WILF'S CONJECTURE FOR NUMERICAL SEMIGROUPS WITH LARGE SECOND GENERATOR

DARIO SPIRITO

ABSTRACT. We study Wilf's conjecture for numerical semigroups S such that the second least generator a_2 of S satisfies $a_2 > \frac{c(S)+\mu(S)}{3}$, where $c(S)$ is the conductor and $\mu(S)$ the multiplicity of S. In particular, we show that for these semigroups Wilf's conjecture holds when the multiplicity is bounded by a quadratic function of the embedding dimension.

1. INTRODUCTION AND PRELIMINARIES

A *numerical semigroup* is a subset $S \subseteq \mathbb{N}$ that contains 0, is closed under addition and such that the complement $\mathbb{N} \setminus S$ is finite. In particular, there is a largest integer not contained in S, which is called the *Frobenius number* of S and is denoted by $F(S)$. The *conductor* of S is defined as $c(S) := F(S) + 1$, and it is the minimal integer x such that $x + \mathbb{N} \subseteq S$. Calculating $F(S)$ is a classical problem (called the *Diophantine Frobenius problem*), introduced by Sylvester [10]; see [7] for a general overview.

Given coprime integers $a_1 < \ldots < a_n$, the numerical semigroup *generated* by $a_1, \ldots, a_n$ is the set

$$\langle a_1, \ldots, a_n \rangle := \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid \lambda_i \in \mathbb{N}\}.$$

Conversely, if S is a numerical semigroup, there are always a finite number of integers $a_1, \ldots, a_n$ such that $S = \langle a_1, \ldots, a_n \rangle$; moreover, there is a unique minimal set of such integers, whose cardinality, called the *embedding dimension* of S, is denoted by $\nu(S)$. The integer a_1, the smallest minimal generator of S, is called the *multiplicity* of S, and is denoted by $\mu(S)$.

In 1978, Wilf [11] suggested a relationship between the conductor and the embedding dimension of S. More precisely, set

$$L(S) := \{x \in S \mid 0 \leq x < c(S)\}.$$

Wilf hypothesized that the inequality

$$\nu(S)|L(S)| \geq c(S)$$

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holds for every numerical semigroup S; this question is known as *Wilf's conjecture*. The conjecture is still unresolved in the general case, although there have been several partial results: for example, it has been proven that Wilf's conjecture holds when $\nu(S) \leq 3$ [10, 3], when $|\mathbb{N} \setminus S| \leq 60$ [4], when $c(S) \leq 3\mu(S)$ [5, 2] and when $\nu(S) \geq \mu(S)/2$ [9].

In this paper, we study Wilf's conjecture when a_2, the second smallest generator of S, is large, in the sense that

$$a_2 > \frac{c(S)+\mu(S)}{3}.$$

In Section 2 we prove some bounds that this condition imposes on S, while in Section 3 we estimate the cardinality of $L(S)$; finally, in Section 4 we show that for these semigroups Wilf's conjecture holds when ν is large and the multiplicity is smaller than a quadratic function of the embedding dimension (Theorem 4.5). The basic idea is to split the generators of S according to whether they are smaller or bigger than $\frac{c(S)+\mu(S)}{2}$, and using this division to estimate the cardinality of $L(S)$.

For general information and results about numerical semigroups, the reader may consult [8].

2. Splitting the generators

From now on, S will be a numerical semigroup, $\mu := \mu(S)$ its multiplicity, $\nu := \nu(S)$ its embedding dimension, and $c := c(S)$ its conductor. We denote by $\mathrm{Ap}(S)$ the *Apéry set* of S with respect to its multiplicity, i.e.,

$$\mathrm{Ap}(S) := \{i \in S \mid i - \mu \notin S\}.$$

We recall that, for every $t \in \{0, \dots, \mu-1\}$, there is a unique $x \in \mathrm{Ap}(S)$ such that $x \equiv t \bmod \mu$; in particular, $\mathrm{Ap}(S)$ has cardinality μ. Note also that, since $c(S)$ is the maximal integer not belonging to S, every element of $\mathrm{Ap}(S)$ is smaller than $c + \mu$.

Let now $P := \{a_1, \dots, a_\nu\}$ be the set of minimal generators of S, with $\mu = a_1 < a_2 < \cdots < a_\nu$. We shall always suppose that $a_2 > \frac{c+\mu}{3}$. Since each $x \in P \setminus \{\mu\}$ belongs to $\mathrm{Ap}(S)$, we can subdivide $P \setminus \{\mu\}$ into the following three sets:

$$P_1 := \left\{a \in P \setminus \{\mu\} \mid \frac{1}{3}(c+\mu) < a < \frac{1}{2}(c+\mu)\right\},$$

$$P_2 := \left\{a \in P \setminus \{\mu\} \mid \frac{1}{2}(c+\mu) \leq a < \frac{2}{3}(c+\mu)\right\},$$

$$P_3 := \left\{a \in P \setminus \{\mu\} \mid \frac{2}{3}(c+\mu) \leq a < c+\mu\right\}.$$

We set $q_i := |P_i|$, for $i \in \{1, 2, 3\}$.

Let $\pi : \mathbb{Z} \longrightarrow \mathbb{Z}/\mu\mathbb{Z}$ be the canonical quotient map, and let $A := \pi(P)$, $A_i := \pi(P_i)$. Given two subsets $X, Y \subseteq \mathbb{Z}/\mu\mathbb{Z}$, the *sumset* of X

and Y is

$$X + Y := \{x + y \mid x \in X, y \in Y\}.$$

Proposition 2.1. *Let $S = \langle a_1, a_2, \ldots, a_\nu \rangle$ be a numerical semigroup with $a_2 > \frac{c(S)+\mu(S)}{3}$. Then, $\mathbb{Z}/\mu\mathbb{Z} = A \cup (A_1 + A_1) \cup (A_1 + A_2)$.*

Proof. Let $x \in \mathrm{Ap}(S)$, $x \neq 0$. Then, $x < c + \mu$ and x is a sum of elements of $P \setminus \{\mu\}$ (since $x - n\mu \notin S$ for $n > 0$). The sum of three elements of $P \setminus \{\mu\}$ is bigger than $c + \mu$, and thus cannot be equal to x; likewise, x cannot be the sum of two elements of $P_2 \cup P_3$, and it also cannot be the sum of an element of P_1 and an element of P_3. Hence, the unique possibilities are $x \in P$, $x \in P_1 + P_1$, or $x \in P_1 + P_2$. The claim follows by projecting onto $\mathbb{Z}/\mu\mathbb{Z}$. □

Using the previous proposition, we can relate quantitatively μ, ν, q_1 and q_2.

Proposition 2.2. *Let $S = \langle a_1, a_2, \ldots, a_\nu \rangle$ be a numerical semigroup with $a_2 > \frac{c(S)+\mu(S)}{3}$. Then:*

(a) $\mu \leq \nu + \dfrac{q_1(q_1+1)}{2} + q_1 q_2$;

(b) $\mu \leq \dfrac{1}{2}\nu(\nu+1)$;

(c) $q_1 \geq \dfrac{2\nu - 1 - \sqrt{(2\nu+1)^2 - 8\mu}}{2}$.

Proof. By Proposition 2.1, we have

$$\mu \leq |A| + |A_1 + A_1| + |A_1 + A_2|.$$

(a) now follows from the inequalities $|A| = \nu$, $|A_1 + A_1| \leq q_1(q_1+1)/2$ (by symmetry) and $|A_1 + A_2| \leq q_1 q_2$.

Using the previous point and the fact that $q_1 + q_2 \leq \nu - 1$, we have

$$\begin{aligned} \mu &\leq \nu + \frac{q_1(q_1+1)}{2} + q_1 q_2 \leq \\ &\leq \nu + \frac{q_1(q_1+1)}{2} + q_1(\nu - 1 - q_1) = \nu - \frac{1}{2}q_1^2 + \left(\nu - \frac{1}{2}\right) q_1, \end{aligned}$$

and thus

$$q_1^2 - (2\nu - 1)q_1 + 2(\mu - \nu) \leq 0. \tag{1}$$

Therefore, the discriminant of the equation is nonnegative, that is,

$$0 \leq (2\nu - 1)^2 - 8(\mu - \nu) = (2\nu + 1)^2 - 8\mu,$$

or equivalently

$$\mu \leq \frac{1}{2}\nu^2 + \frac{1}{2}\nu + \frac{1}{8}.$$

Moreover, since μ and ν are integers, so is $\frac{1}{2}\nu + \frac{1}{2}\nu = \frac{\nu(\nu+1)}{2}$, and thus we can discard the $\frac{1}{8}$. This proves (b).

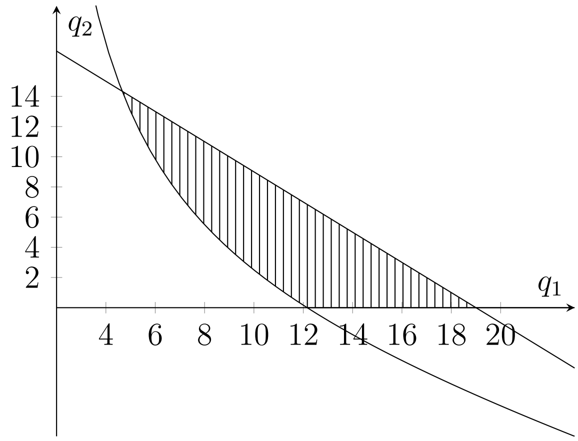

FIGURE 1. The region $\mathcal{A}(100, 20)$.

Under this condition, (1) holds for $q_- \leq q_1 \leq q_+$, where

$$q_- := \frac{2\nu - 1 - \sqrt{(2\nu+1)^2 - 8\mu}}{2} \quad \text{and} \quad q_+ := \frac{2\nu - 1 + \sqrt{(2\nu+1)^2 - 8\mu}}{2}$$

are the solutions of the corresponding equation; hence, (c) follows. □

Remark 2.3. The bound q_- may actually be negative: however, if $q_1 = 0$ then part (a) shows that $\mu \leq \nu$, and thus $\mu = \nu$. In this case, S is of maximal embedding dimension and Wilf's conjecture holds by [3, Theorem 20 and Corollary 2].

Part (a) of Proposition 2.2 can be represented in a graphical way. Fix two integers, μ and ν. The inequalities

- $q_1, q_2 \geq 0$;
- $q_1 + q_2 \leq \nu - 1$;
- $q_1 \left(\frac{1}{2}q_1 + \frac{1}{2} + q_2\right) \geq \mu - \nu$.

define a subset of the plane q_1q_2, which is bounded by two lines and an hyperbola; we denote it by $\mathcal{A}(\mu, \nu)$, or simply $\mathcal{A}$ if there is no danger of confusion. The set is pictured in Figure 1.

Then, if S is a semigroup with multiplicity μ, embedding dimension ν and $a_2 > \frac{c(S)+\mu(S)}{3}$, then the lattice points (q_1, q_2) in $\mathcal{A}(\mu, \nu)$ correspond to the possible cardinalities of the sets P_1 and P_2.

3. Estimates on $|L(S)|$

Lemma 3.1. *Let x, y, b, p be real numbers, with $p > 0$ and $x < y$, and let $A := b + p\mathbb{Z} := \{b + pn \mid n \in \mathbb{Z}\}$. Then:*

(a) $|A \cap [x, y)| \geq \left\lfloor \frac{y-x}{p} \right\rfloor$;

(b) if $x \in A$ and $y \notin A$, then $|A \cap [x, y)| = \left\lfloor \frac{y-x}{p} \right\rfloor + 1$.

Proof. Let $k := \left\lfloor \frac{y-x}{p} \right\rfloor$. Then,

$$x + kp \leq x + \frac{y-x}{p} \cdot p \leq y;$$

hence, the k sets $[x, x+p), [x+p, x+2p), \dots, [x+(k-1)p, x+kp)$ are disjoint subintervals of $[x, y)$. In each $[x+ip, x+(i+1)p)$ there is exactly one element of A; hence, $|A \cap [x, y)| \geq k$.

Moreover, if $x \in A$ then $x + kp \in A$; since $y \notin A$, then $x + kp \neq y$, and thus the interval $[x+kp, y)$ is nonempty and contains exactly one element of A (namely, $x + kp$). Hence, $|A \cap [x, y)| = k + 1$. □

Our goal is to estimate the cardinality of $L := L(S)$. To this end, we introduce the following notation: if x is an integer, let

$$L_x := \{a \in L \mid a \equiv x \bmod \mu\}.$$

Clearly, L_x and L_y are disjoint if $x \not\equiv y \bmod \mu$.

Proposition 3.2. *Let $S = \langle a_1, a_2, \dots, a_\nu \rangle$ be a numerical semigroup with $a_2 > \frac{c(S)+\mu(S)}{3}$. Then,*

$$|L(S)| \geq \left\lfloor \frac{c}{\mu} \right\rfloor + \left(\left\lfloor \frac{1}{2}\frac{c}{\mu} - \frac{1}{2} \right\rfloor + 1 \right) q_1 + \left(\left\lfloor \frac{1}{3}\frac{c}{\mu} - \frac{2}{3} \right\rfloor + 1 \right) q_2. \tag{2}$$

Proof. We have

$$|L(S)| = \sum_{x \in \mathrm{Ap}(S)} |L_x| \geq |L_0| + \sum_{x \in P_1} |L_x| + \sum_{x \in P_2} |L_x|.$$

Suppose $x \in \mathrm{Ap}(S)$. Then, $L_x = (x + \mu\mathbb{Z}) \cap [x, c)$, and by Lemma 3.1 we have $|L_x| \geq \left\lfloor \dfrac{c-x}{\mu} \right\rfloor$. Hence, $|L_0| \geq \left\lfloor \dfrac{c}{\mu} \right\rfloor$, while if $x \in P_1$ then

$$|L_x| \geq \left\lfloor \frac{c - \frac{1}{2}(c+\mu)}{\mu} \right\rfloor = \left\lfloor \frac{1}{2}\frac{c}{\mu} - \frac{1}{2} \right\rfloor$$

and if $x \in P_2$ then

$$|L_x| \geq \left\lfloor \frac{c - \frac{2}{3}(c+\mu)}{\mu} \right\rfloor \geq \left\lfloor \frac{1}{3}\frac{c}{\mu} - \frac{2}{3} \right\rfloor.$$

Hence,

$$|L(S)| \geq \left\lfloor \frac{c}{\mu} \right\rfloor + \left\lfloor \frac{1}{2}\frac{c}{\mu} - \frac{1}{2} \right\rfloor q_1 + \left\lfloor \frac{1}{3}\frac{c}{\mu} - \frac{2}{3} \right\rfloor q_2.$$

Furthermore, applying again Lemma 3.1, for every $x \in \{0\} \cup P_1 \cup P_2$, except possibly one (namely, the x such that $c \equiv x \bmod \mu$), there is a further element in $L_x \cap [x, c)$; hence, we can add $q_1 + q_2$ to the quantity on the right hand side. The claim follows. □

We can get a slightly better version by considering also the relationship between the elements of P_1 and P_2; for this, we shall modify an idea introduced by S. Eliahou in [1]. Say that a pair $(a,b) \in P_1 \times P_2$ is an *Apéry pair* if $a+b \in \mathrm{Ap}(S)$: then, $a+b < c+\mu$, and applying Lemma 3.1 we get

$$\begin{aligned}|L_a|+|L_b| &= \left\lfloor \frac{c-a}{\mu} \right\rfloor + 1 + \left\lfloor \frac{c-b}{\mu} \right\rfloor + 1 \geq \\ &\geq \frac{2c-(a+b)}{\mu} > \frac{c-\mu}{\mu} = \frac{c}{\mu} - 1.\end{aligned} \tag{3}$$

Since $|L_x|+|L_y|$ is an integer, and the inequality is strict, we have $|L_x|+|L_y| \geq \left\lfloor \frac{c}{\mu} \right\rfloor$; in particular, this is better than the number $\left\lfloor \frac{c}{2\mu} \right\rfloor + \left\lfloor \frac{c}{3\mu} \right\rfloor \approx \frac{5}{6}\frac{c}{\mu}$ which we would get by considering the two estimates separately.

Let Σ be the set of Apéry pairs. We say that a subset $\{(a_i,b_i)\}_{i=1}^n \subseteq \Sigma$ is *independent* if $a_i \neq a_j$ and $b_i \neq b_j$ for every $i \neq j$. Denoting by σ the maximal cardinality of an independent set of Apéry pairs, we obtain a slightly better version of Proposition 3.2.

Proposition 3.3. *Let $S = \langle a_1, a_2, \ldots, a_\nu \rangle$ be a numerical semigroup with $a_2 > \frac{c(S)+\mu(S)}{3}$. Then,*

$$|L| \geq \left\lfloor \frac{c}{\mu} \right\rfloor (1+\sigma) + \left(\left\lfloor \frac{1}{2}\frac{c}{\mu} - \frac{1}{2} \right\rfloor + 1 \right)(q_1-\sigma) + \left(\left\lfloor \frac{1}{3}\frac{c}{\mu} - \frac{2}{3} \right\rfloor + 1 \right)(q_2-\sigma). \tag{4}$$

Proof. Take σ independent Apèry pairs $\{(a_t,b_t)\}_{i=1}^{\sigma}$, and write $P_1 = \{a_1, \ldots, a_\sigma, c_1, \ldots, c_r\}$, $P_2 = \{b_1, \ldots, b_\sigma, d_1, \ldots, d_s\}$. Thus, we have

$$|L| \geq |L_0| + \sum_{t=1}^{\sigma}(|L_{a_i}| + |L_{b_i}|) + \sum_{j=1}^{r} |L_{c_j}| + \sum_{k=1}^{s} |L_{d_k}|.$$

Using the estimates in the proof of Proposition 3.2 and the inequality (3) we get our claim. □

Propositions 3.2 and 3.3 can be used to obtain a lower bound on the function $\frac{\nu(S)|L(S)|}{c(S)}$: if this bound is at least 1, then Wilf's conjecture holds for the semigroup S. One problem lies in the floor functions appearing in (2) and (4); the simplest way to get rid of them is to use the inequality $\lfloor x \rfloor \geq x-1$. However, with some additional work we can obtain better estimates.

Indeed, observe that, if $c = (6k-1)\mu$ (where k is an integer), then the quantities $\frac{c}{\mu}$, $\frac{1}{2}\frac{c}{\mu} - \frac{1}{2}$ and $\frac{1}{3}\frac{c}{\mu} - \frac{2}{3}$ appearing in (4) are integers; this suggests to write c as $(6k-1)\mu+\theta\mu$, where k is an integer and $\theta \in [0,6)$ is a rational number. In this way, we have

$$\left\lfloor \frac{c}{\mu} \right\rfloor = \left\lfloor \frac{(6k-1)\mu+\theta\mu}{\mu} \right\rfloor = 6k-1+\lfloor\theta\rfloor = \frac{c}{\mu} - (\theta - \lfloor\theta\rfloor);$$

analogously,

$$\left\lfloor \frac{1}{2}\frac{c}{\mu} - \frac{1}{2} \right\rfloor + 1 = \frac{c}{2\mu} + \frac{1}{2} - \left(\frac{\theta}{2} - \left\lfloor \frac{\theta}{2} \right\rfloor \right)$$

and

$$\left\lfloor \frac{1}{3}\frac{c}{\mu} - \frac{2}{3} \right\rfloor + 1 = \frac{c}{3\mu} + \frac{1}{3} - \left(\frac{\theta}{3} - \left\lfloor \frac{\theta}{3} \right\rfloor \right).$$

Thus, when we multiply (4) by $\frac{\nu}{c}$ we obtain

$$\frac{\nu|L|}{c} \geq \frac{\nu}{\mu}\left[1 - \frac{\mu}{c}(\theta - \lfloor\theta\rfloor)\right](1+\sigma) + \frac{\nu}{\mu}\left[\frac{1}{2} - \frac{\mu}{c}\left(\frac{\theta}{2} - \left\lfloor\frac{\theta}{2}\right\rfloor - \frac{1}{2}\right)\right](q_1 - \sigma) + \\ + \frac{\nu}{\mu}\left[\frac{1}{3} - \frac{\mu}{c}\left(\frac{\theta}{3} - \left\lfloor\frac{\theta}{3}\right\rfloor - \frac{1}{3}\right)\right](q_2 - \sigma).$$

We can write the right hand side of the previous inequality as

$$\ell(q_1, q_2, \sigma) := \frac{\nu}{\mu}[\alpha(1+\sigma) + \beta(q_1 - \sigma) + \gamma(q_2 - \sigma)],$$

where α, β, γ are rational numbers depending on c and μ. By [2], Wilf's conjecture holds when $c \leq 3\mu$; hence, we can suppose, from now on, that $c > 3\mu$. Let now

$$l := \begin{cases} 5 & \text{if } \theta \in [0,4) \\ -1 & \text{if } \theta \in [4,6). \end{cases}$$

Then, $c \geq (l+\theta)\mu$; equivalently, $\frac{\mu}{c} \leq \frac{1}{l+\theta}$. Therefore,

$$\alpha \geq 1 - \frac{\theta - \lfloor\theta\rfloor}{l+\theta}.$$

If $k > -l$, the function $x \mapsto \dfrac{x-k}{x+l}$ is increasing for $x > -l$; hence, in the interval $[k, k+1)$ it is bounded above by its value at $x = k+1$. Thus,

$$\alpha \geq 1 - \frac{1}{l+1+\lfloor\theta\rfloor}.$$

A completely analogous reasoning can be used to estimate β and γ, although in this case the calculations must consider the residue class of $\lfloor\theta\rfloor$ modulo 2 and 3 (for β and γ, respectively). We obtain the following inequalities.

$$\beta \geq \begin{cases} \frac{1}{2} & \text{if } \lfloor\theta\rfloor \equiv 0 \bmod 2 \\ \frac{1}{2} - \frac{1}{2(l+1+\lfloor\theta\rfloor)} & \text{if } \lfloor\theta\rfloor \equiv 1 \bmod 2 \end{cases}$$

$$\gamma \geq \begin{cases} \frac{1}{3} & \text{if } \lfloor\theta\rfloor \equiv 0 \bmod 3 \\ \frac{1}{3} - \frac{1}{3(l+1+\lfloor\theta\rfloor)} & \text{if } \lfloor\theta\rfloor \equiv 1 \bmod 3 \\ \frac{1}{3} - \frac{2}{3(l+1+\lfloor\theta\rfloor)} & \text{if } \lfloor\theta\rfloor \equiv 2 \bmod 3 \end{cases}$$

We now use this estimates to specialize (4) to each possible $\lfloor\theta\rfloor$.

$$\theta\in[0,1)\textbf{:}\quad \frac{\nu|L|}{c}\geq\frac{\nu}{\mu}\left(\frac{5}{6}(1+\sigma)+\frac{1}{2}(q_1-\sigma)+\frac{1}{3}(q_2-\sigma)\right)=$$
$$=\frac{\nu}{\mu}\left(\frac{5}{6}+\frac{1}{2}q_1+\frac{1}{3}q_2\right)=:\frac{\nu}{\mu}\ell_1(q_1,q_2,\sigma).$$

$$\theta\in[1,2)\textbf{:}\quad \frac{\nu|L|}{c}\geq\frac{\nu}{\mu}\left(\frac{6}{7}(1+\sigma)+\frac{3}{7}(q_1-\sigma)+\frac{2}{7}(q_2-\sigma)\right)=$$
$$=\frac{\nu}{\mu}\left(\frac{6}{7}+\frac{3}{7}q_1+\frac{2}{7}q_2+\frac{1}{7}\sigma\right)=:\frac{\nu}{\mu}\ell_2(q_1,q_2,\sigma).$$

$$\theta\in[2,3)\textbf{:}\quad \frac{\nu|L|}{c}\geq\frac{\nu}{\mu}\left(\frac{7}{8}(1+\sigma)+\frac{1}{2}(q_1-\sigma)+\frac{1}{4}(q_2-\sigma)\right)=$$
$$=\frac{\nu}{\mu}\left(\frac{7}{8}+\frac{1}{2}q_1+\frac{1}{4}q_2+\frac{1}{8}\sigma\right)=:\frac{\nu}{\mu}\ell_3(q_1,q_2,\sigma).$$

$$\theta\in[3,4)\textbf{:}\quad \frac{\nu|L|}{c}\geq\frac{\nu}{\mu}\left(\frac{8}{9}(1+\sigma)+\frac{4}{9}(q_1-\sigma)+\frac{1}{3}(q_2-\sigma)\right)=$$
$$=\frac{\nu}{\mu}\left(\frac{8}{9}+\frac{4}{9}q_1+\frac{1}{3}q_2+\frac{1}{9}\sigma\right)=:\frac{\nu}{\mu}\ell_4(q_1,q_2,\sigma).$$

$$\theta\in[4,5)\textbf{:}\quad \frac{\nu|L|}{c}\geq\frac{\nu}{\mu}\left(\frac{3}{4}(1+\sigma)+\frac{1}{2}(q_1-\sigma)+\frac{1}{4}(q_2-\sigma)\right)=$$
$$=\frac{\nu}{\mu}\left(\frac{3}{4}+\frac{1}{2}q_1+\frac{1}{4}q_2\right)=:\frac{\nu}{\mu}\ell_5(q_1,q_2,\sigma).$$

$$\theta\in[5,6)\textbf{:}\quad \frac{\nu|L|}{c}\geq\frac{\nu}{\mu}\left(\frac{4}{5}(1+\sigma)+\frac{2}{5}(q_1-\sigma)+\frac{4}{15}(q_2-\sigma)\right)=$$
$$=\frac{\nu}{\mu}\left(\frac{4}{5}+\frac{2}{5}q_1+\frac{4}{15}q_2+\frac{2}{15}\sigma\right)=:\frac{\nu}{\mu}\ell_6(q_1,q_2,\sigma).$$

4. Wilf's conjecture for large second generator

Proposition 3.3 isn't really better than Proposition 3.2 if we don't have a way to estimate σ. We do it in the following proposition, using a graph-theoretic method; see e.g. [6] for the terminology used in the proof. The proof is inspired by [1].

Proposition 4.1. *Let Σ and σ as in Section 3. Then,* $\sigma\geq\dfrac{|\Sigma|}{\max\{q_1,q_2\}}$.

Proof. Define a graph G by taking the disjoint union $P_1\sqcup P_2$ as the set of vertices and Σ as the set of edges. Then, an independent subset of Σ is exactly an independent subset of edges of G, that is, a matching, and σ is exactly the matching number of G.

Moreover, G is a bipartite graph, and thus (by König's theorem, see e.g. [6, Theorem 1.1.1]) the matching number of G is equal to the its point covering number, i.e., to the cardinality of the smallest set $S \subseteq V(G)$ such that every edge of G has a vertex in S.

For every $v \in V(G)$, the number of edges incident to v is at most q_1 if $v \in P_2$ and at most q_2 if $v \in P_1$; hence, the point covering number of G is at least $|E(G)|/\max\{q_1, q_2\}$. The claim follows. □

We also obtain a slightly better version of Proposition 2.2(a).

Corollary 4.2. *Let $S = \langle a_1, a_2, \dots, a_\nu \rangle$ be a numerical semigroup with $a_2 > \frac{c(S)+\mu(S)}{3}$, and let σ as above. Then,*

$$\frac{q_1(q_1+1)}{2} + \sigma \cdot \max\{q_1, q_2\} + \nu \geq \mu. \tag{5}$$

Proof. Following the proof of Proposition 2.1, we see that if $x \in \mathrm{Ap}(S) \cap (P_1 + P_2)$ then $x = a_1 + b_1$ for some Apéry pair $(a_1, b_1) \in \Sigma$; hence,

$$|\mathrm{Ap}(S) \cap (P_1 + P_2)| \leq \Sigma \leq \sigma \cdot \max\{q_1, q_2\},$$

with the last inequality coming from Proposition 4.1. The claim now follows using the proof of Proposition 2.2(a). □

Before presenting the main theorem, we prove a lemma.

Lemma 4.3. *Let $f(x, y) := \alpha + \beta x + \gamma y$, where α, β, γ are positive real numbers such that $\alpha \leq 1$ and $2\beta \geq \gamma$. For every $\epsilon > 0$ there is a $\nu_0(\epsilon)$ such that, if $\nu \geq \nu_0(\epsilon)$ and μ satisfies*

$$2\nu \leq \mu < 2\gamma(2\beta - \gamma)\nu^2 + (2\alpha - \gamma - 1)\nu - \frac{(2 - 2\alpha + \gamma)^2}{8\gamma(2\beta - \gamma)} - \epsilon, \tag{6}$$

then

$$f(x, y) \geq \frac{\mu}{\nu}$$

for every $(x, y) \in \Omega := \{(x, y) \in \mathbb{R}^2 \mid x > 0, y \geq 0,\ x\left(\frac{1}{2}x + \frac{1}{2} + y\right) + \nu \geq \mu\}$.

Proof. Since f is a linear function, and the components of its gradient are positive, the (eventual) minimum of f on Ω can be reached only on its border $\mathcal{I}$, which is formed by a subset of an hyperbola (say $\mathcal{I}'$) and a subset of the x-axis. Moreover, f is monotone increasing on the x-axis, and thus the minimum can only be reached on $\mathcal{I}'$. On it, f becomes a quadratic function such that $f \to \infty$ when $x \to 0$ (since $y \to \infty$); therefore, f has actually a minimum on $\mathcal{I}'$, and the point (x_0, y_0) where it is reached satisfies, by Lagrange multipliers,

$$\begin{cases} \partial_x f(x_0, y_0) = x_0 + \frac{1}{2} + y_0 = \beta\lambda \\ \partial_y f(x_0, y_0) = x_0 = \gamma\lambda \end{cases}$$

for some $\lambda \in \mathbb{R}$; imposing $(x_0, y_0) \in \mathcal{I}'$ we have

$$\mu - \nu = x_0 \left(\frac{1}{2}x_0 + \frac{1}{2} + y_0\right) =$$
$$= \partial_y f(x_0, y_0) \cdot \left(\partial_x f(x_0, y_0) - \frac{1}{2}\partial_y f(x_0, y_0)\right) = \frac{\gamma(2\beta - \gamma)}{2}\lambda^2$$

and thus

$$\lambda = \sqrt{\frac{2}{\gamma(2\beta - \gamma)}}\sqrt{\mu - \nu}.$$

Substituting in f, we have

$$f(x_0, y_0) = \alpha + \beta\gamma\lambda + \gamma\left[(\beta - \gamma)\lambda - \frac{1}{2}\right] =$$
$$= \alpha - \frac{\gamma}{2} + \gamma(\beta + \beta - \gamma)\sqrt{\frac{2}{\gamma(2\beta - \gamma)}}\sqrt{\mu - \nu} =$$
$$= \alpha - \frac{\gamma}{2} + \sqrt{2\gamma(2\beta - \gamma)}\sqrt{\mu - \nu}.$$

Therefore, if $f(x_0, y_0) \geq \frac{\mu}{\nu}$ then also $f(x, y) \geq \frac{\mu}{\nu}$ for every $(x, y) \in \Omega$. We thus must solve an inequality in the form

$$\zeta + \xi\sqrt{\mu - \nu} \geq \frac{\mu}{\nu}, \tag{7}$$

or equivalently (since $\nu > 0$)

$$\xi\nu\sqrt{\mu - \nu} \geq \mu - \zeta\nu.$$

In our hypothesis, $\zeta = \alpha - \frac{\gamma}{2} \leq \alpha \leq 1$ and $\mu > \nu$; hence, the right hand side is positive and we can square both sides, obtaining

$$\xi^2\nu^2(\mu - \nu) \geq \mu^2 - 2\zeta\nu\mu + \zeta^2\nu^2,$$

or, equivalently,

$$\mu^2 - (2\zeta\nu + \xi^2\nu^2)\mu + \zeta^2\nu^2 + \xi^2\nu^3 \leq 0. \tag{8}$$

Suppose $\mu = 2\nu$: then, the left hand side of (8) is equal to

$$\nu^2(4 - 4\zeta + \zeta^2) + \nu^3(-2\xi^2 + xi^2) = \nu^2[(1 - \zeta)^2 - \nu\xi^2],$$

which is negative for $\nu > \frac{(1-\zeta)^2}{\xi^2}$. Hence, under this condition the left hand side of (8) has two roots, $\mu_- < \mu_+$, and $\mu_- < 2\nu$. On the other hand,

$$\mu_+ = \frac{(2\zeta\nu + \xi^2\nu^2) + \sqrt{\nu^2\xi^2[4(\zeta - 1)\nu + \xi^2\nu^2]}}{2} =$$
$$= \frac{2\zeta\nu + \xi^2\nu^2 + \xi^2\nu^2\sqrt{1 - \frac{4(1-\zeta)}{\xi^2\nu}}}{2}.$$

Expanding $\sqrt{1-\frac{4(1-\zeta)}{\xi^2\nu}}$ as a Taylor series we have

$$\xi^2\nu^2\sqrt{1-\frac{4(1-\zeta)}{\xi^2\nu}} = \xi^2\nu^2\left(1-\frac{1}{2}\cdot\frac{4(1-\zeta)}{\xi^2\nu}-\frac{1}{8}\left(\frac{4(1-\zeta)}{\xi^2\nu}\right)^2+R_2(x)\right) =$$

$$= \xi^2\nu^2 - 2(1-\zeta)\nu - \frac{2(1-\zeta)^2}{\xi^2} + \xi^2\nu^2R_2(x),$$

where R_2 is the remainder and $x = \frac{4(1-\zeta)}{\xi^2\nu}$. In particular, $R_2(x) = O(x^3)$; hence, $\xi^2\nu^2R_2(x)$ is $O(1/\nu)$, and thus it is bigger than $-\epsilon$ for every $\nu \geq \nu_0(\epsilon)$ (for any $\epsilon > 0$). Hence, for $\nu \geq \nu_0(\epsilon)$ (7) holds for $\mu_- \leq \mu \leq \mu_+$, with

$$\mu_+ \geq \xi^2\nu^2 + (2\zeta-1)\nu - \frac{(1-\zeta)^2}{\xi^2} - \epsilon.$$

Substituting ζ and ξ with their definitions we have our claim. □

Remark 4.4. The remainder of the Taylor series can actually be estimated fairly simply. Indeed, using $\frac{\mathrm{d}^3}{\mathrm{d}x^3}\sqrt{1-x} = -\frac{3}{8}\frac{1}{(1-x)^{5/2}}$ and Taylor's theorem, we obtain, putting $\lambda := \frac{4(1-\zeta)}{\xi^2}$,

$$|\xi^2\nu^2R_2(x)| \leq \frac{\lambda^3\xi^2}{16}\cdot\frac{1}{\nu}\left(\frac{\nu}{\nu-\lambda}\right)^{5/2}.$$

As a function of ν, the quantity on the right hand side is decreasing for $\nu > \lambda$; for example, for $\nu \geq 2\lambda$ we have

$$|\xi^2\nu^2R_2(x)| \leq \frac{\sqrt{2}\lambda^3\xi^2}{8\nu}.$$

We will use this estimate in Proposition 4.6.

We are now ready to prove the main theorem.

Theorem 4.5. *For every $\epsilon > 0$ there is a $\nu_0(\epsilon)$ such that, if $S = \langle a_1, a_2, \ldots, a_\nu\rangle$ is a numerical semigroup such that:*

- $a_2 > \frac{c(S)+\mu(S)}{3}$,
- $\nu(S) = \nu \geq \nu_0(\epsilon)$, *and*
- $\mu(S) \leq \frac{8}{25}\nu^2 + \frac{1}{5}\nu - \frac{1}{2} - \epsilon$,

then S satisfies Wilf's conjecture.

Proof. By [2], we need only to consider semigroups S such that $c > 3\mu$. Write $c = (6k-1)\mu + \theta\mu$, where k an integer and $\theta \in [0, 6)$. By the discussion in Section 3, for every $i := \lfloor\theta\rfloor$ there is a linear function $\ell_i(q_1, q_2, \sigma)$, not depending on S, such that

$$\frac{\nu|L|}{c} \geq \frac{\mu}{\nu}\ell_i(q_1, q_2, \sigma).$$

We distinguish two cases.

Suppose $q_1 \geq q_2$. By Corollary 4.2, we have $\frac{q_1(q_1+1)}{2} + \sigma q_1 + \nu \geq \mu$; equivalently, the point (q_1, σ) belongs to the set $\mathcal{A}(\nu, \mu)$ defined at the end of Section 2. Since $q_2 \geq \sigma$, we have $\ell_i(q_1, q_2, \sigma) \geq \ell_i(q_1, \sigma, \sigma) =: \ell'_i(q_1, \sigma)$. By Lemma 4.3 applied to ℓ'_i (and since $\mathcal{A}(\nu, \mu) \subseteq \Omega$), for every $\epsilon > 0$ there is a $\overline{\nu_i}(\epsilon)$ such that $\ell'_i(q_1, \sigma) \geq \frac{\nu}{\mu}$ when $\nu \geq \overline{\nu_i}(\epsilon)$ and

$$\mu \leq A_i\nu^2 + B_i\nu + C_i - \epsilon,$$

where A_i, B_i and C_i are constants depending on i. In particular, A_5 is equal to $\frac{8}{25}$ and smaller than every other A_i; hence, there is a $\nu'_0(\epsilon)$ such that $A_5\nu^2 + B_5\nu + C_5 - \epsilon \leq A_i\nu^2 + B_i\nu + C_i - \epsilon$ for all i and every $\nu \geq \nu'_0(\epsilon)$.

Therefore, $\frac{\nu(S)|L(S)|}{c(S)} \geq 1$ for all semigroups S with

- $a_2 > \frac{c(S)+\mu(S)}{3}$,
- $\mu(S) \leq A_5\nu(S)^2 + B_5\nu(S) + C_5 - \epsilon$ and
- $\nu(S) \geq \nu_0(\epsilon) := \max\{\nu'(\epsilon), \overline{\nu_0}(\epsilon), \ldots, \overline{\nu_5}(\epsilon)\}$.

Since the condition $\frac{\nu(S)|L(S)|}{c(S)} \geq 1$ is equivalent to S satisfying Wilf's conjecture, the claim follows substituting A_5, B_5 and C_5 with their value.

Suppose $q_1 \leq q_2$; by Corollary 4.2, $\frac{q_1(q_1+1)}{2} + \sigma q_2 + \nu \geq \mu$. Then, (q_1, q_2, σ) belongs to the set

$$\Omega' := \left\{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \geq 0, z \leq x \leq y,\ \frac{x(x+1)}{2} + yz \geq \mu - \nu\right\} \subseteq \mathbb{R}^3.$$

As in the proof of Lemma 4.3, the minimum of ℓ_i on Ω' can only belong to the hyperboloid $\left\{\frac{x(x+1)}{2} + yz = \mu - \nu\right\}$; by Lagrange multipliers, the minimum (x_0, y_0, z_0) of ℓ_i on the hyperboloid satisfies

$$\begin{cases} \partial_x\ell_i(x_0, y_0, z_0) = x_0 + \frac{1}{2} = \beta_i\lambda \\ \partial_y\ell_i(x_0, y_0, z_0) = z_0 = \gamma_i\lambda \\ \partial_z\ell_i(x_0, y_0, z_0) = y_0 = \delta_i\lambda. \end{cases}$$

Since $\gamma_i > \delta_i$ for each i, we must have $z_0 > y_0$, which however implies that $(x_0, y_0, z_0) \notin \Omega'$; hence, the minimal point of ℓ_i in Ω' must belong on the intersection between the hyperboloid and one of the planes $\{x = z\}$ and $\{x = y\}$. If it is on the latter, we have $q_1 = q_2$, and we fall back to the case $q_1 \geq q_2$; if it is on the former, then we have to find the minimum of $\ell'_i(\sigma, q_2) := \ell_i(\sigma, q_2, \sigma)$ on

$$\Omega'' := \left\{(z, y) \in \mathbb{R}^2 \mid z > 0, y \geq 0, z \leq y,\ \frac{z(z+1)}{2} + yz \geq \mu - \nu\right\}.$$

This set is contained in the domain Ω of Lemma 4.3; hence, we can apply the lemma and, as in the proof of the case $q_1 \geq q_2$, we obtain that $\frac{\nu(S)|L(S)|}{c(S)} \geq 1$ for all semigroups S with $\nu(S) \geq \nu_0(\epsilon)$ and $\mu(S) \leq A_j\nu^2 + B_j\nu + C_j - \epsilon$ (where $\nu_0(\epsilon), A_j, B_j, C_j$ are different from the previous case). However, a direct calculation shows that all A_i are strictly bigger than $\frac{8}{25}$; hence, this case does not give any further restriction on the semigroups on which Wilf's conjecture holds (except perhaps the need to pass from $\nu_0(\epsilon)$ to a larger number). Hence, the claim holds. □

The same reasoning can yield a more explicit version.

Proposition 4.6. *Let $S = \langle a_1, a_2, \dots, a_\nu \rangle$ be a numerical semigroup with $\nu(S) = \nu \geq 10$. If $a_2 > \frac{c(S)+\mu(S)}{3}$ and*

$$\mu(S) \leq \frac{8}{25}\nu(S)^2 + \frac{1}{5}\nu(S) - \frac{5}{4},$$

then S satisfies Wilf's conjecture.

Proof. The proof is akin to the one of Theorem 4.5; we employ the same notation. Suppose $q_1 \geq q_2$, and let A_i, B_i, C_i be the coefficients of the polynomial in ν which is on the right hand side of (7) when $f = \ell_i'$. When $\nu \geq 10$, for each i the left hand side of (8) is negative when $\mu = 2\nu$; furthermore, $A_5\nu^2 + B_5\nu + C_5 \leq A_i\nu^2 + B_i\nu + C_i$ for each i when $\nu \geq 10$. Moreover, in the notation of Remark 4.4, the largest λ and $\lambda^3\xi^2$ appear again when $\theta \in [5, 6)$, when their value is, respectively, 5 and 40; hence, the error term is at most

$$\frac{\sqrt{2}\lambda^3\xi^2}{8\nu} \leq \frac{\sqrt{2}\cdot 40}{80} = \frac{\sqrt{2}}{2} < \frac{3}{4}.$$

Therefore, in this case Wilf's conjecture holds when

$$\mu(S) \leq \frac{8}{25}\nu^2 + \frac{1}{5}\nu - \frac{1}{2} - \frac{3}{4},$$

as claimed.

In the case $q_1 \leq q_2$ the functions ℓ_i' we obtain putting $q_1 = \sigma$ are always bigger than the corresponding functions for the case $q_1 \geq q_2$; hence, also in this case Wilf's conjecture holds when $\nu \geq 10$ and μ verifies the above inequality. The claim is proved. □

To conclude the paper, we give three variants of Theorem 4.5 that can be proved with arguments very similar to the proof of the theorem. The first one looks at case $c \equiv 0 \bmod \mu$, the second one strengthens the coefficients $\frac{8}{25}$ and the third one weakens Wilf's conjecture.

Proposition 4.7. *If $S = \langle a_1, a_2, \dots, a_\nu \rangle$ is a numerical semigroup such that*

- $a_2 > \frac{c(S)+\mu(S)}{3}$,
- $\nu(S) \geq 10$ *and*
- $c \equiv 0 \bmod \mu$,

then S satisfies Wilf's conjecture.

Proof. Using the same reasoning of the proof of Theorem 4.5, the worst bound of μ with respect to ν happens in the case $q_1 \geq q_2 = \sigma$; under this condition, we have

$$\frac{\nu|L|}{c} \geq \frac{\nu}{\mu}[\alpha(1+q_2) + \beta(q_1 - q_2)]$$

where

$$\alpha := 1 - \frac{\mu}{c}(\theta - \lfloor \theta \rfloor) = 1$$

(using the condition $c \equiv 0 \bmod \mu$, which is equivalent to θ being an integer). Likewise,

$$\beta := \frac{1}{2} - \frac{\mu}{c}\left(\frac{\theta}{2} - \left\lfloor \frac{\theta}{2} \right\rfloor - \frac{1}{2}\right) \geq \frac{1}{2}$$

because $\frac{\theta}{2} - \left\lfloor \frac{\theta}{2} \right\rfloor \leq \frac{1}{2}$. Hence,

$$\frac{\nu|L|}{c} \geq \frac{\nu}{\mu}\left[(1+q_2) + \frac{1}{2}(q_1 - q_2)\right] = 1 + \frac{1}{2}q_1 + \frac{1}{2}q_2 =: \frac{\nu}{\mu}\ell(q_1, q_2).$$

By Lemma 4.3, Remark 4.4 and the proof of Proposition 4.6, if $\nu(S) \geq 10$ then $\ell(q_1, q_2) \geq \mu/\nu$ when $(q_1, q_2) \in \mathcal{A}(\mu, \nu)$ and μ satisfies

$$2\nu \leq \mu < \frac{1}{2}\nu^2 + \frac{1}{2}\nu - \frac{1}{4} - \frac{\sqrt{2}}{2}.$$

Since μ is an integer and $\frac{1}{4} + \frac{\sqrt{2}}{2} < 1$, this means that Wilf's conjecture holds when $\mu < \frac{1}{2}\nu^2 + \frac{1}{2}\nu$.

By Proposition 2.2(b), the only case left to consider is $\mu = \frac{1}{2}\nu^2 + \frac{1}{2}\nu = \frac{\nu(\nu+1)}{2}$. Under this condition, we have, by Proposition 2.2(c),

$$q_1 \geq \frac{2\nu - 1 - 1}{2} = \nu - 1;$$

since also $q_1 \leq \nu - 1$ we must have $q_1 = \nu - 1$ and $q_2 = 0$. In this case,

$$\frac{\nu|L|}{c} \geq \frac{\nu}{\mu}\left[1 + \frac{1}{2}(\nu - 1)\right] = \frac{\nu}{\mu} \cdot \frac{\nu+1}{2} = \frac{\nu(\nu+1)}{2\mu} = 1$$

and thus S satisfies Wilf's conjecture. □

Proposition 4.8. *There is an integer N such that, for every $\nu \geq N$, there are only finitely many numerical semigroups $S = \langle a_1, a_2, \ldots, a_\nu \rangle$ with*

- $a_2 > \frac{c(S)+\mu(S)}{3}$,
- $\nu = \nu(S)$, *and*
- $\mu(S) \leq \frac{4}{9}\nu^2$,

and that do not *satisfy Wilf's conjecture.*

Proof. Fix any $\chi \in (0, 1/3)$, and consider the function

$$f(q_1, q_2) := 1 - \chi + \frac{1}{2}q_1 + \frac{1}{3}q_2.$$

By Lemma 4.3, for every $\epsilon > 0$ there is an $N_1(\chi, \epsilon)$ such that, for every point $(q_1, q_2) \in \mathcal{A}(\mu, \nu)$, with $\nu \geq N_1(\chi, \epsilon)$, we have $f(q_1, q_2) \geq \mu/\nu$ whenever

$$\mu \leq \frac{4}{9}\nu^2 + \left(\frac{2}{3} - 2\chi\right)\nu - \frac{1}{16} - \epsilon.$$

Let $N_2(\chi, \epsilon) := \left(\epsilon + \frac{1}{16}\right)\left(\frac{2}{3} - 2\chi\right)^{-1}$: then, for $\nu \geq N_2(\chi, \epsilon)$, we have

$$\left(\frac{2}{3} - 2\chi\right)\nu - \frac{1}{16} - \epsilon \geq 0.$$

Therefore, for every $\nu \geq N := N(\chi, \epsilon) := \max\{N_1(\chi, \epsilon), N_2(\chi, \epsilon)\}$ we have $f(q_1, q_2) \geq \mu/\nu$ whenever $\mu \leq \frac{4}{9}\nu^2$. Equivalently, we have

$$1 + \frac{1}{2}q_1 + \frac{1}{3}q_2 \geq \frac{\mu}{\nu} + \chi.$$

Using the inequality $\lfloor x \rfloor > x - 1$ on Proposition 3.2, we have

$$\frac{\nu|L|}{c} \geq \frac{\nu}{\mu}\left(1 + \frac{1}{2}q_1 + \frac{1}{3}q_2\right) - \frac{\nu}{c}\left(1 + \frac{1}{2}q_1 + \frac{2}{3}q_2\right)$$

which for $\nu \geq N$ is bigger than

$$\frac{\nu}{\mu}\left(\frac{\mu}{\nu} + \chi\right) - \frac{\nu}{c}\left(\frac{\mu}{\nu} + \chi + \frac{1}{3}q_2\right) \geq 1 + \frac{\nu}{\mu}\chi - \frac{1}{c}\left(\mu + \chi\nu + \frac{\nu(\nu-1)}{3}\right),$$

using also the fact that $q_2 \leq \nu - 1$. The quantity on the right hand side is bigger than 1 when

$$\frac{\nu}{\mu}\chi - \frac{1}{c}\left(\mu + \chi\nu + \frac{\nu(\nu-1)}{3}\right) \geq 0;$$

since c, ν, μ and χ are positive, this is equivalent to

$$c \geq \frac{\mu}{\chi\nu}\left(\mu + \chi\nu + \frac{\nu(\nu-1)}{3}\right), \tag{9}$$

and all semigroups satisfying this inequality satisfy Wilf's conjecture.

In particular, for any value of ν, μ and χ, there are only a finite number of semigroups that do not satisfy this condition. For any ν, there are also a finite number of multiplicities μ satisfying $\mu \leq \frac{4}{9}\nu^2$; hence, for any fixed $\nu \geq N$ there are only finitely many numerical semigroups that verify the hypothesis of the theorem and that do not satisfy Wilf's conjecture. □

We note that the right hand side of (9) is very large: for example, if $\nu = 10$, $\mu = 50$ and $\chi = \frac{1}{6}$, then it is equal to 26050. The strategy used in the proof of Theorem 4.5 (i.e., writing $c = (6k-1)\mu + \theta\mu$ and using different estimates for different $\lfloor\theta\rfloor$) can be employed to obtain numerically better bounds (but still with the hypothesis $\mu \leq \frac{4}{9}\nu^2$).

Proposition 4.9. *For every* $\lambda < \frac{4}{5}$ *there is a* $\nu_0(\lambda)$ *such that, if* $S = \langle a_1, a_2, \dots, a_\nu\rangle$ *is a numerical semigroup such that* $a_2 > \frac{c(S)+\mu(S)}{3}$ *and* $\nu \geq \nu_0(\lambda)$*, then*

$$\nu(S)|L(S)| \geq \lambda \cdot c(S). \tag{10}$$

Proof. Fix a $\lambda < \frac{4}{5}$. Let $c = (6k-1)\mu + \theta\mu$, with k an integer and $\theta \in [0,6)$. For any fixed $\lfloor\theta\rfloor$, we have

$$\frac{\nu|L|}{c} \geq \frac{\mu}{\nu}(\alpha + \beta q_1 + \gamma q_2 + \delta\sigma),$$

for some $\alpha, \beta, \gamma, \delta$ depending on $\lfloor\theta\rfloor$. Therefore, (10) holds if

$$\lambda^{-1}\alpha + \lambda^{-1}\beta q_1 + \lambda^{-1}\gamma q_2 + \lambda^{-1}\delta\sigma \geq \frac{\mu}{\nu}$$

which, by Lemma 4.3, holds for

$$\mu \leq [2(\lambda^{-1}\gamma)(2\lambda^{-1}\beta - \lambda^{-1}\gamma) - \epsilon]\nu^2 = \left(\frac{2\gamma(2\beta-\gamma)}{\lambda^2} - \epsilon\right)\nu^2.$$

for $\nu \geq \nu_0'(\epsilon)$. By Theorem 4.5, $2\gamma(2\beta-\gamma)$ is at least $\frac{8}{25}$; if $\lambda < \frac{4}{5}$, then

$$\frac{2\gamma(2\beta-\gamma)}{\lambda^2} > \frac{8}{25}\cdot\frac{25}{16\cdot 2} = \frac{1}{2}.$$

Therefore, we can choose an ϵ satisfying

$$0 < \epsilon < \frac{2\gamma(2\beta-\gamma)}{\lambda^2} - \frac{1}{2},$$

and for such an ϵ there is a $\nu_0''(\epsilon,\lambda)$ such that

$$\left(\frac{2\gamma(2\beta-\gamma)}{\lambda^2} - \epsilon\right)\nu^2 > \frac{1}{2}\nu^2 + \frac{1}{2}\nu$$

for all $\nu \geq \nu_0''(\epsilon,\lambda)$. Setting $\nu_0(\lambda) := \max\{\nu_0'(\epsilon), \nu_0''(\epsilon,\lambda)\}$, we have that the inequality (10) holds for $\nu \geq \nu_0(\lambda)$ and $\mu \leq \frac{1}{2}\nu^2 + \frac{1}{2}\nu$. Since every semigroup with $a_2 > \frac{c(S)+\mu(S)}{3}$ satisfies the latter condition (by Proposition 2.2(b)), the claim holds. □

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E-mail address: spirito@mat.uniroma3.it

Dipartimento di Matematica e Fisica, Università degli Studi "Roma Tre", Roma, Italy