explicitly as a map into \mathbb{R}^2 (see [18, Theorem 4.10]).

We denote the set of all the V_E as \mathcal{V} : then, \mathcal{V} is a regular space in both the Zariski and the constructible topology [18, Theorem 7.3], but the two topologies agree on \mathcal{V} if and only if the residue field of V is

²A *pseudo-valuation* on K is a map $v : K \longrightarrow \Gamma_v \cup \{\infty\}$ (where $(\Gamma_v, +)$ is a totally ordered abelian group) such that $v(a+b) \geq \min\{v(a), v(b)\}$ and $v(ab) = v(a) + v(b)$ for all $a, b \in K$; that it, it is a valuation without the hypothesis that only 0 goes to ∞ . It is *not* linked with the notion of pseudo-valuation domain used in Section 2.

finite [18, Proposition 6.8]. There is also a map

$$\begin{aligned} \mathcal{W} &\longrightarrow \mathcal{V} \\ W_E &\longmapsto V_E \end{aligned}$$

that, under the Zariski topology, is continuous and injective, but *not* a topological embedding [18, Proposition 6.10].

There are two natural ways to partition \mathcal{V} , either by fixing the breadth of the sequences or by fixing a pseudo-limit.

Let $\delta \in \mathbb{R} \cup \{\infty\}$, and define $\mathcal{V}(\bullet, \delta) := \{V_E \in \mathcal{V} \mid \delta_E = \delta\}$. Then, the Zariski and the constructible topology agree on $\mathcal{V}(\bullet, \delta)$ [18, Theorem 8.7]; furthermore, this topology is also generated by the ultrametric distance

$$d_\delta(V_E, V_F) := \lim_{n \rightarrow \infty} \max\{d(s_n, t_n) - e^{-\delta}, 0\},$$

where $E := \{s_n\}_{n \in \mathbb{N}}$ and $F := \{t_n\}_{n \in \mathbb{N}}$. Under this metric, $\mathcal{V}(\bullet, \delta)$ is complete, and is the completion of the subspace [18, Propositions 8.5 and 8.6]

$$\mathcal{V}_K(\bullet, \delta) := \{V_E \in \mathcal{V}(\bullet, \delta) \mid E \text{ has a pseudo-limit in } K\}.$$

When $\delta = \infty$, the space $\mathcal{V}(\bullet, \infty)$ is canonically isomorphic to the completion \widehat{K} , and d_∞ reduces to the distance induced by \widehat{v} ; furthermore, $\mathcal{V}_K(\bullet, \infty)$ corresponds to K . Hence, $\mathcal{V}(\bullet, \delta)$ can be seen as a generalization of the completion of V , with the elements of $\mathcal{V}(\bullet, \delta)$ corresponding to the closed balls of V of radius $e^{-\delta}$. Note that the various d_δ *cannot* be unified to a metric on the whole \mathcal{V} (since otherwise they would define closed subspaces of \mathcal{V} , but the $\mathcal{V}(\bullet, \delta)$ are not closed) [18, Proposition 8.11].

Let $\beta \in \overline{K}$, fix an extension u of v to \overline{K} and let

$$\mathcal{V}^u(\beta, \bullet) := \{V_E \in \mathcal{V} \mid \beta \text{ is a pseudo-limit of } E \text{ w.r.t. } u\}.$$

Then, each $\mathcal{V}^u(\beta, \bullet)$ is a closed subspace of \mathcal{V} [18, Proposition 8.13], and the Zariski and the constructible topology agree on $\mathcal{V}^u(\beta, \bullet)$ [18, Proposition 8.17]; furthermore, the elements of $\mathcal{V}^u(\beta, \bullet)$ are parametrized by the breadth, and so there is a bijection between $\mathcal{V}^u(\beta, \bullet)$ and $(-\infty, \delta(\beta, K)]$ (given by $E \mapsto \delta_E$), where $\delta(\beta, K) := \sup\{u(\beta - x) \mid x \in K\}$ represent (the valuation relative to) the distance between β and K . The topology induced by $\mathcal{V}^u(\beta, \bullet)$ on $(-\infty, \delta(\beta, K)]$ is generated by the sets $(a, b]$, with $b \in \mathbb{Q}\Gamma_v$ [18, Theorem 8.15]. This topology is metrizable if and only if Γ_v is countable; in particular, we have the following.

Proposition 3.1. [18, Proposition 8.18] *If Γ_v is not countable, then $\text{Zar}(K(X)|V)^{\text{cons}}$ is not metrizable.*

To conclude, we list some open problems on the topological properties of \mathcal{V} , \mathcal{W} and their subsets.

- Is \mathcal{V} zero-dimensional?
- Is \mathcal{V} a normal space?

- Are $\mathcal{V}(\delta_1, \bullet)$ and $\mathcal{V}(\delta_2, \bullet)$ homeomorphic for $\delta_1 \neq \delta_2$? (This is true if $\delta_1 - \delta_2 \in \Gamma_v$ [18, Proposition 8.4].)
- If Γ_v is countable, are \mathcal{V} and \mathcal{W} metrizable?
- If Γ_v is countable, is $\text{Zar}(K(X)|V)^{\text{cons}}$ metrizable?
- More generally, when is $\text{Zar}(K|D)^{\text{cons}}$ metrizable?
- If any of them is metrizable, can we find an *ultrametric* distance?
- What happens to \mathcal{V} when the rank of V is not 1?

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