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Spaces of closure operations on rings and numerical semigroups

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Sets of ideals and modules	
$\mathcal{F}(S)$, $\mathcal{F}(D)$	set of fractional ideals of the numerical semigroup S or the domain D [Section 1.1]
$\mathcal{F}_0(S)$	set of fractional ideals of the numerical semigroup S comprised between S and $\mathbb{N}$ [Section 1.1]
$\mathcal{F}^*(S)$	set of closed ideals for $* \in \mathrm{Star}(S)$ [Section 1.2]
$\mathcal{F}_0^*(S)$	$= \mathcal{F}^*(S) \cap \mathcal{F}_0(S)$ [Section 1.2]
$\mathcal{G}_0(S)$	ideals of $\mathcal{F}_0(S)$ that are not divisorial [Section 1.2]
$\mathcal{G}_1(S)$	$= \{I \in \mathcal{F}_0(S) : \lvert I^v \setminus I\rvert = 1\}$ [Corollary 1.40]
$\mathcal{I}(D)$	set of integral ideals of the ring D [Section 2.2]
$\mathcal{I}_f(D)$	set of finitely-generated integral ideals of D [Section 2.2]
$\mathcal{F}_f(D)$	set of finitely-generated fractional ideals of D [Section 2.2]
$\mathbf{F}(D)$	set of D-submodules of the quotient field of D [Section 2.2]
$\mathrm{QMax}^*(D)$	set of the quasi-$*$-maximal ideal of D [Definition 2.28]
$\mathrm{QSpec}^*(D)$	set of the quasi-$*$-prime ideal of D [Definition 2.28]
$\mathrm{Inv}^*(R)$	set of the $*$-invertible ideals of R [Section 3.3.1]
$\mathrm{Cl}^*(R)$	$*$-class group of R [Section 3.3.1]
$X^1(R)$	set of the height-1 prime ideals of R
Notations for closures	
$*_I$	star operation generated by the ideal I [Definition 1.4]
$*_\Delta$	star operation generated by $\Delta \subseteq \mathcal{F}(S)$ [Section 1.2]
$\wedge_I$	principal semistar operation generated by I [Section 1.8.1.6]
$\wedge_{\{T\}}$	extension to the overring T [Example 2.12(2)]
$\wedge_\Delta$	semistar operation induced by the family Δ of overrings of D [Example 2.12(4)]
I^b	b-closure of the ideal I [Example 2.12(6)]
$*_f$	semistar (or star) operation of finite type associated to $*$ [Definition 2.16]
Ψ_f	map $* \mapsto *_f$ [Section 2.2.1]
$\overline{*}$	stable semistar (or star) operation associated to $*$ [Section 2.2.2]
Ψ_{st}	map $* \mapsto \overline{*}$ [Section 2.2.2]
s_Δ	spectral semistar operation induced by $\Delta \subseteq \mathrm{Spec}(R)$ [Definition 2.26]
$\widetilde{*}$	spectral semistar (or star) operation associated to $*$ [Section 2.2.3]
Ψ_w	map $* \mapsto \widetilde{*}$ [Section 2.2.3]

$*_a$	eab semistar (or star) operation of finite type associated to $*$ [Section 2.3.2.3]
Ψ_a	map $* \mapsto *_a$ [Section 2.3.2.3]
$\lambda_{R,T}$	extension map of star operations from R to T [Section 3.1.1]
λ_Θ	extension map of star operations from R to the family Θ [Section 3.1.1]
$\rho_{R,T}$	restriction map of star operations from T to R [Section 3.1.2]
ρ_Θ	restriction map of star operations from the family Θ to R [Section 3.1.2]
Sets of closures	
$\mathrm{Clos}(\mathcal{P})$	set of closure operation on the partially ordered set $\mathcal{P}$ [Section 1.2 and Definition 2.1]
$\mathrm{Star}(S)$, $\mathrm{Star}(R)$	set of star operations on the numerical semigroup S [Definition 1.1] or of the ring R [Definition 2.11]
$\mathrm{SStar}(D)$	set of the semistar operations on the integral domain D [Definition 2.11]
$\mathrm{(S)Star}(D)$	set of the (semi)star operations on the integral domain D [Definition 2.11]
$\mathrm{SStar}_f(D)$	set of the semistar operations of finite type on D [Definition 2.16]
$\mathrm{SStar}_{st}(D)$	set of the stable semistar operations on D [Definition 2.23]
$\mathrm{SStar}_{sp}(D)$	set of the spectral semistar operations on D [Definition 2.26]
$\mathrm{SStar}_{f,sp}(D)$	set of the spectral semistar operations of finite type on D [Definition 2.26]
$\mathrm{Std}(D)$	set of the standard operations on D [Definition 2.56]
$\mathrm{Sp}(D)$	set of the semiprime operations on D [Definition 2.56]
$\mathrm{SStar}_{val}(D)$	set of valutative semistar operations on D [Section 2.3.2]
$\mathrm{SStar}_{f,val}(D)$	set of valutative semistar operations of finite type on D [Section 2.3.2]
$\mathrm{SStar}_{eab}(D)$	set of eab semistar operations on D [Section 2.3.2.2]
$\mathrm{ExtStar}(R;T)$	set of the star operations on R extendable to T [Definition 3.1]
$\mathrm{ExtStar}(R)$	set of the totally extendable star operations on R [Definition 3.10]
$\mathcal{C}(\Theta)$	set of the star operations on R compatible on Θ [Definition 3.19]
$\mathrm{Star}^\Delta(R)$	set of the star operations of finite type $*$ on R such that $\mathrm{QMax}^*(R) = \Delta$ [Definition 3.34]
$\mathrm{Star}_{\mathrm{noeth}}(R)$	set of the Noetherian star operations on R [Section 3.1]
$\mathrm{PStar}(R)$	set of the principal star operations on R [Section 3.3.2]
$\mathrm{IrrStar}(R)$	set of the irriducible star operations on R [Section 3.3.2]
$\mathrm{PrimeStar}(R)$	set of the prime star operations on R [Section 3.3.2]
$\mathrm{FStar}(R)$	set of the fractional star operations on R [Definition 3.123]
$\widehat{\mathrm{FStar}}(R)$	$= \mathrm{FStar}(R) \cup \{\infty\}$ [Section 3.5.5]
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$\mathrm{Over}(D)$	set of the overrings of the integral domain D [Section 2.3]
$\mathrm{LocOver}(D)$	set of the local overrings of D [Section 2.3]

$\mathrm{Zar}(D)$	Zariski space of D [Section 2.3.2]
$\mathrm{Zar}_{\min}(D)$	set of the minimal elements of $\mathrm{Zar}(D)$ [Section 2.3.2]
$\mathrm{Over}_{\mathrm{ic}}(D)$	set of the integrally closed overrings of D [Section 2.3.2.3]
$\mathrm{Loc}(D)$	set of the localizations of D on prime ideals [Section 2.5]
$\mathrm{Over}_{\mathrm{qr}}(D)$	set of the localizations of D [Section 2.5]
$\mathrm{Over}_{\mathrm{flat}}(D)$	set of the flat overrings of D [Section 2.5]
$\mathrm{Over}_{\mathrm{sloc}}(D)$	set of the sublocalizations of D [Section 2.5]
$\mathrm{FOver}(D)$	set of the fractional overrings of D [Definition 3.118]
Other constructions	
$\mathrm{LS}(D)$	set of the localizing sistems of D [Section 2.2.3.1]
$\mathrm{LS}_{\mathrm{f}}(D)$	set of the localizing sistems of finite type of D [Section 2.2.3.1]
$\boldsymbol{\mathcal{X}}(X)$	space of nonempty inverse-closed subsets of X [Section 2.4]
$\boldsymbol{\mathcal{U}}(\Omega)$	subbasic open set of $\boldsymbol{\mathcal{X}}(X)$ associated to Ω [Section 2.4]
$\boldsymbol{\mathcal{Z}}(X)$	space of the nonempty subsets of X closed by generization [Section 2.4.2]
$\boldsymbol{\mathcal{S}}(R)$	space of the semigroup primes of R [Section 2.5.1]
Notations for numerical semigroups	
$\langle a_1, \ldots, a_n \rangle$	numerical semigroup generated by $a_1, \ldots, a_n$
$\{0, b_1, \ldots, b_n, \rightarrow\}$	numerical semigroup containing $0, b_1, \ldots, b_n$ and all integers bigger than b_n
$g(S)$	Frobenius number of the numerical semigroup S [Section 1.1]
$\delta(S)$	degree of singularity of the numerical semigroup S [Section 1.1]
M_S	maximal ideal of the numerical semigroup S [Section 1.1]
$\mu(S)$	multiplicity of the numerical semigroup S [Section 1.1]
$T(S)$	$= (S - M_S) \setminus S$, for S a numerical semigroup [Section 1.1]
$t(S)$	type of the numerical semigroup S [Section 1.1]
M_a	biggest ideal in $\mathcal{F}_0(S)$ not containing a [Definition 1.13]
$\leq_*$	$*$-order on $\mathcal{F}(S)$ and $\mathcal{G}_0(S)$ [Definition 1.27]
$\mathcal{Q}_a$	$= \{I \in \mathcal{F}_0(S) : a = \sup(\mathbb{N} \setminus I), a \in I^v\}$ [Definition 1.41]
$\mathrm{qm}(*)$	biggest integer x such that there is an $I \in \mathcal{Q}_x$ such that I is $*$-closed [Definition 1.56]
$\mathrm{Star}_x(S)$	set of star operations such that $\mathrm{qm}(*) = x$ [Definition 1.56]
$\xi(n)$	number of numerical semigroup with exactly n star operations [Section 1.7.1]
$\Xi(n)$	number of nonsymmetric numerical semigroup with less than n star operations [Section 1.7.1]
$\xi_\mu(n)$	number of numerical semigroup with multiplicity μ and exactly n star operations [Section 1.7.1]
$\Xi_\mu(n)$	number of nonsymmetric numerical semigroup with multiplicity μ and less than n star operations [Section 1.7.1]
Notations related to semigroup rings	
$K[[S]]$	semigroup ring associated to S [Section 1.8.2]
$\mathrm{Mon}(I)$	biggest monomial ideal contained in I [Definition 1.138]

$\mathrm{vMon}(I)$	valuation of $\mathrm{Mon}(I)$ [Definition 1.138]
$\mathfrak{C}(V)$	set of the analytically irreducible rings with integral closure V [Section 1.8.3]
$\mathfrak{V}(V)$	subset of $\mathfrak{C}(V)$ of residually rational rings [Section 1.8.3]
Topological notations	
V_I	subbasic open set of the Zariski topology on $\mathrm{SStar}(D)$ [Definition 2.14]
$\mathcal{D}(I)$	open set of $\mathrm{Spec}(R)$ induced by the ideal I
$\mathcal{V}(I)$	closed set of $\mathrm{Spec}(R)$ induced by the ideal I
X^{inv}	X endowed with the inverse topology [Section A.3]
$\mathrm{Cl}^{\mathrm{inv}}(Y)$	closure of Y in the inverse topology [Section A.3]
X^{cons}	X endowed with the constructible topology [Section A.4]
$\mathrm{Cl}^{\mathrm{cons}}(Y)$	closure of Y in the constructible topology [Section A.4]
$A_{\mathscr{U}}$	ultrafilter limit point of the ultrafilter $\mathscr{U}$ [Proposition 2.111]
Miscellanea	
$\Omega(\mathcal{P})$	set of antichains of the partially ordered set $\mathcal{P}$ [Definition 1.28]
$\omega(\mathcal{P})$	cardinality of $\Omega(\mathcal{P})$
$\omega(n)$	n-th Dedekind number [Section 1.3.2]
$\omega_{\mathrm{i}}(\mathcal{Q}_a)$	$\omega((\mathcal{Q}_a, \subseteq))$ [Section 1.3.2]
$\mathcal{R}(a, b)$	rectangle $a \times b$ [Section 1.5.4]
$\mathrm{Ass}(I)$	set of associated primes of the ideal I
$\mathrm{Kr}(D)$	Kronecker function ring of D [Definition 2.83]
$\mathrm{Na}(D)$	Nagata ring of D [Section 2.5.1.2]
$\mathrm{Pic}(D)$	Picard group of D

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Introduction

A *closure operation* is a map c from a partially ordered set X to itself that verifies three properties: it is *extensive* ($x \leq c(x)$ for every $x \in X$), *idempotent* ($c(c(x)) = c(x)$, i.e., $c \circ c = c$) and *order-preserving* (if $x \leq y$, then $c(x) \leq c(y)$). This definition is very general: for example, if X is a set with some additional algebraic structure (e.g. a group, a ring, or a vector space) then it is possible to define a closure operation on the power set of X by sending every subset Y into the substructure that Y generates, i.e., into the smallest substructure containing Y. Despite this example, probably the most known example of closure operations in general mathematics is the closure between subsets of a topological space; indeed, it is possible to define topological spaces in terms of closure operations (via the *Kuratowski closure axioms*).

Dealing with closure operations in commutative algebra, there are three usual choices for the partially ordered set where the closure acts: the set of ideals of a ring, the set of fractional ideals of a domain or the set of submodules of the quotient field of a domain. In each case, to obtain significant ring-theoretic properties, it is not enough to take simply closure operations, but it is necessary to impose additional restrictions, usually relating the closure of the products $x \cdot I$ with the closure of the ideal I: in the case of the ideals of a ring R, a popular choice is imposing that the closure c verifies $x \cdot c(I) \subseteq c(xI)$ for every $x \in R$ and every ideal I (obtaining the *semiprime operations*), while in the other two cases above the usual additional axiom is that $x \cdot c(I) = c(xI)$ for every x in the quotient field and every I (obtaining, respectively, *star* and *semistar operations*).

Historically, the study of these classes of closure operations have followed two routes: the first is trying to find properties that hold for all closure operations of a given class, or at least for some distinguished subset; the second is studying specific examples of closure operations. In the latter case, typical examples are the v- and the t-operation (in part due to their link with factorization properties [107, 18]), tight closure and its several variations (in the characteristic p setting [65, 32]) and integral closure (that links quite a great number of topics [72]). Both ways to consider this subject tend to consider closure operations as somewhat isolated one from each other, at most connecting one closure c to a few other that can be constructed – more or less explicitly – from c.

This thesis tries instead to study closure operations from a *global* perspective; that is, to study the set of closure operations (or rather, some distinguished set of closure operation) as a whole. This has been done, for example, in the context of counting the set of star or semistar operations [67, 68, 70], trying to represent the set of semiprime operations of certain one-dimensional local domains as union of monoids [112], or establishing bijections between two different sets of closure operations [33]. To accomplish this objective, we focus on some structures that can introduced on these sets.

The first, obvious, structure is what we can call the *set structure*: that is, we want to study the cardinality of some classes of closure operations. This kind of structure is often not very interesting: indeed, "almost always" (in a non-technical sense) the set of semistar operations is infinite, and the same happens (although less frequently) for the set of star operations. We shall study this structure mainly in Chapter 1, for the set of star operations on a numerical semigroup; most of the results in Chapter 3 can also be read through this lens.

The second structure is the *order structure*: that is, the set of closure operation is naturally a partially ordered set, under the relation $\leq$ such that $c \leq d$ if and only if $c(x) \leq d(x)$ for every x in the original ordered set. This structure has received some attention; indeed, part of the interest of the v- and the t-operation is due to the fact that they are maxima in some precisely-defined set of star operations, and the b-operation (a semistar analogue of integral closure) can also be characterized – albeit not in a very enlightening way – as a minimum. We shall not study this structure extensively; rather, it will be a background over which we will build many of our results. Yet, two results are mostly concerned with it, namely Corollary 1.96 (that characterizes numerical semigroups with a linearly ordered set of star operations) and Theorem 3.130 (dealing with the relation between the spectrum and the set of semistar operations of a semilocal finite-dimensional Prüfer domain).

The third structure is a *topological structure* or, more precisely, a way to build possibily infinitely many topological structures. While it is technically convenient to work with a more general definition, the topology we will mostly consider is the following: given a ring R and a set Δ of closures on the ideals (or fractional ideals, or submodules of the quotient field), the *Zariski topology* on Δ is the topology having, as a subbasis of open sets, the sets of the form

$$V_{I,x} := \{c \in \Delta \mid x \in c(I)\}$$

as x ranges in the ring (or in the quotient field) and I ranges in the whole set of ideals (or modules). Chapter 2 mostly deals with the analysis of this topology and of its consequences; on the other hand, the results of Chapter 3 are stated in topological terms (since the topological structure behaves very naturally) but the ideas and the results are often more algebraic in nature.

The thesis is divided into three chapters: the first deals with star operations in the setting of numerical semigroups (and some related classes of rings), the second with the topological structure of the set of semistar operations and of the set of overrings, and the third with the possibility of expressing the set of star operations of a ring as a product, and with the consequences of this possibility.

Chapter 1 is mostly about star operations on numerical semigroups.

A *numerical semigroup* is a set $S \subseteq \mathbb{N}$ such that:

- $0 \in S$;
- if $a, b \in S$, then $a + b \in S$;
- $\mathbb{N} \setminus S$ is finite.

The numerical semigroup generated by $a_1, a_2, \ldots, a_n \in \mathbb{N}$ is linked to the possibility of expressing a number as a linear combination (with nonnegative integer coefficients) of

$a_1, \ldots, a_n$; therefore, numerical semigroups are related to the *Frobenius problem* (finding thelargest integer that is not a nonnegative linear combination of $a_1, \ldots, a_n$) and to the *Sylvester problem* (finding how many integers are not exprimible as a nonnegative linear combination of $a_1, \ldots, a_n$) [104].

Later, numerical semigroups were connected to algebraic geometry, and in particular to the study of curves. From an algebraic point of view, this leads to the study of *semigroup rings* (i.e., rings of the form $K[[X^{a_1}, \ldots, X^{a_n}]]$, where K is a field) or, more generally, *residually rational* rings. In particular, many algebraic properties of these rings are in a very natural relation to properties of a corresponding numerical semigroup [13]. Moreover, semigroup rings are a rich source of examples in the theory of star or semiprime operations over one-dimensional Noetherian rings [68, 70, 112].

In view of this, the concept of star operation can be generalized to numerical semigroups (or even to general semigroups [78]), hoping to relate such closures with star operations on rings. Precisely, a *star operation* on the semigroup S is a closure operation $*$ on the set of fractional ideals of S such that $S^* = S$ and $d + I^* = (d + I)^*$ for every $d \in \mathbb{Z}$ and every ideal I. (From now on, we adopt the usual convention of writing the closure of an ideal I under the closure c as I^c.)

A star operation $*$ on the numerical semigroup S can be characterized by the set $\mathcal{F}_0^*(S) := \{I \mid I \text{ is a fractional ideal, } S \subseteq I \subseteq \mathbb{N} \text{ and } I = I^*\}$ of $*$-*closed* ideals, and the sets that can be written in this way can be characterized explicitly. This means both that the set $\mathrm{Star}(S)$ of star operations on S is finite, and that it can be determined algorithmically in finite time, although this calculation is quite slow and may take a lot of time. From our point of view, it means that even the "coarsest" of the three structures we consider, i.e., the set structure of $\mathrm{Star}(S)$, is of interest.

The main question we study in this chapter is how the size of $\mathrm{Star}(S)$ correlates with the semigroup, in particular with the various invariants that describe it (like the Frobenius number, the degree of singularity, or the multiplicity). In particular, we are interested in finding estimates for $|\mathrm{Star}(S)|$, or, if possible, a quicker way (or even a formula) for this cardinality, without having to use the general algorithm.

The main tool we use is the possibility to associate to any ideal I of S a star operation $*_I$ (Definition 1.4) by defining

$$L^{*_I} := (S - (S - L)) \cap (I - (I - L))$$

for any ideal L. Characterizing when $*_I = *_J$ (Theorem 1.9) leads to the estimate $|\mathrm{Star}(S)| \geq |\mathbb{N} \setminus S| + 1$ (Theorem 1.25) and thus to the fact that, if $n > 1$, there are only a finite number of semigroups with exactly n star operations (Theorem 1.26).

The next step is studying star operations generated by sets of ideals: Sections 1.3 and 1.4 are devoted to understanding when two sets generate the same closure. In particular, we consider on the set of the ideals of S the antichains of an order defined from the order on the set of star operations. We are able to find several conditions under which two sets generate different operations, although not an easily applicable general criterion. However, they are sufficient to obtain several inequalities, culminating in Theorem 1.101: the number $\Xi(n)$ of numerical semigroups S such that $2 \leq |\mathrm{Star}(S)| \leq n$ is $O(n^{(A+\epsilon)\log\log(n)})$ for any $\epsilon > 0$, where $A := \frac{2}{\log(2)}$.

Section 1.5 completely settles the case of semigroups of multiplicity 3: if $S = \langle 3, 3a + 1, 3b + 2\rangle$, then $|\mathrm{Star}(S)| = \binom{a+b+1}{2a-b}$ (Theorem 1.88), and we can actually describe the partially ordered set of the non-divisorial ideals of S (Theorem 1.86). In particular, we obtain that there are $O(\log(n))$ semigroups of multiplicity 3 with exactly n star operations, and that the number of semigroups of multiplicity 3 such that $2 \leq |\mathrm{Star}(S)| \leq n$ is $\frac{2}{3}n + O(\sqrt{n}\log(n))$ (Proposition 1.106).

Section 1.8 tries to apply the methods and the results of the numerical semigroup case back to the ring case. Section 1.8.1 studies star operations generated by ideals, while 1.8.2 deals with semigroup rings and Section 1.8.3 replays the previous sections in the setting of residually rational rings.

Chapter 2 focuses on the topological study of the set $\mathrm{SStar}(D)$ of semistar operations on a domain D.

A *semistar operation* on D is a closure operation $*$ on the set of the D-submodules of the quotient field K of D such that $x \cdot I^* = (xI)^*$ for every $x \in K$ and every submodule I, and the *Zariski topology* we put on $\mathrm{SStar}(D)$ is generated by the subbasic open sets $V_I := \{* \in \mathrm{SStar}(D) \mid 1 \in I^*\}$, as I ranges among the D-submodules of K.

This topology is linked with the space $\mathrm{Over}(D)$ of *overrings* of D, i.e., of rings contained between D and K. Indeed, $\mathrm{Over}(D)$ can be endowed with a topology whose subbasic open sets are those in the form $B_F := \{T \in \mathrm{Over}(D) \mid F \subseteq T\}$, as F ranges among the finite subsets of K. The interest of this topology lies in the fact that it generalizes both the Zariski topology on the spectrum of D (since the map $P \mapsto D_P$ becomes a topological inclusion – see [28, Lemma 2.4] or Proposition 2.67) and the Zariski topology on the Zariski-Riemann space $\mathrm{Zar}(D)$ of D (i.e., the set of valuation overrings of D). The latter space, in particular, was introduced by Zariski during its proof of resolution of singularities for low-dimensional algebraic varieties in characteristic 0 (see [119, Chapter VI], [117] and [118]), and, over the years, has been studied in greater detail from several points of view [71, 97, 37]. There is a natural topological inclusion of $\mathrm{Over}(D)$ into $\mathrm{SStar}(D)$, where the overring T is associated to the semistar operation $I \mapsto IT$ (Proposition 2.63); this means that $\mathrm{SStar}(D)$ can be seen as an extension of $\mathrm{Over}(D)$, and thus a further extension of $\mathrm{Spec}(D)$ and $\mathrm{Zar}(D)$.

One of the questions we study about $\mathrm{SStar}(D)$ (and some of its subspaces) is whether it is a *spectral space*, i.e., whether it is homeomorphic to the spectrum of a commutative ring (endowed with the Zariski topology). Spectral spaces can be characterized in a purely topological way [64]; in particular, we use a characterization based on ultrafilters [36, Corollary 3.3]. The Zariski space $\mathrm{Zar}(D)$ has long been known to be a spectral space [28], and the Kronecker function ring provides a construction that represent explicitly $\mathrm{Zar}(D)$ as the spectrum of a ring [29]. More recently, $\mathrm{Over}(D)$ has also been proved to be a spectral space [36, Proposition 3.5]. In this context, it is also useful to introduce two topologies that can be defined on a spectral space X, namely the *inverse* and the *constructible* topology: the former has the property of reversing the order induced by the original topology, while the latter is Hausdorff. Both topologies make X into a spectral space. Several definitions and known results about spectral spaces have been collected in the appendix.

Sections 2.2 and 2.3.2 analyze topologically a few distinguished subspaces of $\mathrm{SStar}(D)$: those of finite-type, stable, spectral, valutative and eab operations. We show that the spaces of finite-type, finite-type spectral and finite-type valutative semistar operations are spectral spaces, and that the spaces of spectral and valutative semistar operations behave – from the topological point of view – very similarly, having a point of contact in the Kronecker function ring.

Sections 2.3.1 and 2.3.3, moving beyond some results in previous sections (like Propositions 2.32 and 2.80) are devoted to the study of the algebraic consequences of compactness of families of semistar operations. Along the way, we prove that the intersection of a compact family of overrings commutes with the multiplication by a flat module (Theorem 2.76). Section 2.4 presents a construction giving, in a purely topological way, the space of spectral semistar operations on D starting from $\mathrm{Spec}(D)$; this construction is also extendable to arbitrary spectral spaces and thus to arbitrary rings (not necessarily domains).

Section 2.3.4 studies which subspaces of $\mathrm{Over}(D)$ are *proconstructible*, i.e., closed in the constructible topology; we show that this is true for several natural spaces, while in some other cases we need to suppose beforehand some other property, like compactness. Section 2.5 is centered on the study of the set of localizations at the prime ideals of a domain D, as a subspace of $\mathrm{Over}(D)$, and to three related spaces: the set of all localizations, non necessarily at prime ideals (through the use of *prime semigroups*), the set of flat overrings and the set of *sublocalizations* of D (i.e., intersection of localizations of D). In all cases, the main question is under what hypothesis these spaces are spectral or proconstructible.

Most of the content of this chapter arises from joint investigations with Marco Fontana and Carmelo Finocchiaro.

Chapter 3 is based on the idea of *extending* star operations, and on the possibility of describing the space $\mathrm{Star}(R)$ of the star operations on a domain R as a product of spaces of type $\mathrm{Star}(T)$, as T ranges among a family of overrings of R.

The concept of localization of a star operation (of which extension is a slight generalization, passing from quotient rings to flat overrings) has been introduced in [67] as a way to expand the characterization of domains with only one star operation given in [58] to domains with exactly two star operations. Subsequently, it has been employed, explicitly or implicitly, to study the cardinality of $\mathrm{Star}(R)$ in several cases, like Noetherian domains [68, Theorem 2.3] or Prüfer domains [69]. Section 3.1 analyzes a few cases where we can prove that the extension of a star operation exists, and some ways to go back – when possible – from extensions to the original star operations.

Section 3.2 studies *Jaffard families*, i.e., families of overrings of a domain R that are locally finite, independent and such that every ideal is an intersection of ideals of rings of the family. Jaffard families were introduced in [41, Section 6.3] as a generalization of h-local domains, that is, locally finite domains such that no nonzero prime ideal is contained in more than one maximal ideal. These hypotheses imply that localization behaves very well in h-local domains, giving rise to a good number of useful module-theoretic properties [94], that are partially inherited (with some modifications) by Jaffard families.

In Section 3.3, these properties are used to show that, if Θ is a Jaffard family of R, then $\mathrm{Star}(R)$ is homeomorphic to the product of $\mathrm{Star}(T)$, as T ranges in Θ (Theorem 3.67). It is then shown that this correspondence respects many properties of star operations, and moreover that it offers a way to calculate the class group of R (with respect to a star operation $*$) in terms of the class group of the $T \in \Theta$ (with respect to the extensions $*_T$).

In Section 3.5, these methods are complemented with a technique allowing to relate the star operations on a Prüfer domain R with the star operations on a quotient R/P, provided that P is a nonzero prime ideal contained in every maximal ideal of R. Applied together with the previous results about Jaffard families, we get a method to study systematically some properties of star operations on a Prüfer domain R: find a Jaffard family Θ of R, and then for every $T \in \Theta$ pass to the quotient ring T/P, of which we can find a Jaffard family, and then repeat the process. Under some hypothesis on R (for example, if R is semilocal or R is finite-dimensional and with Noetherian spectrum), this method can be applied in full force, ending after a finite number of steps at valuation domains. Therefore, under these hypotheses, we can proceed in an inductive way: in particular, we show that, if R is a finite-dimensional Prüfer domain of finite character, then the class group of R under $*$ (or rather, its quotient modulo the Picard group) is the direct sum of a family of quotients of $\mathbb{R}$, considered as an additive group (Corollary 3.111 and [11, Corollaries 3.6 and 3.7], proved here as Proposition 3.114).

In Section 3.5.5, we use the previous theory to show that, if two semilocal and finite-dimensional Prüfer domains R and T have homeomorphic prime spectra, and if PR_P is principal if and only if $\phi(P)T_{\phi(P)}$ is principal (where ϕ is the homeomorphism between $\mathrm{Spec}(R)$ and $\mathrm{Spec}(T)$), then the sets $\mathrm{Star}(R)$ and $\mathrm{Star}(T)$ are in bijective correspondence, and this correspondence extends to a bijection between $\mathrm{SStar}(R)$ and $\mathrm{SStar}(T)$

1. Star operations on numerical semigroups

1.1. Notation and preliminaries

Since our study of numerical semigroups is intimately connected with ring theory, we will follow the notation of [13]. For a general background about numerical semigroups, and for further informations, the interested reader may consult [106].

A *numerical semigroup* S is a subset $S \subseteq \mathbb{N}$ such that the following properties hold:

(a) $0 \in S$;
(b) the set $\mathbb{N} \setminus S$ is finite;
(c) if $a, b \in S$, then $a + b \in S$.

A submonoid M of $\mathbb{N}$ is a numerical semigroup if and only if $\gcd(M) = 1$ (where $\gcd(M)$ denotes the greatest common divisor of the elements of M); if $\gcd(M) = m > 1$, then $\frac{M}{m} := \{\frac{x}{m} \mid x \in M\}$ is a numerical semigroup.

If $a_1, \ldots, a_n$ are coprime natural numbers, we denote by $\langle a_1, \ldots, a_n \rangle$ the numerical semigroup *generated* by $a_1, \ldots, a_n$, that is, the minimal numerical semigroup containing all of them; more explicitly, $\langle a_1, \ldots, a_n \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid \lambda_i \in \mathbb{N}\}$. The notation $S = \{0, b_1, \ldots, b_n, \rightarrow\}$ indicates that S is composed of 0, b_1, ..., b_n and all the integers bigger than b_n. Every numerical semigroup is generated by a finite number of coprime integers; the smallest number of integers needed to generate the semigroup is called the *embedding dimension* of S.

Since $\mathbb{N} \setminus S$ is finite, it has a maximal element: this integer is called the *Frobenius number* of S, and is denoted by $g(S)$ or $F(S)$ (we will use the former notation). The cardinality of $\mathbb{N} \setminus S$ is called the *degree of singularity* (or the *genus*) of S; we will denote it by $\delta(S)$ (it is also denoted by $g(S)$). When $S = \mathbb{N}$, we define $g(S) := -1$ and $\delta(S) := 0$.

An *ideal* of S is a nonempty subset $I \subseteq S$ such that $i + s \in I$ for every $i \in I$ and every $s \in S$; the set of proper ideals of S has a maximal element, $M_S := S \setminus \{0\}$, which is called the *maximal ideal* of S. The minimal element of M_S is called the *multiplicity* of the semigroup, and it is denoted by $\mu(S)$ or $m(S)$ (we will use the former notation). The union and the intersection of an arbitrary family of ideals (if nonempty) is again an ideal.

A *fractional ideal* of S is a subset $I \subseteq \mathbb{Z}$ such that $d + I$ is an ideal of S; a fractional ideal contained in S is an ideal. In particular, $d + I \subseteq S$, and thus every fractional ideal is bounded below; hence, every fractional ideal has a minimal element. We denote by $\mathcal{F}(S)$ the set of fractional ideals of S, and by $\mathcal{F}_0(S)$ the set of fractional ideals whose minimal element is 0; equivalently, $\mathcal{F}_0(S)$ is the set of fractional ideals of S comprised between S and $\mathbb{N}$. If $I \in \mathcal{F}_0(S)$, then $S \subseteq I \subseteq \mathbb{N}$; therefore, I is characterized by a subset of $\mathbb{N} \setminus S$. In particular, $\mathcal{F}_0(S)$ is finite. Note that, for every fractional ideal I, we have $I - \min(I) \in \mathcal{F}_0(S)$. The intersection of a family of fractional ideals is again a fractional ideal, provided it is nonempty; on the other hand, the union is a fractional ideal if and only if there is an integer that is smaller than every element of every ideal of the family. In particular, the union of a family of fractional ideals contained in $\mathbb{N}$ (and so of a family of ideals in $\mathcal{F}_0(S)$) is a fractional ideal. We will often refer to fractional ideals simply as “ideals”, using “integral ideal” to denote ideals contained into S.

Given two (fractional) ideal I, J of S, we define $(I - J) := \{x \in \mathbb{Z} : x + J \subseteq I\}$. This set is an ideal of S. The set $(S - M_S) \setminus S$ is denote by $T(S)$; its cardinality is denoted by $t(S)$ and called the *type* of S. For every numerical semigroup S, $g(S) \in T(S)$, and hence $t(S)$ is positive.

1.2. Principal star operations

Definition 1.1. *Let S be a numerical semigroup. A* star operation *on S is a map $* : \mathcal{F}(S) \longrightarrow \mathcal{F}(S)$, $I \mapsto I^*$, such that, for any $I, J \in \mathcal{F}(S)$, $a \in \mathbb{Z}$, the following properties hold:*

(a) $I \subseteq I^*$;
(b) *if* $I \subseteq J$, *then* $I^* \subseteq J^*$;
(c) $(I^*)^* = I^*$;
(d) $a + I^* = (a + I)^*$;
(e) $S^* = S$.

We denote the set of star operations on S by $\mathrm{Star}(S)$.

An ideal I of S is said to be $*$-*closed* if $I = I^*$; we denote the set of $*$-closed ideals of S by $\mathcal{F}^*(S)$ (or simply $\mathcal{F}^*$ if the semigroup S is understood from the context).

The set $\mathrm{Star}(S)$ inherits from the set $\mathrm{Clos}(\mathcal{F}(S))$ (see Section 2.1) an order relation, such that $*_1 \leq *_2$ if $I^{*_1} \subseteq I^{*_2}$ for every $I \in \mathcal{F}(S)$ or, equivalently, if $\mathcal{F}^{*_1}(S) \supseteq \mathcal{F}^{*_2}(S)$. This order makes $\mathrm{Star}(S)$ a complete lattice; the infimum of a set $\{*_\lambda\}_{\lambda \in \Lambda}$ is the star operation $*$ defined by $I^* := \bigcap_{\lambda \in \Lambda} I^{*_\lambda}$ for each $I \in \mathcal{F}(S)$, while the supremum is the closure operation $\sharp$ such that $\mathcal{F}^\sharp = \bigcap_{\lambda \in \Lambda} \mathcal{F}^{*_\lambda}$. We denote the infimum of $*_1, \ldots, *_n$ as $*_1 \wedge \cdots \wedge *_n$. In particular, $\mathrm{Star}(S)$ has a minimum (the identity star operation $d : I \mapsto I$) and a maximum (the v-operation $v : I \mapsto (S - (S - I))$). Ideals that are closed by the v-operation are said to be *divisorial*; S and $\mathbb{N}$ (as a S-fractional ideal) are always divisorial, and thus are $*$-closed for any star operation $*$. The set of ideals of $\mathcal{F}_0(S)$ that are *not* divisorial is denoted by $\mathcal{G}_0(S)$.

Our approach to star operations on semigroups will mainly be based on the study of the possible sets $\mathcal{F}^*(S)$, or, more precisely, on the sets $\mathcal{F}_0^*(S) := \mathcal{F}^*(S) \cap \mathcal{F}_0(S)$. Note that every $\mathcal{F}_0^*(S)$ is finite, since so is $\mathcal{F}_0(S)$. The next proposition characterizes these sets.

Proposition 1.2. *Let S be a numerical semigroup; for any $* \in \mathrm{Star}(S)$, let $\mathcal{F}_0^*(S) := \mathcal{F}^*(S) \cap \mathcal{F}_0(S)$.*

(a) For any $* \in \mathrm{Star}(S)$, the set $\mathcal{F}^*(S)$ is closed by arbitrary intersections.
(b) $\mathcal{F}^*(S) = \mathbb{Z} + \mathcal{F}_0^*(S) = \{d + I : d \in \mathbb{Z}, I \in \mathcal{F}_0^*(S)\}$; therefore, $*$ is uniquely determined by $\mathcal{F}_0^*(S)$.
(c) $\mathrm{Star}(S)$ is finite.
(d) Let $\Delta \subseteq \mathcal{F}_0(S)$. Then, $\Delta = \mathcal{F}_0^*(S)$ for some star operation $*$ on S if and only if $S \in \Delta$, Δ is closed by intersection and $(-\alpha + I) \cap \mathbb{N} \in \Delta$ for every $I \in \Delta$, $\alpha \in I$.

Proof. (a) is a general property of closure operations (see e.g. [90], [16, Chapter IV, Theorem I] or Lemma 2.2(a)), while (b) follows from the fact that I is $*$-closed if and only if $d + I$ is $*$-closed for some $d \in \mathbb{Z}$. The number of sets in the form $\mathcal{F}_0(S) \cap \mathcal{F}^*(S)$ is finite (since $\mathcal{F}_0(S)$ is finite), so (c) follows too.

For (d), the necessity follows from the previous points. For the sufficiency, consider $\mathbb{Z} + \Delta := \{d + L : d \in \mathbb{Z}, L \in \Delta\}$; the hypotheses imply that $(\mathbb{Z} + \Delta) \cap \mathcal{F}_0(S) = \Delta$. Let I be an ideal such that $I = \bigcap_{J \in \mathcal{J}} J$ for a family $\mathcal{J} \subseteq \mathbb{Z} + \Delta$. Then,

$$I - \min(I) = \bigcap_{J \in \mathcal{J}} (J - \min(I)) = \bigcap_{J \in \mathcal{J}} ((J - \min(I)) \cap \mathbb{N})$$

and, in particular, $0 \in J - \min(I)$ for every $J \in \mathcal{J}$. Hence, $((J - \min(I)) \cap \mathbb{N}) \in \mathcal{F}_0(S)$ for every $J \in \mathcal{J}$. Moreover, every $J - \min(I)$ is in $\mathbb{Z} + \Delta$, and thus $(J - \min(I)) \cap \mathbb{N} \in (\mathbb{Z} + \Delta) \cap \mathcal{F}_0(S) = \Delta$ for every $J \in \mathcal{J}$. Since Δ is closed by intersections, also $I - \min(I) \in \Delta$, and thus $I \in \mathbb{Z} + \Delta$. It follows that $\mathbb{Z} + \Delta$ is closed by intersections. Defining $I^* := \bigcap\{J : I \subseteq J, J \in \mathbb{Z} + \Delta\}$, we have that $*$ is a star operation on S, and that $\mathbb{Z} + \Delta = \mathcal{F}^*(S)$. $\square$

Note that, if $I \in \Delta$ and $n > g := g(S)$, then $(-n + I) \cap \mathbb{N} = \mathbb{N}$. Hence, part (d) of the above proposition shows that, given a semigroup S, we can calculate explicitly, in finite time, the star operations on S: there are only a finite number of possible Δ, and testing each of them requires only finitely many calculations. However, this approach requires to find explicitly the set $\mathcal{G}_0(S)$ of non-divisorial ideals in $\mathcal{F}_0(S)$, and then to test all the $2^{|\mathcal{G}_0(S)|}$ possible subsets: since the cardinality $|\mathcal{G}_0(S)|$ can be much larger than the degree of singularity $\delta(S)$, and that testing some of the subsets may not be short, the direct application by hand of this criterion is unrealistic, except for semigroups which are "very close" to $\mathbb{N}$; moreover, it also quickly becomes too long even for a computer.

To show explicitly these problems, we present an example of a calculation done with this method (coupled with another observation).

Example 1.3. Consider the semigroup $S := \langle 3, 5, 7 \rangle = \{0, 3, 5, 6, 7, \ldots\}$. Then, $\mathcal{F}_0(S) = \{S, \mathbb{N}, I_1, I_2, I_4, J\}$, where $I_1 := S \cup \{1, 4\} = \mathbb{N} \setminus \{2\}$, $I_2 := S \cup \{2\}$, $I_4 := S \cup \{4\}$ and $J := S \cup \{2, 4\} = \mathbb{N} \setminus \{1\}$. Suppose $\Delta \subseteq \mathcal{F}_0(S)$. If $\Delta = \mathcal{F}_0^*(S)$ for some star operation $*$, then Δ contains all the divisorial ideals of S; therefore, $S, \mathbb{N} \in \Delta$, and also $J \in \Delta$, since J is divisorial. Moreover, there are no other divisorial ideals in $\mathcal{F}_0(S)$, since $I_1^v = \mathbb{N}$ and $I_2^v = J = I_4^v$. Therefore, $\{S, \mathbb{N}, J\} = \mathcal{F}_0(S) \cap \mathcal{F}^v(S)$. Let us consider the other subsets Δ contained between $\{S, \mathbb{N}, J\}$ and $\mathcal{F}_0(S)$.

1. $\Delta = \{S, \mathbb{N}, J, I_1\}$ is not acceptable, since $I_1 \cap J = I_4 \notin \Delta$.
2. $\Delta = \{S, \mathbb{N}, J, I_1, I_2\}$ is not acceptable, as above.
3. $\Delta = \{S, \mathbb{N}, J, I_1, I_2, I_4\} = \mathcal{F}_0(S)$ is acceptable, and corresponds to the identity star operation.
4. $\Delta = \{S, \mathbb{N}, J, I_1, I_4\}$ is acceptable, since
   - $I_1 \cap I_4 = I_1 \cap J = I_4 \cap J = I_4$;
   - $(I_1 - 1) \cap \mathbb{N} = J$;
   - $(I_1 - 4) \cap \mathbb{N} = (I_1 - 3) \cap \mathbb{N} = \mathbb{N}$;
   - $(I_4 - 3) \cap \mathbb{N} = (I_4 - 4) \cap \mathbb{N} = \mathbb{N}$.
5. $\Delta = \{S, \mathbb{N}, J, I_2\}$ is not acceptable, since $(I_2 - 2) \cap \mathbb{N} = I_1 \notin \Delta$.
6. $\Delta = \{S, \mathbb{N}, J, I_2, I_4\}$ is not acceptable, as above.
7. $\Delta = \{S, \mathbb{N}, J, I_4\}$ is acceptable, since $I_4 \cap J = I_4$ and $(I_4 - 3) \cap \mathbb{N} = (I_4 - 4) \cap \mathbb{N} = \mathbb{N}$.

Therefore, $|\mathrm{Star}(S)| = 4$. Note also that, since each pair of acceptable Δ is comparable, $\mathrm{Star}(S)$ is linearly ordered (we shall return on this property in Propositions 1.94-1.96).

Our main tool for describing star operations will be generating them through ideals. Our first definition involves a single ideal.

Definition 1.4. *Let S be a numerical semigroup. For every $I \in \mathcal{F}(S)$, the star operation* generated by I, *denoted by $*_I$, is the supremum of all the star operations $*$ on S such that I is $*$-closed (that is, $I = I^*$). If $* = *_I$ for some ideal I, we say that $*$ is a* principal *star operation.*

Since $\mathcal{F}^{*_I} = \bigcap\{\mathcal{F}^* : I = I^*\} = \bigcap\{\mathcal{F}^* : I \in \mathcal{F}^*\}$, the ideal I is $*_I$-closed, and thus the supremum in the above definition is, in fact, a maximum.

Principal star operation can be used to generate all star operations, as the following proposition shows.

Proposition 1.5. *Let S be a numerical semigroup, and let $* \in \mathrm{Star}(S)$. Then,*

$$* = \inf\{*_I \mid I \in \mathcal{F}_0^*(S)\}. \tag{1.1}$$

Proof. Let $\mathcal{F}_0^*(S) := \{I_1, \ldots, I_n\}$, and define $\sharp := *_{I_1} \wedge \cdots \wedge *_{I_n}$. By Definition 1.4, $* \leq *_I$ for each $I \in \mathcal{F}_0(S)$, and thus $* \leq \inf *_I = \sharp$; thus, $\mathcal{F}_0^\sharp(S) \subseteq \mathcal{F}_0^*(S)$. Conversely, if $J \in \mathcal{F}_0^*(S)$, then $\sharp \leq *_J$; since J is $*_J$-closed, it is also $\sharp$-closed, i.e., $J \in \mathcal{F}_0^\sharp(S)$. It follows that $\mathcal{F}_0^\sharp(S) = \mathcal{F}_0^*(S)$ and, by Proposition 1.2(b), $* = \sharp$. □

The representation (1.1), despite being canonical, is essentially equivalent to describing star operations through the set of closed ideals, and thus it is not easier to work with. However, the advantage of this way of thinking is that we do not need the whole $\mathcal{F}_0^*(S)$ to generate $*$: indeed, if $* = *_I$ is a principal star operation, then we have another "natural" representation ($\{I\}$ itself), which almost always does not coincide with the whole $\mathcal{F}_0^*(S)$. In general, if Δ is a subset of $\mathcal{G}_0(S)$, we say that $*_\Delta := \inf\{*_I \mid I \in \Delta\}$ is the star operation *generated* by Δ; more explicitly, $J^{*_\Delta} := \bigcap_{I \in \Delta} J^{*_I}$.

Before studying general star operation, we would wish is to understand principal star operations; in particular, we should like to see in what cases two ideals generate the same closure.

Lemma 1.6. *Let S be a numerical semigroup and $I \in \mathcal{F}(S)$.*

(a) $*_I = v$ if and only if I is divisorial.
(b) $*_I \leq *_J$ if and only if J is $*_I$-closed.
(c) $*_I = *_J$ if and only if I is $*_J$-closed and J is $*_I$-closed; in particular, $*_I = *_{\alpha+I}$ for every $\alpha \in \mathbb{Z}$.

Proof. (a) If I is divisorial, then by definition v closes I, and being v the maximal star operation we have $*_I = v$. Conversely, if I is non-divisorial, then $I \in \mathcal{F}^{*_I} \setminus \mathcal{F}^v$, and thus $*_I \neq v$.
(b) If $*_I \leq *_J$, then $J^{*_I} \subseteq J^{*_J} = J$, and thus J is $*_I$-closed. Conversely, if J is $*_I$-closed, then $*_J$ is the supremum of a set containing $*_I$, and thus $*_J \geq *_I$.
(c) Immediate from (b). □

The definition of principal star operation isn't explicit enough to allow a deep study. The next result takes care of this aspect.

Proposition 1.7. *Let S be a numerical semigroup and I be an ideal of S. For every $J \in \mathcal{F}(S)$,*
$$J^{*_I} = J^v \cap \bigcap_{\alpha\in(I-J)} (-\alpha + I) = J^v \cap (I - (I - J)).$$

See also Remark 1.18.

Proof. For the first equality, let $*$ be the map $J \mapsto J^v \cap \bigcap_{\alpha\in(I-J)}(-\alpha+I)$. It is clear that $*$ is a star operation (since both v and the intersection are extensive, order-preserving, idempotent, both respect translation and $S^v = S$), and that $I^* = I$ (since $0 \in (I - I)$), so that $* \leq *_I$. Moreover, if $*'$ is another star operation that closes I, then J^v and every $-\alpha + I$ are $*'$-closed, and thus $I^{*'} \subseteq I^*$, that is, $*' \leq *$. By definition of $*_I$, we have $* = *_I$.

To show the second equality, it is sufficient to prove that
$$\bigcap_{\alpha\in(I-J)} (-\alpha + I) = (I - (I - J)).$$
We merely translate the proof of [60, Lemma 3.1] into the language of semigroups. If $x \in (I - (I - J))$ and $J \subseteq -\alpha + I$, then $x + (I - J) \subseteq I$, so that $x + \alpha \in I$ and $x \in -\alpha + I$. Conversely, if $x \in \bigcap(-\alpha + I)$, then $x + \alpha \in I$ for every $\alpha \in (I - J)$, that is, $x + (I - J) \subseteq I$, which means $x \in (I - (I - J))$. □

Lemma 1.8. *Let S be a numerical semigroup and $I, J \in \mathcal{F}(S)$. If $*_I = *_J$ then*
$$I = I^v \cap \bigcap_{\gamma\in(I-J)+(J-I)} (-\gamma + I).$$

Proof. By Lemma 1.6(c) and Proposition 1.7,
$$J = J^v \cap \bigcap_{\alpha\in(I-J)} (-\alpha + I), \quad \text{and} \quad I = I^v \cap \bigcap_{\beta\in(J-I)} (-\beta + J).$$

Thus,

$$I = I^v \cap \bigcap_{\beta\in(J-I)} (-\beta + J) = I^v \cap \bigcap_{\beta\in(J-I)} -\beta + \left(J^v \cap \bigcap_{\alpha\in(I-J)} (-\alpha + I)\right) =$$

$$= I^v \cap \bigcap_{\beta\in(J-I)} (-\beta + J^v) \cap \bigcap_{\beta\in(J-I)} \left(-\beta + \bigcap_{\alpha\in(I-J)} (-\alpha + I)\right) =$$

$$= I^v \cap \bigcap_{\beta\in(J-I)} (-\beta + J^v) \cap \bigcap_{\gamma\in(I-J)+(J-I)} (-\gamma + I).$$

However, for every $\beta \in (J - I)$, we have $I \subseteq -\beta + J \subseteq -\beta + J^v$, and so $I^v \subseteq -\beta + J^v = (-\beta + J)^v$. Therefore, the second term can be dropped. □

Theorem 1.9. *Let S be a numerical semigroup and $I, J \in \mathcal{G}_0(S)$. Then $*_I = *_J$ if and only if $I = J$.*

Proof. The sufficiency is trivial.

Assume $*_I = *_J$ and suppose $I \neq J$. Let $\psi := \sup(I^v \setminus I)$. Since $(I - J) + (J - I) \subseteq (I - I)$, for every $\gamma \in (I - J) + (J - I)$ we have $\gamma + I \subseteq I$, and thus $\gamma + I^v \subseteq I^v$; in particular, $\gamma + \psi \in I^v$. However, since $I, J \in \mathcal{F}_0$, both $(I - J)$ and $(J - I)$ are contained in $\mathbb{N}$. Moreover, $0 \in (I - J)$ if and only if $J \subseteq I$ and thus, if $I \neq J$, each member of $(I - J) + (J - I)$ is positive. Therefore, $\gamma + \psi > \psi$, and thus $\gamma + \psi$ must be in I. This shows that

$$\psi \in I^v \cap \bigcap_{\gamma\in(I-J)+(J-I)} (-\gamma + I).$$

However, we have chosen $\psi \notin I$, and therefore $*_I \neq *_J$ by Lemma 1.8. □

Corollary 1.10. *Let S be a numerical semigroup and $I, J \in \mathcal{F}(S)$ be non-divisorial ideals. Then $*_I = *_J$ if and only if $I = \alpha + J$ for some $\alpha \in \mathbb{Z}$.*

As a corollary, we also get a first (albeit not very explicit) estimate for the number of star operations.

Corollary 1.11. *For every numerical semigroup S, $|\mathcal{G}_0(S)| + 1 \leq |\mathrm{Star}(S)| \leq 2^{|\mathcal{G}_0(S)|}$.*

Proof. Each non-divisorial ideal generates a different star operation on S. Moreover, we have the v-operation, which is different from all these star operations. Thus, $|\mathcal{G}_0(S)|+1 \leq |\mathrm{Star}(S)|$.

The other estimate is just a numerical translation of Proposition 1.2(d), since each star operation is determined by a subset of $\mathcal{G}_0(S)$. □

A first test of non-divisoriality, useful in some special cases, is the following result.

Proposition 1.12. *Let S be a numerical semigroup and $S \subsetneq I \in \mathcal{F}_0(S)$. Then $(S - M) \subseteq I^v$.*

Proof. Since $S \subseteq I$, $(S - I) \subseteq (S - S) = S$ and, since $S \neq I$, we have $0 \notin (S - I)$, so that $(S - I) \subsetneq S$ and $(S - I) \subseteq M$. Thus $I^v = (S - (S - I)) \supseteq (S - M)$. □

1.2.1. The ideals M_a

Definition 1.13. *Let S be a numerical semigroup and $a \in \mathbb{N} \setminus S$. We define M_a as the biggest ideal in $\mathcal{F}_0(S)$ not containing a. More explicitly, $M_a := \bigcup\{I \in \mathcal{F}_0(S) : a \notin I\}$.*

Note that, if $a \neq b$, then $M_a \neq M_b$, since the ideal $S \cup \{x \in \mathbb{Z} : x > a\}$ does not contain a, and thus $\max(\mathbb{N} \setminus M_a) = a$.

Lemma 1.14. *Let S be a numerical semigroup and let $a, b \in \mathbb{N} \setminus S$, $a < b$. Then:*

(a) $M_a = \{b \in \mathbb{N} : a - b \notin S\}$;
(b) $M_a = (a - b + M_b) \cap \mathbb{N}$.
(c) M_a *is* $*_{M_b}$*-closed; in particular, if* M_a *is not divisorial, also* M_b *is not divisorial.*

Proof. (a) Let $b \in \mathbb{N}$. If $a - b \in S$, then $a \in b + S$ and thus $b \notin M_a$, while, if $a - b \notin S$, then $a \notin S \cup (b + S)$, that is, there is an ideal of $\mathcal{F}_0(S)$ containing b but not a, and thus $b \in M_a$.

(b) Let $c \in \mathbb{N}$. Then,

$$c \in a - b + M_b \iff b - a + c \in M_b \iff b - (b - a + c) \notin S \iff$$

$$\iff a - c \notin S \iff c \in M_a$$

and thus $M_a = (a - b + M_b) \cap \mathbb{N}$.

(c) Since M_b and $\mathbb{N}$ are $*_{M_b}$-closed, so is $(a - b + M_b) \cap \mathbb{N}$, which by the previous point is equal to M_a. If M_b were divisorial, $*_{M_b}$ would be equal to v, and thus also M_a would be divisorial; but this contradicts the hypothesis M_a not divisorial. □

Thus, an easy way to find nondivisorial ideals is by finding appropriate ideals M_a. The following proposition gives a simple test to see if M_a is divisorial.

Proposition 1.15. *Let S be a numerical semigroup, let $g = g(S)$ and let $a \in \mathbb{N} \setminus S$. The following statements are equivalent:*

(i) $M_a = M_a^v$;
(ii) $M_a = (-\gamma + S) \cap \mathbb{N}$ *for some* $\gamma \in S$;
(iii) $M_a = (a - g + S) \cap \mathbb{N}$.

Proof. (i $\Longrightarrow$ ii) Since M_a is divisorial,

$$M_a = \bigcap_{\gamma \in (S - M_a)} (-\gamma + S) = \bigcap_{\gamma \in (S - M_a)} ((-\gamma + S) \cap \mathbb{N}).$$

If $M_a \neq (-\beta + S) \cap \mathbb{N}$ for some $\beta \in (S - M_a)$, then, by maximality of M_a, we have $a \in (-\beta + S) \cap \mathbb{N}$. Hence, if $M_a \neq (-\gamma + S) \cap \mathbb{N}$ for all $\gamma \in (S - M_a)$, we would have $a \in \bigcap_{\gamma \in (S - M_a)} (-\gamma + S) = M_a^v$, and in particular $M_a \neq M_a^v$, against the hypothesis.

(ii $\Longrightarrow$ iii) The greatest element in $\mathbb{N} \setminus M_a$ is a, while the the greatest element in $\mathbb{N} \setminus ((-\gamma + S) \cap \mathbb{N})$ is $-\gamma + g$. Hence $a = -\gamma + g$ and $-\gamma = a - g$.

(iii $\Longrightarrow$ i) Trivial, since both $\mathbb{N}$ and $a - g + S$ are divisorial. □

Corollary 1.16. *Let S be a numerical semigroup, $g = g(S)$. Then, $*_{M_g}$ is the identity.*

Proof. For every $I \in \mathcal{F}_0(S)$, we have $I = \bigcap_{b \in \mathbb{N} \setminus I} M_b$. In fact, by definition of M_b we have $I \subseteq M_b$ for every $b \notin I$, while, if $a \in \mathbb{N} \setminus I$, then $a \notin M_a$ and thus a is not in the intersection.

Since $b \leq g$ for every $b \notin S$, each M_b is $*_{M_g}$-closed, and thus I is $*_{M_g}$-closed. Therefore, $*_{M_g}$ is the identity. □

Proposition 1.17. *Let S be a numerical semigroup, $g = g(S)$. Then:*

(a) $(M_g - M_g) = S$;
(b) if Δ is a set of semigroups contained properly between S and $\mathbb{N}$ such that $\bigcap_{T \in \Delta} T = S$, then the map $* : I \mapsto \bigcap_{T \in \Delta} I + T$ is a star operation different from the identity.

Proof. (a) $T := (M_g - M_g)$ is a semigroup contained between S and $\mathbb{N}$; note that, since $0 \in M_g$, we have $T \subseteq M_g$ and in particular $g \notin T$. If $a \in T \setminus S$, then $g - a \notin T$, and thus $a, g - a \notin S$: hence, $a, g - a \in M_g$, and thus $g = a + g - a \in T + M_g \subseteq M_g$, which is absurd. Hence, $T = S$.

(b) It is straightforward to see that $*$ is a star operation. For every $T \in \Delta$, the set $M_g + T$ is a S-ideal, and is bigger than M_g since $T \nsubseteq (M_g - M_g)$. Hence, by definition, $g \in T + M_g$ and thus $g \in M_g^*$. In particular, $*$ is not the identity on S. □

Remark 1.18. By Corollary 1.16 and Proposition 1.17, the ideal M_g has the property that $(M_g - (M_g - I)) = I$ for every fractional ideal I of S. Hence, M_g is what is usually called the *canonical ideal* (or *standard canonical ideal*) of S, and is usually denoted by $K(S)$ (see for instance [74] or [23]). Therefore, these two results are essentially a reformulation (in the language of star operations) of [74, Satz 4 and Hillsatz 5].

In terms of ring theory (which we shall approach in Section 1.8), an ideal J on a domain R such that $(J : (J : I)) = I$ for every I is usually called *m-canonical*, following Heinzer-Huckaba-Papick [60]; indeed, part of the proof of Proposition 1.7 follows closely [60, Lemma 3.1].

Lemma 1.19. *Let S be a numerical semigroup, $I \in \mathcal{F}_0(S)$ and $a := \sup(\mathbb{N} \setminus I)$. If $g - a \notin S$, then $a \in I^v$, and in particular I is not divisorial.*

Positive integers a such that $a, g - a \notin S$ are known as *holes* of S, or *gaps of the second type* (while, if $a \in \mathbb{N} \setminus S$ and $g - a \in S$, then a is called a *gap of the first type*).

Proof. Let $I \subseteq -\gamma + S$ for some $\gamma \in \mathbb{Z}$. Since I contains all the integers bigger than a, so does $-\gamma + S$; hence $\gamma \geq g - a$. If $\gamma = g - a$, then $0 \notin -\gamma + S$ (since, by hypothesis, $g - a \notin S$), against $0 \in I$; hence $\gamma > g - a$, and $a \in -\gamma + S$. However, $I^v = \bigcap(-\gamma + S)$, where the intersection ranges among the integers γ such that $I \subseteq -\gamma + S$. In particular, each of these contains a, and so does I^v. □

Corollary 1.20. *Let S be a numerical semigroup, and let $a \in \mathbb{N} \setminus S$ be a hole of S. If $b \in \mathbb{N} \setminus S$ and $b \geq a$, then M_b is not divisorial.*

Proof. By Lemma 1.19, M_a is not divisorial. By Lemma 1.14(c), it follows that neither M_b is divisorial. $\square$

Example 1.21. Consider the semigroup $S := \langle 4, 5, 7\rangle = \{0, 4, 5, 7, \ldots\}$. Then, $g = 6$ and $\mathbb{N} \setminus S = \{1, 2, 3, 6\}$, and so the unique hole of S is 3. Hence, $M_3 = S \cup \{1, 2, 6\}$ and $M_6 = S \cup \{3\}$ are not divisorial.

On the other hand, we have $M_1 = S \cup \{2, 3, 6\} = (-5+S) \cap \mathbb{N}$ and $M_2 = S \cup \{1, 3, 6\} = (-4 + S) \cap \mathbb{N}$; hence, both M_1 and M_2 are divisorial.

Example 1.22. Let $S := \langle 3, 10, 11\rangle = \{0, 3, 6, 9, 10, 11, \ldots\}$. Then $g = 8$ and $\mathbb{N} \setminus S = \{1, 2, 4, 5, 7, 8\}$, and the holes of S are 1, 4 and 7. Thus, no M_a is divisorial.

Example 1.23. Preserve the notation of Lemma 1.19. Then, it is possible that I is not divisorial but $a \notin I^v$: for example, if $S = \langle 3, 8, 13\rangle$ and $I := S \cup \{10\}$, then $\max(\mathbb{N} \setminus I) = 7$, but $I^v = (S - M) = S \cup \{5, 10\}$. More generally, the same happens when there is a $\tau \in T(S)$ such that $\tau < g - \mu$: if $I := S \cup \{g\}$, we have $I^v = (S - M) = S \cup T(S)$, but $g - \mu > g/2$ does not belong to I.

Not every semigroup has holes: semigroups without holes are said to be *symmetric*, and can be characterized as those numerical semigroups of type 1 or, equivalently, those such that $T(S) = \{g\}$ [49, Proposition 2]. All semigroups generated by two integers, and in particular all semigroups of multiplicity 2, are symmetric (see for instance [49]).

Lemma 1.19 allows to re-prove another characterization of symmetric semigroups.

Proposition 1.24 [13, Proposition I.1.15]. *Let S be a numerical semigroup. The following are equivalent:*

(i) S is symmetric;
(ii) $I^v = I$ for each fractional ideal I of S (that is, $d = v$);
(iii) $I^v = I$ for each integral ideal I of S;
(iv) $T^v = T$ for each semigroup $T \supseteq S$.

Proof. (ii $\Longleftrightarrow$ iii) and (ii $\Longrightarrow$ iv) are clear (since a semigroup $T \supseteq S$ is a fractional ideal of S).

(i $\Longleftrightarrow$ ii). By Corollary 1.16, $d = v$ if and only if M_g is divisorial; by Proposition 1.15 and Lemma 1.6(a), this happens if and only if $M_g = S$, if and only if S is symmetric.

(iv $\Longrightarrow$ i). If not, let $\{a, g - a\} \subseteq \mathbb{N} \setminus S$. Then $T := S \cup \{x \in \mathbb{N} : x > a\}$ is a semigroup containing S such that $g(T) = a$ and thus, by Lemma 1.19, it is not divisorial (as an ideal of S). $\square$

Despite its seemingly innocuous looking, Corollary 1.20 can be used to prove a simple and powerful estimate on $|\mathrm{Star}(S)|$.

Theorem 1.25. *Let S be a non-symmetric semigroup. Then, $|\mathcal{G}_0(S)| \geq \delta(S)$, and thus $|\mathrm{Star}(S)| \geq \delta(S) + 1$.*

Proof. Let $g := g(S)$. Since S is not symmetric, there is a $\tau \in T(S) \setminus \{g\}$; let $\lambda := \min\{\tau, g - \tau\}$ (note that it may be $\tau = g - \tau$). Consider the three sets

$$A := \{x \in \mathbb{N} \setminus S \mid x < \lambda, \lambda - x \notin S\},$$

$$B := \{x \in \mathbb{N} \setminus S \mid x < \lambda, \lambda - x \in S\},$$

$$C := \{x \in \mathbb{N} \setminus S \mid x \geq \lambda\}.$$

Since $\mathbb{N} \setminus S = A \cup B \cup C$ and the three sets are disjoint, we have $\delta(S) = |A| + |B| + |C|$; we will define for every $x \in \mathbb{N} \setminus S$ a different non-divisorial ideal I_x, whose definition depends on whether $x \in A$, $x \in B$ or $x \in C$.

If $x \in C$, then define $I_x := M_x$; since $x \geq \lambda$ and $g - \lambda \notin S$, by Corollary 1.20 $I_x \in \mathcal{G}_0(S)$.

If $x \in A$, then $x \in M_\lambda$ (Lemma 1.14); we define $I_x := S \cup \{z \in \mathbb{N} \mid z > x, z \in M_\lambda\}$; then, $\sup(M_\lambda \setminus I_x) = x$ (so that I_x is not divisorial by Lemma 1.19), and thus $I_x \neq I_y$ if $x \neq y$ are in A.

If $x \in B$, consider $y := g - \lambda + x$. Then, $y = g - (\lambda - x)$, and since $\lambda - x \in S$, we have $y \notin S$; moreover, $g - \lambda < y < g$. Let $I_x := S \cup \{z \in \mathbb{N} \mid z > y\}$; then, g belongs to I_x while τ does not, and thus I_x is not divisorial by Proposition 1.12. Moreover, $\sup(\mathbb{N} \setminus I_x) = y$ (so that $I_x \neq I_z$ if $x \neq z$ are in B) and M_y contains $g - \lambda$ (since $x \notin S$); hence, $I_x \neq M_y$.

It is straightforward to see that $I_x \neq I_y$ if x and y belong to different subsets; therefore, $\{I_x \mid x \in \mathbb{N} \setminus S\}$ is a set of $\delta(S)$ non-divisorial ideals. In particular, $|\mathcal{G}_0(S)| \geq \delta(S)$, and $|\mathrm{Star}(S)| \geq \delta(S) + 1$ (using Theorem 1.9, and since we can consider also the v-operation). □

Theorem 1.26. *For each $n > 1$, there are only a finite number of numerical semigroups S such that $|\mathrm{Star}(S)| = n$.*

Proof. By Theorem 1.25, if S is not symmetric then $|\mathrm{Star}(S)| \geq \delta(S) + 1$; it follows that if S is a numerical semigroups with at n star operations, then $\delta(S) \geq n - 1$. Hence, the claim will follow from the fact that there are only a finite number of numerical semigroups with degree of singularity less or equal than a fixed δ.

However, this is clear: indeed, for an arbitrary S, we have $\delta(S) \geq g(S)/2$, and since all integers bigger than $g(S)$ are in S, every integer bigger than $2\delta(S)$ must be in S. Hence, a numerical semigroup S with degree of singularity less or equal than δ is characterized by the set $\{1, \ldots, 2\delta\} \cap S$; in particular there are only a finite number of possibilities. The claim is proved. □

The proof of Theorem 1.26 can also be used to give an explicit bound on the number of numerical semigroups S with n star operations: indeed, an inspection of the proof shows that this number is at most $2^{2(n-1)} = 4^{n-1}$. More precisely, it has been proved that there is a constant C such that the number $d(n)$ of numerical semigroups S such that $\delta(S) = n$ satisfies

$$\lim_{n \to \infty} \frac{d(n)}{\phi^n} = C,$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio [120]; thus, there is a constant D such that $d(n) \leq D\phi^n$, and so the number of numerical semigroups S with n star operations is at most

$$\sum_{i=1}^{n-1} D\phi^i \leq \frac{D\phi}{\phi - 1}\phi^{n-1} = D'\phi^n.$$

We shall study the problem of finding better effective versions of Theorem 1.26 in Section 1.7.

1.3. Ordering and antichains

In general, principal star operations are only a small minority of whole set of star operations on a given numerical semigroup: most closures are only generated by two or more ideals. This means that, while estimating the cardinality of $\mathcal{G}_0(S)$ is enough to prove Theorem 1.26, using subsets can improve dramatically the estimates on the number of semigroups with n star operations.

However, unlike principal star operations (Theorem 1.9), two different subsets of $\mathcal{G}_0(S)$ often generate the same star operation. For example, if I and J are non-divisorial ideal and J is $*_I$-closed, then any $*_J$-closed ideal is also $*_I$-closed: it follows that the sets $\{I\}$ and $\{I, J\}$ generate the same closure. On the other hand, if I is not $*_J$-closed and J is not $*_I$-closed, then $\{I, J\}$ generate a star operation that closes both I and J, and thus it is different from both $*_I$ and $*_J$: it follows that a crucial part of this problem is how the ideals compare with respect to the order of the set of star operations.

To study these relationships, we first note that the order on the set of closure operations allows to define a preorder on the set of fractional ideals of S.

Definition 1.27. *Let S be a numerical semigroup and let $I, J \in \mathcal{F}(S)$. We say that I is $*$-minor than J, and we write $I \leq_* J$, if $*_I \geq *_J$ or, equivalently, if I is $*_J$-closed.*

The relation $\leq_*$ is not an order on $\mathcal{F}(S)$, both because $*_I = *_{a+I}$ for every $a \in \mathbb{Z}$ (and so both $I \leq_* a + I$ and $a + I \leq_* I$) and because $*_I = *_J$ for every divisorial ideal. However, if we restrict to the set $\mathcal{G}_0(S)$ of nondivisorial ideals of S that are in $\mathcal{F}_0(S)$, then Theorem 1.9 guarantees that $(\mathcal{G}_0(S), \leq_*)$ is a partial order. Moreover, by Corollary 1.16, $(\mathcal{G}_0(S), \leq_*)$ has a maximum, the canonical ideal M_g.

The reasoning above shows that $\{I, J\}$ generates a "new" star operation (that is, different from $*_I$ and $*_J$) if and only if I and J are not comparable in the order $\leq_*$. To make this idea more general, we will use the following definition.

Definition 1.28. *Let $(\mathcal{P}, \leq)$ be a partially ordered set. An* antichain *of $\mathcal{P}$ is a set $\Delta \subseteq \mathcal{P}$ such that no two members of Δ are comparable. We denote by $\Omega(\mathcal{P})$ the set of antichains of $\mathcal{P}$, and by $\omega(\mathcal{P})$ its cardinality.*

Consider a set $\Delta \subseteq \mathcal{G}_0(S)$. Then, the set $\max_* \Delta := \{I \in \Delta \mid I \not\leq_* J \text{ for every } J \in \Delta\}$ is an antichain of $(\mathcal{G}_0(S), \leq_*)$; therefore, we have a canonically defined map

$$\begin{aligned} \mathcal{A}\colon \mathrm{Star}(S) &\longrightarrow \Omega(\mathcal{G}_0(S)) \\ * &\longmapsto \max_*(\mathcal{F}^* \cap \mathcal{G}_0). \end{aligned}$$

Moreover, $*_\Delta = *_{\max_* \Delta}$: indeed, it is clear that $*_{\max_* \Delta} \leq *_\Delta$, while if I is $*_\Delta$-closed then (since Δ is finite) there is a $J \in \max_* \Delta$ such that $I \leq_* J$, and so $I = I^{*_J}$ is also $*_{\max_* \Delta}$-closed. Therefore,

$$\begin{aligned} *\colon \Omega(\mathcal{G}_0(S)) &\longrightarrow \mathrm{Star}(S) \\ \Delta &\longmapsto *_\Delta \end{aligned}$$

(which can be seen as the restriction of the analogous map from the power set of $\mathcal{G}_0(S)$ to $\mathrm{Star}(S)$) is surjective. Moreover, in view of Proposition 1.5, we have that $\mathcal{A} \circ *(\Delta) = \Delta$, i.e., $* \circ \mathcal{A}$ is the identity on $\Omega(\mathcal{G}_0(S))$. In particular, we have $|\mathrm{Star}(S)| \leq \omega(\mathcal{G}_0)$; note also that $\omega(\mathcal{G}_0)$ is finite, because $\mathcal{G}_0$ is finite.

It is natural to ask what happens to the other composition, that is, to $* \circ \mathcal{A}$. For example, if $\Delta = \{I\}$ is a single ideal, then $\mathcal{A}(*_I) = \{I\}$, and $* \circ \mathcal{A}(*_I) = *_I$. Similarly, if $\Delta = \emptyset$, then $*_\emptyset = v$ and $* \circ \mathcal{A}(v) = v$. In general, the composition $* \circ \mathcal{A}$ is bijective if and only if $\mathcal{A}$ is surjective, if and only if $*$ is injective; that is, it is equivalent to the fact that $*_\Delta \neq *_\Lambda$ whenever $\Delta \neq \Lambda$ are antichains of $\mathcal{G}_0(S)$. Unfortunately, this does not always happen, as the next example shows.

Example 1.29. Let $S := \langle 6, 7, 8, 9, 10, 11 \rangle = \{0, 6, \rightarrow\}$, $I := S \cup \{3, 4, 5\}$, $J := S \cup \{1, 3, 5\}$, $L := S \cup \{4, 5\}$. Calculations show that $\Delta := \{I, J\}$ is an antichain of $\mathcal{G}_0$, and that $L^{*_I} = L \cup \{3\} = I$, $L^{*_J} = L \cup \{2\}$, so that L is nor $*_I$ nor $*_J$-closed. However,

$$L^{*_\Delta} = L^{*_I} \cap L^{*_J} = L$$

and hence $\mathcal{A}(*_\Delta)$ must contain an ideal $\geq_* L$. Therefore, $\Delta \neq \mathcal{A}(*)$ for every $* \in \mathrm{Star}(S)$.

In particular, this means that (for general semigroups) the structure of the set $\mathrm{Star}(S)$ does not depend exclusively on the order structure of $\mathcal{G}_0(S)$, but also on the specific ideals.

1.3.1. Prime star operations

Definition 1.30. *A star operation $*$ of a numerical semigroup is* prime *if, whenever $* \geq *_1 \wedge *_2$, we have $* \geq *_1$ or $* \geq *_2$.*

Proposition 1.31. *A prime star operation is principal.*

Proof. Suppose it is not, and consider the antichain $\mathcal{A}(*) := \{I_1, \ldots, I_n\}$. Then, $* = *_{I_1} \wedge \cdots \wedge *_{I_n}$, and in particular $* \leq *_{I_i}$ for every $i \in \{1, \ldots, n\}$.

However, an inductive argument applied to the definition of prime star operation shows that $* \geq *_I$ for some $I \in \mathcal{A}(*)$; hence, $*_I \leq * \leq *_I$, and $* = *_I$, that is, $*$ is a principal star operation. $\square$

Remark 1.32. Similarly to prime operations, we can also define *irreducible* star operations as the closures $*$ such that, whenever $* = *_1 \wedge \cdots \wedge *_n$, then $* = *_i$ for some i. Since every star operation is the infimum of a family of principal operations (Proposition 1.5), an irreducible star operation must be principal, and it is easy to see that every prime operation is irreducible. However, I do not have any example of a principal star operation that is not irreducible. We will see that such an example do exist in the case of rings (see Remark 3.80).

Definition 1.33. *If I is an ideal of S such that $*_I$ is prime, we say that I is an* atom *of $\mathcal{G}_0(S)$.*

To simplify some of the statements of the following results, we are not imposing – with what can be seen as a slight abuse of terminology – that atoms of $\mathcal{G}_0(S)$ are actually in $\mathcal{G}_0(S)$; they can be divisorial, or even not in $\mathcal{F}_0(S)$. Indeed, the v-operation is prime, so every divisorial ideal I is an atom (being, in this case, $*_I = v$).

Proposition 1.34. *Let S be a numerical semigroup and $I \in \mathcal{G}_0(S)$. The following are equivalent:*

(i) I is an atom of $\mathcal{G}_0(S)$;
(ii) for every $*_1, *_2 \in \mathrm{Star}(S)$, I is $*_1 \wedge *_2$-closed if and only if I is $*_1$- or $*_2$-closed;
(iii) for every $J_1, J_2 \in \mathcal{F}_0(S)$ such that $*_I \geq *_{J_1} \wedge *_{J_2}$, we have $*_I \geq *_{J_1}$ or $*_I \geq *_{J_2}$;
(iv) if $I = J_1 \cap J_2$, then I is $*_{J_1}$- or $*_{J_2}$-closed;
(v) for every $*_1, \dots *_n \in \mathrm{Star}(S)$, I is $*_1 \wedge \dots \wedge *_n$-closed if and only if I is $*_i$-closed for some $i \in \{1, \dots, n\}$;
(vi) for every $\Delta \subseteq \mathcal{F}(S)$, $I = I^{*_\Delta}$ if and only if $I \leq_* J$ for some $J \in \Delta$.

Proof. (ii) is just a restatement of the definition of atom, so it is equivalent to (i). Clearly (ii $\Longrightarrow$ iii), while (iii $\Longrightarrow$ iv) since if $I = J_1 \cap J_2$ then $*_I \geq *_{J_1} \wedge *_{J_2}$. Suppose (iv) holds and suppose that I is $*_1 \wedge *_2$-closed. Then, $I = I^{*_1 \wedge *_2} = I^{*_1} \cap I^{*_2}$, and thus, if $J_i := I^{*_i}$, then I is $*_{J_1}$- or $*_{J_2}$-closed. However, $*_{J_i} \geq *_i$, and thus I is $*_1$- or $*_2$-closed. Hence, (iv $\Longrightarrow$ ii).

(ii $\Longrightarrow$ v) follows by induction: the case $n = 2$ is the hypothesis, while if I is $*_1 \wedge \cdots \wedge *_n$-closed then it is $((*_1 \wedge \cdots \wedge *_{n-1}) \wedge *_n)$-closed, so that $* \geq *_1 \wedge \cdots \wedge *_{n-1}$ (and thus $* \geq *_i$ by induction) or $* \geq *_n$.

To show (v $\Longrightarrow$ vi), we can suppose $\Delta \subseteq \mathcal{F}_0(S)$; since $\mathcal{F}_0(S)$ is finite, so is Δ. Hence, since $*_\Delta = \inf_{J \in \Delta} *_J$, if $I = I^{*_\Delta}$ then I is $*_J$-closed for some $J \in \Delta$.

(vi $\Longrightarrow$ i) Suppose $*_I \geq *_1 \wedge *_2$, and let $\Delta_1 := \{J \in \mathcal{G}_0(S) : J = J^{*_1}\}$, $\Delta_2 := \{J \in \mathcal{G}_0(S) : J = J^{*_2}\}$, $\Delta := \Delta_1 \cup \Delta_2$. Then $I = I^{*_\Delta}$, and thus $I \leq_* J$ for some $J \in \Delta$: if $J \in \Delta_1$ (say), then $*_I \geq *_1$, and I is an atom. □

Corollary 1.35. *Let S be a numerical semigroup and $\Gamma \subseteq \mathcal{G}_0(S)$ a set of atoms of $\mathcal{G}_0(S)$. If $\Delta \neq \Lambda$ are nonempty antichains of Γ, then $*_\Delta \neq *_\Lambda$.*

Proof. We claim that $\Delta = \mathcal{F}^{*_\Delta}(S) \cap \Gamma$. Indeed, let L be a maximal element of $\mathcal{F}^{*_\Delta} \cap \Gamma$, with respect to the $*$-order. Then, by Proposition 1.34(vi), there is a $J \in \Delta$ such that $L \leq_* J$. However, $J = J^{*_\Delta}$, i.e., $J \in \mathcal{F}^{*_\Delta}$; by maximality, $J = L$, and in particular $L \in \Delta$. Conversely, if $J \in \Delta$ but J is not maximal in $\mathcal{F}^{*_\Delta} \cap \Gamma$, there would be a $J' \in \mathcal{F}^{*_\Delta}(S) \cap \Gamma$ such that $J <_* J'$; then, there would be a $J'' \in \Delta$ such that $J' \leq_* J''$ (again by Proposition 1.34(vi)). It would follow that $J <_* J''$, against the hypothesis that Δ is an antichain; therefore, $\max_*(\mathcal{F}^{*_\Delta} \cap \Gamma) = \Delta$.

In particular, if $\Delta \neq \Lambda$, then $\mathcal{F}^{*_\Delta} \neq \mathcal{F}^{*_\Lambda}$, and thus $*_\Delta \neq *_\Lambda$. □

Corollary 1.36. *Let S be a numerical semigroup and $\Gamma \subseteq \mathcal{G}_0(S)$ be the set of atoms of $\mathcal{G}_0(S)$. Then, $|\mathrm{Star}(S)| \geq \omega(\Gamma)$.*

Proof. Apply Corollary 1.35: every nonempty antichain generates a different star operation, and the empty antichain generates the v-operation. $\square$

Thus, a way to estimate $|\mathrm{Star}(S)|$ is through finding atoms. The next proposition establishes a useful criterion.

Proposition 1.37. *Let S be a numerical semigroup and $I \in \mathcal{G}_0(S)$.*

(a) If, for every $*_1, *_2 \in \mathrm{Star}(S)$, we have $I^{*_1} \subseteq I^{*_2}$ or $I^{*_2} \subseteq I^{*_1}$, then I is an atom.
(b) If I^* is an atom for every $* \in \mathrm{Star}(S)$, then I^{*_1} and I^{*_2} are comparable for every pair $*_1, *_2$ of star operations.

Proof. (a) Suppose I is not an atom. By Proposition 1.34, there are star operations $*_1, *_2$ such that $*_I \geq *_1 \wedge *_2$ but $*_I \not\geq *_1$ and $*_I \not\geq *_2$. Then, $I \neq I^{*_1}$ and $I \neq I^{*_2}$, but $I = I^{*_1 \wedge *_2} = I^{*_1} \cap I^{*_2}$, so that I^{*_1} and I^{*_2} are not comparable.

(b) Let $J := I^{*_1} \cap I^{*_2} = I^{*_1 \wedge *_2}$. Then, $I^{*_i} \subseteq J^{*_i} \subseteq (I^{*_i})^{*_i} = I^{*_i}$, and thus $I^{*_i} = J^{*_i} =: J_i$. By hypothesis, J is an atom; by Proposition 1.34(iv), J is $*_{J_i}$-closed for some i (say $i = 1$). Then, since J_1 is $*_1$-closed, we have $*_1 \leq *_{J_1}$ and

$$J_1 = J^{*_1} \subseteq J^{*_{J_1}} = J,$$

and thus $J = J_1$. In particular, $J_1 \subseteq J_2$, and I^{*_1} and I^{*_2} are comparable. $\square$

Condition (a) of the previous proposition is sometimes simple to verify explicitly. A result similar to the next result will be stated in Proposition 1.43.

Proposition 1.38. *Let S be a numerical semigroup and $I \in \mathcal{F}_0(S)$. If $|I^v \setminus I| = 1$, then I is an atom of $\mathcal{G}_0(S)$.*

Proof. Immediate from Proposition 1.37(a), since I^* is contained between I and I^v, and there are no ideals properly in between. $\square$

Proposition 1.39. *Let S be a numerical semigroup. The following are equivalent:*

(i) every ideal of S is an atom of $\mathcal{G}_0(S)$;
(ii) for every ideal I and every $*_1, *_2 \in \mathrm{Star}(S)$, the ideals I^{*_1} and I^{*_2} are comparable;
(iii) the map $\mathcal{A} : \mathrm{Star}(S) \longrightarrow \Omega(\mathcal{G}_0(S))$, $* \mapsto \mathcal{A}(*)$, is bijective;
(iv) $\mathcal{A} \circ *$ is the identity on $\Omega(\mathcal{G}_0(S))$;
(v) for every antichain Δ of $\mathcal{G}_0(S)$, $\mathcal{A}(*_\Delta) = \Delta$;
(vi) $|\mathrm{Star}(S)| = \omega(\mathcal{G}_0(S))$.

Proof. (i $\Longrightarrow$ ii) follows from Proposition 1.37(b), since each I^* is an atom; (ii $\Longrightarrow$ i) is a direct consequence of Proposition 1.37(a).

(i $\Longrightarrow$ iii) Since $\mathcal{A}$ is injective, it is enough to show that it is surjective. Let Δ be a nonempty antichain of $\mathcal{G}_0(S)$, and consider the star operation $*_\Delta$: if $\mathcal{A}(*_\Delta) = \Lambda \neq \Delta$, then $*_\Lambda = *_\Delta$, against Corollary 1.35.

(iii $\iff$ iv $\iff$ v) follows from the discussion after Definition 1.28.

(iv $\implies$ i) Suppose $I \in \mathcal{F}_0(S)$ is not an atom: then I is not divisorial, and there are ideals J_1, J_2 such that $I = J_1 \cap J_2$ but I is not $*_{J_1}$- nor $*_{J_2}$-closed. The ideals J_1 and J_2 are not $*$-comparable: if $J_1 \leq_* J_2$ (say), then $J_1 = J_1^{*_{J_2}}$ and thus I would be $*_{J_2}$-closed, which is impossible. Hence, $\Delta := \{J_1, J_2\}$ is an antichain, and thus $\mathcal{A}(*_\Delta) = \Delta$ (since iv $\iff$ v).

Since I is $*_\Delta$-closed, $*_\Delta = *_\Delta \wedge *_I = *_{\Delta \cup \{I\}}$, and thus $\Delta \cup \{I\}$ cannot be an antichain. However, I is not $*$-minor than each J_i, and thus $I \geq_* J_i$ for some i. This would imply that J_i is not $*$-maximal in $\mathcal{F}^{*_\Delta}$, that is, $J_i \notin \mathcal{A}(*_\Delta)$, a contradiction; therefore, I is an atom.

(iii $\iff$ vi) is a simple consequence of the finiteness of $\mathrm{Star}(S)$ and $\Omega(\mathcal{G}_0(S))$. $\square$

Corollary 1.40. *Let* $\mathcal{G}_1 := \mathcal{G}_1(S) := \{I \in \mathcal{F}_0(S) : |I^v \setminus I| = 1\}$. *Then,* $|\mathrm{Star}(S)| \geq \omega(\mathcal{G}_1)$ *and, if* $\mathcal{G}_0 = \mathcal{G}_1$*, then* $|\mathrm{Star}(S)| = \omega(\mathcal{G}_1)$.

Proof. It follows directly from Corollary 1.36 and Proposition 1.39. $\square$

1.3.2. The $\mathcal{Q}_a$

The methods of Section 1.3.1 provide a new way to estimate the number of star operations on a numerical semigroup S: simply find the set Γ atoms of S and determine the number of $*$-antichains of Γ. The problem is that finding atoms if often time-consuming, and not easy to do in a general way.

However, it is not actually needed that we work with atoms: indeed, is is enough to find a set of ideals Δ such that different antichains of Δ generate different star operations. (Under this point of view, the case $\Delta = \mathcal{G}_0(S)$ corresponds to the best possible scenario, where each ideal is an atom.)

Definition 1.41. *Let S be a numerical semigroup. For every $a \in \mathbb{N} \setminus S$, let* $\mathcal{Q}_a(S) := \{I \in \mathcal{F}_0(S) : a = \sup(\mathbb{N} \setminus I), a \in I^v\}$.

Proposition 1.42. *Let S be a numerical semigroup and $\mathcal{Q}_a := \mathcal{Q}_a(S)$. Then:*

(a) $\mathcal{Q}_a$ is nonempty if and only if M_a is not divisorial;
(b) if $\mathcal{Q}_a$ is nonempty, M_a is its $*$-maximum;
(c) if $b \leq a$, then $M_b \leq_* M_a$;
(d) if $\mathcal{Q}_a = \emptyset$ then $\mathcal{Q}_b = \emptyset$ for every $b \leq a$.

Proof. (a) If M_a is not divisorial, then M_a^v is an ideal in $\mathcal{F}_0(S)$ properly containing M_a; by the definition of M_a, it follows that $a \in M_a^v$, and thus $M_a \in \mathcal{Q}_a$. Conversely, if M_a is divisorial, let $I \in \mathcal{F}_0(S)$ be an ideal such that $a \notin I$. Then, $I \subseteq M_a$, and thus $I^v \subseteq M_a^v = M_a$, and in particular $a \notin I^v$. Hence, $I \notin \mathcal{Q}_a$, which therefore is empty.

(b) follows from noting that $I = \bigcap_{b \in \mathbb{N} \setminus I} M_b$, and that each M_b is $*_{M_a}$-closed when $b \leq a$; (c) follows from the equality $M_b = (b - a + M_b) \cap \mathbb{N}$ (Lemma 1.14(b)).

(d) If $\mathcal{Q}_a = \emptyset$, then M_a is divisorial, and $*_{M_b} \geq *_{M_a} = v$. Thus, $*_{M_b} = v$, M_b is divisorial and $\mathcal{Q}_b = \emptyset$ by point (a). $\square$

Note that, if $a \in T(S)$ and S is not symmetric, then $a \in M_a^v$, and thus $\mathcal{Q}_a \neq \emptyset$.

Proposition 1.43. *Let S be a numerical semigroup, and suppose that $I \in \mathcal{Q}_a$. If $|M_a \setminus I| \leq 1$, then I is an atom of S.*

Proof. Suppose $I = J_1 \cap J_2$. Since $a \notin I$, without loss of generality we can suppose $a \notin J_1$; moreover, if $b > a$ then $b \in I$, and so $b \in J_1$. Therefore, $I \subseteq J_1 \subseteq M_a$, and since $|M_a \setminus I| \leq 1$ we have $J_1 = I$ or $J_1 = M_a$. In the former case I is trivially $*_{J_1}$-closed; in the latter, we have $I \leq_* M_a$ by Proposition 1.42(b), and thus I is again $*_{J_1}$-closed. The claim follows applying condition (iv) of Proposition 1.34. □

On the other hand, if $I \in \mathcal{Q}_a$ but $|M_a \setminus I| \geq 2$, it is possible that $*_I$ is not prime. We digress to establish a general lemma.

Lemma 1.44. *Let S, U be numerical semigroups, and I be an ideal of S such that $S \subseteq I \subseteq U$; let v be the divisorial closure of the S-ideals. Then, $I^{*_U} = I^v \cap U$.*

Proof. Suppose $I \subseteq -\alpha + U$. Then, $\alpha \in U$; however, since U is a semigroup, $U = (U - U)$, and thus $U \subseteq -\alpha + U$. Therefore,

$$I^{*_U} = I^v \cap \bigcap_{\alpha \in (U-I)} (-\alpha + U) \supseteq I^v \cap U.$$

Since $I^{*_U} \subseteq U^{*_U} = U$, we have $I^{*_U} \subseteq U \cap I^v$, and thus the two sides are equal. □

Example 1.45. Consider the semigroup $S := \langle 4, 6, 7, 9 \rangle = \{0, 4, 6, \rightarrow\}$, and let $I := S \cup \{5\}$. Then, I is a semigroup and $I^v = (S - M) = S \cup \{2, 3, 5\}$; in particular, $I \in \mathcal{Q}_3$. Let $J_1 := I \cup \{2\}$ and $J_2 := I \cup \{3\}$: both J_1 and J_2 are semigroups containing I, so that $I^{*_{J_i}} = J_i$, and in particular I is not $*_{J_1}$- nor $*_{J_2}$-closed. However, $J_1 \cap J_2 = I$, and thus I is $(*_{J_1} \wedge *_{J_2})$-closed. Hence, I is not an atom of S.

This example could be generalized.

Corollary 1.46. *Let S be a numerical semigroup, $t := t(S)$, $\mu := \mu(S)$, $g := g(S)$; suppose $t \geq 3$ and $g \leq 2\mu - 2$. Then, $S \cup \{g\}$ is an atom of S if and only if $S = \langle 4, 5, 6, 7 \rangle$.*

Proof. If $S = \langle 4, 5, 6, 7 \rangle$, then $M_2 = S \cup \{1, 3\}$, and thus $S \cup \{g\} = S \cup \{3\}$ is an atom by Proposition 1.43 (see Example 1.54 for a deeper analysis of this semigroup).

Suppose $S \neq \langle 4, 5, 6, 7 \rangle$, and let $I := S \cup \{g\}$. Since $\mu > t \geq 3$, we have $\mu \geq 4$. If $g < \mu$ (i.e., $S = \{0, \mu, \rightarrow\}$ and so $g = \mu - 1$), consider the ideals $T_2 := S \cup \{\mu - 1, \mu - 2\}$ and $T_3 := S \cup \{\mu - 1, \mu - 3\}$: since $S \neq \langle 4, 5, 6, 7 \rangle$, $\mu > 4$, so that $2(\mu - 3) \geq \mu - 1$ and both T_2 and T_3 are semigroups. By Lemma 1.44, $I^{*_{T_i}} = I^v \cap T_i = \mathbb{N} \cap T_i = T_i$, while $I = T_2 \cap T_3$; by Proposition 1.34, I is not an atom of S.

Suppose $\mu < g < 2\mu - 2$. Then, $\mu - 1, \mu - 2 \in T(S)$; let $T_1 := S \cup \{g, \mu - 1\}$ and $T_2 := S \cup \{g, \mu - 2, 2\mu - 4\}$. Then, both T_1 and T_2 are semigroups, $T_1 \cap T_2 = I$ but $I^{*_{T_i}} = T_i \cap (S - M)$ contains $\mu - i$ and thus it is different from I. Hence, I is not an atom of S.

Suppose $g = 2\mu - 2$. If $\{\mu+1, \dots, 2\mu-3\} \subseteq S$, then $T(S) = \{g, \mu-1\}$, and thus $t = 2$; therefore, under our hypothesis, there is a $\tau \in \{\mu+1, \dots, 2\mu-3\} \setminus S$. Then, $\tau \in T(S)$ and $2\tau > g$, and thus $T_1 := S \cup \{g, \tau\}$ is a semigroup contained in $S \cup T(S) = (S - M)$, and the same happens for $T_2 := S \cup \{\mu - 1, g\}$. Again, $I = T_1 \cap T_2$ but $I^{*_{T_i}} = T_i$, so that I is not an atom of S. □

We resume the analysis of the $*$-order on $\mathcal{Q}_a$.

Proposition 1.47. *Let S be a numerical semigroup, $a \in \mathbb{N} \setminus S$, and $\mathcal{Q}_a := \mathcal{Q}_a(S)$. Let $I, J \in \mathcal{Q}_a$ and $\Delta \subseteq \mathcal{Q}_a$.*

(a) If $I \nsubseteq J$ then $a \in I^{*_J}$.
(b) If $I \nsubseteq J$ for every $J \in \Delta$ then $a \in I^{*_\Delta}$.
(c) The $*$-order on $\mathcal{Q}_a$ is coarser than the inclusion, i.e., if $I \leq_* J$ then $I \subseteq J$.
(d) Let $\Delta \neq \Lambda$ be two nonempty subsets of $\mathcal{Q}_a$ that are antichains with respect to inclusion. Then, $*_\Delta \neq *_\Lambda$.

Proof. (a) By definition,

$$I^{*_J} = I^v \cap \bigcap_{\gamma \in (J-I)} (-\gamma + J).$$

If $I \nsubseteq J$, then $0 \notin (J - I)$. Thus, for each $\gamma \in (J - I)$, $a \in -\gamma + J$ and, since $I \in \mathcal{Q}_a$, $a \in I^v$. Therefore, $a \in I^{*_J}$.

(b) is immediate from the above point, since $I^{*_\Delta} = \bigcap_{J \in \Delta} I^{*_J}$; (c) is just a reformulation of point (a).

(d) Suppose $*_\Delta = *_\Lambda$; suppose, without loss of generality, that there is a $I \in \Delta \setminus \Lambda$. If $I \nsubseteq J$ for every $J \in \Lambda$, then $a \in I^{*_\Lambda}$, which is different from $I = I^{*_\Delta}$. Hence, there is a $J \in \Lambda$ such that $J \supseteq I$. Similarly, if there is no $I' \in \Delta$ containing J, then J^{*_Δ} contains a, a contradiction; therefore, $I \subseteq J \subseteq I'$ for some I', and since Δ is an antichain with respect to the containment we must have $I = I'$, and thus $I = J$. But this is impossible, since $I \notin \Lambda$; hence, $\Delta = \Lambda$. □

Remark 1.48. Note that the $*$-order on $\mathcal{Q}_a$ may really be different from the containment: for example, consider $S := \{0, 5, \rightarrow\}$ and let $I := S \cup \{1\}$, $J := S \cup \{1, 3\}$. Both I and J are in $\mathcal{Q}_4$, and $I \subseteq J$; we claim that $I \nleq_* J$.

Indeed, $I^v = \mathbb{N}$; suppose $I \subseteq -\gamma + J$. Then, $\gamma \in J$, and thus $\gamma \in \{0, 1, 3\}$ or $\gamma \geq 5$. If $\gamma = 1$ or $\gamma = 3$, then $1 \notin -\gamma + J$; but if $\gamma \geq 5$, then $\mathbb{N} \subseteq -\gamma + J$. It follows that $I^{*_J} = \mathbb{N} \cap J = J \neq I$.

When $\mathcal{P}$ is the power set $\mathscr{P}(\{1, \dots, n\})$ of the finite set in n elements, ordered by inclusion, we denote the number of antichains of $\mathcal{P}$ simply as $\omega(n)$. These numbers are called *Dedekind numbers*; their sequence grows super-exponentially, since each family of subsets of $\{1, \dots, n\}$ of size $\lfloor n/2 \rfloor$ is an antichain. More precisely, $\omega(n)$ is bounded as follows (see [79]):

$$\binom{n}{\lfloor n/2 \rfloor} \leq \log_2 \omega(n) \leq \binom{n}{\lfloor n/2 \rfloor} \left(1 + O\left(\frac{\log n}{n}\right)\right).$$

If n is small, $\omega(n)$ can be calculated by hand.

- If $n = 0$, then the antichains of $\mathscr{P}(\emptyset)$ are the empty antichain and the antichain $\{\emptyset\}$ composed of the only empty set
- If $n = 1$, then $\mathscr{P}(\{1\}) = \{\emptyset, \{1\}\}$, and thus the antichains are the empty antichain, $\{\emptyset\}$ and the one formed by the set $\{1\}$.
- If $n = 2$, then we have the empty antichain, $\{\emptyset\}$, $\{\{1\}\}$, $\{\{2\}\}$, $\{\{1\}, \{2\}\}$ and $\{\{1, 2\}\}$.

Hence, $\omega(0) = 2$, $\omega(1) = 3$ and $\omega(2) = 6$.

However, the quick growth of these numbers means that calculating them precisely becomes difficult already for low n: indeed, $\omega(n)$ has been calculated explicitly only for $n \leq 8$ [113, 109].

Corollary 1.49. *Let S be a numerical semigroup. Then, $|\mathrm{Star}(S)| \geq \omega(t(S) - 1) - 1$.*

Proof. Let $t := t(S)$. Consider the ideals $I_A := S \cup A$, with $A \subseteq T(S) \setminus \{g\}$. If $A \neq \emptyset$, then $I_A \neq S$, and so $T(S) \subseteq I_A^v$; it follows that, in this case, $I_A \in \mathcal{Q}_g$. By Proposition 1.47(d), each nonempty antichain (with respect to inclusion) of $\{I_A : A \subseteq T(S) \setminus \{g\}, A \neq \emptyset\}$ generates a different star operation; however, the inclusion order is nothing but the order of the power set of $T(S) \setminus \{g\}$, which has $\omega(t-1)$ antichains. We must exclude the empty antichain and the antichain corresponding to the empty set, so that we have $\omega(t-1)-2$ star operations. Moreover, each of these operations is different from the v-operation, and thus $|\mathrm{Star}(S)| \geq \omega(t - 1) - 1$. □

Since $I \in \mathcal{Q}_a$ does not imply that $*_I$ is prime, we cannot in general use concurrently the ideals of $\mathcal{Q}_a$ and the atoms, that is, we cannot simply sum the estimates obtained in these two ways. However, we can compare ideals belonging to different $\mathcal{Q}_a$.

Lemma 1.50. *Let S be a numerical semigroup, $I, J \in \mathcal{G}_0(S)$ such that $J \leq_* I$. If $I \in \mathcal{Q}_a$ and $J \in \mathcal{Q}_b$, then $a \geq b$.*

Proof. The proof is the same as the proof of Proposition 1.47(a): if $a < b$, then b belongs to both J^v and $-\alpha + I$, (for every $\alpha \in I - J$) and so $b \in J^{*_I}$, and in particular $J \neq J^{*_I}$, against the hypothesis $J \leq_* I$. □

The next step is generalizing Proposition 1.47 to antichains generated by different $\mathcal{Q}_a$.

Proposition 1.51. *Let S be a numerical semigroup. Let $\Delta \subseteq \mathcal{Q}_a$, $\Lambda \subseteq \mathcal{Q}_b$ be two nonempty sets that are antichains with respect to inclusion. If $\Delta \neq \Lambda$ (in particular, if $a \neq b$) then $*_\Delta \neq *_\Lambda$.*

Proof. The case $a = b$ is just Proposition 1.47. Suppose (without loss of generality) that $a > b$.

Let $I \in \Delta$, and let $\gamma \in \mathbb{N}$, $J \in \Lambda$ such that $I \subseteq -\gamma + J$. Since $\gamma + a \geq a > b$, we have $a \in -\gamma + J$, and thus $a \in I^{*_\Lambda} \setminus I$, and $I^{*_\Lambda} \neq I = I^{*_\Delta}$. Hence, $*_\Lambda \neq *_\Delta$. □

We will denote by $\omega_{\mathrm{i}}(\mathcal{Q}_a)$ the number of antichains of $(\mathcal{Q}_a, \subseteq)$, that is, the number of antichains of $\mathcal{Q}_a$ with respect to inclusion. By Proposition 1.47(c), every antichain with respect to inclusion is also an antichain with respect to the $*$-order, so that $\omega_{\mathrm{i}}(\mathcal{Q}_a) \leq \omega(\mathcal{Q}_a)$.

Corollary 1.52. *Let S be a numerical semigroup. Then,*

$$|\mathrm{Star}(S)| \geq 1 + \sum_{a\in\mathbb{N}\setminus S}(\omega_{\mathrm{i}}(\mathcal{Q}_a) - 1) \geq 1 + \sum_{a\in\mathbb{N}\setminus S}|\mathcal{Q}_a|.$$

Proof. It is enough to apply Proposition 1.51 to the nonempty antichains of the $\mathcal{Q}_a$, and then add the v-operation. For the second inequality, note that every ideal of $\mathcal{Q}_a$ is an antichain of $\mathcal{Q}_a$. □

We can also prove a limited form of the above results for "mixed" antichains, i.e., antichains whose elements come from different $\mathcal{Q}_a$.

Proposition 1.53. *Let S be a numerical semigroup, and let $x < y$ be two positive integers such that:*

1. *$x, y \notin S$;*
2. *every integer w such that $x < w < y$ is in S;*
3. *M_x and M_y are not divisorial.*

Let Λ, Δ be nonempty subsets of $\mathcal{Q}_y$ that are antichains with respect to inclusion, and suppose $M_y \notin \Lambda$. Then:

(a) $*_{\Lambda\cup\{M_x\}} \neq *_\Delta$;
(b) if $\Lambda \neq \Delta$ then $*_{\Lambda\cup\{M_x\}} \neq *_{\Delta\cup\{M_x\}}$.

Proof. Claim 1: $y - x$ is the minimal element of $M_y \setminus \{0\}$.

Indeed, $y - x \in M_y$ because $y - (y - x) = x \notin S$; on the other hand, if $0 < \beta < y - x$, then $y > y - \beta > y - (y - x) = x$, and thus, by hypothesis, $y - \beta \in S$, so that $\beta \notin M_y$.

Claim 2: Let $I \in \mathcal{Q}_y \setminus \{M_y\}$. Then, $x \in M_x^{*_I}$.

Suppose $x \notin M_x^{*_I}$. Then, there is an α such that $M_x \subseteq -\alpha + I$ while $x \notin -\alpha + I$. We distinguish four cases:

1. $\alpha = 0$: then, $M_x \subseteq I$, against the fact that $y \in M_x \setminus I$;
2. $0 < \alpha < y - x$: then, $x < x + \alpha < y$; however, $x + \alpha \in S \subseteq I$, contradicting $x + \alpha \notin I$;
3. $\alpha > y - x$: then, x would be contained in $-\alpha + I$, since I contains each element bigger than y, but this is absurd;
4. $\alpha = y - x$: in this case,

$$x = \sup(\mathbb{N} \setminus (-\alpha + I)) = \sup(\mathbb{N} \setminus ((-\alpha + I) \cap \mathbb{N})),$$

   so that $(-\alpha + I) \cap \mathbb{N} \subseteq M_x$; since $M_x \subseteq -\alpha + I$, it follows that $(-\alpha + I) \cap \mathbb{N} = M_x = (-\alpha + M_y) \cap \mathbb{N}$. Since $I \neq M_y$, there is a $\beta \in M_y \setminus I$; if $\beta > \alpha$, then

$$-\alpha + \beta \in [(-\alpha + M_y) \cap \mathbb{N}] \setminus [(-\alpha + I) \cap \mathbb{N}],$$

   against the hypothesis. Thus $\alpha > \beta$; this means that $y > y - \beta > y - \alpha = x$, and thus $y - \beta \in S$. But this contradicts the fact that $\beta \in M_y$ while $y \notin M_y$.

We are now ready to prove (a). Since Λ is a nonempty antichain of $\mathcal{Q}_y \setminus \{M_y\}$, we have $x \in M_x^{*_\Lambda} = \bigcap_{I \in \Lambda} M_x^{*_I}$. If Δ does not contain M_y, then by Claim 2 we have $x \in M_x^{*_\Delta}$, while $M_x^{*_{\Lambda \cup \{M_x\}}} = M_x$; assume now that $M_y \in \Delta$. Then, M_y is $*$-bigger than M_x and than every $I \in \mathcal{Q}_y \setminus \{M_y\}$, and thus M_y is not $*_I$-closed for every $I \in \Lambda \cup \{M_x\}$. Since M_y is an atom, it follows that M_y is not $*_{\Lambda \cup \{M_x\}}$-closed, while it is $*_\Delta$-closed. Therefore, $*_{\Lambda \cup \{M_x\}} \neq *_\Delta$.

To show (b) we can proceed like in the proof of Proposition 1.47(d), using the fact that $y \in I^{*_{M_x}}$ for every $I \in \mathcal{Q}_y$. $\square$

We end this section by using the methods we developed to calculate the number of star operations in two particular cases.

Example 1.54. The star operations of $S := \langle 4, 5, 6, 7 \rangle = \{0, 4, \rightarrow\}$.

The ideals of $\mathcal{F}_0(S)$ are in the form $S \cup A$, where $A \subseteq \{1, 2, 3\}$, and every such A is acceptable. Moreover, $S \cup A$ is divisorial if and only if $A = \emptyset$ or $A = \{1, 2, 3\}$. To ease the notation, we set $I(a) := S \cup \{a\}$ and $I(a, b) := S \cup \{a, b\}$.

Since $I^v = \mathbb{N}$ if $I \in \mathcal{F}_0(S)$ and I is not divisorial, every ideal of $\mathcal{G}_0(S)$ belongs to $\mathcal{Q}_a$, for some a: to be specific,

- $\mathcal{Q}_3 = \{I(1, 2), I(1), I(2)\}$;
- $\mathcal{Q}_2 = \{I(1, 3), I(3)\}$;
- $\mathcal{Q}_1 = \{I(2, 3)\}$.

Since $M_a = \mathbb{N} \setminus \{a\}$, we have $I(1, 2) = M_3$, $I(1, 3) = M_2$ and $I(2, 3) = M_1$. Hence, $I(1, 2)$ is the maximum of $\mathcal{G}_0(S)$ and $I(1, 2) \geq_* I(1, 3) \geq_* I(2, 3)$. Since $I(3) = I(2, 3) \cap I(1, 3)$, we also have $I(1, 3) \leq_* I(3)$. If I is equal either to $I(2, 3)$ or to $I(3)$, and $0 \in -a + I$, then either $a = 0$ or $\mathbb{N} \subseteq -a + I$; therefore, $I(2, 3)$ and $I(3)$ are minimal elements of $(\mathcal{G}_0, \leq_*)$.

By Proposition 1.47, $I(1)$ and $I(2)$ are not $*$-comparable. If $(-a + I(1)) \cap \mathbb{N} \in \mathcal{G}_0(S)$, then a is equal either to 0 or to 1; therefore $I(3) \leq_* I(1)$, and since $I(1) \cap I(3) = S$ there are no other $*_{I(1)}$-closed ideals. In the same way, the unique $*_{I(2)}$-closed ideals in $\mathcal{G}_0(S)$ are $I(2)$ and $I(2, 3)$. The last ideal to be considered is $I(1, 3)$. By the proof of Proposition 1.51, $I(1, 3)$ is not $*$-bigger than $I(1)$ and $I(2)$ and, by the above reasoning, nor is $*$-minor than them. In conclusion, we get the Hasse diagram of $(\mathcal{G}_0(S), \leq_*)$, which is pictured in Figure 1.1.

Every $I(a)$ is in $\mathcal{Q}_b$, for some b, and $|M_b \setminus I(a)| = 1$; therefore, applying Proposition 1.43, every principal star operation is prime, and by Proposition 1.39 the number of star operations on S is equal to the number of antichains of $(\mathcal{G}_0(S), \leq_*)$. Counting, we see that $\mathcal{G}_0(S)$ contains 7 antichains with two or more elements: adding 6 principal star operations and the empty antichain (corresponding to the v-operation), we get $|\mathrm{Star}(S)| = 14$.

Example 1.55. The star operations of $S := \langle 4, 5, 11 \rangle = \{0, 4, 5, 8, \rightarrow\}$.

Let $I(a_1, \ldots, a_k)$ be the ideal $S \cup \{a_1, \ldots, a_k\}$. Define also $\rho_\alpha(I)$ as the ideal $(-\alpha + I) \cap \mathbb{N}$. There are 13 ideals in $\mathcal{F}_0(S)$:

$$\begin{tikzcd}
 & I(1,2) \arrow[dl] \arrow[d] \arrow[dr] & \\
I(2) \arrow[dr] & I(1,3) \arrow[d] \arrow[dr] & I(1) \arrow[d] \\
 & I(2,3) & I(3)
\end{tikzcd}$$

Figure 1.1: Hasse diagram of $\mathcal{G}_0(\langle 4,5,6,7\rangle)$.

- S
- $I(1,6)$
- $I(1,2,6,7)$
- $I(1,2,3,6,7) = \mathbb{N}$
- $I(1,3,6,7)$
- $I(1,6,7)$
- $I(2,6,7)$
- $I(2,3,6,7)$
- $I(3,7)$
- $I(3,6,7)$
- $I(6)$
- $I(6,7) = (S - M)$
- $I(7)$

We have $\rho_4(S) = I(1,6,7)$ and $\rho_5(S) = I(3,6,7)$; thus, they are divisorial ideal, and so is their intersection $I(6,7) = (S - M)$. Since $\rho_l(I(6,7)) = \mathbb{N}$ if $l > 0$ and $0 \in \rho_l(I(6,7))$, these (along with S and $\mathbb{N}$) are all the divisorial ideals of $\mathcal{F}_0(S)$. From this (and from the fact that I^v is the smallest divisorial ideal containing I) we see that all the ideals belong to some $\mathcal{Q}_a$: we have, respectively,

- $\mathcal{Q}_7 = \{I(1,6) = M_7, I(6)\}$;
- $\mathcal{Q}_6 = \{I(3,7) = M_6, I(7)\}$;
- $\mathcal{Q}_3 = \{I(1,2,6,7) = M_3, I(2,6,7)\}$;
- $\mathcal{Q}_2 = \{I(1,3,6,7)\}$;
- $\mathcal{Q}_1 = \{I(2,3,6,7)\}$.

By Lemma 1.14(c), we have $I(1,6) \geq_* I(3,7) \geq_* I(1,2,6,7) \geq_* I(1,3,6,7) \geq_* I(1,2,6,7)$; thus, we only need to establish the position of $I(6)$, $I(7)$ and $I(2,6,7)$.

Start from $I(2,6,7)$. Since $I(2,6,7) \in \mathcal{Q}_3$, $I(1,2,6,7) = M_3 \geq_* I(2,6,7)$; by Proposition 1.53, $I(2,6,7)$ is not $*$-comparable with $M_2 = I(1,3,6,7)$. Moreover, $\rho_2(I(2,6,7)) = I(2,3,6,7)$, so that $I(2,6,7) \geq_* I(2,3,6,7)$.

In the same way, we see that $I(3,7) \geq_* I(7) \geq_* I(1,3,6,7)$ and $I(6)$ and $I(3,7)$ are incomparable, and that $I(1,6) \geq_* I(6) \geq_* I(1,2,6,7)$ and $I(7)$ and $I(1,2,6,7)$ are incomparable; thus, we only need to determine the $*$-relations between $I(6)$, $I(7)$ and $I(2,6,7)$. Clearly $I(6) \geq_* I(2,6,7)$ since $I(6) \geq_* I(1,2,6,7) \geq_* I(2,6,7)$.

Suppose I is $*_{I(6)}$-closed: then $I = I^v \cap \bigcap(-\alpha + I) = I^v \cap \bigcap \rho_\alpha(I)$ for $\alpha \in (I - I(6))$. However, $\min(I(6) \setminus \{0\}) = 4$, and $\rho_4(I(6)) = I(1,2,6,7)$; since $\rho_{a+b}(I) = \rho_a(\rho_b(I))$, and $\rho_a(I) \in \mathcal{F}_0(S)$ only if $a \in I$, it follows that if I is $*_{I(6)}$ closed then it is either contained in $I(6)$ (i.e., $0 \in (I - I(6))$) or I is also $*_{I(1,2,6,7)}$-closed. Since $I(6) = S \cup \{6\}$ cannot properly contain any element of $\mathcal{G}_0(S)$, it follows that $I(6)$ is not $*$-bigger with $I(7)$,

$$\begin{tikzcd}
 & I(6) \arrow[dr] & & I(2,6,7) \arrow[dr] & \\
I(1,6) \arrow[r] \arrow[ur] & I(3,7) \arrow[r] \arrow[dr] & I(1,2,6,7) \arrow[r] \arrow[ur] & I(1,3,6,7) \arrow[r] & I(2,3,6,7) \\
 & & I(7) \arrow[ur] & &
\end{tikzcd}$$

Figure 1.2: Hasse diagram of $\mathcal{G}_0(\langle 4,5,11\rangle)$.

since $I(7)$ is not comparable with $I(1,2,6,7)$. Moreover, $I(7)$ cannot be $*$-bigger than $I(6)$ by Lemma 1.50, and so $I(6)$ and $I(7)$ are not $*$-comparable. In the same way, $I(7)$ and $I(2,6,7)$ are not $*$-comparable. In the end, the Hasse diagram of $\mathcal{G}_0(S)$ is the one pictured in Figure 1.2.

Now we see that all the ideals of $\mathcal{G}_0$ are in $\mathcal{G}_1$, with the exception of $I(2,6,7)$; however, $|M_3 \setminus I(2,6,7)| = 1$, so that we can apply Proposition 1.43. Hence, every ideal of S is an atom, and $|\mathrm{Star}(S)| = \omega(\mathcal{G}_0(S))$. A direct computation shows that $\mathcal{G}_0(S)$ has 14 antichains: 8 composed of a single element, the empty antichain and five with more than one ideal: $\{I(3,7), I(7)\}$, $\{I(7), I(1,2,6,7)\}$, $\{I(1,3,6,7), I(2,6,7)\}$, $\{I(6), I(7)\}$ and $\{I(7), I(2,6,7)\}$. Hence, $|\mathrm{Star}(S)| = 14$.

1.3.3. Translating antichains into estimates

The previous section allows much better control over the number of star operations on S than simply counting nondivisorial ideals. To keep track of the antichains we find (and to ensure they really generate different closures) we introduce the following notation.

Definition 1.56. *Let $*$ be a star operation on a numerical semigroup S. Then,* $\mathrm{qm}(*)$ *is defined as the biggest integer x such that there is an $I \in \mathcal{Q}_x$ such that I is $*$-closed; if x does not exist, set* $\mathrm{qm}(*) := 0$. *Moreover, for an integer x, let* $\mathrm{Star}_x(S)$ *is the set of star operations such that* $\mathrm{qm}(*) = x$.

Lemma 1.57. *Let S be a numerical semigroup and let $\Delta \subseteq \mathcal{G}_0(S)$ be nonempty.*

(a) If $* \in \mathrm{Star}(S)$, then either $\mathrm{qm}(*) = 0$ or $\mathrm{qm}(*) \in \mathbb{N} \setminus S$.
(b) If $\Delta \subseteq \bigcup_{x\in X} \mathcal{Q}_x$ for some set X, then $\mathrm{qm}(*_\Delta) = \max\{x \in X \mid \mathcal{Q}_x \cap \Delta \neq \emptyset\}$.
(c) If $\Delta \subseteq \mathcal{Q}_x$, then $\mathrm{qm}(*_\Delta) = x$.
(d) $\mathrm{qm}(v) = 0$.

Proof. (a) If $x := \mathrm{qm}(*) \neq 0$, then there is an $I \in \mathcal{Q}_x$ such that $I = I^*$; however, $\mathcal{Q}_x$ is nonempty if and only if M_x is nondivisorial (Proposition 1.42) and in particular $x \in \mathbb{N} \setminus S$.

(b) Let $y := \max\{x \mid \mathcal{Q}_x \cap \Delta \neq \emptyset\}$. If $I \in \Delta \cap \mathcal{Q}_y$, then $I = I^*$, so $\mathrm{qm}(*_\Delta) \geq y$; on the other hand, if $J \in \mathcal{Q}_z$ for some $z > y$, then $z \in J^{*_\Delta}$, since $z \in J^v$ (by definition of $\mathcal{Q}_z$) and $z \in (-\alpha + I)$ for any $I \in \mathcal{Q}_x$ with $x < z$ and every $\alpha \geq 0$.

(c) follows directly from the previous point. For (d) it is enough to note that, if $I \in \mathcal{Q}_x$, then by definition $I \neq I^v$. $\square$

To simplify the statement of the next corollary, we say that a nonempty subset $\Lambda \subseteq \mathcal{G}_0(S)$ is *good* if one of the following two conditions holds:

1. Λ is an antichain, with respect to inclusion, of $\mathcal{Q}_y$ (for some $y \in \mathbb{N} \setminus S$);
2. $\Lambda = \Delta \cup \{M_x\}$, where Δ is a nonempty antichain of $\mathcal{Q}_y \setminus \{M_y\}$ with respect to inclusion, and x, y are as in Proposition 1.53.

The next corollary can be seen as the "completion" of Propositions 1.51 and 1.53.

Corollary 1.58. *Let S be a numerical semigroup, and let $\Lambda_1, \Lambda_2 \subseteq \mathcal{G}_0(S)$ be two good sets. If $*_{\Lambda_1} = *_{\Lambda_2}$, then $\Lambda_1 = \Lambda_2$.*

Proof. The case in which both Λ_i are antichains of some $\mathcal{Q}_{y_i}$ is Proposition 1.51.

Suppose $\Lambda_1 = \Delta_1 \cup \{M_x\}$, with $\Delta_1 \subseteq \mathcal{Q}_y \setminus \{M_y\}$; then, by Lemma 1.57(b), $\mathrm{qm}(*_{\Delta_1 \cup \{M_x\}}) = \sup\{x, y\} = y$. Since $*_{\Lambda_1} = *_{\Lambda_2}$, it must be $\mathrm{qm}(*_{\Lambda_2}) = y$; since Λ_2 is good, still by Lemma 1.57, either $\Lambda_2 \subseteq \mathcal{Q}_y$ or $\Lambda_2 = \Delta_2 \cup \{M_x\}$ for some antichain Δ_2 of $\mathcal{Q}_y \setminus \{M_y\}$. By Proposition 1.53, the former case is impossible, while the latter implies $\Delta_2 = \Delta_1$, i.e., $\Lambda_2 = \Lambda_1$. The claim is proved. $\square$

Note, however, that Corollary 1.58 can't be further extended to cover the case of the antichains Δ that are composed of arbitrary ideals in different $\mathcal{Q}_a$. Indeed, let $S := \langle 5, 6, 7, 8, 9 \rangle = \{0, 5, \rightarrow\}$. For every $I \in \mathcal{G}_0(S)$, $I^v = \mathbb{N}$, and thus $\mathcal{G}_0(S) = \mathcal{Q}_4 \cup \mathcal{Q}_3 \cup \mathcal{Q}_2 \cup \mathcal{Q}_1$. However, $S \cup \{4\}$ is not an atom (Corollary 1.46) and so, by Proposition 1.39, there are antichains $\Delta \neq \Lambda$ such that $*_\Delta = *_\Lambda$.

A direct consequence of Definition 1.56 is that $|\mathrm{Star}(S)| = \sum_x |\mathrm{Star}_x(S)|$. Therefore, estimating $|\mathrm{Star}(S)|$ can be done through estimating the size of the various $\mathrm{Star}_x(S)$.

Proposition 1.59. *Let S be a numerical semigroup and let $T(S) = \{\tau_1 < \cdots < \tau_t\}$; let $x, y, a \in \mathbb{N} \setminus S$.*

(a) If $x < y$ and M_x is not divisorial, then $|\mathrm{Star}_y(S)| \geq 2\omega_{\mathrm{i}}(\mathcal{Q}_y) - 3$.
(b) If $i \neq 1, t$, then $|\mathrm{Star}_{\tau_i}(S)| \geq 2\omega(i-1) - 3$.
(c) $|\mathrm{Star}_g(S)| \geq 2\omega(t-1) - 5$.
(d) If $\mu < a < g$ and $g - a \notin S$, then $\omega_{\mathrm{i}}(\mathcal{Q}_a) \geq \omega(t-1)$.
(e) $|\mathrm{Star}_0(S)| \geq 1$.

Proof. (a) The existence of x implies the existence of a $x' \in \mathbb{N} \setminus S$ such that $x' < y$ and all integers between x' and y are in S. We have $\omega_{\mathrm{i}}(\mathcal{Q}_y) - 1$ nonempty antichains (with respect to inclusion) of $\mathcal{Q}_y$, each of which induces a different star operation; by Proposition 1.53 and Corollary 1.58, if $\Lambda \neq \{M_y\}$ is one of these, then $\Lambda \cup \{M_{x'}\}$ gives a new star operation $*$ with $\mathrm{qm}(*) = y$, so we can add other $\omega_{\mathrm{i}}(\mathcal{Q}_y) - 2$ star operations.

(b) Consider the ideals of the form $S \cup \{x \in \mathbb{N} : x > \tau_i\} \cup A$, for $A \subseteq \{\tau_1, \dots, \tau_{i-1}\}$. Since $\tau_i \neq g$, all these are strictly bigger than S and so are not divisorial, and they are un $\mathcal{Q}_{\tau_i}$; therefore, by Proposition 1.47, $\omega_{\mathrm{i}}(\mathcal{Q}_{\tau_i}) \geq \omega(i-1)$. By part (a), $|\mathrm{Star}_{\tau_i}(S)| \geq 2\omega(i-1) - 3$.

(c) We can use the same proof of the previous point, only noting that the antichain composed of $A = \emptyset$ generates the v-operation, which is not in $\mathrm{Star}_g(S)$ but rather in $\mathrm{Star}_0(S)$. In the same way, $\{\emptyset\} \cup \{M_{x'}\}$ generates a star operation in $\mathrm{Star}_{x'}(S)$ rather than a star operation in $\mathrm{Star}_g(S)$.

(d) Suppose $\mu < a$. Let i be such that $\tau_{i-1} < a \leq \tau_i$ (with $\tau_0 := 0$). If $j < i$, define $\eta_j := \tau_j$. If $j > i$, define $\eta_j := \tau_j - k_j\mu$, where $k_j \in \mathbb{N}$ is such that $a - \mu < \tau_j - k_j\mu < a$. For every $A \subseteq \{\eta_1, \dots, \eta_t\}$, the set $I_A := A \cup S \cup \{x \in \mathbb{N} : x > a\}$ is an ideal, $I_A \in \mathcal{Q}_a$ and $I_A \subseteq I_B$ if and only if $A \subseteq B$; therefore, $\omega_{\mathrm{i}}(\mathcal{Q}_a) \geq \omega(t-1)$.

(e) follows from the fact that $v \in \mathrm{Star}_0(S)$. □

Corollary 1.60. *Let S be a numerical semigroup. Then,*

$$|\mathrm{Star}(S)| \geq 2\left[\sum_{i=1}^{t-1}\omega(i)\right] - 3(t-1). \tag{1.2}$$

Proof. If $t = 2$, then the right hand side of (1.2) is equal to $2\omega(1) - 3 = 3$; since S admits the three (different) star operations v, $*_{M_g}$ and $*_{M_\tau}$, the inequality is proved.

Suppose $t > 2$ and let $T(S) := \{\tau_1, \dots, \tau_t = g\}$. If $1 < i < t$, then by the previous proposition we have $|\mathrm{Star}_{\tau_i}(S)| \geq 2\omega(i-1) - 3$, while $|\mathrm{Star}_{\tau_t}(S)| \geq 2\omega(t-1) - 5$. Moreover, $\mathrm{Star}_{\tau_1}(S)$ and $\mathrm{Star}_0(S)$ are nonempty, so that

$$|\mathrm{Star}(S)| \geq \sum_x |\mathrm{Star}_x(S)| \geq |\mathrm{Star}_0(S)| + |\mathrm{Star}_{\tau_1}(S)| + |\mathrm{Star}_g(S)| + \sum_{i=2}^{t-1}|\mathrm{Star}_{\tau_i}(S)| \geq$$

$$\geq 2 + 2\omega(t-1) - 5 + \sum_{i=2}^{t-1}(2\omega(i-1) - 3) = 2\omega(t-1) - 3 + \sum_{i=1}^{t-2}(2\omega(i) - 3).$$

After a rearrangement, we obtain our claim. □

The proof above shows that the previous corollary does not give an useful estimate in the case $t = 2$. However, yet when $t = 3$ we get

$$|\mathrm{Star}(S)| \geq 2(\omega(2) + \omega(1)) - 3 \cdot 2 = 2(6+3) - 6 = 12,$$

and when $t = 4$ we already have $|\mathrm{Star}(S)| \geq 49$.

Corollary 1.61. *Let S be a numerical semigroup, and let $t := t(S)$. If $\tau > \mu$ for every $\tau \in T(S)$, then*

$$|\mathrm{Star}(S)| \geq (2t-1) \cdot \omega(t-1) - 3t + 1.$$

Proof. Let $T(S) := \{\tau_1, \dots, \tau_t = g\}$, with τ_1 being the smallest element. As in the proof of Corollary 1.60, we have

$$|\mathrm{Star}(S)| \geq |\mathrm{Star}_0(S)| + \sum_{i=1}^{t} |\mathrm{Star}_{\tau_i}(S)|.$$

Clearly, $|\mathrm{Star}_0(S)| \geq 1$, while $|\mathrm{Star}_g(S)| \geq 2\omega(t-1) - 5$ by Proposition 1.59(c). If $i \neq t$, then by Proposition 1.59(d) we have $\omega_{\mathrm{i}}(\mathcal{Q}_{\tau_i}) \geq \omega(t-1)$; hence, $|\mathrm{Star}_{\tau_1}(S)| \geq \omega(t-1)-1$ by Proposition 1.47(d). On the other hand, if $i \neq 1$, then Proposition 1.59(a) implies that $|\mathrm{Star}_{\tau_i}(S)| \geq 2\omega_{\mathrm{i}}(\mathcal{Q}_{\tau_i}) - 3 \geq 2\omega(t-1) - 3$. Therefore,

$$|\mathrm{Star}(S)| \geq 1 + [\omega(t-1) - 1] + [2\omega(t-1) - 5] + (t-2)[2\omega(t-1) - 3] =$$

$$= (1 + 2 + 2t - 4)\omega(t-1) - 5 - 3t + 6 = (2t-1)\omega(t-1) - 3t + 1.$$

The claim is proved. □

1.4. Estimates in μ

Theorem 1.25 and Proposition 1.59 are two different methods with which we can estimate the number of star operations on a numerical semigroup S: the former is quite simple, and can be applied uniformly to every semigroup, while the latter, when applied to any given semigroup, yields better estimates. In this section we develop a third way to estimate $\mathrm{Star}(S)$: while it is not as powerful as Theorem 1.25 (for example, it is not enough to prove, alone, Theorem 1.26), it can be applied in a more general way than Proposition 1.59. In particular, we find a very good bound on the multiplicity $\mu(S)$, depending on the cardinality of $\mathrm{Star}(S)$.

We need to consider separately two cases: when we can find a hole $a < \mu$ and when we can find a hole $a > \mu$. Note that the two cases are not mutually exclusive.

Proposition 1.62. *Let S be a numerical semigroup; suppose there is a positive integer a such that $a < \mu$ and $g - a \notin S$. Then:*

(a) $\omega_{\mathrm{i}}(\mathcal{Q}_a) \geq \omega(a-1)$;
(b) if $a < s < \mu$, then $\omega_{\mathrm{i}}(\mathcal{Q}_s) \geq \omega(s-2)$.

Note that, if $a < \mu$, then $a \notin S$ and $g - a \in T(S)$.

Proof. (a) Define $I := \{0\} \cup \{x \in \mathbb{N} : x > a\}$. For each subset $A \subseteq \{1, \dots, a-1\}$, $I \cup A$ is a nondivisorial ideal of S, and it belongs to $\mathcal{Q}_a$. Hence, $\mathcal{Q}_a$ has at least $\omega(a-1)$ antichains, with respect to inclusion.

(b) Let $s \in \mathbb{N}$ such that $a < s < \mu$, and define $A_s := \{1, \dots, s-1\} \setminus \{s-a\}$ and $I_s := S \cup \{x \in \mathbb{N} : x > s\}$. We claim that, for every $B \subseteq A_s$, the ideal $J := I_s \cup B \cup \{s-a\}$ belongs to $\mathcal{Q}_s$.

Indeed, suppose $s \notin J^v$. Then, there is a $\gamma \in \mathbb{N}$ such that $J \subseteq -\gamma + S$ but $s \notin -\gamma + S$. In particular, since $s = \sup(\mathbb{N} \setminus S)$, it must be $\gamma = g - s$; thus, $-\gamma + (g-a) = s - a \notin -\gamma + S$. However, this would imply $J \nsubseteq -\gamma + S$, against the hypothesis. Therefore, $J \in \mathcal{Q}_s$. Hence, $\omega(\mathcal{Q}_s) \geq \omega(s-2)$. □

Next we turn to the case $a > \mu$.

Lemma 1.63. *Let S be a numerical semigroup and let $\mu = \mu(S)$. For every $a \in \mathbb{N} \setminus S$, let $B_a := \{n \in \mathbb{N} : a - \mu \leq n < a\}$ and $B'_a := B_a \setminus \{a - \mu\}$. Suppose that $\mu < a \leq g/2$.*

(a) $|B_a \setminus S| \geq \left\lceil \dfrac{\mu}{2} \right\rceil$.

(b) If $g - a \notin S$, then $|B'_a \setminus S| \geq \left\lceil \dfrac{\mu - 1}{2} \right\rceil$.

Proof. Let $a \notin S$, $\mu < a \leq g/2$. For every integer m, let $[m]_\mu^{B_a}$ be the (necessarily unique) element of B_a congruent to m modulo μ: the existence of $[m]_\mu^{B_a}$ is guaranteed since B_a is a complete system of residues modulo μ. Define

$$\begin{aligned} \phi\colon B_a &\longrightarrow B_a \\ x &\longmapsto [g - x]_\mu^{B_a}. \end{aligned}$$

The map ϕ is well-defined, and it is a bijection since $g - x \not\equiv g - y \bmod \mu$ whenever $x \not\equiv y \bmod \mu$, and in particular if $x, y \in B_a$ and $x \neq y$.

If now $x \in S \cap B_a$, then $g - x \notin S$; but since $a \leq g/2$, we have $g - x > g/2 \geq \phi(x)$, and thus $\phi(x) = g - x - k\mu$ for some $k \in \mathbb{N}$ (depending on x). Hence, $\phi(x) \notin S$, that is, $\phi(B_a \cap S) \subseteq B_a \setminus S$. In particular, $|B_a \cap S| \leq |B_a \setminus S|$, and thus $|B_a \setminus S| \geq \frac{|B_a|}{2} = \frac{\mu}{2}$.

Suppose $g - a \notin S$. Since $B_a \setminus S = (B'_a \setminus S) \cup \{a - \mu\}$, we have $\phi(B'_a \cap S) \subseteq (B'_a \setminus S) \cup \{a - \mu\}$. If $\phi(x) = a - \mu$, then $g - x \equiv a - \mu \bmod \mu$, and thus $x \equiv g - a \bmod \mu$. Since $g - a \geq g/2$ and $g - a \notin S$, then $x \notin S$, and thus $\phi(B'_a \cap S) \subseteq B'_a \setminus S$. In particular, $|B'_a \cap S| \leq |B'_a \setminus S|$, and thus $|B'_a \setminus S| \geq \frac{|B'_a|}{2} = \frac{\mu - 1}{2}$. □

Proposition 1.64. *Let S be a numerical semigroup, and let $\nu := \left\lceil \frac{\mu - 1}{2} \right\rceil$; let $a \leq g/2$ be a positive integer such that $a, g - a \notin S$.*

(a) If $a > \mu$, then $\omega_{\mathrm{i}}(\mathcal{Q}_a) \geq \omega(\nu)$.

(b) If $a > 2\mu$, then $\omega_{\mathrm{i}}(\mathcal{Q}_a) \geq 2\omega(\nu) - 2$.

Proof. With the notation of Lemma 1.63, let $X := B'_a \setminus S = \{x_1, \ldots, x_\eta\}$; we have $\eta \geq \nu$.

(a) Each set $A \subseteq X$ generates an ideal $S \cup \{x \in \mathbb{N} : x > a\} \cup A$, and all of these are in $\mathcal{Q}_a$ (since $g - a \notin S$). Thus, the number of antichains in $\mathcal{Q}_a$, with respect to inclusion, is at least $\omega(\eta) \geq \omega(\nu)$.

(b) For every $x_i \in X$, we have $x_i > \mu$, since $a > 2\mu$. Let $y_i := a - x_i$; then, $y_i < \mu$, so that $y_i \notin S$ and $X \cap Y = \emptyset$. Let $Y := \{y_1, \ldots, y_\eta\}$ and let $I := S \cup \{x \in \mathbb{N} : x > a\}$. For each $A \subseteq X$ (respectively, $A \subseteq Y$), $I_A := I \cup (A + S)$ is an ideal not containing a, and thus (by Lemma 1.19, since $g - a \notin S$) $I_A \in \mathcal{Q}_a$; moreover, $I_A \cap X = A$ (resp., $I_A \cap Y = A$), so that if $I_A \subseteq I_B$ then $A \subseteq B$.

Therefore, each antichain of the power set of X, and each antichain of the power set of Y (both with respect to inclusion), give rise to an antichain of $\mathcal{Q}_a$ (with respect to inclusion). Moreover, the empty antichain and the antichain composed of the empty set belong to both power sets, while all the others are different; therefore, $\omega_{\mathrm{i}}(\mathcal{Q}_a) \geq 2\omega(\eta) - 2 \geq 2\omega(\nu) - 2$. □

Putting together Propositions 1.62 and 1.64, we get a quite powerful estimate.

Proposition 1.65. *For every $\epsilon > 0$ there is an integer $n_0(\epsilon)$ such that, for every $n \geq n_0(\epsilon)$, if S is a nonsymmetric numerical semigroup such that $|\mathrm{Star}(S)| \leq n$, then*

$$\mu(S) \leq \left[\frac{2}{\log(2)} + \epsilon\right] \log\log(n). \tag{1.3}$$

Proof. Let S be a nonsymmetric semigroup; then, there is a x such that $x, g - x \notin S$. If $x < \mu$ we have $|\mathrm{Star}(S)| \geq \omega(\mu - 3)$ (by Proposition 1.62), while if $x > \mu$, we have $|\mathrm{Star}(S)| \geq \omega(\nu)$ (by Proposition 1.64, where $\nu := \left\lceil \frac{\mu-1}{2} \right\rceil$).

The quantity on the right hand side of (1.3) goes to infinity; therefore, for large n, we can restrict ourselves to $\mu(S) \geq 5$, so that $\nu \leq \mu - 3$ and $|\mathrm{Star}(S)| \geq \omega(\nu)$.

For any integer k, no two different subsets of $\{1, \dots, k\}$ of cardinality $\lceil k/2 \rceil$ are comparable; therefore, every family of such subsets is an antichain of $\mathcal{P}(\{1, \dots, k\})$. Hence,

$$\log_2 \omega(k) \geq \binom{k}{\lceil k/2 \rceil}.$$

For large a, the binomial coefficient $\binom{2a}{a}$ is asymptotic to $\frac{2^{2a}}{\sqrt{\pi a}}$; in particular, for every ϵ_0 and large enough a (where "large enough" depends on ϵ_0) we have $\binom{2a}{a} > 2^{a(2-\epsilon_0)}$. Thus, for every ϵ_1 there is a ν_0 such that, if $\nu \geq \nu_0$, we have

$$\log_2(\omega(\nu)) \geq 2^{\frac{\nu}{2}(2-2\epsilon_1)} = 2^{\nu(1-\epsilon_1)}.$$

Fix an ϵ, and take an $\epsilon_1 < \frac{\epsilon}{A+\epsilon}$, where $A := \frac{2}{\log(2)}$; find ν_0 as above, let $n_0' := \omega(\nu_0)$, and take a $n \geq n_0'$. Moreover, choose the maximal $\overline{\mu}$ such that $n \geq \omega\left(\left\lceil \frac{\overline{\mu}-1}{2} \right\rceil\right)$, so that $\overline{\nu} := \left\lceil \frac{\overline{\mu}-1}{2} \right\rceil \geq \nu_0$. For any semigroup S such that $|\mathrm{Star}(S)| \leq n$, we must have $\mu(S) \leq \overline{\mu}$ and $\nu(S) \leq \overline{\nu}$. Hence,

$$\log_2(n) \geq \log_2(\omega(\overline{\nu})) \geq 2^{\overline{\nu}(1-\epsilon_1)}$$

i.e., $\log(n) \geq \log(2) \cdot 2^{\overline{\nu}(1-\epsilon_1)}$. Taking logarithms,

$$\log\log(n) \geq \log(2) \cdot \left[\log_2(\log(2) \cdot 2^{\overline{\nu}(1-\epsilon_1)})\right] = \log\log(2) + \log(2)(\overline{\nu}(1-\epsilon_1)).$$

Isolating $\overline{\nu}$, we have

$$\overline{\nu} \leq \frac{1}{(1-\epsilon_1)\log(2)}(\log\log(n) - \log\log(2)),$$

and substituting $\overline{\nu}$ with $\frac{\overline{\mu}-1}{2}$ we have

$$\mu(S) \leq \overline{\mu} \leq \frac{2}{\log(2)(1-\epsilon_1)}(\log\log(n) - \log\log(2)) + 1 =$$

$$= \frac{A}{1-\epsilon_1} \log\log(n) + \left[1 - \frac{A}{1-\epsilon_1} \log\log(2)\right].$$

The inequality $\epsilon_1 < \frac{\epsilon}{A+\epsilon}$ implies that

$$\epsilon > \epsilon_1(A+\epsilon) \Longrightarrow \epsilon > \frac{\epsilon_1 A}{1-\epsilon_1};$$

therefore,

$$A+\epsilon-\frac{A}{1-\epsilon} > A+\frac{\epsilon_1 A}{1-\epsilon_1}-\frac{A}{1-\epsilon} = A+\frac{\epsilon_1-1}{1-\epsilon_1}A = 0,$$

or equivalently $A+\epsilon > \frac{A}{1-\epsilon}$. Hence, there is a $n_0 \geq n_0'$ such that, whenever $n \geq n_0$, we have

$$(A+\epsilon)\log\log(n) \geq \frac{A}{1-\epsilon_1}\log\log(n) + \left[1-\frac{A}{1-\epsilon_1}\log\log(2)\right].$$

In particular, for $n \geq n_0$, we have

$$\mu(S) \leq (A+\epsilon)\log\log(n) = \left[\frac{2}{\log(2)}+\epsilon\right]\log\log(n),$$

as claimed. □

Remark 1.66. The fact that Proposition 1.65 is an asymptotic result, with the presence of the unknown quantity $n_0(\epsilon)$, prevents the possibility of finding an explicit bound for $\mu(S)$. On the other hand, since $\omega(a) \geq 2^a$ for all integers a, we could have avoided the use of ϵ by employing the inequality

$$|\mathrm{Star}(S)| \geq 2^{\frac{\mu-1}{2}}, \quad \text{that is,} \quad \mu(S) \leq 2\log_2|\mathrm{Star}(S)|+1, \tag{1.4}$$

which is valid for $\mu \geq 5$. This is a worse result than the one obtained above, but allows to have an explicit bound (which, for small n, can be also fine-tuned using actual values of $\omega(a)$).

We shall come back to finding explicit versions of Theorem 1.26 in Section 1.7.

1.5. The case of multiplicity 3

1.5.1. The graphical representation

This section will deal exclusively with semigroups of multiplicity 3. The following trivial observation is the basis of all our method.

Proposition 1.67. *Let S be a numerical semigroup of multiplicity 3, and I a fractional ideal of S. Then, there are uniquely determined $a,b,c \in \mathbb{Z}$ such that $I = (3a+1+3\mathbb{N}) \cup (3b+2+3\mathbb{N}) \cup (3c+3\mathbb{N})$. If $I \in \mathcal{F}_0(S)$, then $c = 0$.*

Proof. Since I is a fractional ideal of S, I is bounded below. Let a',b',c' be the minimal element of I congruent (respectively) to 1, 2 and 0 modulo 3: defining a,b,c as the integers such that $a' = 3a+1$, $b' = 3b+2$ and $c' = 3c$ we obtain what we need, since $3 \in S$ implies that if $x \in I$ then also $x+3 \in I$. If moreover $I \in \mathcal{F}_0(S)$, then $0 \in I$, so that $c \leq 0$, but $I \subseteq \mathbb{N}$, and thus $c \geq 0$. □

In particular, the above proposition applies when $I = S$: in this case, we use α and β instead of a and b, that is, we will put $S = (3\alpha + 1 + 3\mathbb{N}) \cup (3\beta + 2 + 3\mathbb{N}) \cup 3\mathbb{N}$. In particular, we have $S = \langle 3, 3\alpha + 1, 3\beta + 2\rangle$.

Let $I \in \mathcal{F}_0(S)$. If $I = (3a + 1 + 3\mathbb{N}) \cup (3b + 2 + 3\mathbb{N}) \cup 3\mathbb{N}$, then we set $[a, b] := I$. We note that $\mathbb{N} = [0, 0]$ and $S = [\alpha, \beta]$.

Proposition 1.68. *Let $S = \langle 3, 3\alpha + 1, 3\beta + 2\rangle$ be a numerical semigroup of multiplicity 3, and suppose that $\alpha \le \beta$.*

(a) If $I = [a, b] \in \mathcal{F}_0(S)$, then $0 \le a \le \alpha$, $0 \le b \le \beta$ and $-\alpha \le b - a \le \alpha$.
(b) Conversely, if a, b are integers, $0 \le a \le \alpha$, $0 \le b \le \beta$ and $b - a \le \alpha$, then $I = [a, b]$ for some $I \in \mathcal{F}_0(S)$.

Proof. (a) Suppose $I = [a, b]$. Since $I \subseteq \mathbb{N}$, $a, b \ge 0$ and, since $S \subseteq I$, we have $3\alpha + 1, 3\beta + 2 \in I$, and thus $a \le \alpha$, $b \le \beta$. In particular, $b - a \ge 0 - \alpha = -\alpha$. If $b - a > \alpha$, then

$$3a + 1 + 3\alpha + 1 = 3(a + \alpha) + 2 < 3(a + b - a) + 2 < 3b + 2$$

and thus $3a+1+3\alpha+1 \notin I$, while we should have $3a+1+3\alpha+1 \in 3a+1+S \subseteq I+S \subseteq I$. Hence $b - a \le \alpha$.

(b) Let $I := (3a + 1 + 3\mathbb{N}) \cup (3b + 2 + 3\mathbb{N}) \cup \mathbb{N}$; we have to prove that I is indeed an ideal, and to do this it is enough to show that $I + 3$, $I + 3\alpha + 1$ and $I + 3\beta + 2$ belong to I. Clearly $I + 3 \subseteq I$; for $3\alpha + 1$, note that

$$3b + 2 + 3\mathbb{N} + 3\alpha + 1 = 3(b + \alpha + 1) + 3\mathbb{N} \subseteq S$$

since $b + \alpha + 1 \ge \alpha + 1 \ge 0$, while $3\alpha + 1 + 3\mathbb{N} \subseteq I$ since $a \ge \alpha$. Moreover,

$$3a + 1 + 3\mathbb{N} + 3\alpha + 1 = 3(a + \alpha) + 2 + 3\mathbb{N} \subseteq I$$

since $a + \alpha \ge a + b - a = b$. Analogously, $3a + 1 + 3\mathbb{N} + 3\beta + 2 \subseteq I$ and $3\mathbb{N} + 3\beta + 2 \subseteq I$, while

$$3b + 2 + 3\mathbb{N} + 3\beta + 2 = 3(b + \beta + 1) + 1 + 3\mathbb{N} \subseteq I$$

since $b + \beta + 1 \ge \beta \ge \alpha \ge a$. □

Symmetrically, we have:

Proposition 1.69. *Let $S = \langle 3, 3\alpha + 1, 3\beta + 2\rangle$ be a numerical semigroup of multiplicity 3, and suppose that $\alpha \ge \beta$.*

1. *If $I = [a, b] \in \mathcal{F}_0(S)$, then $0 \le a \le \alpha$, $0 \le b \le \beta$ and $-\beta \le a - b \le \beta + 1$.*
2. *Conversely, if a, b are integers, $0 \le a \le \alpha$, $0 \le b \le \beta$ and $a - b \le \beta + 1$, then $I = [a, b]$ for some $I \in \mathcal{F}_0(S)$.*

Proof. It is enough to repeat the proof of Proposition 1.68. □

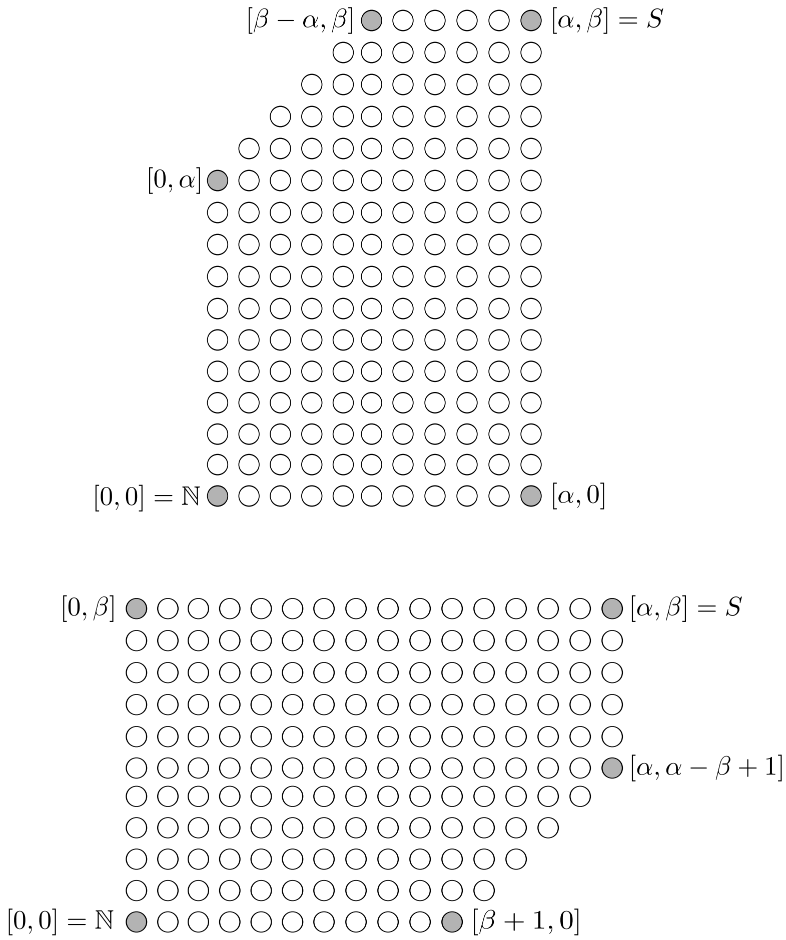


Figure 1.3: Graphical representation of the ideals of a semigroup of multiplicity 3: above, the case $\alpha \leq \beta$; below, the case $\alpha \geq \beta$.

Suppose S is a numerical semigroup of multiplicity 3. If $I = [a,b] \in \mathcal{F}_0(S)$, then we can represent I by the point (a,b) in the lattice $\mathbb{Z}\times\mathbb{Z}$ of the integral points of the plane, and Propositions 1.68 and 1.69 determines the image of $\mathcal{F}_0(S)$: firstly, the bounds $0 \leq a \leq \alpha$ and $0 \leq b \leq \beta$ shows that it will be contained in the rectangle whose vertices are $[0,0]$, $[0,\beta]$, $[\alpha,0]$ and $[\alpha,\beta]$. Moreover, since each “ascending” diagonal (i.e., each diagonal going from the lower left to the upper right of the rectangle) is characterized by the quantity $b-a$, we see that if $\alpha \leq \beta$ then the image of $\mathcal{F}_0(S)$ will lack the upper left corner of the rectangle (the points with $b-a > \alpha$) while if $\alpha \geq \beta$ then we have to “cut” the lower right corner. In the case $\alpha = \beta$, $\mathcal{F}_0(S)$ will be represented by the whole rectangle (that will, indeed, be a square). Thus, $\mathcal{F}_0(S)$ will be represented by a polygon vaguely similar to a trapezoid, like the one showed in Figure 1.3; we will often identificate an ideal with the point corresponding to it in this graphical representation.

Proposition 1.70. *Let S be a numerical semigroup of multiplicity 3 and let $[a,b]$, $[a',b']$ be ideals in $\mathcal{F}_0(S)$. Then:*

(a) $[a,b] \subseteq [a',b']$ *if and only if* $a \geq a'$ *and* $b \geq b'$*;*

(b) $[a,b] \cap [a',b'] = [\max\{a,a'\}, \max\{b,b'\}]$;
(c) $[a,b] \cup [a',b'] = [\min\{a,a'\}, \min\{b,b'\}]$.

Proof. Straightforward. □

Definition 1.71. *Let $S = \langle 3, 3\alpha+1, 3\beta+2\rangle$.*

- *Σ^0 is the ascending diagonal that contains $S = [\alpha,\beta]$, i.e., the diagonal such that $b-a=\beta-\alpha$.*
- *$\Sigma^+ := \{[a,b] \in \mathcal{F}_0(S) : b-a > \beta-\alpha\}$*
- *$\Sigma^- := \{[a,b] \in \mathcal{F}_0(S) : b-a < \beta-\alpha\}$*

The notation Σ^+ and Σ^- is chosen to highlight the position of the two sets in the graphical representation.

Lemma 1.72. *Let S be a numerical semigroup of multiplicity 3. The sets Σ^+, Σ^-, Σ^0, $\Sigma^+ \cup \Sigma^0$ and $\Sigma^- \cup \Sigma^0$ are closed by intersections.*

Proof. Σ^0 is linearly ordered, so this case is trivial.

Let $[a,b],[a',b'] \in \Sigma^+$, and suppose without loss of generality $a \le a'$, $b \ge b'$ (if $b \le b'$, then $[a,b] \supseteq [a',b']$). Then $[a,b] \cap [a',b'] = [a,b']$, and $b'-a \ge b'-a' > \beta-\alpha$, and thus $[a,b'] \in \Sigma^+$.

For Σ^-, in the same way, if $[a,b] \cap [a',b'] = [a,b']$, then $b'-a \le b-a < \beta-\alpha$ and $[a,b'] \in \Sigma^-$.

If $[a,b] \in \Sigma^+$ and $[a',b'] \in \Sigma^0$, then $b' = a'+\beta-\alpha$ and $b > a+\beta-\alpha$; hence $\min\{b,b'\} \ge \min\{a,a'\}+\beta-\alpha$ and $[a,b] \cap [a',b'] \in \Sigma^+ \cap \Sigma^0$.

Analogously, if $[a,b] \in \Sigma^-$ and $[a',b'] \in \Sigma^0$, then $\min\{b,b'\} \le \min\{a,a'\}+\beta-\alpha$ and $[a,b] \cap [a',b'] \in \Sigma^- \cap \Sigma^0$. □

1.5.2. Shifting ideals

Definition 1.73. *If $I \in \mathcal{F}_0(S)$ and $k \in I$, the k-shift of I, denoted by $\rho_k(I)$, is the ideal $(I-k) \cap \mathbb{N}$.*

It is clear that, if $\rho_k(I)$ is defined, then it is contained in $\mathcal{F}_0(S)$, since 0 is the minimum of $\rho_k(I)$. Since $3k \in S \subseteq I$ for every $k \in \mathbb{N}$, the $3k$-shift (and in particular the 3-shift) is always defined.

It is straightforward to see that, if $a, a+b \in I$, then $\rho_b(\rho_a(I)) = \rho_{a+b}(I)$. Therefore, applying repeatedly the 3-shift, we can always write $\rho_k(I)$ as $\rho_r \circ \rho_3^q(I)$, where $r \in \{0,1,2\}$ is congruent to k modulo 3. Hence, the study of the shifts reduces to the study of ρ_1, ρ_2 and ρ_3.

Lemma 1.74. *Let S be a numerical semigroup of multiplicity 3 and let $I = [a,b]$ be an ideal in $\mathcal{F}_0(S)$.*

(a) $\rho_3(I) = [\max\{0,a-1\}, \max\{0,b-1\}]$; *in particular, if $a,b > 0$, then* $\rho_3(I) = [a-1,b-1]$.

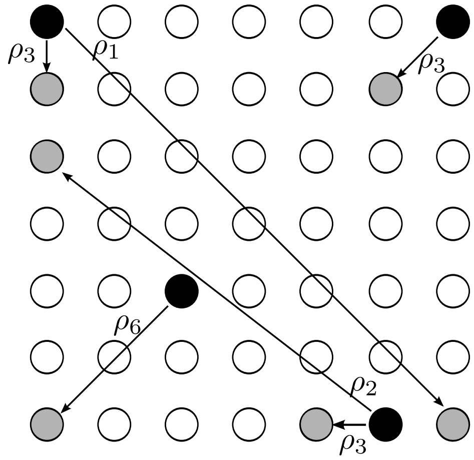


Figure 1.4: Action of the shifts.

(b) *$\rho_1(I)$ is defined if and only if $a = 0$, and in this case $\rho_1(I) = [b, 0]$.*
(c) *$\rho_2(I)$ is defined if and only if $b = 0$, and in this case $\rho_2(I) = [0, a-1]$.*

In terms of the graphical representation, this means that ρ_1 and ρ_2 swap the x-axis $\{[a, 0] : 0 \leq a \leq \min\{\alpha, \beta+1\}\}$ and the y-axis $\{[0, b] : 0 \leq b \leq \min\{\alpha, \beta\}\}$. Meanwhile, ρ_3 moves the ideals one step closer to the origin.

Proof. Write $I = 3\mathbb{N} \cup (3a+1+3\mathbb{N}) \cup (3b+2+3\mathbb{N})$. Then,

- $I - 3 = (-3+3\mathbb{N}) \cup (3(a-1)+1+3\mathbb{N}) \cup (3(b-1)+2+3\mathbb{N})$,
- $I - 1 = 3a\mathbb{N} \cup (3b+1+3\mathbb{N}) \cup (2+3\mathbb{N})$,
- $I - 2 = 3b\mathbb{N} \cup (1+3\mathbb{N}) \cup (3(a-1)+2+3\mathbb{N})$.

If $\rho_1(I)$ (respectively, $\rho_2(I)$) is defined, then we must have $0 \in 3a\mathbb{N}$, and thus $a = 0$ (resp., $0 \in 3b\mathbb{N}$, and thus $b = 0$). The lemma now follows from the definition of $[x, y]$. □

The following proposition is actually a part of Proposition 1.2(d); we state it here separetely to highlight a property we will be using many times.

Proposition 1.75. *Let S be a numerical semigroup of multiplicity 3, $I \in \mathcal{F}_0(S)$, $k \in I$ and $* \in \mathrm{Star}(S)$. If I is $*$-closed, so is $\rho_k(I)$.*

1.5.3. Principal star operations

Proposition 1.76. *Let $S = \langle 3, 3\alpha+1, 3\beta+2\rangle$ be a numerical semigroup of multiplicity 3. Then:*

(a) *if $\alpha \leq \beta$, then $\mathcal{F}^v(S) \cap \mathcal{F}_0(S) = \Sigma^0 \cup \{[a, b] \in \Sigma^- : a \leq \beta - \alpha\}$;*
(b) *if $\alpha \geq \beta$, then $\mathcal{F}^v(S) \cap \mathcal{F}_0(S) = \Sigma^0 \cup \{[a, b] \in \Sigma^+ : b \leq \alpha - \beta - 1\}$.*

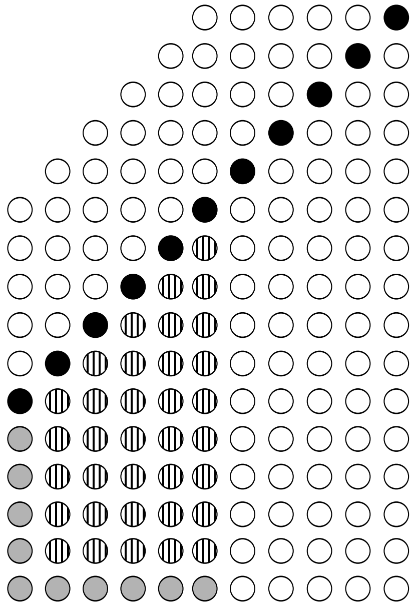


Figure 1.5: Divisorial and nondivisorial ideals. Black circles represent ideals of Σ^0, gray circles other ideals in the form $\rho_x(S)$, striped circles are intersections of black and gray ideals. White circles represent non-divisorial ideals.

Proof. We will prove only the case $\alpha \leq \beta$; the proof for $\alpha \geq \beta$ is entirely analogous.

Let Δ be the set on the right hand side. We will show that Δ verifies the hypotheses of Proposition 1.2(d) (so that $\Delta = \mathcal{F}^*(S) \cap \mathcal{F}_0(S)$ for some star operation $*$), and that each $I \in \Delta$ is divisorial: since $v \geq *$ for every $* \in \mathrm{Star}(S)$, the claim will follow.

If $[a,b] \in \Sigma^0$, then $[a,b] = [\alpha - k, \beta - k] = \rho_{3k}(S)$ for some $k \in \mathbb{N}$, so that $[a,b]$ is divisorial. In particular, $[0, \beta - \alpha] \in \mathcal{F}^v(S)$. Therefore, $[0, \beta - \alpha - x] = \rho_{3x}([0, \beta - \alpha])$ is divisorial for every $x \geq 0$, and so is $[\beta - \alpha - x, 0] = \rho_1([0, \beta - \alpha - x])$. Let $[a,b] \in \Sigma^-$ such that $a \leq \beta - \alpha$. If $b \leq \beta - \alpha$, then $[a,b] = [a,0] \cap [0,b]$ is the intersection of two divisorial ideals; if $b > \beta - \alpha$, then $[a,b] = [a,0] \cap [b - (\beta - \alpha), b]$, and the latter is divisorial since it belongs to Σ^0. Hence $\Delta \subseteq \mathcal{F}^v$.

Let now $[a,b], [a',b'] \in \Delta$; if they are both in Σ^0 they are comparable, and thus the intersection is in Δ. If $[a,b] \in \Sigma^-$, then by Lemma 1.72 its intersection with $[a',b']$ is in $\Sigma^- \cup \Sigma^0$; moreover, $\min\{a,a'\} \leq a \leq \beta - \alpha$, and thus $[a,b] \cap [a',b'] \in \Delta$.

It is clear that $\rho_3(I) \in \Delta$ whenever $I \in \Delta$, since $\rho_3([a,b]) \in \Sigma^0$ if $[a,b] \in \Sigma^0$ and $a > 0$, while $\rho_3([0, \beta - \alpha]) = [0, \beta - \alpha - 1] \in \Delta$; if $[a,b] \in \Delta \setminus \Sigma^0$, then $\rho_3([a,b]) = [\max\{a-1,0\}, \max\{b-1,0\}]$, and $\max\{a-1,0\} \leq a$, so that $\rho_3([a,b]) \in \Delta$.

By the discussion in Section 1.5.2, we only need to show that $\rho_1([0,c]), \rho_2([c,0]) \in \Delta$ if $[0,c]$ or $[c,0]$ are in Δ. However, excluding the case $c = 0$ (which is trivial), we have $\rho_1([0,c]) = [c,0]$ and $\rho_2([c,0]) = [0,c-1]$, and since $c \leq \beta - \alpha$ we have $[c,0], [0,c-1] \in \Delta$. Hence, Δ verifies the hypothesis of Proposition 1.2(d), and since $\Delta \subseteq \mathcal{F}^v$ we must have $\Delta = \mathcal{F}^v$, as claimed. $\square$

Lemma 1.77. *Let S be a semigroup of multiplicity 3, and let $I \in \mathcal{F}(S)$. Then, the set of ideals between I and I^v is linearly ordered.*

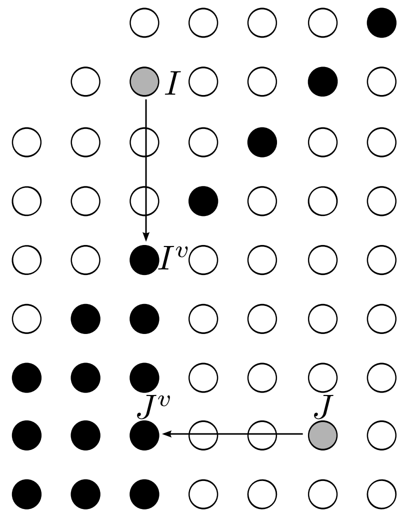


Figure 1.6: Divisorial closure of ideals.

Proof. If $[a,b]\in\Sigma^0$, then it is divisorial.

Suppose $[a,b]\in\Sigma^+$. Then, $\rho_{3(\alpha-a)}([\alpha,\beta])=[a,\min\{\beta-\alpha+a,0\}]$. However, $\beta-\alpha+a\leq b-a+a=b$, and thus $[a,b]\subseteq[a,b']=\rho_{3(\alpha-a)}(S)$. However, the ideals between $[a,b]$ and $[a,b']$ are linearly ordered, and $\rho_{3x}(S)$ is always divisorial (by Proposition 1.75); hence $[a,b]^v\subseteq[a,b']$ and the ideals between $[a,b]$ and $[a,b]^v$ are linearly ordered.

If $[a,b]\in\Sigma^-$, then in the same way $[a,b]^v\subseteq\rho_{3(\beta-b)}([\alpha,\beta])=[a',b]$ for some $a'\leq a$, and the claim follows. □

Corollary 1.78. *Let S be a semigroup of multiplicity 3. Then, the maps $\mathcal{A}$ and $*$ (defined at the end of Section 3) are bijections, and $|\mathrm{Star}(S)|$ is equal to the number of antichains of $(\mathcal{G}_0(S),\leq_*)$.*

Proof. Be the previous lemma, the set of ideals between I and I^v is linearly ordered; hence, for any $*_1,*_2\in\mathrm{Star}(S)$, I^{*_1} and I^{*_2} are comparable. Applying Proposition 1.39 we get the claim. □

Corollary 1.79. *Let S be a semigroup of multiplicity 3 and let $I,J\in\mathcal{F}_0(S)\cap\mathcal{F}^*(S)$ for some $*\in\mathrm{Star}(S)$. Then, $I\cup J$ is $*$-closed.*

Proof. Let $I=[a,b]$ and $J=[a',b']$. Without loss of generality, we can suppose $a<a'$ and $b>b'$ (if $b\leq b'$, then $I\supseteq J$ and $I\cup J=J$). Then, $I\cup J=[a,b']$.

Suppose $I\cup J\in\Sigma^+$. Then, since $a-b<a-b'$, it follows that $I\in\Sigma^+$. Hence, $[a,b']=\rho_{3(b-b')}(I)\cap I^v$, and thus $[a,b']\in\Sigma^+$. Analogously, if $I\cup J\in\Sigma^-$, then $J\in\Sigma^-$ and $[a,b']=\rho_{3(a'-a)}(J)\cap J^v$. In both cases, $I\cup J$ is $*_I$- or $*_J$-closed, and in particular, since $*\leq *_I\wedge *_J$, it is $*$-closed. □

Note that the hypothesis $I,J\in\mathcal{F}_0(S)$ is necessary: for example, if $S=\langle 3,5,7\rangle$, $I=S$, $J=4+\mathbb{N}$, then both I and J are divisorial, but $I\cup J=S\cup\{4\}$ while $(I\cup J)^v=(S-M)=S\cup\{2,4\}$.

Lemma 1.80. *Let S be a numerical semigroup of multiplicity 3, and let $I, J \in \mathcal{F}(S)$ such that J is $*_I$-closed. There are $\gamma_0, \gamma_1, \gamma_2 \in \mathbb{N}$, $\gamma_i \equiv i \bmod 3$, such that $J^{*_I} = J^v \cap (-\gamma_0 + I) \cap (-\gamma_1 + I) \cap (-\gamma_2 + I)$. In particular, if $I, J \in \mathcal{F}_0(S)$, then there are γ_i such that $J^{*_I} = J^v \cap \rho_{\gamma_0}(I) \cap \rho_{\gamma_1}(I) \cap \rho_{\gamma_2}(I)$.*

Proof. Since J is $*_I$-closed, we have $J = J^v \cap \bigcap_{\gamma \in (I-J)} -\gamma + I$; separating the γ according to their residue class modulo 3 we have

$$J = J^v \cap \bigcap_{\gamma \in \Gamma_0} (-\gamma + I) \cap \bigcap_{\gamma \in \Gamma_1} (-\gamma + I) \cap \bigcap_{\gamma \in \Gamma_2} (-\gamma + I),$$

where $\Gamma_i := (I - J) \cap (i + 3\mathbb{Z})$. Since $(I - J) \subseteq \mathbb{N}$, each Γ_i has a minimum. However, if $\gamma, \delta \in \Gamma_i$, then either $-\gamma + I \subseteq -\delta + I$ or $-\delta + I \subseteq -\gamma + I$; therefore it is enough to take $\gamma_i := \min \Gamma_i$.

For the "in particular" statement, note that both J and J^v are contained in $\mathbb{N}$, so that the intersection does not change substituting $-\gamma_i + I$ with $(-\gamma_i + I) \cap \mathbb{N} = \rho_{\gamma_i}(I)$. $\square$

Proposition 1.81. *Let S be a numerical semigroup of multiplicity 3, and let $I = [a, b]$ be an ideal.*

- *If $[a, b] \in \Sigma^+$, then $\mathcal{F}^{*_I} \cap \Sigma^+ = \{[c, d] : d \leq b,\ d - c \leq b - a\}$.*
- *If $[a, b] \in \Sigma^-$, then $\mathcal{F}^{*_I} \cap \Sigma^- = \{[c, d] : c \leq a,\ d - c \geq b - a\}$.*

Proof. Suppose $[a, b] \in \Sigma^+$, and let $[c, d] \in \Sigma^+$ such that $d \leq b$ and $d - c \leq b - a$. Then, $\rho_{3(b-d)}([a, b]) = [a - (b - d), b - (b - d)] = [a - b + d, d]$ is $*_{[a,b]}$-closed; moreover, $a - b + d \geq c - d + d = c$, and thus $[c, d] = [a - b + d, d] \cap [c, c']$, where $c' - c = \beta - \alpha$ (i.e., $c' = c + \beta - \alpha$), so that $[c, c'] \in \Sigma^0$ is divisorial, and $[c, d]$ is $*_{[a,b]}$-closed.

Conversely, let $\Delta := (\mathcal{F}^{*_I} \cap \Sigma^+) \setminus \{[c, d] : d \leq b,\ d - c \leq b - a\}$ and suppose $\Delta \neq \emptyset$. Note that, by Proposition 1.76, $\mathcal{F}^v(R) \cap \Delta = \emptyset$. Let B be the maximum b' such that $[a', b'] \in \Delta$ for some a', and let A be the minimum a' such that $[a', B] \in \Delta$. Let $J := [A, B]$.

By Lemma 1.80, $J = J^v \cap I_0 \cap I_1 \cap I_2$, where $I_i := \rho_{\gamma_i}(I) = [a_i, b_i]$. Since $J^v = [A, b'']$ for some $b'' < B$, at least one of the b_i must be equal to B. We have $I_i \in \Sigma^+$: indeed, if $I \in \Sigma^0$ it is divisorial, while if $I_i \in \Sigma^-$ then $L := [B - \beta + \alpha, B] \in \Sigma^0$ is divisorial and is contained between J and I_i: in both cases, $J^v \subseteq I_i$, so that $J^v \subseteq [A, b''] \cap [a_i, B] = [A, B] = J$, and J is divisorial, against $J \in \Delta$. Since $J \subseteq [a_i, B]$, we have $a_i \leq A$. Suppose $a_i < A$: then, by definition of A, $I_i \notin \Delta$. However, I_i is $*_I$-closed: hence, $B \leq b$ and $B - a_i \leq b - a$. But $B - a_i \geq B - A$, so that $B - A \leq b - a$; this would imply $J \notin \Delta$, against its definition. Therefore $a_i = A$, and $J = I_i$. However:

1. if $i = 0$, then $b_i \leq b$, and $b_i - a_i = b - a$;
2. if $i = 1$, then $I_i \in \Sigma^-$;
3. if $i = 2$, then $[a_i, b_i] = [0, 0]$ (since $J \in \Sigma^+$).

Therefore, $\Delta = \emptyset$.

If $[a, b] \in \Sigma^-$, we can use the same method reversing the rôle of a and b: we choose first A as the maximum a' such that $[a', b'] \in \Delta$ for some b', and then B as the minimum

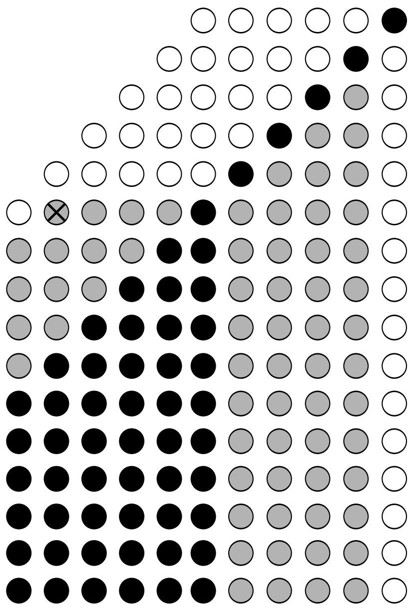


Figure 1.7: The set of divisorial ideals (in black) and of non-divisorial $*_I$-closed ideals (in gray), where I is the marked ideal.

b' such that $[A, b'] \in \Delta$. It follows as above that $[a_i, b_i] = [A, B]$ for some i, and $I_i \in \Sigma^-$; moreover, if $i = 0$ then $[a_i, b_i] \notin \Delta$, if $i = 1$ then $[a_i, b_i] = [0, 0]$ and if $i = 2$ then $[a_i, b_i] \in \Sigma^+$. None of this cases is acceptable, and $\Delta = \emptyset$. $\square$

Proposition 1.82. *Let S be a numerical semigroup of multiplicity 3, and let $I = [a, b]$ be an ideal.*

- *If $[a, b] \in \Sigma^+$, then $\mathcal{F}^{*_I} \cap \Sigma^- = \mathcal{F}^{*_{[b-a,0]}} \cap \Sigma^-$.*
- *If $[a, b] \in \Sigma^-$, then $\mathcal{F}^{*_I} \cap \Sigma^+ = \mathcal{F}^{*_{[0,b-a-1]}} \cap \Sigma^+$*

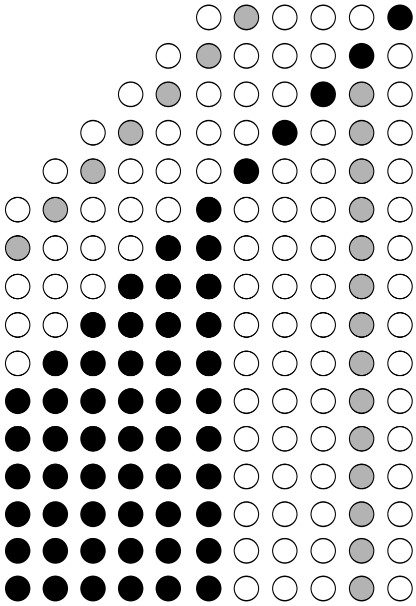
In particular, both depends only on $b - a$.

Proof. Suppose $[a, b] \in \Sigma^+$. Since $[a, b]$ is closed, so is $[0, b-a]$, and thus also $[b-a, 0] = \rho_1([0, b-a])$ is closed. Hence $\mathcal{F}^{*_{[b-a,0]}} \cap \Sigma^- \subseteq \mathcal{F}^{*_I} \cap \Sigma^-$.

Let $\Delta := (\mathcal{F}^{*_I} \cap \Sigma^-) \setminus \mathcal{F}^{*_{[b-a,0]}}$ and suppose it is nonempty; as in the proof of the previous proposition, let A be the maximum a' such that $[a', b'] \in \Delta$ for some b' and let B be the minimum b' such that $[A, b'] \in \Delta$. Observe that $A > b - a$ since $[a', 0]$ is $*_{[b-a,0]}$-closed for every $a' \leq b - a$. Then $J := [A, B] \in \Delta$, and $J = \rho_\gamma(I)$ for some γ such that $\rho_\gamma(I) \in \Sigma^-$, and the unique possibility is $\gamma \equiv 1 \bmod 3$; let $\gamma = 3k + 1$. Then $\rho_{3k}([a, b]) = [0, c]$ for some $c \leq b - a$, and thus $\rho_\gamma(I) = [c - 1, 0]$, with $c - 1 \leq b - a$, which is impossible.

The case $[a, b] \in \Sigma^-$ is treated in the same manner. $\square$

$[k, \beta]$



$[0, \beta - k]$

$[\beta - k, 0]$

Figure 1.8: A $\mathcal{L}_k$ in the case $\alpha \leq \beta$.

1.5.4. The number of star operations

Let $S = \langle 3, 3\alpha + 1, 3\beta + 2\rangle$ be a numerical semigroup, and suppose that $\alpha \leq \beta$; let k be an integer such that $\beta - \alpha \leq k < \alpha$. We define:

- $\mathcal{L}_k^+ := \{[k, \beta], [k-1, \beta-1], \ldots, [0, \beta - k]\}$;
- $\mathcal{L}_k^- := \{[\beta - k, 0], [\beta - k, 1], \ldots, [\beta - k, 2\beta - \alpha - k - 1]\}$;
- $\mathcal{L}_k := \mathcal{L}_k^+ \cup \mathcal{L}_k^-$.

Equivalently, $\mathcal{L}_k^+$ is the set of ideals $[a, b]$ such that $b - a = \beta - k$, while $\mathcal{L}_k^-$ is the set of ideals $[a, b] \in \Sigma^-$ such that $a = \beta - k$. Note that, since $k < \alpha$, each element of $\mathcal{L}_k^+$ is in Σ^+. Figure 1.8 represents a $\mathcal{L}_k$.

Proposition 1.83. *Preserve the notation above. Then:*

(a) $\mathcal{L}_k \cap \mathcal{L}_j = \emptyset$ *if* $k \neq j$*;*
(b) $\bigcup_{k=\beta-\alpha}^{\alpha-1} \mathcal{L}_k = \mathcal{G}_0(S)$*;*
(c) $|\mathcal{L}_k| = 2\beta - \alpha + 1$*;*
(d) each $\mathcal{L}_k$ *is linearly ordered (in the* $*$*-order).*

Proof. (a) Suppose $[a, b] \in \mathcal{L}_k \cap \mathcal{L}_j$. If $[a, b] \in \Sigma^+$, then $\beta - k = b - a = \beta - j$; if $[a, b] \in \Sigma^-$, then $\beta - k = a = \beta - j$. In both cases, $k = j$.

(b) Suppose $[a, b] \in \mathcal{L}_k$ for some k. If $[a, b] \in \Sigma^+$, then it is not divisorial by Proposition 1.76; if $[a, b] \in \Sigma^-$, then $a = \beta - k > \beta - \alpha$ and thus $[a, b] \neq [a, b]^v$, again by Proposition 1.76.

Conversely, suppose $[a,b] \neq [a,b]^v$. If $[a,b] \in \Sigma^+$, then $\beta - \alpha \leq b - a < \alpha$, and thus $[a,b] \in \mathcal{L}_{\beta-(b-a)}$; if $[a,b] \in \Sigma^-$, then by Proposition 1.76 we have $a > \beta - \alpha$, so that $\beta - a < \alpha$ and thus $[a,b] \in \mathcal{L}_{\beta-a}$.

(c) We have $|\mathcal{L}_k^+| = k+1$ and $|\mathcal{L}_k^-| = 2\beta - \alpha - k$; since $\mathcal{L}_k^+$ and $\mathcal{L}_k^-$ are disjoint, $|\mathcal{L}_k| = 2\beta - \alpha + 1$.

(d) By Lemma 1.74, if $j \geq j'$ then $[k-j', \beta - j'] = \rho_{3(j-j')}([k-j, \beta-j])$, so that $\mathcal{L}_j^+$ is totally ordered, with minimum $[0, \beta - k]$; analogously, if $l \geq l'$, then $[a,l] = [a,l'] \cap [a,l]^v$ (see the proof of Lemma 1.77) and thus $[a,l] \leq_* [a,l']$, i.e., $\mathcal{L}_j^-$ is linearly ordered, with maximum $[\beta - k, 0]$. Moreover, $[\beta - k, 0] = \rho_1([0, \beta - k])$, and thus $\mathcal{L}_k$ is totally ordered. □

When $\alpha \geq \beta$, we can reason in a completely analogous way, but we have to reverse the rôle of Σ^+ and Σ^-: we choose an integer k such that $\alpha - \beta + 1 \leq k < \beta$, and define

- $\mathcal{L}_k^- := \{[\alpha, k], [\alpha - 1, k-1], \ldots, [0, \alpha - k]\}$;
- $\mathcal{L}_k^+ := \{[0, \alpha - k - 1], [1, \alpha - k - 1], \ldots, [2\alpha - \beta - k - 2, \alpha - k - 1]\}$;
- $\mathcal{L}_k := \mathcal{L}_k^+ \cup \mathcal{L}_k^-$.

Then, the elements of $\mathcal{L}_k^-$ are in Σ^- and are characterized by $b - a$, while the elements of $\mathcal{L}_k^+$ are the ideals in Σ^+ with the same b. Proposition 1.83 becomes:

Proposition 1.84. *Preserve the notation above. Then:*

(a) $\mathcal{L}_k \cap \mathcal{L}_j = \emptyset$ *if* $k \neq j$;
(b) $\bigcup_{k=\alpha-\beta+1}^{\beta-1} \mathcal{L}_k = \mathcal{G}_0(S)$;
(c) $|\mathcal{L}_k| = 2\alpha - \beta$;
(d) each $\mathcal{L}_k$ *is linearly ordered (in the* $*$*-order).*

Corollary 1.85. *Let* $S = \langle 3, 3\alpha+1, 3\beta+2 \rangle$ *be a numerical semigroup. Then,* $|\mathcal{G}_0(S)| = (2\alpha - \beta)(2\beta - \alpha + 1)$.

By a *rectangle* $a \times b$, indicated with $\mathcal{R}(a,b)$, we denote the Cartesian product $\{1, \ldots, a\} \times \{1, \ldots, b\}$, endowed with the *reverse* product order (that is, $(x,y) \geq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$).

Theorem 1.86. *Let* $S = \langle 3, 3\alpha+1, 3\beta+2 \rangle$ *be a numerical semigroup. Then,* $(\mathcal{G}_0(S), \leq_*)$ *is isomorphic (as an ordered set) to* $\mathcal{R}(2\alpha - \beta, 2\beta - \alpha + 1)$.

Proof. Suppose $\alpha \leq \beta$, and let $I \in \mathcal{G}_0(S)$. If $I \in \mathcal{L}_k$, define $\psi_1(I) := k - (\beta - \alpha) + 1$. Moreover, if there are exactly $j - 1$ ideals in $\mathcal{L}_k$ strictly bigger (in the $*$-order) than I, then define $\psi_2(I) := j$. Explicitly, if $[a,b] \in \Sigma^+$ then $\psi_2([a,b]) = \beta - b + 1$, while if $[a,b] \in \Sigma^-$ then $\psi_2([a,b]) = k + 1 + b = \beta + 1 + b - a$ (using $a = \beta - k$). By Proposition 1.83, the map

$$\begin{aligned} \Psi\colon \mathcal{G}_0(S) &\longrightarrow \mathcal{R}(2\alpha - \beta, 2\beta - \alpha + 1) \\ [a,b] &\longmapsto (\psi_1(I), \psi_2(I)) \end{aligned}$$

is a bijection.

For a partially ordered set $\mathcal{P}$, and a subset $\Delta \subseteq \mathcal{P}$, denote by $\overline{\Delta}$ the lower set of Δ: i.e., let $\overline{\Delta} := \{x \in \mathcal{P} : \exists y \in \Delta : x \leq y\}$. To show that Ψ is order-preserving, it is enough to show that $\Psi\left(\overline{\{I\}}\right) = \overline{\Psi(I)}$ for every ideal $I \in \mathcal{G}_0(S)$. Since $\overline{\{I\}} = \mathcal{G}_0(S) \cap \mathcal{F}^{*_I}$, we need to show that J is $*_I$-closed if and only if $\Psi(J) \leq \Psi(I)$.

Let $I = [a, b]$ and $J = [c, d]$ be ideals. If $I, J \in \Sigma^+$, then by Proposition 1.81 J is $*_I$-closed if and only if $d \leq b$ and $d-c \leq b-a$. We have $d \leq b$ if and only if $\psi_2(J) \geq \psi_2(I)$; on the other hand, $x-y = \beta-k$ if $[y, x] \in \mathcal{L}_k$, and thus $\psi_1([y, x]) = \beta-x+y$. Therefore, $d-c \leq b-a$ if and only if $\psi_1(J) \geq \psi_1(I)$. Hence (remember that the order on the rectangle is the reverse product order), $J \in \overline{\{I\}}$ if and only if $\Psi(J) \leq \Psi(I)$. On the other hand, if $I, J \in \Sigma^-$, then $J \in \overline{\{I\}}$ if and only if $c \leq a$ and $d - c \leq b - a$; the first condition if equivalent to the requirement that $\psi_1(J) \geq \psi_1(I)$, while the second is equivalent to $\psi_2(J) \geq \psi_2(I)$. Again, $J \in \overline{\{I\}}$ if and only if $\Psi(J) \leq \Psi(I)$.

Suppose $I \in \Sigma^+$ and $J \in \Sigma^-$. If J is $*_I$-closed, then by Proposition 1.82 it is $*_{[b-a,0]}$-closed, and, by the previous paragraph, this happens if and only if $\Psi(J) \leq \Psi([b-a, 0])$. However, $[b-a, 0]$ and I belong to the same $\mathcal{L}_k$ (since $[b-a, 0] = \rho_1\rho_{3(b-a)}([a, b])$), and thus $\Psi([b-a, 0]) \leq \Psi(I)$; hence $\Psi(J) \leq \Psi(I)$. Conversely, if $\Psi(J) \leq \Psi(I)$ then $J = [c, d]$ belongs to $\mathcal{L}_j$ for some $j \geq k$ (where $I = [a, b] \in \mathcal{L}_k$) and thus $c \leq a$, and J is $*_I$-closed (applying again Proposition 1.82). If $I \in \Sigma^-$ and $J \in \Sigma^+$, the same reasoning applies; therefore, in all cases, $J \in \overline{\{I\}}$ if and only if $\Psi(J) \leq \Psi(I)$, that is, if and only if $\Psi(J) \in \overline{\Psi(I)}$. Hence Ψ is an order isomorphism.

If $\alpha \geq \beta$, then we can apply the same method: we define a map

$$\begin{aligned}\Psi\colon \mathcal{G}_0(S) &\longrightarrow \mathcal{R}(2\beta-\alpha+1, 2\alpha-\beta)\\ [a,b] &\longmapsto (\psi_1(I), \psi_2(I))\end{aligned}$$

where, if $I \in \mathcal{L}_k$, then $\psi_1(I) = k - (\alpha - \beta + 1) + 1$, and $\psi_2(I) = j$ if there are exactly $j-1$ elements of $\mathcal{L}_k$ $*$-bigger than I. Proposition 1.84 shows that Ψ is a bijection, and (as before) the use of Propositions 1.81 and 1.82 shows that it is an order isomorphism. Since $\mathcal{R}(2\beta-\alpha+1, 2\alpha-\beta) \simeq \mathcal{R}(2\alpha-\beta, 2\beta-\alpha+1)$, the theorem is proved. □

Lemma 1.87. *The number of antichains in $\mathcal{R}(a, b)$ is* $\binom{a+b}{a} = \binom{a+b}{b}$.

Proof. Let $A := \{1, \ldots, a\}$ and $B := \{1, \ldots, b\}$.

For each antichain Δ, let $\overline{\Delta}$ be the lower set of Δ; clearly $\Delta = \max\overline{\Delta}$, so that the number of antichains is equal to that of the sets that are downward closed (i.e., sets Λ such that $\Lambda = \overline{\Lambda}$). When restricted to a single row $A \times \{c\}$, $\overline{\Delta}$ becomes a segment $\{a_c, \ldots, a\} \times \{c\}$; moreover, if $d \leq c$, then $a_d \leq a_c$. Thus the number of antichains is equal to the number of sequences $\{1 \leq a_1 \leq \cdots \leq a_b \leq a+1\}$ (where $a_i = a+1$ if and only if $(A \times \{i\}) \cap \overline{\Delta} = \emptyset$), that in turn is equal to the number of combinations with repetitions of b elements of $\{1, \ldots, a+1\}$. This is equal to $\binom{a+1+b-1}{b} = \binom{a+b}{b} = \binom{a+b}{a}$. □

Theorem 1.88. *Let $S = \langle 3, 3\alpha+1, 3\beta+2\rangle$ be a numerical semigroup of multiplicity 3, $g := g(S)$, $\delta := \delta(S)$. Then,*

$$|\mathrm{Star}(S)| = \binom{\alpha+\beta+1}{2\alpha-\beta} = \binom{\alpha+\beta+1}{2\beta-\alpha+1} = \binom{\delta+1}{g-\delta+2}.$$

Proof. By Corollary 1.78, $|\mathrm{Star}(S)|$ is equal to the number of antichains of $\mathcal{G}_0(S)$, which is equal (by Theorem 1.86) to the number of antichains of $\mathcal{R}(2\alpha-\beta, 2\beta-\alpha+1)$. Lemma 1.87 now completes the reasoning.

To show the last equality, note that an element in $\mathbb{N}\setminus S$ can be written as $3a+1$ or $3b+2$, where $0\leq a<\alpha$ or $0\leq b<\beta$, and thus $\delta=\alpha+\beta$. On the other hand, if $\alpha>\beta$ then $g=3\alpha-2$, and thus $2\alpha-\beta=g-\delta+2$, while if $\alpha\leq\beta$ then $g=3\beta-1$, and again $2\beta-\alpha+1=g-\delta+2$. □

1.6. Pseudosymmetric semigroups

Definition 1.89. *A semigroup S is called* pseudosymmetric *if $g(S)$ is even and $T(S)=\{g,g/2\}$ or, equivalently, if $g(S)$ is even and $g-a\in S$ for every $a\in\mathbb{N}\setminus S$, $a\neq g/2$.*

Proposition 1.90. *Let S be a pseudosymmetric semigroup. The unique minimal element of $\mathcal{G}_0(S)$ is $S\cup\{g\}$.*

Proof. Let $I:=S\cup\{g\}$ and let $\tau:=g/2$. It is enough to show that I is $*_J$-closed for each nondivisorial ideal $J\in\mathcal{F}_0(S)$. If $g\notin J$, then $J=S\cup\{\tau\}=M_g$ is the maximum of $(\mathcal{G}_0,\leq_*)$.

Suppose $g\in J$. If $\tau\notin J$, then $I=J\cap(S-M)$; since $(S-M)$ is divisorial, I is $*_J$-closed. Suppose $\tau\in J$ and consider the ideal $L:=(J-(J-I))$; note that it contains g since J contains all the integers greater or equal than g. If $\tau\notin L$, then $I=L\cap(S-M)$ is $*_J$-closed. Otherwise, $\tau+(J-I)\subseteq J$. However,

$$(J-I)=(J-(S\cup\{g\}))=(J-S)\cap(J-g)=J$$

(the last equality coming from $(J-S)=J$ and $g\in J$); therefore, $\tau+J\subseteq J$. By [13, Proposition I.1.16], this would imply that J is divisorial, against our assumption. Therefore, I must be $*_J$-closed. □

Proposition 1.91. *Let S be a pseudosymmetric semigroup, and let $\tau:=g/2$. Then:*

(a) if $I\in\mathcal{F}_0(S)$, $I\neq S$ and $\tau\notin I$, then $I^v=I\cup\{\tau\}$;
(b) $\mathcal{Q}_\tau(S)\subseteq\mathcal{G}_1(S)$;
(c) if $I,J\in\mathcal{Q}_\tau$, then $I\geq_* J$ if and only if $I\supseteq J$.

Proof. (a) By [13, Proposition I.1.16], and since $\tau\in T(S)$ (so that $I\neq I^v$ by Proposition 1.12), it is enough to show that $\tau+(I\cup\{\tau\})\subseteq(I\cup\{\tau\})$. However,

$$\tau+(I\cup\{\tau\})=\tau+(\{0\}\cup M\cup(I\setminus S)\cup\{\tau\})=$$

$$=\{\tau,g\}\cup(\tau+M)\cup(\tau+(I\setminus S)).$$

The first two sets are contained in $I\cup\{\tau\}$ because $\tau\in(S-M)$. If now $x\in I\setminus S$, then either $x>\tau$ (and so $x+\tau>g$ and $x+\tau\in S$) or $x<\tau$, and so $\tau-x\notin S$ (otherwise $\tau\in I$); in the latter case, $g-(\tau-x)\in S$, but $g-(\tau-x)=\tau+x$, and thus $x+\tau\in S\subseteq I$.

(b) follows directly from (a).

(c) If $I \geq_* J$, then $I \supseteq J$ by Proposition 1.47. Suppose $J \subseteq I$. Then,

$$J^{*_I} \subseteq J^v \cap I = (J \cup \{\tau\}) \cap I = J$$

since $\tau \notin I$. Hence, $*_J \geq *_I$ and $J \leq_* I$. □

A direct consequence of Proposition 1.91 is a direct formula for the number of star operations on a particular class of semigroups.

Proposition 1.92. *Let $S := \{0, \mu, \mu+1, \dots, 2\mu-3, 2\mu-1, \rightarrow\}$, where $\mu \geq 3$. Then, $|\mathrm{Star}(S)| = 1 + \omega(\mu-2)$.*

Proof. It is clear that $g := g(S) = 2\mu - 2$. Let $\tau := g/2 = \mu - 1$; then, $T(S) = \{g, \tau\}$, so that S is pseudosymmetric.

If $I \in \mathcal{F}_0(S)$ is an ideal not containing g, then I is either S or $S \cup \{\tau\}$. Moreover, if $(S - M) \subseteq I$, then every element greater than τ is in I and thus $\tau + I \subseteq I$, and it follows from [13, Proposition I.1.16] that any such I is divisorial. By Proposition 1.91, if I contains g but not τ, then $I^v = I \cup \{\tau\}$. Define $I_A := S \cup \{g\} \cup A$, where $A \subseteq \mathbb{N}$. Then,

$$\mathcal{G}_0 = \mathcal{G}_1 = \{S \cup \{\tau\}\} \cup \{I_A : A \subseteq \{1, \dots, \mu-2\}\}.$$

By Corollary 1.40, $|\mathrm{Star}(S)| = \omega(\mathcal{G}_0)$.

The ideal $M_g = S \cup \{\tau\}$ generates the identity. Moreover, each I_A is in $\mathcal{Q}_\tau$; by Proposition 1.91(c), $*_{I_A} \geq *_{I_B}$ if and only if $I_A \supseteq I_B$, i.e., if and only if $A \supseteq B$.

Therefore, if Δ is an antichain of $\mathcal{G}_0$, then either $\Delta = \{M_g\}$ or Δ is an antichain of $\mathscr{P}(\{1, \dots, \mu-2\})$. Hence,

$$|\mathrm{Star}(S)| = \omega(\mathcal{G}_0) = 1 + \omega(\mathscr{P}\{1, \dots, \mu-2\}) = 1 + \omega(\mu-2),$$

as claimed. □

Proposition 1.93. *Let S be a pseudosymmetric semigroup such that $g > 2\mu$. Let b be an integer such that $\tau < b < \tau+\mu$ and $b \notin S$; let $Y_b := \{a \in \mathbb{N} \setminus S : b-\mu < a < b, a \neq \tau\}$. Then, $\omega_{\mathrm{i}}(\mathcal{Q}_b) \geq \omega(|Y_b|)$.*

Proof. Consider the ideal $I_b := S \cup \{\tau\} \cup \{x \in \mathbb{N} : x > b\} \cup (b - \tau + S)$; being a finite union of ideals, I_b is an ideal, and $b \notin I_b$ since $\tau \notin S$. Let J be an ideal such that $b \notin J$ but $I_b \subseteq J$: we claim that $J \in \mathcal{Q}_b$.

Indeed, let $l \in (S - J)$; since $J \supsetneq S$, we have $l > 0$. Moreover, $l + b - \tau \in S$, and since $b > \tau$ we have $l+b-\tau > 0$, i.e., $l+b-\tau \in M$. But then $l+b = \tau+(l+b-\tau) \in \tau+M \subseteq M$, and $b \in (S - (S - J)) = J^v$.

Let $Z_b := \{b - y : y \in Y_b\}$; then, $Z_b \cap I_b = \emptyset$. For every $B \subseteq Y_b$, the ideal $W_B := I_b \cup (B + S)$ does not contain b (since $b \notin Y_b$ and $Y_b \cap S = \emptyset$) and $b \in I_b^v \subseteq W_B^v$, so that $W_B \in \mathcal{Q}_b$. Moreover, $W_B \cap \{1, \dots, \mu-1\} = B \cup \{b - \tau\}$, so that $W_B \subseteq W_{B'}$ if and only if $B \subseteq B'$. By Proposition 1.47, $\omega_{\mathrm{i}}(\mathcal{Q}_b) \geq \omega(|Y_b|)$. □

Using Proposition 1.91 together with the result of Section 1.5, we can characterize when $(\mathcal{G}_0(S), \leq_*)$ is totally ordered.

Proposition 1.94. *Let $S = \langle 3, 3\alpha+1, 3\beta+2\rangle$ be a nonsymmetric numerical semigroup of multiplicity 3. Then, the following are equivalent:*

(i) *S is pseudosymmetric;*
(ii) *$\alpha = 2\beta$ or $\beta = 2\alpha - 1$;*
(iii) *$(\mathcal{G}_0(S), \leq_*)$ is linearly ordered;*
(iv) $\mathrm{Star}(S)$ *is linearly ordered.*
(v) *every star operation on S is principal.*

Proof. (i $\iff$ ii) Let $a := 3\alpha + 1 - 3 = 3\alpha - 2$ and $b := 3\beta + 2 - 3 = 3\beta - 1$: then, $a, b \notin S$ but $a+3, b+3 \in S$. Hence, S is pseudosymmetric if and only if $a = 2b$ or $b = 2a$.

If $\alpha \geq \beta$, then $a \geq b$, and thus S is pseudosymmetric if and only if $3\alpha - 2 = 2(3\beta - 1)$, that is, if and only if $\alpha = 2\beta$. Analogously, if $\beta \geq \alpha$, S is pseudosymmetric if and only if $3\beta - 1 = 2(2\alpha - 2)$, that is, if and only if $\beta = 2\alpha + 1$.

(ii $\iff$ iii) $\mathcal{G}_0(S)$ is linearly ordered if and only if $\mathcal{R}(2\alpha - \beta, 2\beta - \alpha + 1)$ is linearly ordered; but this happens if and only if one of the sides of the rectangle has length 1, that is, if and only if $2\alpha - \beta = 1$ (i.e., $\beta = 2\alpha - 1$) or $2\beta - \alpha + 1 = 1$ (i.e., $\alpha = 2\beta$).

(iv $\Longrightarrow$ iii) is obvious.

(iii $\Longrightarrow$ iv,v) Let $*$ be a star operation. Then, $* = *_{I_1} \wedge \cdots \wedge *_{I_n}$ for some $I_1, \ldots, I_n$; since $\mathcal{G}_0(S)$ is linearly ordered, $* = *_{I_j}$ for some j. Hence each star operation is principal, and $\mathrm{Star}(S)$ is linearly ordered.

(v $\Longrightarrow$ ii) Suppose $\alpha \neq 2\beta$ and $\beta \neq 2\alpha - 1$. Then, the length of both sides of the rectangle $\mathcal{R}(2\alpha - \beta, \beta - 2\alpha + 1)$ is 2 or more; consider the set Δ composed of $(1,2)$ and $(2,1)$. Then, Δ is an antichain; therefore, so is $\Psi^{-1}(\Delta)$, where Ψ is the isomorphism defined in the proof of Theorem 1.86. By hypothesis, $*_{\Psi^{-1}(\Delta)}$ is principal, i.e., $*_{\Psi^{-1}(\Delta)} = *_I$ for some $I \in \mathcal{G}_0(S)$; however, by Corollary 1.78, this would imply $\Psi^{-1}(\Delta) = \{I\}$, which is absurd. Hence S is pseudosymmetric. $\square$

Lemma 1.95. *Let S be a nonsymmetric numerical semigroup. If $(\mathcal{G}_0(S), \leq_*)$ is linearly ordered, then $\mu(S) = 3$.*

Proof. Let a be a hole of S such that $a \leq g/2$ (it exists because S is not symmetric). If $a \geq 3$, then (by Lemma 1.63) there are $x_1, x_2 \in \mathbb{N} \setminus S$ such that $a - \mu < x_i < a$; consider $I_i := S \cup \{x \in \mathbb{N} \mid x > a\} \cup \{x_i\}$. Then, I_1 and I_2 are noncomparable elements (with respect to inclusion) of $\mathcal{Q}_i$; by Proposition 1.47 it follows that I_1 and I_2 are not comparable in the $*$-order.

In the same way, if $a < 3$ and $\mu \geq 5$, consider $b := 4$. Then, the set $\{1, 2, 3\} \setminus \{3 - a\}$ contains two different elements, say x_1 and x_2, and the two ideals $I_i := S \cup \{x \in \mathbb{N} \mid x > 3\} \cup \{3 - a, x_i\}$ are noncomparable elements of $\mathcal{Q}_3$ (by the proof of Proposition 1.62).

Thus, we can suppose $\mu = 4$ and $a < 3$. If $a = 1$, then one between g and $g - 1$ is even; call it e. Then, $e/2$ is a hole of S which is not bigger then $g/2$; in particular, if $\frac{e}{2} \geq 3$ we are in the case above. However, $\frac{e}{2} \leq 2$ implies $g \leq 5$, and thus the unique possibilities are $g = 3$ or $g = 5$. The latter case is impossible since $g - 1 = 4$ would be in S; the former is Example 1.54, whose star operations are not linearly ordered.

If $a = 2$, consider the ideals $I_1 := S \cup \{g - 2\}$ and $I_2 := S \cup (2 + S)$. Both are elements of $\mathcal{Q}_g$, and $g - 2 \notin I_2$ (otherwise $g - 2 - 2 = g - 4 = g - \mu \in S$, which is absurd). The equality $2 = g - 2$ would imply $g = 4$; therefore, $2 \neq g - 2$ and $I_1 \neq I_2$ are noncomparable ideals, and thus are not comparable also in the $*$-order. □

Corollary 1.96. *Let S be a nonsymmetric numerical semigroup. Then, the following are equivalent:*

(i) *S is pseudosymmetric and $\mu(S) = 3$;*
(ii) *$S = \langle 3, 3\alpha + 1, 3\beta + 2\rangle$ with $\alpha = 2\beta$ or $\beta = 2\alpha - 1$;*
(iii) *$(\mathcal{G}_0(S), \leq_*)$ is linearly ordered;*
(iv) $\mathrm{Star}(S)$ *is linearly ordered.*

Proof. It is enough to join Proposition 1.94 and Lemma 1.95. □

Proposition 1.97. *Let S be a pseudosymmetric semigroup of multiplicity 4, and let $\tau := g/2 := 4k \pm 1$. Then, $\mathcal{Q}_\tau \simeq \mathcal{R}(k + 1, k + 1)$.*

Proof. Let $I := S \cup \{x \in \mathbb{N} : x > \tau\}$, and suppose $\tau = 4k + 1$. Note that every $J \in \mathcal{Q}_\tau$ must contain I. Since $\nu = 2$, there are two elements in $\{4k - 2, 4k - 1, 4k\}$ out of S, and they must be $4k - 2$ and $4k - 1$; it follows that every ideal in $\mathcal{Q}_\tau$ must be in the form $J(a, b) := I \cup (4a + 3 + S) \cup (4b + 2 + S)$, for some $0 \leq a, b \leq k$. Since $J(a, b) \subseteq J(a_1, b_1)$ if and only if $a \leq a_1$ and $b \leq b_1$, and since (by Proposition 1.91) the order on $\mathcal{Q}_\tau$ coincide with the order indued by the containment, it follows that $\mathcal{Q}_\tau \simeq \mathcal{R}(k + 1, k + 1)$.

The case $\tau = 4k - 1$ is done in the same way. □

Lemma 1.98. *Let S be a pseudosymmetric semigroup and $I \in \mathcal{G}_0(S)$. Then,* $\sup(\mathbb{N} \setminus I) \geq \tau$.

Proof. If $\sup(\mathbb{N} \setminus I) < \tau$, then each element bigger than τ is in I. Hence, $\tau + i \geq \tau$ for every $i \in I$, and $\tau + I \subseteq I$, i.e., $\tau \in (I - I)$. Thus, $I = I^v$ by [13, Proposition I.1.16], and $I \notin \mathcal{G}_0(S)$. □

1.6.1. The case $g = 2\mu + 2$

Suppose now that S is a pseudosymmetric semigroup of multiplicity μ with Frobenius number $g = 2\mu + 2$. Then, $\tau = \mu + 1$; since the unique positive integer belonging to S and smaller than τ is μ, we have $\mathbb{N} \setminus S = \{1, \ldots, \mu - 1, \tau, \tau + 1, g\}$. By Lemma 1.98, it follows that if $I \in \mathcal{G}_0(S)$ then $\sup(\mathbb{N} \setminus I) \in \{\tau, \tau + 1, g\}$. We distinguish four cases:

1. $\sup(\mathbb{N} \setminus I) = g$. Then, $I = S \cup \{\tau\} = M_g$.
2. $\sup(\mathbb{N} \setminus I) = \tau$. Then, $I = S \cup \{\tau + 1, g\} \cup X$, where $X \subseteq \{2, \ldots, \mu - 1\}$ ($1 \notin I$ since otherwise $1 + \mu = \tau \in I$). Each subset X defines a nondivisorial ideal, and each of these ideals is in $\mathcal{Q}_\tau$. Define $A_X := S \cup \{\tau + 1, g\} \cup X$.

3. $\sup(\mathbb{N} \setminus I) = \tau + 1$ and $\tau \notin I$. Since $\tau = \mu + 1$ and $\tau + 1 = \mu + 2$ are not in I, we have $1, 2 \notin I$. Therefore, $I = S \cup \{g\} \cup Y$, with $Y \subseteq \{3, \dots, \mu - 1\}$. Each subset Y defines a nondivisorial ideal; if $B_Y := S \cup \{g\} \cup Y$ then by Proposition 1.91 we have $B_Y^v = B_Y \cup \{\tau\}$.
4. $\sup(\mathbb{N} \setminus I) = \tau + 1$ and $\tau \in I$. Then, $2 \notin I$; moreover, if $1 \notin I$ then $I + \tau \subseteq I$, and I would be divisorial. Therefore, $I = S \cup \{1, \tau, g\} \cup Z$, with $Z \subseteq \{3, \dots, \mu - 1\}$; each Z defines a nondivisorial ideal, and if $C_Z := S \cup \{1, \tau, g\} \cup Z$ then $C_Z^v = C_Z \cup \{\tau + 1\}$ (since $C_Z \cup \{\tau + 1\}$ is divisorial by Lemma 1.98).

In particular, since $|I^v \setminus I| = 1$ for each $I \in \mathcal{G}_0(S)$, every principal star operation of S is prime by Proposition 1.38. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be the sets of ideals A_X, B_Y and C_Z, respectively. Note that $\mathcal{A} = \mathcal{Q}_\tau$ and $\mathcal{C} = \mathcal{Q}_{\tau+1}$. Applying the same reasoning of the proof of Proposition 1.91(c), we see that the $*$-order on $\mathcal{B}$ and $\mathcal{C}$, like the one on $\mathcal{A}$, coincides with the containment order. In particular, we get the following.

Corollary 1.99. *Preserve the notation above. Then,* $|\mathrm{Star}(S)| \geq \omega(\mu - 2) + 2\omega(\mu - 3) - 2$.

Proof. The order on $\mathcal{A}$ (resp., $\mathcal{B}$, $\mathcal{C}$) gives $\omega(\mu - 2) - 1$ antichains (resp., $\omega(\mu - 3) - 1$ antichains). Adding the v-operation we get at least $\omega(\mu - 2) + 2\omega(\mu - 3) - 2$ star operations. □

We can go further. Indeed, suppose $\gamma \geq 0$ is such that $A_X \subseteq -\gamma + B_Y$. Then we have three cases:

1. $\gamma = 0$: impossible since $\tau + 1 \notin A_X$;
2. $\gamma = 1$: impossible, since $1 \notin B_Y$;
3. $\gamma > 1$: then, $\tau \in -\gamma + B_Y$. Hence, $\bigcap_{\gamma \in (B_Y - A_X)} -\gamma + B_Y \supseteq A_X^v$.

Therefore, $A_X^{*_{B_Y}} = A_X^v$, and $A_X \not\leq_* B_Y$. In a similar way, we can see that, for every X, Y, Z, we have $C_Z \not\leq_* B_Y$ and $C_Z \not\leq_* A_X$.

On the other hand, we claim that $B_Y \leq_* A_X$ if and only if $Y \subseteq X$. Indeed, if $Y \subseteq X$ then $B_Y = B_Y^v \cap A_X$; however, if $B_Y \subseteq -\gamma + A_X$, then either $\gamma = 0$ (and $Y \subseteq X$) or $\gamma > 0$ (in which case $B_Y^v \subseteq -\gamma + A_X$ would not be significant for the calculation of $B_Y^{*_{A_X}}$).

To study the case of $\mathcal{C}$, we need a notation. For each $Z \subseteq \{3, \dots, \mu - 1\}$, let $\rho_1(Z) := \{z - 1 : z \in Z \cup \{\mu\}\}$. Then, we claim that $A_X \leq_* C_Z$ if and only if $X \subseteq \rho_1(Z)$: indeed, if the latter is true then $A_X = (-1 + C_Z) \cap A_X^v$. On the other hand, if $A_X \subseteq -\gamma + C_Z$, then $\gamma = 0$ is impossible (since $\mu + 2 \in C_Z \setminus A_X$) while $\gamma \geq 2$ implies $A_X^v \subseteq -\gamma + C_Z$, and thus it is not significant. If A_X is $*_{C_Z}$-closed, then $\gamma = 1$ must be acceptable, and thus $X \subseteq \rho_1(Z)$. The same reasoning shows that $B_Y \leq_* C_Z$ if and only if $Y \subseteq \rho_1(Z)$.

This relations are enough to determine explicitly, for a given μ, the order on $\mathcal{G}_0(S)$; since each ideal is an atom, the number of star operations is equal to the number of antichains of $\mathcal{G}_0(S)$, and thus we can find $|\mathrm{Star}(S)|$. For example, if $\mu = 4$, then $S = \langle 4, 7, 9 \rangle$, and the Hasse diagram of $\mathcal{G}_0(S)$ is the one pictured in Figure 1.9. We see that $\mathcal{G}_0(S)$ has 15 antichains, and thus $|\mathrm{Star}(S)| = 15$.

$$\begin{tikzcd}
& & A_{2,3} \arrow[r] \arrow[ddr] & A_2 \arrow[r] & A_\emptyset \arrow[dr] & \\
M_g \arrow[r] & C_3 \arrow[ur] \arrow[dr] & & & & B_\emptyset \\
& & C_\emptyset \arrow[r] & A_3 \arrow[r] \arrow[uur] & B_3 \arrow[ur] &
\end{tikzcd}$$

Figure 1.9: Hasse diagram of $\mathcal{G}_0(\langle 4, 7, 9\rangle)$.

1.7. The number of numerical semigroups with exactly n star operations

1.7.1. Estimating $\Xi(n)$

Let $\xi(n)$ denote the number of numerical semigroups with exactly n star operations. Theorem 1.26 can be reparaphrased by saying that $\xi(n) < \infty$ for every $n > 1$; moreover, we have seen that its proof can be adapted to show that $\xi(n) \leq 4^{n-1}$, and by using [120] we obtain the estimate $\xi(n) \leq D'\phi^n$, where D' is a constant and ϕ is the golden ratio.

Denote now by $\Xi(n)$ the number of numerical semigroups S with $2 \leq |\mathrm{Star}(S)| \leq n$; i.e., let $\Xi(n) = \sum_{i=2}^{n} \xi(i)$. The reasoning after Theorem 1.26 actually shows that the estimates above hold for Ξ; that is, $\Xi(n) \leq D'\phi^n$ for some D'.

Another way to express this estimate is through the big-O notation: recall that, given two function f and g, the notation $f(n) = O(g(n))$ means that $\limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty$. Thus, we have $\Xi(n) = O(\phi^n)$.

Let now $\Xi_\mu(n)$ denote the number of numerical semigroups with $\mu(S) = \mu$ such that $2 \leq |\mathrm{Star}(S)| \leq n$. Clearly, for every n, we have $\Xi(n) = \sum_n \Xi_\mu(n)$, where n ranges from 3 to infinity (semigroups of multiplicity 2 are symmetric, so they don't contribute to $\Xi(n)$).

Proposition 1.100. *Let n and μ be integers. Then,*

$$\Xi_\mu(n) \leq \binom{n-1}{\mu-1} \leq (n-1)^{\mu-1}.$$

Proof. A semigroup S of multiplicity μ can be described by its *Apéry set* $\mathrm{Ap}(S, \mu) := \{0, a_1, \ldots, a_{\mu-1}\}$, where $a_i := k_i\mu + i$ is the minimal element of S congruent to i modulo μ (see for example [106, Chapter 1] for a deeper discussion of Apéry sets). In particular, it is uniquely described by the ordered sequence $(k_1, \ldots, k_{\mu-1})$.

Each k_i is a positive integer (since there are no elements in S smaller than μ) and the sum $k_1 + \cdots + k_{\mu-1}$ is equal to $\delta(S)$: indeed, if $x \in \mathbb{N} \setminus S$ then $x = y_i\mu + i$, with $0 \leq y_i < k_i$. The number of sequences $(\alpha_1, \ldots, \alpha_q)$ such that $\alpha_1 + \cdots + \alpha_q \leq \delta$ is equal to the number of ordered partitions of $\delta + 1$ into $q + 1$ positive integers, or equivalently to the number of ways to divide a line of $\delta + 1$ points into $q + 1$ nonempty lines, which in turn is equal to the number of ways to place q separators among δ holes; that is, it is equal to the number of subsets of $\{1, \dots, \delta\}$ with q elements, i.e., it is equal to $\binom{\delta}{q}$.

Since $|\mathrm{Star}(S)| \leq \delta(S) + 1$, by Theorem 1.26 $\Xi_\mu(n)$ is smaller than the number of semigroups S such that $\delta(S) \geq n - 1$; substituing in the above reasoning we get our claim. □

Theorem 1.101. *For any $\epsilon > 0$,*

$$\Xi(n) = O[n^{(A+\epsilon)\log\log(n)}] = O\Big[\exp\Big((A+\epsilon)\log(n)\log\log(n)\Big)\Big],$$

where $A := \frac{2}{\log(2)}$.

Proof. Let $A_\epsilon := A + \epsilon$. For every ϵ, and large enough n, we have $A_\epsilon \log\log(n) > 4$; therefore, by Proposition 1.65 we have, for large n,

$$\Xi(n) = \sum_{\mu=3}^{\infty} \Xi_\mu(n) = \sum_{\mu=3}^{A_\epsilon \log\log(n)} \Xi_\mu(n).$$

Using Proposition 1.100, this becomes

$$\Xi(n) \leq \sum_{\mu=3}^{A_\epsilon \log\log(n)} \Xi_\mu(n) \leq \sum_{\mu=3}^{A_\epsilon \log\log(n)} n^{\mu-1} \leq n^{A_\epsilon \log\log(n)},$$

and the claim follows. □

1.7.2. Multiplicity 3

When $\mu = 3$, the methods of the previous section can be specialized to give

$$\Xi_3(n) \leq \binom{n-1}{2} = \frac{(n-1)(n-2)}{2}.$$

However, under this condition we have a much more precise way of counting star operations, namely Theorem 1.88; using it, we will obtain a much better control over $\Xi_3(n)$. We will denote by $\xi_3(n)$ be the number of numerical semigroups of multiplicity 3 with exactly n star operations, so that $\Xi_3(n) = \sum_{k=2}^{n} \xi_3(k)$.

Proposition 1.102. $\xi_3(n) = |\{\binom{a}{b} : \binom{a}{b} = n, a + b \equiv 1 \bmod 3\}|$.

Proof. If $S = \langle 3, 3\alpha + 1, 3\beta + 2\rangle$, then $|\mathrm{Star}(S)| = \binom{\alpha+\beta+1}{2\alpha-\beta}$ and $\alpha + \beta + 1 + 2\alpha - \beta = 3\alpha + 1 \equiv 1 \bmod 3$; conversely, if $a + b \equiv 1 \bmod 3$, then the linear system

$$\begin{cases} \alpha + \beta + 1 = a \\ 2\alpha - \beta = b \end{cases}$$

has solutions $\alpha = \frac{a+b-1}{3}$, $\beta = \frac{2a-2b-1}{3}$, that are integers if $a + b \equiv 1 \bmod 3$, and verify $\alpha \leq 2\beta + 1$ and $\beta \leq 2\alpha$. Hence to each semigroup we can attach a binomial coefficient and to each coefficient a semigroup, these maps are inverses and the two sets have the same cardinality. □

Thus, to find all numerical semigroups of multiplicity 3 with exactly n star operations, we only need to determine the binomial coefficients $\binom{a}{b}$ equal to n. Since $\binom{a}{b} \geq a$ if $\binom{a}{b} \neq 1$, this means that we only need to inspect the case $a \leq n$.

Removing the congruence condition, we get the function $\eta(n) := |\{\binom{a}{b} : \binom{a}{b} = n\}|$, that has been studied in [108] and [1]. It is straightforward to see that $\eta(n)$ is finite for every $n > 1$, and it is also quick to show (quantifying the previous reasoning) that $\eta(n) \leq 2 + 2\log_2 n$ [108]. A deeper analysis, using results about the distribution of the primes, proves that $\eta(n) = O(\log n/\log\log n)$ [1]; these results are however weaker than the expected, since in [108] it is conjectured that η is bounded for $n > 1$.

Clearly, $\xi_3(n) \leq \eta(n)$, and thus $\xi_3(n) < \infty$ for every $n > 1$. (In particular, this gives a different proof of Theorem 1.26 when restricted to the case $\mu = 3$.) Note also that $\xi_3(1) = \infty$, because $|\mathrm{Star}(S)| = 1$ whenever $\alpha = 2\beta + 1$ or $\beta = 2\alpha$.

Proposition 1.103. *For every $n \in \mathbb{N}$, $\xi_3(n) \leq \frac{\eta(n)}{2}$.*

Proof. If $n = 1$, then both sides of the equality are infinite; suppose $n > 1$. Then, $\eta(n) = \xi_3(n) + \xi_3^{(0)}(n) + \xi_3^{(2)}(n)$, where $\xi_3^{(i)}$ is the number of binomial coefficients $\binom{a}{b}$ such that $\binom{a}{b} = n$ and $a + b \equiv i \bmod 3$. We will show that $\xi_3(n) = \xi_3^{(2)}$, from which the claim follows.

Suppose $\binom{a}{b} = n$ and $a+b \equiv 1 \bmod 3$. Then also $\binom{a}{a-b} = n$, and $a+(a-b) = 2a-b \equiv 2a + 2b \bmod 3 \equiv 2 \bmod 3$. Therefore, $\xi_3(n) = \xi_3^{(2)}(n)$, as claimed. □

Proposition 1.104. *Let $Z(x) := \{n : 1 < n \leq x, \xi_3(n) > 1\}$.*

(a) $|Z(x)| = O(\sqrt{x})$.
(b) There are an infinite number of integers n such that $\xi_3(n) = 0$.

Proof. Following the proof of [1, Theorem 1], let $g(x) := \{n : 1 < n \leq x, \eta(n) > 2\}$. If $\xi_3(n) > 1$, then $\eta(n) \geq 2\xi_3(n) > 2$. Therefore, $Z(x) \leq g(x) = O(\sqrt{x})$, applying again the proof of [1, Theorem 1].

Take an $n \in \mathbb{N}$ such that $\eta(n) = 2$. Then, the only binomial coefficients such that $\binom{a}{b} = n$ are $\binom{n}{1}$ and $\binom{n}{n-1}$. It follows that $\xi_3(n) = 1$ if $n+1$ or $n+(n-1)$ are congruent to 1 modulo 3, i.e., if $n \equiv 0 \bmod 3$ or $n \equiv 1 \bmod 3$, while $\xi_3(n) = 0$ otherwise, i.e., if $n \equiv 2 \bmod 3$. (Compare the following Proposition 1.105.)

Suppose that $\xi_3(n) = 0$ only for $n \in \{n_1, \ldots, n_k\}$. For every $m \equiv 2 \bmod 3$ such that $m \neq n_i$ for every i, there is a binomial coefficient $\binom{a}{b}$ such that $\binom{a}{b} = m$ and $a+b \equiv 1 \bmod 3$. The last condition implies that $a-b \neq b$ (otherwise, $a+b = a-b+2b = 3b \equiv 0 \bmod 3$); if $b = 1$ or $b = a-1$, then $\binom{a}{b} = a = m$, and so $a+b \equiv m+1 \equiv 0 \bmod 3$ or $a + b \equiv 2m - 1 \equiv 0 \bmod 3$, against the congruence condition. Therefore, $\binom{a}{b} = \binom{a}{a-b} = \binom{m}{1} = \binom{m}{m-1} = m$, and the four coefficients are different from each other, so that $\eta(m) \geq 4$. Thus, $g(x) \geq \frac{1}{3}x - k$, against the fact that $g(x) = O(\sqrt{x})$. Hence, $\xi_3(n) = 0$ infinitely often. □

Before estimating $\Xi_3(n)$, we single out the case of pseudosymmetric semigroups.

Proposition 1.105. *If $n \equiv 0, 1 \bmod 3$, $n > 1$, then there is a unique pseudosymmetric semigroup of multiplicity 3 such that $|\mathrm{Star}(S)| = n$; if $n \equiv 2 \bmod 3$, there is no such S.*

Proof. Let $S = \langle 3, 3\alpha + 1, 3\beta + 2\rangle$ be a pseudosymmetric semigroup of multiplicity 3.

If $\alpha \geq \beta$, then by Proposition 1.94 we have $\beta = 2\alpha - 1$; hence $|\mathrm{Star}(S)| = \binom{\alpha+\beta+1}{2\beta-\alpha+1} = \alpha + \beta + 1 = 3\beta + 1$; for each $n \equiv 1 \bmod 3$ there is a unique β and thus a unique pseudosymmetric semigroup.

Analogously, if $\beta \geq \alpha$, then $\alpha = 2\beta$, and $|\mathrm{Star}(S)| = \binom{\alpha+\beta+1}{2\beta-\alpha+1} = \alpha + \beta + 1 = 3\alpha$, and every $n \equiv 0 \bmod 3$ can be (uniquely) obtained this way. $\square$

Proposition 1.106. *Let $\Xi_3(n)$ as above. Then,*

$$\Xi_3(n) = \frac{2}{3}n + O(\sqrt{n}\log(n))$$

and, if ξ_3 is bounded (equivalently, if η is bounded), then

$$\Xi_3(n) = \frac{2}{3}n + O(\sqrt{n}).$$

Proof. Let $\alpha(n)$ (respectively, $\beta(n)$) be the number of semigroups S of multiplicity 3 such that $2 \leq |\mathrm{Star}(S)| \leq n$ and that are pseudosymmetric (respectively, that are not pseudosymmetric). Clearly, $\Xi_3(n) = \alpha(n) + \beta(n)$.

By Proposition 1.105, we have

$$\alpha(n) = \begin{cases} \frac{2}{3}n - 1 & \text{if } n \equiv 0 \bmod 3 \\ \left\lfloor \frac{2}{3}n \right\rfloor = \frac{2}{3}n - \frac{2}{3} & \text{if } n \equiv 1 \bmod 3 \\ \left\lfloor \frac{2}{3}n \right\rfloor = \frac{2}{3}n - \frac{4}{3} & \text{if } n \equiv 2 \bmod 3 \end{cases}$$

or, more shortly, $\alpha(n) = \frac{2}{3}n + O(1)$.

Consider now $\beta(n)$. We have $\beta(n) = \sum_{2\leq k\leq n} \xi_3'(k)$, where $\xi_3'(k)$ counts the non-pseudosymmetric semigroups of multiplicity 3 with exactly k star operations. If $\xi_3'(k) > 0$, then there must be (by Propositions 1.94 and 1.106) a binomial coefficient $\binom{a}{b} = k$ such that $b \neq 1, a-1$; in particular, using the notation of the proofs of Proposition 1.104 and of [1, Theorem 1], $g(k)$ must be at least 2. But, again by [1, Theorem 1], this can happen at most $O(\sqrt{n})$ times; hence,

$$\beta(n) = \sum_{\substack{2\leq k\leq n \\ g(k)>1}} \xi_3'(k) \leq O(\sqrt{n})\log(n) = O(\sqrt{n}\log(n))$$

since $\xi_3'(k) \leq \xi_3(k) \leq \log(k) + 1 \leq \log(n) + 1$ if $k \leq n$, by Proposition 1.103 and the discussione just before. Therefore,

$$\Xi_3(n) = \alpha(n) + \beta(n) = \frac{2}{3}n + O(1) + O(\sqrt{n}\log(n)) = \frac{2}{3}n + O(\sqrt{n}\log(n))$$

as claimed. If moreover ξ_3 is bounded, then the same reasoning gives $\beta(n) = O(\sqrt{n})$, and thus $\Xi_3(n) = \frac{2}{3}(n) + O(\sqrt{n})$. $\square$

1.7.3. Explicit calculation

The goal of this section is to determine all the numerical semigroups S such that $2 \leq |\mathrm{Star}(S)| \leq 15$.

Case 1. $\mu(S) = 3$.

By Theorem 1.88 and Proposition 1.102, numerical semigroups of multiplicity 3 with exactly n star operations are in bijective correspondence with binomial coefficients $\binom{a}{b}$ such that $\binom{a}{b} = n$ and $a + b \equiv 1 \bmod 3$.

Suppose $x := \binom{a}{b}$ is a binomial coefficient such that $x \leq 15$. Then, $a \leq 15$; the unique possibilities with $a + b \equiv 1 \bmod 3$ are the following.

- $\binom{\alpha+\beta+1}{2\alpha-\beta} = \binom{3}{1} = 3$: then, $\alpha = 1$ and $\beta = 1$, so $S = \langle 3, 4, 5\rangle$.
- $\binom{\alpha+\beta+1}{2\alpha-\beta} = \binom{4}{3} = 4$: then, $\alpha = 2$ and $\beta = 1$, so $S = \langle 3, 5, 7\rangle$.
- $\binom{\alpha+\beta+1}{2\alpha-\beta} = \binom{5}{2} = 10$: then, $\alpha = 2$ and $\beta = 2$, so $S = \langle 3, 7, 8\rangle$.
- $\binom{\alpha+\beta+1}{2\alpha-\beta} = \binom{6}{1} = 6$: then, $\alpha = 2$ and $\beta = 3$, so $S = \langle 3, 7, 11\rangle$.
- $\binom{\alpha+\beta+1}{2\alpha-\beta} = \binom{6}{4} = 15$: then, $\alpha = 3$ and $\beta = 2$, so $S = \langle 3, 8, 10\rangle$.
- $\binom{\alpha+\beta+1}{2\alpha-\beta} = \binom{7}{6} = 7$: then, $\alpha = 4$ and $\beta = 2$, so $S = \langle 3, 8, 13\rangle$.
- $\binom{\alpha+\beta+1}{2\alpha-\beta} = \binom{9}{1} = 9$: then, $\alpha = 3$ and $\beta = 5$, so $S = \langle 3, 10, 17\rangle$.
- $\binom{\alpha+\beta+1}{2\alpha-\beta} = \binom{10}{9} = 10$: then, $\alpha = 6$ and $\beta = 3$, so $S = \langle 3, 11, 19\rangle$.
- $\binom{\alpha+\beta+1}{2\alpha-\beta} = \binom{12}{1} = 12$: then, $\alpha = 4$ and $\beta = 7$, so $S = \langle 3, 13, 23\rangle$.
- $\binom{\alpha+\beta+1}{2\alpha-\beta} = \binom{13}{12} = 13$: then, $\alpha = 8$ and $\beta = 4$, so $S = \langle 3, 14, 25\rangle$.
- $\binom{\alpha+\beta+1}{2\alpha-\beta} = \binom{15}{1} = 15$: then, $\alpha = 5$ and $\beta = 9$, so $S = \langle 3, 16, 29\rangle$.

Case 2. $t(S) \geq 4$. By Corollary 1.60,

$$|\mathrm{Star}(S)| \geq 2(\omega(3) + \omega(2) + \omega(1)) - 3 \cdot 3 = 49.$$

Hence no semigroups arise from this case.

Case 3. $t(S) = 3$. Let $T(S) = \{\lambda, \tau, g\}$, with $\lambda < \tau < g$. By Proposition 1.59, we have $|\mathrm{Star}_g(S)| \geq 2\omega(2) - 5 = 7$, while $|\mathrm{Star}_\lambda(S)|$ and $|\mathrm{Star}_0(S)|$ are both at least 1. Consider τ: if $\tau > \mu$, then by Proposition 1.59(a) $|\mathrm{Star}_\tau(S)| \geq 2\omega(2) - 3 = 9$; suppose $\tau < \mu$. If $\tau \geq 3$, then by Propositions 1.62 and 1.59 (which is applicable since $\lambda < \tau$) we have again $|\mathrm{Star}_\tau(S)| \geq 2\omega(2) - 3 = 9$. Hence, in these cases we have

$$|\mathrm{Star}(S)| \geq |\mathrm{Star}_g(S)| + |\mathrm{Star}_\tau(S)| + |\mathrm{Star}_\lambda(S)| + |\mathrm{Star}_0(S)| \geq 7 + 9 + 1 + 1 = 18.$$

On the other hand, $\tau = 1$ is impossible ($0 < \lambda < \tau$) and $\tau = 2$ implies $g = 3$ and $S = \langle 4, 5, 6, 7\rangle$. By Example 1.54, in this case we have $|\mathrm{Star}(S)| = 14$.

Therefore, we can suppose $t(S) = 2$, i.e., $T(S) = \{\tau, g\}$ for some integer τ. Recall that, by definition, $\nu = \lceil \frac{\mu-1}{2} \rceil$.

Case 4. S is pseudosymmetric and $\mu > 3$.
If $g < \mu$, then the unique possibility is $\mu = 3$, which we have already considered.
If $\mu < g < 2\mu - 2$, then $\mu - 1 \in T(S)$, and $\mu - 1$ is different from g and τ, against $t(S) = 2$.
If $g = 2\mu - 2$, then $\tau = \mu - 1$; we can apply Proposition 1.92, obtaining $|\mathrm{Star}(S)| = 1 + \omega(\mu - 2)$. If $\mu = 4$ we have $|\mathrm{Star}(S)| = 1 + \omega(2) = 7$, while if $\mu = 5$ we have $|\mathrm{Star}(S)| = 1 + \omega(3) = 21$ and for bigger values of μ the cardinality of $\mathrm{Star}(S)$ is even greater. Therefore, we get the unique possibility $\mu = 4$, when $S = \langle 4, 5, 7\rangle$.
Suppose $g > 2\mu$. By Proposition 1.64, we have $\omega_{\mathrm{i}}(\mathcal{Q}_\tau) \geq \omega(\nu)$, and thus (if $|\mathrm{Star}(S)| < 16$) $\nu = 2$ and $\mu \in \{4, 5\}$.
Suppose $\mu = 4$, and let $\tau = 4k \pm 1$. By Proposition 1.97, $\mathcal{Q}_\tau \simeq \mathcal{R}(k+1, k+1)$, and so

$$\omega(\mathcal{Q}_\tau) = \omega(\mathcal{R}(k+1, k+1)) = \binom{2k+2}{k+1}$$

using Lemma 1.87. If $k = 2$, this means that $|\mathrm{Star}_\tau(S)| = \binom{6}{3} - 1 = 19 > 15$; hence $k \leq 1$. Since $k = 0$ is not possible, we must have $k = 1$, that is, $\tau = 3$ or $\tau = 5$. In the former case $g = 2\mu - 2$, so we are in the case of Proposition 1.92, which gives $S = \langle 4, 5, 7\rangle$ and $|\mathrm{Star}(S)| = 7$. In the latter, we are in the case $g = 2\mu + 2$; by the end of Section 1.6, $S = \langle 4, 7, 9\rangle$ and $|\mathrm{Star}(S)| = 15$.
Suppose now $\mu = 5$. Let $X := \{b \in \mathbb{N} \setminus S : \tau - \mu < b < \tau - \mu\}$ and $Y := \{b \in \mathbb{N} \setminus S : \tau < b < \tau + \mu\}$; we have $|X| \geq 2$, and since S is pseudosymmetric $|X| + |Y| = \mu - 1$=4. If $|X| = 3$, then by Proposition 1.64 $|\mathrm{Star}_\tau(S)| \geq \omega(3) - 1 = 19$ and $|\mathrm{Star}(S)| > 15$. Hence $|X| = |Y| = 2$; let $Y = \{b, b'\}$, with $b < b'$. Both Y_b and $Y_{b'}$ are nonempty: indeed, $b \in Y_{b'}$ and $b' - \mu \in Y_b$. Hence, by Propositions 1.93 and 1.59, both $|\mathrm{Star}_b(S)|$ and $|\mathrm{Star}_{b'}(S)|$ are at least $2\omega(1) - 3 = 3$.
By Proposition 1.64, if $\tau \geq 9$ then $|\mathrm{Star}_\tau(S)| \geq (2\omega(\nu) - 2) - 1 = 9$; hence, in this case we have

$$\begin{aligned} |\mathrm{Star}(S)| &\geq |\mathrm{Star}_\tau(S)| + |\mathrm{Star}_b(S)| + |\mathrm{Star}_{b'}(S)| + |\mathrm{Star}_g(S)| + |\mathrm{Star}_0(S)| \geq \\ &\geq 9 + 3 + 3 + 1 + 1 = 17 > 15. \end{aligned}$$

It remains to analyze the cases $\tau = 6$, $\tau = 7$ and $\tau = 8$.

- If $\tau = 6$, then $X = \{2, 3, 4\}$, which is impossible for the previous reasoning.
- If $\tau = 8$, then $6 \notin S$ (since $10 \in S$); let $Z := \{2, 4, 6\}$. For every $C \subseteq Z$, the set $J_C := S \cup \{x : x > \tau\} \cup \{7\} \cup C$ is an ideal of S which does not contain τ, and $J_C \subseteq J_{C'}$ if and only if $C \subseteq C'$; it follows (using the same proof of Proposition 1.64) that $\omega_{\mathrm{i}}(\mathcal{Q}_\tau) \geq \omega(3)$ and $|\mathrm{Star}(S)| \geq |\mathrm{Star}_\tau(S)| \geq \omega(3) - 1 = 19 > 15$.

- If $\tau = 7$, then $g = 14$ and, since 11, 12 and 13 must be in S, the semigroup must be $\langle 5, 6, 13\rangle = \{0, 5, 6, 10, 11, 12, 13, 15, \rightarrow\}$. We claim that there are (at least) 13 atoms in $\mathcal{G}_0(S)$. As in Example 1.55, let $I(a_1, \dots, a_k)$ be the ideal $S \cup \{a_1, \dots, a_k\}$. Then, the ideals

  - $I(3, 8, 9, 14)$
  - $I(3, 4, 8, 9, 14)$
  - $I(4, 9, 14)$
  - $I(4, 8, 9, 14)$
  - $I(8, 14)$
  - $I(8, 9, 14)$
  - $I(9, 14)$
  - $I(14)$

  do not contain τ and so are in $\mathcal{G}_1$ (and hence are atoms) by Proposition 1.91. By the proof of Proposition 1.93, $I_8 := I(1, 7, 9, 14)$ is not divisorial, and is in $\mathcal{Q}_8$; since $M_8 = I(1, 4, 7, 9, 14) = I_8 \cup \{4\}$, by Proposition 1.43 also I_8 is an atom. Analogously, $I_9 := I(2, 7, 8, 14)$ and $M_9 = I_9 \cup \{1\}$ are atoms. The 13th atom is $M_{14} = I(7)$. This gives 14 star operations (the $*_I$ plus the v-operation); moreover, by Corollary 1.35 every antichain with respect to the $*$-order induces a star operation different from the principal ones. By Proposition 1.53, $\{I_8, M_7\}$ and $\{I_9, M_8\}$ are antichains, and so $|\mathrm{Star}(S)| \geq 16 > 15$.

Let now $a := \min\{\tau, g - \tau\}$; then, $a \leq g/2$. If S is not pseudosymmetric, then $a \neq g - a$; thus, $\mathcal{Q}_g$ contains at least two elements, $S \cup \{\tau\}$ and $S \cup (g - \tau + S)$ (note that we can't suppose that the latter is different from M_g, nor that it does not contain τ). Hence, $\omega(\mathcal{Q}_g) \geq 3$ and $|\mathrm{Star}_g(S)| \geq 3$. We can also suppose $\mu > 3$, and $a < g/2$.

<u>**Case 5.**</u> $\mu < a < g/2$ and $\mu > 3$.

By Proposition 1.64, $\omega_{\mathrm{i}}(\mathcal{Q}_a) \geq \omega(\nu)$. Moreover, $\mathcal{Q}_g$ contains at least 2 ideals ($S \cup (a+S)$ and $S \cup (g - a + s)$) so that $|\mathrm{Star}_g(S)| \geq 2 \cdot 2 - 1 = 3$. We claim that $|\mathcal{Q}_{g-a}| \geq 3$.

Indeed, let $I_{g-a} := S \cup \{x : x > g - a\}$. Then, $I_{g-a} \in \mathcal{Q}_{g-a}$. There is a k such that $0 < g - a - \mu < g - k\mu < g - a$; hence, $\mathcal{Q}_{g-a}$ contains also $I_{g-a} \cup \{g - k\mu\}$ and $I_{g-a} \cup ((g - a) - (g - k\mu) + S)$. If the two latter ideals are equal, then $g - k\mu < \mu$, and so in particular $g - k\mu < a$. Thus, $(g - a) - a < \mu$, and thus $I_{g-a} \cup \{a\}$ is a new ideal in $\mathcal{Q}_{g-a}$ (note that in this case $a \neq g - k\mu$ since $a > \mu$). Hence, $|\mathcal{Q}_a| \geq 3$ and thus $|\mathrm{Star}_{g-a}(S)| \geq 2 \cdot 3 - 1 = 5$, and

$$|\mathrm{Star}(S)| \geq |\mathrm{Star}_g(S)| + |\mathrm{Star}_{g-a}(S)| + |\mathrm{Star}_a(S)| + |\mathrm{Star}_0(S)| \geq 3 + 5 + 5 + 1 = 14.$$

We also have $g - \mu > g - a$ (again, since $a > \mu$) and so $M_{g-\mu}$ generates another star operation. If now $(g - a) - a \geq \mu$, then $g - 2\mu > a$ (and $g - 2\mu \neq g - a$) so that $M_{g-2\mu}$ generates the 16th star operation; otherwise, $g - a \notin S \cup (a + S)$ and so $S \cup (a + S) \neq S \cup (g - a + S)$. This implies that $|\mathcal{Q}_g| \geq 3$ and $|\mathrm{Star}_g(S)| \geq 5$, and thus $|\mathrm{Star}(S)| \geq 17$. Thus, no semigroup arise from this case.

<u>**Case 6.**</u> $a < \mu$, $\mu > 3$ and S is not pseudosymmetric.

If $a = \tau$, then $g - \tau > g - \mu$; but this implies $g - \tau \in T(S)$, and since $T(S) = \{\tau, g\}$, it would follow that $\tau = g - \tau$, i.e., that S is pseudosymmetric. Since we have explicitly excluded that case, we have $a \neq \tau$, i.e., $a = g - \tau < \tau$.

By Proposition 1.62 we have $\omega_{\mathrm{i}}(\mathcal{Q}_a) \geq \omega(a-1)$, while $\omega_{\mathrm{i}}(\mathcal{Q}_x) \geq \omega(x-2)$ if $a < x < \mu$. Since $\omega(3) = 20$ and $|\mathrm{Star}(S)| \geq \omega_{\mathrm{i}}(\mathcal{Q}_y) - 1$ for all y, it follows that, if $|\mathrm{Star}(S)| \leq 16$, we have $a \leq 3$ and $x \leq 4$; in particular, $\mu \leq 5$.

If $\mu = 5$, then (using Proposition 1.59)

$$|\mathrm{Star}(S)| \geq |\mathrm{Star}_4(S)| + |\mathrm{Star}_3(S)| + |\mathrm{Star}_g(S)| + |\mathrm{Star}_0(S)| \geq$$

$$\geq [2\omega_{\mathrm{i}}(\mathcal{Q}_4) - 3] + [\omega_{\mathrm{i}}(\mathcal{Q}_3) - 1] + 2 = 2\omega(2) - 3 + \omega(2) - 1 + 2 = 16;$$

therefore, $\mu = 4$.

If $\tau < \mu$, then $a > 1$ (otherwise $g = \tau + a \leq \mu$, and $t(S) = 2$ would imply $S = \langle 3, 4, 5\rangle$); thus, $\tau \geq 3$. If $\tau \geq 4$, then $|\mathrm{Star}_\tau(S)| \geq 2\omega(\tau - 1) - 3 \geq 2\omega(3) - 3 \geq 37$; if $\tau = 3$ and $a = 2$, then $g = 5$ and $S = \{0, 4, 6, \rightarrow\}$. However, in this case, $t(S) = 3$, against our hypothesis. Thus, $\tau > \mu$, and $|\mathrm{Star}_\tau(S)| \geq 2\omega(1) - 3 = 3$. Hence,

$$|\mathrm{Star}(S)| \geq |\mathrm{Star}_0(S)| + |\mathrm{Star}_g(S)| + |\mathrm{Star}_\tau(S)| + |\mathrm{Star}_a(S)| \geq$$

$$\geq 1 + 3 + 3 + |\mathrm{Star}_a(S)| \geq 7 + |\mathrm{Star}_a(S)|.$$

If $a = 2$, then since $\mu = 4$ the integers g and τ must be odd. However, if $4l + 2$ is the minimal integer equivalent to 2 modulo 4 belonging to S, then $4l - 2 \in T(S)$: indeed, using $\mu = 4$, we have:

- $g - 2 + 4 + 4l - 2 = g + 4l > g$;
- $g + 4 + 4l - 2 > g$;
- $4l - 2 + 6 = 4l + 4 \in S$.

Thus, $t(S) = 3$, against our standing hypothesis.

Suppose that a is 1 or 3. First note that, if $g = 2\lambda$ is even and $\lambda > \mu$, then $\lambda \neq a, g - a$ and, by Corollary 1.52 we have $|\mathrm{Star}_\lambda(S)| \geq 2\omega(\nu) - 3 = 9$. If g is even and $\lambda < \mu$ the unique case is $g = 6$, but in that case both 3 and one among 1 and 5 are in $T(S)$, so that $t(S) = 3$.

On the other hand, if $k\mu < g < (k+1)\mu$, then $M_{g-\mu}, \ldots, M_{g-(k-1)\mu}$ generates other $k - 1$ star operations; the same happens if $j\mu < g - a < (j+1)\mu$ (we are excluding $g - k\mu$ since it may already be counted in some $\mathcal{Q}_z$ with $z < \mu$). Note that $j \geq k - 1$, so that if $g > k\mu$ then we get other $k - 1 + k - 2 = 2k - 3$ operations. Let thus k be the integer such that $k\mu < g < (k+1)\mu$.

If $a = 3$, then $\omega(\mathcal{Q}_a) = \omega(2)$, and $|\mathrm{Star}(S)| \geq 7 + 6 - 1 = 12$, and if g is even then $|\mathrm{Star}(S)| \geq 21$ (using $\mathrm{Star}_\lambda(S)$). If $k \geq 4$, then $|\mathrm{Star}(S)| \geq 12 + (2 \cdot 4 - 3) = 12 + 5 = 17 > 16$; hence $g < 4\mu$. Note also that $g \not\equiv 3 \bmod 4$ (otherwise $g - a \equiv 0 \bmod 4$ and so $g - a \in S$); hence $g \equiv 1 \bmod 4$ and thus $g \in \{5, 9, 13\}$.

- The case $g = 5$ is impossible (otherwise $g - a = 2 < 5$).

- If $g = 9$, then either $7 \in S$ (which implies $3 \in T(S)$) or $7 \notin S$ (which implies $7 \in T(S)$). In both cases $t(S) = 3$, which is absurd.
- If $g = 13$, then $k = 3$ and so $|\mathrm{Star}(S)| \geq 12+3 = 15$. If $11 \notin S$, then $11 \in T(S)$ and $t(S) = 3$; the same happens if $7 \notin S$. Therefore, $7 \in S$, and so $5 \notin S$. However, $\omega(\mathcal{Q}_5) \geq 5$ (since 2 and 3 are between $5 - \mu = 1$ and 5) instead of the 1 we had from the existence of $M_5 = M_{13-2\mu}$, and so $|\mathrm{Star}(S)| \geq 15 - 1 + (2 \cdot 5 - 1) = 23$, above the limit of 16.

If $a = 1$, then $\omega(\mathcal{Q}_a) = 2$, but we have also $|\mathrm{Star}_3(S)| \geq 3$ and $|\mathrm{Star}_2(S)| \geq 2\omega(0)-3 = 1$; hence $|\mathrm{Star}(S)| \geq 7 + (2 - 1) + 3 + 1 = 12$, and if g is even then $|\mathrm{Star}(S)| \geq 21$. As before, we must have $g \equiv 3 \bmod 4$; in this case, we also have the nondivisorial ideal $M_{g-2-k\mu}$ (since $g - 2 - k\mu > 1$), and so $|\mathrm{Star}(S)| \geq 12 + 2k - 2$, which is 16 or more if $k \geq 3$. Thus, $k \in \{1, 2\}$ and $g \in \{7, 11\}$.

- If $g = 7$ then $S = \langle 4, 5, 11\rangle$, and $|\mathrm{Star}(S)| = 14$ by Example 1.55.
- Let $g = 11$. Then, $S = \langle 4, 5, 13, 14\rangle$; we observe that $\mathcal{Q}_{10}$ contains three ideals – $S \cup \{11\}$, $S \cup \{11, 7\}$ and $S \cup \{11, 7, 3\}$ – so that $|\mathrm{Star}_{10}(S)| \geq 5$ instead of 3. Hence, $|\mathrm{Star}(S)| \geq 12 + 2 + 2 = 16$.

We have proved the following:

Theorem 1.107. *Let S be a numerical semigroup which is not symmetric. Then, $|\mathrm{Star}(S)| \leq 15$ if and only if one of the following holds:*

(a) $S = \langle 3, 4, 5\rangle$, *and* $|\mathrm{Star}(S)| = 3$*;*
(b) $S = \langle 3, 5, 7\rangle$, *and* $|\mathrm{Star}(S)| = 4$*;*
(c) $S = \langle 3, 7, 11\rangle$, *and* $|\mathrm{Star}(S)| = 6$*;*
(d) $S = \langle 3, 8, 13\rangle$, *and* $|\mathrm{Star}(S)| = 7$*;*
(e) $S = \langle 4, 5, 7\rangle$, *and* $|\mathrm{Star}(S)| = 7$*;*
(f) $S = \langle 3, 10, 17\rangle$, *and* $|\mathrm{Star}(S)| = 9$*;*
(g) $S = \langle 3, 7, 8\rangle$, *and* $|\mathrm{Star}(S)| = 10$*;*
(h) $S = \langle 3, 11, 19\rangle$, *and* $|\mathrm{Star}(S)| = 10$*;*
(i) $S = \langle 3, 13, 23\rangle$, *and* $|\mathrm{Star}(S)| = 12$*;*
(j) $S = \langle 3, 14, 25\rangle$, *and* $|\mathrm{Star}(S)| = 13$*;*
(k) $S = \langle 4, 5, 6, 7\rangle$, *and* $|\mathrm{Star}(S)| = 14$*;*
(l) $S = \langle 4, 5, 11\rangle$, *and* $|\mathrm{Star}(S)| = 14$*;*
(m) $S = \langle 3, 16, 29\rangle$, *and* $|\mathrm{Star}(S)| = 15$*;*
(n) $S = \langle 3, 8, 10\rangle$, *and* $|\mathrm{Star}(S)| = 15$*;*
(o) $S = \langle 4, 7, 9\rangle$, *and* $|\mathrm{Star}(S)| = 15$.

1.8. Applications to ring theory

1.8.1. Principal star operations

The concept of star operations was born in the setting of integral domains; therefore, it is natural to ask how much of the theory developed in Section 1.2 and in Section 1.3 holds also for rings. The theory has been somewhat sketched in [60], where the main point of interest was the study of m-canonical ideals, that is, ideals I such that $(I:(I:J))=J$ for every ideal J. (See Proposition 1.17 for a semigroup analogue.) The main difference between [60] and here is that we study $*_I$ even when $(I:I)\neq R$.

While the present section is thematically tied to the rest of this chapter (and is necessary for Sections 1.8.2 and 1.8.3), it uses some terminology and some results that we shall prove and discuss in more detail in Chapters 2 and 3. We will also return on the subject of principal star operations, from another point of view, in Section 3.3.2.

Definition 1.108. *Let R be an integral domain. For every $I\in\mathcal{F}(R)$, the star operation* generated by I, *denoted by $*_I$, is the supremum of all the star operations $*$ on R such that I is $*$-closed (that is, $I=I^*$). More explicitly,*

$$J^{*_I}:=J^v\cap(I:(I:J))=J^v\cap\bigcap_{\alpha\in(I:J)\setminus\{0\}}\alpha^{-1}I. \tag{1.5}$$

If $*=*_I$ for some ideal I, we say that $*$ is a principal *star operation.*

Note that the equality of the two representations in (1.5) follows from [60, Lemma 3.1]. It also follows that, if $(I:I)=R$, then $J^{*_I}=(I:(I:J))=\bigcap\alpha^{-1}I$.

Analogously, the star operation generated by a set Δ of fractional ideals of R is just the biggest star operation that closes all the members of Δ or, equivalently, the infimum of the $*_I$, as I ranges among Δ. With the same proof of Proposition 1.5, it can be seen that every star operation is the infimum of a family of principal star operations.

Definition 1.109. *Let I,J be ideals of the integral domain R. We say that I is $*$-minor than J, and we write $I\leq_* J$, if $*_I\geq *_J$, or, equivalently, if I is $*_J$-closed.*

Lemma 1.6 holds like in the case of semigroups, with the only difference that $\alpha+I$ must be changed to αI. Moreover, by essentially repeating the proof of Theorem 1.9, we get the following.

Proposition 1.110. *Let R be an integral domain and I,J be non-divisorial ideals of R. If $*_I=*_J$ then*

$$I=I^v\cap\bigcap_{\gamma\in(I:J)(J:I)\setminus\{0\}}(\gamma^{-1}I).$$

As in the case of semigroups, if I is an ideal of R and $a\in K\setminus\{0\}$ (where K is the quotient field of R), then $*_{aI}=*_I$, so that $aI\leq_* I$ and $I\leq_* aI$; in particular, $\leq_*$ is not a partial order on $\mathcal{F}(R)\setminus\mathcal{F}^v(R)$. Taking care of this equivalence is not enough:

Lemma 1.111. *Let I and L be ideals of a domain R, and suppose that L is invertible.*

(a) For every star operation $*$, $I^*L=(IL)^*$.
(a) $*_I=*_{IL}$.

Proof. (a) $I^*L \subseteq (I^*L)^* = (IL)^*$; conversely, if $L^{-1} := (R : L)$, then

$$(IL)^*L^{-1} \subseteq ((IL)^*L^{-1})^* = (ILL^{-1})^* = I^*$$

and thus $(IL)^* \subseteq I^*L$.

(a) By the previous point, $(IL)^{*_I} = I^{*_I}L = IL$, and thus $*_{IL} \leq *_I$; but L^{-1} is invertible, and hence $*_I = *_{ILL^{-1}} \leq *_{IL}$. □

We shall see later (Example 1.137) that even multiplication by invertible ideals (and even with the exclusion of divisorial ideals) is not enough to cover all the cases where $*_I = *_J$.

Another problem is the lack of a general "good" set of representatives for the classes of fractional ideals: while in the case of numerical semigroups we can restrict to the ideals I such that $S \subseteq I \subseteq \mathbb{N}$, there is no canonical way to find an analogue for an arbitrary ring, so we have to fall back to using the whole $\mathcal{F}(S)$ – with all the redundancies it involves.

We shall study in more detail (albeit briefly) five cases: local rings, intersection of primary ideals, unique factorization domains, pseudo-valuation domains and Noetherian domains. We also give an extension to semistar operations.

1.8.1.1. Local rings

The following result can be considered a generalization of [60, Lemma 2.2(e)].

Lemma 1.112. *Let I be an ideal of a domain R such that $(I : I) = R$. Let $\{J_\alpha \mid \alpha \in A\}$ be $*_I$-ideals such that $\bigcap_{\alpha\in A} J_\alpha \neq (0)$. Then,*

$$\left(I : \bigcap_{\alpha\in A} J_\alpha\right) = \left(\sum_{\alpha\in A}(I : J_\alpha)\right)^{*_I}.$$

Proof. Let $J := \sum_{\alpha\in A}(I : J_\alpha)$. Since $(I : I) = R$, we have $L^{*_I} = (I : (I : L))$ for every ideal L; therefore,

$$(I : J) = \left(I : \sum_{\alpha\in A}(I : J_\alpha)\right) = \bigcap_{\alpha\in A}(I : (I : J_\alpha)) = \bigcap_{\alpha\in A} J_\alpha^{*_I} = \bigcap_{\alpha\in A} J_\alpha$$

and thus

$$J^{*_I} = (I : (I : J)) = \left(I : \bigcap_{\alpha\in A} J_\alpha\right),$$

as claimed. □

The following definition abstracts a property proved, for m-canonical ideals of local domains, in [60, Lemma 4.1].

Definition 1.113. *Let $*$ be a star operation on a domain R. We say that an ideal I of R is* strongly $*$-irreducible *if $I = I^*$ and $I \neq \bigcap\{J \in \mathcal{F}(R) \mid J = J^*, I \subsetneq J\}$.*

Lemma 1.114. *Let R be a domain and I be a nondivisorial ideal of R. If I is strongly $*_I$-irreducible and $*_I = *_J$, then $I = uJ$ for some $u \in K$.*

Proof. Suppose $*_I = *_J$. Then

$$I = I^{*_J} = I^v \cap \bigcap_{\alpha \in (J:I)\setminus\{0\}} \alpha^{-1}J. \tag{1.6}$$

and both I^v and each $\alpha^{-1}J$ is a $*_I$-ideal. Hence either $I = I^v$ (which is impossible since I is not divisorial) or $I = \alpha^{-1}J$ for some $\alpha \in K$. □

Proposition 1.115. *Suppose (R, M) is a local ring and $R = (I : I)$. If M is $*_I$-closed, then I is strongly $*_I$-irreducible.*

Proof. Let $\{J_\alpha\}$ be a family of $*_I$-ideals such that $I = \bigcap J_\alpha$. Then

$$R = (I : I) = \left(I : \bigcap_\alpha J_\alpha\right) = \left(\sum_\alpha (I : J_\alpha)\right)^{*_I}$$

by Lemma 1.112.

Hence $(I : J_\alpha) \subseteq R$ for every α; suppose $I \subsetneq J_\alpha$ for all α. Then, $1 \notin (I : J_\alpha)$ and thus $(I : J_\alpha) \subseteq M$; therefore, $\sum(I : J_\alpha) \subseteq M$ and, since M is $*_I$-closed, also $(\sum_\alpha (I : J_\alpha))^{*_I} \subseteq M$, a contradiction. Therefore, we must have $J_\alpha = I$ for some α, and I is strongly $*_I$-irreducible. □

Corollary 1.116. *Let (R, M) be a local domain and I an ideal of R such that $(I : I) = R$. If $M = M^{*_I}$ (in particular, if M is divisorial), then $*_I = *_J$ for some ideal J if and only if $I = uJ$ for some $u \in K$.*

Corollary 1.117. *Let (R, M) be a local domain. If $(M : M) = R$ and M is not divisorial, then $*_M = *_I$ for some ideal I if and only if $I = uM$ for some $u \in K$.*

Proposition 1.118. *Suppose that R is a local ring with maximal ideal M and that $(R : M)$ is the complete integral closure of R. Let I, J be ideals of R such that $*_I = *_J$. If I and J are properly contained between R and $(R : M)$, then $(I : I) = (J : J)$.*

Proof. For every fractional ideal $R \subsetneq L \subsetneq (R : M)$, we have $(R : I) \subsetneq R$, and thus $(R : I) \subseteq M$; it follows that $L^v = (R : (R : L)) \supseteq (R : M) \supseteq L^v$, the last inequality coming from the fact that $(R : M)$ is always divisorial. Thus, $L^v = (R : M)$.

Let now $T_1 := (I : I)$ and $T_2 := (J : J)$; note that T_1 and T_2 are contained in $(R : M)$, since $(R : M)$ is the complete integral closure of R and both T_i are almost integral over R (being fractional ideals). For $i \in \{1, 2\}$, define $*_i$ as the star operation $*_i : L \mapsto L^v \cap LT_i$, for every $L \in \mathcal{F}(R)$. If $R \subsetneq L \subsetneq (R : M)$, then L is $*_i$-closed if and only if it is a T_i-ideal: indeed, by the previous paragraph, $L^{*_i} = LT_i \cap (R : M)$, and $LT_i \subseteq (R : M)$ since T_i is contained in $(R : M)$. Hence, $L^{*_i} = LT_i$.

Since $*_I = *_J$, for any star operation $*$ the ideal I is $*$-closed if and only if J is $*$-closed; therefore, I and J are both T_1- and T_2-ideals. But $(I : I)$ (respectively, $(J : J)$) is the maximal overring of R in which I (respectively, J) is an ideal; thus $(I : I) = (J : J)$. □

1.8.1.2. Intersection of primary ideals

While in general sums and intersections of ideals are not reflected in the principal operations generated, in case of coprime ideals there are some results that can be proved.

Proposition 1.119. *Let R be an integral domain, let $Q_1, \ldots, Q_n$ be ideals and let $I := Q_1 \cap \cdots \cap Q_n$.*

(a) If each Q_i is primary, $Q_i^v = R$ for every i and $Q_i + Q_j = R$ if $i \neq j$, then $*_I = *_{Q_1} \wedge \cdots \wedge *_{Q_n}$.
(b) If each Q_i is maximal, $*_I = *_{Q_1} \wedge \cdots \wedge *_{Q_n}$.

Proof. (a) Let $P_i := \mathrm{rad}(Q_i)$. Note that $P_i \neq P_j$ if $i \neq j$. By definition, I is $(*_{Q_1} \wedge \cdots \wedge *_{Q_n})$-closed, and thus $*_I \geq *_{Q_1} \wedge \cdots \wedge *_{Q_n}$. To prove the converse, we only need to show that every Q_i is $*_I$-closed.

Without loss of generality, consider $i = 1$, and let $\widehat{P} := P_2 \cap \cdots \cap P_n$ and $\widehat{Q} := Q_2 \cap \cdots \cap Q_n$. We claim that $\widehat{Q} = (I : Q_1)$. Indeed, let $x \in \widehat{Q}$ and $y \in Q_1$: then, $xy \in xR \subseteq Q_j$ for $j > 1$, and $xy \in yR \subseteq Q_1$, so $xy \in \widehat{Q} \cap Q_1 = I$ and $x \in (I : Q_1)$.

Conversely, let $x \in (I : Q_1)$. Since $(I : Q_1) \subseteq (R : Q_1) = R$ (using $Q_1^v = R$), we have $x \in R$. The ideal Q_1 is not contained in P_j if $j > 1$ (for otherwise $Q_1 + Q_j \subseteq P_j$, against the hypotheses), and thus Q_1 is not contained in $\widehat{P}$. Choose a $q \in Q_1 \setminus \widehat{P}$: then, for every $j > 1$, $xq \in Q_j$. If $x \notin Q_j$, then $q^t \in Q_j$ for some integer t; but this would imply $q \in \mathrm{rad}(Q_j) = P_j$ and $q \in \widehat{P}$, against our choice. Therefore, $x \in Q_j$, and $(I : Q_1) \subseteq Q_j$ for every j, i.e., $(I : Q_1) \subseteq \widehat{Q}$. Hence, $(I : Q_1) = \widehat{Q}$, as claimed. Therefore,

$$Q_1^{*_I} = Q_1^v \cap (I : (I : Q_1)) = R \cap (I : \widehat{Q}).$$

Suppose $q \in Q_1^{*_I}$. Then, $q\widehat{Q} \subseteq I \subseteq Q_1$; by prime avoidance, we can choose an $x \in \widehat{Q} \setminus P_1$. Then, $qx \in Q_1$, and if $q \notin Q_1$ there is an integer t such that $x^t \in Q_1$, and thus $x \in \mathrm{rad}(Q_1) = P_1$, against the choice of x. Hence, $q \in Q_1$, and $Q_1^{*_I} = Q_1$.

(b) As before, I is $*_{Q_1} \wedge \cdots \wedge *_{Q_n}$-closed, and the claim will follow if we prove that each Q_i is $*_I$-closed. We can also suppose that $Q_i \neq Q_j$ if $i \neq j$, for otherwise we merely discard one of the copies without affecting I nor $*_{Q_1} \wedge \cdots \wedge *_{Q_n}$.

Suppose that $Q_1, \ldots, Q_m$ are not divisorial while $Q_{m+1}, \ldots, Q_n$ are. Then, Q_k is $*_I$-closed if $k > m$, and $*_{Q_1} \wedge \cdots \wedge *_{Q_n} = *_{Q_1} \wedge \cdots \wedge *_{Q_m}$; but in the latter case, we can apply the previous point to $Q_1, \ldots, Q_m$, obtaining the claim. □

Remember that an integral domain is said to be *h-local* if every ideal is contained in a finite number of maximal ideals and every prime ideal is contained in only one maximal ideal.

Corollary 1.120. *Let R be an h-local Prüfer domain, and let $\mathcal{M}$ be the set of nondivisorial maximal ideals of R. Suppose that $\mathcal{M}$ is finite. Then, every star operation is principal, and there is a bijective correspondence between* $\mathrm{Star}(R)$ *and the set $\mathcal{G}$ of finite intersection of elements of $\mathcal{M}$.*

Proof. By the proof of [67, Theorem 3.1] (see also Proposition 3.100), any star operation $*$ on R is determined by its action on the elements of $\mathcal{M}$, and for every subset $A \subseteq \mathcal{M}$ there is a star operation $*^{(A)}$ such that, if $N \in \mathcal{M}$, then $N = N^{*^{(A)}}$ if and only if $N \in A$. In particular, for every $M \in \mathcal{M}$ there is a unique star operation $*^{(M)}$ such that $N = N^{*^{(M)}}$ if and only if $N = M$. Therefore, $*^{(M)}$ must be the star operation generated by M, i.e., $*^{(M)} = *_M$; since, for every $N \in \mathcal{M}$, we have $N^* = R$ if $N \neq N^*$, it follows every star operation $*$ can be written as $*_{M_1} \wedge \cdots \wedge *_{M_n}$, where $M_1, \ldots, M_n$ are the elements of $\mathcal{M}$ that are $*$-closed.

However, by Proposition 1.119, $*_{M_1} \wedge \cdots \wedge *_{M_n} = *_{M_1 \cap \cdots \cap M_n}$; hence, every star operation on R is principal, and there is a bijection between $\mathcal{G}$ and $\mathrm{Star}(R)$. □

Note that the above result fails if $\mathcal{M}$ is infinite: indeed, we always obtain that the star operations can be represented as an infimum of a family $\{*_{M_\alpha} \mid \alpha \in A\}$ for some $A \subseteq \mathcal{M}$, but if A is infinite then the intersection of the members of A is (0), and there is no ideal that generates $*$.

1.8.1.3. v-trivial ideals and unique factorization domains

Definition 1.121. *An ideal I of a domain R is* v-trivial *if $I^v = R$.*

A star operation $*$ is *semifinite* if, for every ideal $I = I^* \subsetneq R$, there is a prime ideal $P = P^*$ such that $I \subseteq P$. (We shall study more deeply semifinite operations from Definition 2.28 onwards.)

Lemma 1.122. *Let R be an integral domain, and let I, J be v-trivial ideals of R.*

(a) If $J \subseteq I$, then $J^{*_I} = I$.

Suppose v is semifinite on R.

(b) $I \cap J$ is v-trivial.
(c) $I \subseteq J^{*_I}$
(d) If $I \neq J$, then $*_I \neq *_J$.

Proof. (a) Since $J^v = R$, $(R : J) = R$; hence $R \subseteq (I : J) \subseteq (R : J) = R$, and thus

$$J^{*_I} = J^v \cap (I : (I : J)) = R \cap (I : R) = R \cap I = I.$$

(b) If $(I \cap J)^v \neq R$, then by semifiniteness there is a prime ideal P such that $I \cap J \subseteq P = P^v$: But this would imply $I \subseteq P$ or $J \subseteq P$, against the hypothesis that I and J are v-trivial.

(c) Since $J \subseteq J^{*_I}$, it follows that J^{*_I} is v-trivial, and by the previous point so it $J^{*_I} \cap I$. If $I \nsubseteq J^{*_I}$, it would follow that $J^{*_I} \cap I \subsetneq I$; but $J^{*_I} \cap I$ is $*_I$-closed, against (a). Hence $I \subseteq J^{*_I}$.

(d) If both I and J are $*_I$-closed, then so is $I \cap J$; by (b), $(I \cap J)^v = R$, and thus by (a) $(I \cap J)^{*_I} = I$ while $(I \cap J)^{*_J} = J$. It follows that $I = J$. □

Corollary 1.123. *Let R be a unique factorization domain. Then:*

(a) *for every principal star operation $* \neq v$ there is a proper ideal I such that $h(I) > 1$ and $* = *_I$;*
(b) *if I, J are fractional ideals of R, $*_I = *_J$ if and only if $I = uJ$ for some $u \in K$.*

Proof. Let $* = *_I$ for some ideal I. By [50, Corollary 44.5], every v-closed ideal of R is principal; hence, let $I^v = pR$. Then, $(p^{-1}I)^v = R$, i.e., $p^{-1}I$ is v-trivial. It follows that $h(p^{-1}I) > 1$ (since the v-maximal ideals of R are the height-1 primes) with $*_I = *_{p^{-1}I}$. Moreover, if $*_I = *_J$, then we can choose v-trivial ideals $p^{-1}I$ and $q^{-1}J$; since $*_{p^{-1}I} = *_{q^{-1}J}$, by Lemma 1.122(d) we have $p^{-1}I = q^{-1}J$, i.e., $I = (pq^{-1})J$. □

Proposition 1.124. *Let R be a domain such that v is semifinite, and let $I_1, \dots, I_n$ be v-trivial ideals; let $* := *_{I_1} \wedge \cdots \wedge *_{I_n}$. Then, the ideal $J(*) := I_1 \cap \cdots \cap I_n$ is the minimal v-trivial ideal that is $*$-closed.*

Proof. By Lemma 1.122(b), $J(*)$ is v-trivial. Clearly $J(*)$ is $*$-closed. Suppose L is v-trivial; then, applying Lemma 1.122(c),

$$L^* = L^{*_{I_1}} \cap \cdots \cap L^{*_{I_n}} \supseteq I_1 \cap \cdots I_n = J(*).$$

Therefore, $J(*)$ is the minimum among v-trivial $*$-closed ideals. □

Corollary 1.125. *Let R be a Noetherian unique factorization domain and let $* \in$ Star(R), $* \neq v$. If, for every $*$-closed ideal J, J^2 is $*$-closed, then $*$ is not the infimum of a finite family of principal star operations.*

Proof. Since R is Noetherian, v is semifinite; since it is a UFD, every principal star operation can be generated by a v-trivial ideal. If $*$ were to be finitely generated, $J(*)$ would be the minimal v-trivial $*$-closed ideal; but, by hypothesis, $J(*)^2$ is $*$-closed, and $J(*)^2 \subsetneq J(*)$. Therefore, $*$ is not finitely generated. □

Corollary 1.126. *Let R be a Noetherian unique factorization domain of dimension $d > 1$. Then, the b-operation is not the infimum of a finite family of principal star operations.*

Proof. Suppose that the b-operation is finitely generated, and let J be the minimal v-trivial ideal that is b-closed. Clearly, $J^2 \subseteq (J^2)^b \subseteq J = J^b$; by minimality, it follows that $(J^2)^b = J$. Let V be a discrete valuation overring of R such that $JV \neq V$; then, $J^2V \subsetneq JV$. But, for every ideal L, $LV = L^bV$; hence, we have $JV \neq J^2V = (J^2)^bV = JV$, a contradiction. □

A star operation on R is *spectral* if $I^* = \bigcap\{IR_P \mid P \in \Delta\}$ for some nonempty $\Delta \subseteq \mathrm{Spec}(R)$. We shall study spectral operations in Section 2.2.3.

Corollary 1.127. *Let R be a Noetherian unique factorization domain. If $* \neq v$ is a spectral star operation, then $*$ is not the infimum of a finite family of principal star operations.*

Proof. On a unique factorization domain, the v-operation is spectral; therefore, by Proposition 2.29, there is a prime ideal P such that $P = P^* \neq P^v$. For every $n \in \mathbb{N}$, $P^{(n)} = P^nR_P \cap R$ is $*$-closed, and thus $J(*) \subseteq P^{(n)}$. However, $\bigcap_{n \geq 1} P^{(n)} = (0)$, which would imply $J(*) \subseteq (0)$, which is absurd. □

1.8.1.4. Pseudo-valuation domains

Throughout this section, we let R be a pseudo-valuation domain (PVD) with valuation overring V, let F be the residue field of R, L the residue field of V and K the quotient field of R and V. We denote by M the maximal ideal of R, which coincides with that of V.

Let $\mathcal{F}_0(R) := \{I \in \mathcal{F}(R) \mid R \subseteq I \subseteq V\}$; we will start by showing that it represents all the fractional ideals of R. The following is a “dual” of [56, Corollary 2.15].

Lemma 1.128. *Preserve the notation defined at the beginning of this section. An ideal $I \in \mathcal{F}(R)$ is nondivisorial if and only if $I = aJ$ for some $a \in K$ and some $J \in \mathcal{F}_0(R)$ different from R and V.*

Proof. Let I be an ideal of R, and let $\Delta := \mathbf{v}(I)$, where $\mathbf{v}$ is the valuation associated to V. If Δ has not an infimum in the value group of V, then $I = \bigcap xV$, where x ranges among the elements of K such that $v(x) < \delta$ for each $\delta \in \Delta$; since $V = (R : M)$ is divisorial, so would be I.

On the other hand, if Δ has an infimum, then $\mathbf{v}(x) = \inf \Delta$ for some $x \in I$, and $J := x^{-1}I \in \mathcal{F}_0(R)$. If $J = R$ or $J = V$, then J is divisorial and so is I; if $R \subsetneq J \subsetneq V$, then $(R : J) \subseteq M$ and thus $J^v \supseteq (R : M) = V$; however, V is divisorial, and thus $J^v = V$. In particular, J is not be divisorial and thus also I is not divisorial. □

Proposition 1.129. *Let R be a pseudo-valuation domain with quotient field K. Let I, J be non-divisorial ideals of R and suppose that either*

1. *I or J is finitely generated;*
2. *$F \subseteq L$ is algebraic.*

Then $*_I = *_J$ of and only if $I = uJ$ for some $u \in K$.

Proof. Suppose $*_I = *_J$. Then, since $(R : M) = V$ is the complete integral closure of R, we can apply Proposition 1.118, and thus $(I : I) = (J : J) =: T$.

Let now $\sharp_I$ be the star operation generated by I on T, i.e., the biggest elements $*$ of $\mathrm{Star}(T)$ such that I is $*$-closed. Take a T-ideal L such that $R \subsetneq L \subsetneq V$; we claim that $L^{*_I} = L^{\sharp_J}$. Indeed, by Lemma 1.128 the unique divisorial ideals in $\mathcal{F}_0(R)$ are R and V; in particular, $L^v = V$ (where v is the v-operation on R). Moreover, $(I : L) \supseteq (I : R) = I$, and thus $(I : (I : L)) \subseteq (I : I) \subseteq V$; hence,

$$L^{*_I} = L^v \cap (I : (I : L)) = V \cap (I : (I : L)) = (I : (I : L)) = L^{\sharp_I},$$

the last equality coming from the fact that $(I : I) = T$. Thus, $*_I = *_J$ implies $\sharp_I = \sharp_J$.

Under one of the two hypotheses, $(I : I)$ is integral over R; hence it satisfies incomparability and by [56, Theorem 1.7], $(I : I) =: T$ is a PVD. In particular, the maximal ideal of T is divisorial over T; therefore, we can apply Corollary 1.116 to obtain $I = uJ$ for some $u \in K$. □

1.8.1.5. Noetherian domains

For this last case, we will use freely some notations which we introduce and study more deeply in Chapters 2 and 3.

Proposition 1.130. *Let R be a domain and I an ideal of R. Then,* $\mathrm{QSpec}^{*_I}(R) \supseteq \mathrm{QSpec}^v(R) \cup \mathrm{Ass}(I)$*, and if R is Noetherian the two sets are equal.*

Proof. If $P \in \mathrm{Ass}(I)$, then $P = (I :_R x) = x^{-1}I \cap R$ for some $x \in R$, and thus it is $*_I$-closed; if $P \in \mathrm{QSpec}^v(R)$ then $P = P^v$ and thus $P = P^{*_I}$.

Conversely, suppose R is Noetherian and $P = P^{*_I}$. Then $P = P^v \cap (I : (I : P)) = P^v \cap (I : J)$, where $J = (I : P)$; let $J = j_1R + \cdots + j_nR$. We have

$$\begin{aligned} P &= P^v \cap (I : J) = P^v \cap R \cap (I : J) = P^v \cap (I :_R J) = \\ &= P^v \cap (I :_R j_1R + \cdots + j_nR) = P^v \cap \textstyle\bigcap_{i=1}^n (I :_R j_iR), \end{aligned}$$

and, since P is prime, this implies that $P^v = P$ or $(I :_R j_iR) = P$ for some i. In the latter case, since $j_i \in K$, $j_i = a/b$ for some $a, b \in R$; hence $(I :_R j_iR) = (I : ab^{-1}R) \cap R = (bI :_R aR)$, and thus P is associated to bI. There is an exact sequence

$$0 \longrightarrow \frac{bR}{bI} \longrightarrow \frac{R}{bI} \longrightarrow \frac{R}{bR} \longrightarrow 0$$

and, since R is a domain, $bR/bI \simeq R/I$ and thus $\mathrm{Ass}(bI) \subseteq \mathrm{Ass}(I) \cup \mathrm{Ass}(bR)$ [17, Chapter IV, Proposition 3]; therefore, P is associated to I or it is divisorial (since an associated prime of a divisorial ideal – in this case, bR – is divisorial). □

Remark 1.131. Note that, if $P^v = R$, then $(I : P) \subseteq (R : P) = R$, and thus $j_i \in R$; in this case, the last part of the proof could be cut and, since $*_{bI} = *_I$, we could use it to deduce that, if $P^v = R$ and $P \in \mathrm{Ass}(I)$, then $P \in \mathrm{Ass}(bI)$ for every $b \in R$.

Proposition 1.130 allows to determine, in the Noetherian case, all the spectra of the principal star operations. We need a lemma.

Lemma 1.132. *Let R be a Noetherian ring and $\Delta \subseteq \mathrm{Spec}(R) \setminus \{(0)\}$ be a finite set. There is an ideal I of R such that $\mathrm{Ass}(I) = \Delta$.*

Proof. We proceed by induction on $n = |\Delta|$. If $n = 1$ and $\Delta = \{P\}$ we can take $I = P$.

Suppose $n > 1$ and let $\Delta = \{P_1, \ldots, P_n\}$; without loss of generality we can suppose $P_i \nsubseteq P_j$ for every $i > j$. Let I_0 be an ideal such that $\mathrm{Ass}(I_0) = \{P_1, \ldots, P_{n-1}\}$, and let $I_0 = Q_1 \cap \cdots \cap Q_{n-1}$ be a primary decomposition, where $P_i := \mathrm{rad}(Q_i)$. Since the intersection of all P_n-primary ideals is (0), there is a P_n-primary ideal Q_n such that $Q_n \nsubseteq I_0$; let $I := I_0 \cap Q_n$. To show that $\mathrm{Ass}(I) = \Delta$, it is enough to prove that $Q_a \cap \cdots \cap Q_n$ is an irredundant intersection.

Suppose Q_i is redundant. By construction, $i \neq n$; moreover, if $i = 1$, then $Q_2 \cap \cdots \cap Q_n \subseteq Q_1$ and thus, passing to the radical, $P_2 \cap \cdots \cap P_n \subseteq P_1$, and $P_j \subseteq P_1$ for some $j > 1$, against the hypothesis. Hence suppose $1 < i < n$, and let $L_1 := Q_1 \cap \cdots \cap Q_{i-1}$ and $L_2 := Q_{i+1} \cap \cdots \cap Q_n$. By inductive hypothesis, $Q_1 \cap \cdots \cap Q_i = L_1 \cap Q_i$ is irredundant, and thus $L_1 \nsubseteq Q_i$; let $x \in L_1 \setminus Q_i$. For every $a \in L_2$, $xa \in L_1L_2 \subseteq L_1 \cap L_2 \subseteq Q_i$ (since Q_i is redundant), and thus $L_2 \subseteq (Q_i :_R x)$. However, $\mathrm{rad}((Q :_R x)) \neq R$, and thus $\mathrm{rad}((Q_i :_R x)) = \mathrm{rad}(Q_i) = P_i$; hence, $\mathrm{rad}(L_2) \subseteq \mathrm{rad}(Q_i)$, i.e., $P_{i+1} \cap \cdots \cap P_n \subseteq P_i$. However, this implies that $P_j \subseteq P_i$ for some $j > i$, which still is against the hypothesis. Therefore, no Q_i can be redundant. □

Corollary 1.133. *Let R be a Noetherian domain, and let $\Delta \subseteq \mathrm{Spec}(R)$. Then $\Delta = \mathrm{QSpec}^{*_I}(R)$ for some ideal I if and only if $\Delta = \mathrm{QSpec}^v(R) \cup \Delta'$, where Δ' is a finite set.*

Proof. If $\Delta = \mathrm{QSpec}^{*_I}(R)$, then $\Delta = \mathrm{QSpec}^v(R) \cup \mathrm{Ass}(I)$, and $\Delta' = \mathrm{Ass}(I)$ is finite.

Conversely, if $\Delta = \mathrm{QSpec}^v(R) \cup \Delta'$, with Δ' finite, then by Lemma 1.132 there is an ideal I of R such that $\mathrm{Ass}(I) = \Delta'$, and $\Delta = \mathrm{QSpec}^{*_I}(R)$ by Proposition 1.130. □

We can associate to every star operation $\sharp$ a spectral star operation $\widetilde{\sharp}$ defined by $I^{\widetilde{\sharp}} := \bigcap_{P \in \mathrm{QMax}^{*_f}(R)} IP$. See Section 2.2.3 for more details.

Corollary 1.134. *Let $*$ be a spectral operation on a Noetherian domain R. Then $* = \widetilde{*_I}$ for some I if and only if $\mathrm{QMax}^*(R) \setminus \mathrm{QMax}^v(R)$ is finite.*

Proof. If $* = \widetilde{*_I}$, then $\mathrm{QMax}^*(R) \subseteq \mathrm{QMax}^v(R) \cup \mathrm{Ass}(I)$, and thus $\mathrm{QMax}^*(R) \setminus \mathrm{QMax}^v \subseteq \mathrm{Ass}(I)$ is finite. Conversely, if $\Delta := \mathrm{QMax}^*(R) \setminus \mathrm{QMax}^v(R)$ is finite, it is enough to apply the previous lemma to Δ; then $\mathrm{QSpec}^{*_I}(R) = \mathrm{QSpec}^v(R) \cup \Delta$ and thus $\mathrm{QMax}^{\widetilde{*_I}}(R)$ is the set of maximal elements of $\Delta \cup \mathrm{QMax}^v$, so that $\widetilde{*_I} = *$. □

Theorem 1.135. *Let R be a Noetherian domain, and let I, J be non-divisorial ideals such that $(I : I) = (J : J) = R$. Then, $*_I = *_J$ if and only if $\mathrm{Ass}(I) \cup \mathrm{QSpec}^v(R) = \mathrm{Ass}(J) \cup \mathrm{QSpec}^v(R)$ and, for every $P \in \mathrm{Ass}(I) \cup \mathrm{QSpec}^v(R)$, there is an $a_P \in K$ such that $IR_P = a_P JR_P$.*

Proof. ($\Longleftarrow$). By Proposition 1.130, $\mathrm{Ass}(I) \cup \mathrm{QSpec}^v(R) = \mathrm{QSpec}^{*_I}(R)$, and thus $\mathrm{QSpec}^{*_I}(R) = \mathrm{QSpec}^{*_J}(R) =: \Delta$. With the same proof of Lemma 3.76, for any I the localization $(*_I)_{R_P}$ is equal to $*_{IR_P}$; using Proposition 3.33, we have, for any ideal L,

$$L^{*_I} = \bigcap_{P \in \Delta} L^{*_I} R_P = \bigcap_{P \in \Delta} (LR_P)^{*_{IR_P}} = \bigcap_{P \in \Delta} (LR_P)^{*_{JR_P}} = \bigcap_{P \in \Delta} L^{*_J} R_P = L^{*_J}$$

and hence $*_I = *_J$.

($\Longrightarrow$). Suppose $*_I = *_J =: *$; then $\mathrm{QSpec}^*(R)$ is equal to both $\mathrm{Ass}(I) \cup \mathrm{QSpec}^v(R)$ and $\mathrm{Ass}(J) \cup \mathrm{QSpec}^v(R)$, which thus are equal. Note also that $(I : I) = R$ implies that $R_P = (I : I)R_P = (IR_P : IR_P)$ for every prime ideal P.

Let now $P \in \mathrm{QSpec}^*(R)$. Since $*_I = *_J$, clearly $(*_I)_{R_P} = (*_J)_{R_P}$, so that $*_{IR_P} = *_{JR_P}$. However, PR_P is $*_{IR_P}$-closed because P is $*_I$-closed; it follows, by Corollary 1.116, that $IR_P = a_P JR_P$ for some $a_P \in K$, as claimed. □

We denote by $X^1(R)$ the set of height-1 prime ideals of R.

Corollary 1.136. *Let R be an integrally closed Noetherian domain, and let I, J be non-divisorial ideals. Then, $*_I = *_J$ if and only if* $\mathrm{Ass}(I) \cup X^1(R) = \mathrm{Ass}(J) \cup X^1(R)$ *and for every $P \in \mathrm{Ass}(I)$ there is an $a_P \in R_P$ such that $IR_P = a_P JR_P$.*

Proof. It is enough to apply Theorem 1.135, noting that $(I : I) = (J : J) = R$ for every I, J, that the divisorial primes are the height 1 primes (this follows, for example, from [77, Theorems 94 and 95] and Krull's principal ideal theorem, or from the fact that $R = \bigcap\{R_P \mid P \in X^1(R)\}$), and that IR_P and JR_P are isomorphic for $P \in X^1(R)$ since R_P is a DVR. □

Example 1.137. Let R be a Noetherian integrally closed ring, and suppose that R_M is not a UFD for some maximal ideal M. Let P be an height 1 prime contained in M such that PR_M is not principal, and let Q be a prime ideal of height bigger than 1 such that $P + Q = R$ (in particular, $Q \nsubseteq M$). We claim that $*_{PQ} = *_Q$ but PQ and Q are not locally isomorphic.

In fact, since they are coprime, $PQ = P \cap Q$, and thus $\mathrm{Ass}(PQ) = \{P, Q\}$ while $\mathrm{Ass}(Q) = \{Q\}$; moreover, $P \nsubseteq Q$ and thus $PQR_Q = QPR_Q = QR_Q$. Since $P \in X^1(R)$, by Corollary 1.136 it follows that $*_{PQ} = *_Q$. However, $QR_M = R_M$ is principal, while $PQR_M = PR_M$, by hypothesis, is not: therefore, Q and PQ are not locally isomorphic. In particular, there cannot be an invertible ideal L such that $Q = LPQ$, because LR_M would be principal and thus Q and PQ would be locally isomorphic.

1.8.1.6. Semistar operations

The definition of principal star operations works in the same way in the context of semistar operations; that is, for every R-submodule I of the quotient field K we can define $\wedge_I$ as the biggest semistar operation that closes I or, equivalently, we can define $L^{\wedge_I} := (I : (I : L))$ for every $L \in \mathbf{F}(R)$. When $(I : I) = R$, then $\wedge_I$ coincides with $*_I$ on the fractional ideals of R, while it sends every other submodule to the whole quotient field K; more generally, the star operation $*_I$ can be thought of as the restriction to the fractional ideals of the meet between $\wedge_I$ and $\wedge_R = v$.

As in the star operation case, the set of principal semistar operations generates, by taking infima, all semistar operations; however, since principal operations are trivial on non-fractional ideals, they are in general even farther from the semistar operations that are commonly studied, and there is in general very little hope to obtain rings such that principal semistar operations, or the infima of finite sets of principal operations, represent a significant fraction of all semistar operations.

1.8.2. Semigroup rings

In the last two sections of this chapter, we study two classes of rings whose theory is very close to the theory of numerical semigroups.

Among rings, the closest analogue to numerical semigroups are *semigroup rings*, that is, rings of the form $K[[S]] := K[[X^S]] := K[[\{X^s : s \in S\}]]$, where K is a field and S is a numerical semigroup. Each semigroup ring is a Noetherian local domain of dimension

1, and its integral closure is the power series ring $K[[X]]$, which is a fractional ideal of $K[[S]]$. Throughout the rest of this section, we denote by R any ring of this form.

Let $\mathbf{v}$ be the canonical valuation associated to $K[[X]]$. For every subset A of the quotient field $K((X))$ of $K[[X]]$, let $\mathbf{v}(A) := \{\mathbf{v}(a) : a \in A, a \neq 0\}$. Note that, if $\phi \in K((X))$, $\phi \neq 0$, then $\phi = X^{\mathbf{v}(\phi)}\phi_1$, where $\phi_1 = a_0 + a_1X + \cdots$ is a unit of $K[[X]]$ (and in particular $a_0 \neq 0$).

Definition 1.138. *Let $R = K[[S]]$, where S is a numerical semigroup. A* monomial ideal *of R is a fractional ideal generated by a set $\{X^n : n \in N\}$ of monomials, where $N \subseteq \mathbb{Z}$. For each fractional ideal I of R, the* monomial ideal associated to I, *denoted by* $\mathrm{Mon}(I)$, *is the biggest monomial ideal contained in I, or, explicitly, the ideal generated by all the X^n contained in I. We denote $\mathbf{v}(\mathrm{Mon}(I))$ as $\mathrm{vMon}(I)$.*

To each fractional ideal U of a numerical semigroup S we can associate the monomial ideal X^U of $K[[S]]$, generated by all the X^u with $u \in U$. Since $X^uX^s \in X^U$ for every $s \in S$, an element $\phi = \sum a_iX^i \in K((X))$ is in X^U if and only if $a_i = 0$ for every $i \notin U$. In particular, $\mathrm{Mon}(I) = X^{\mathrm{vMon}(I)}$, and thus the monomial ideals of R are in bijective correspondence with the ideals of S. The following two propositions, whose proofs are straightforward, collect some properties of this correspondence.

Proposition 1.139. *Let $T, U, \{U_\alpha : \alpha \in A\}$ be fractional ideals of S, $n \in \mathbb{Z}$, $R = K[[S]]$.*

- *(a) If $\bigcup_{\alpha\in A} U_\alpha$ is a fractional ideal of S, then $X^{\bigcup_{\alpha\in A} U_\alpha} = \sum_{\alpha\in A} X^{U_\alpha}$.*
- *(b) $X^{\bigcap_{\alpha\in A} U_\alpha} = \bigcap_{\alpha\in A} X^{U_\alpha}$.*
- *(c) $X^{(U-T)} = (X^U : X^T)$.*
- *(d) $X^{n+T} = X^nX^T$.*
- *(e) $X^{T+U} = X^TX^U$.*

Proposition 1.140. *Let $\{I_\alpha\}_{\alpha\in A}$ be ideals of $R = K[[S]]$.*

- *(a) $\mathrm{Mon}\left(\bigcap_{\alpha\in A} I_\alpha\right) = \bigcap_{\alpha\in A} \mathrm{Mon}(I_\alpha)$.*
- *(b) $\mathrm{Mon}\left(\sum_{\alpha\in A} I_\alpha\right) \supseteq \bigcup_{\alpha\in A} \mathrm{Mon}(I_\alpha)$, and equality holds if all the I_α are monomial ideals.*

However, the relation between $\mathrm{Mon}(I)$ and $\mathrm{Mon}(\psi I)$, where $\psi \in K((X))$, is not so simple.

Definition 1.141. *Let $\psi := \sum a_iX^i \in K((X))$, $\psi \neq 0$. The* support $\mathrm{supp}(\psi)$ *of ψ is the set of $i \in \mathbb{Z}$ such that $a_i \neq 0$. We denote by T_ψ the S-ideal generated by the support of S, that is, $T_\psi := \mathrm{supp}(\psi) + S$.*

Proposition 1.142. *Let $R = K[[S]]$, S a numerical semigroup, $U \in \mathcal{F}(S)$ and $\phi \in K((X)) \setminus \{0\}$. Then $\mathrm{vMon}(\phi(X^U)) = (U - T_{\phi^{-1}})$ and $\mathrm{Mon}(\phi(X^U)) = (X^U : X^{T_{\phi^{-1}}})$.*

Proof. Note that

$$s \in \mathrm{vMon}(\phi(X^U)) \iff X^s \in \phi(X^U) \iff \phi^{-1}X^s \in X^U.$$

Since X^U is composed of the elements of the form $\sum a_i X^{u_i}$, with $u_i \in U$, this happens if and only if $X^t X^s \in X^U$ for each $t \in \mathrm{supp}(\phi^{-1})$. Hence $s \in \mathrm{vMon}(\phi(X^U))$ if and only if $X^{t+s} \in (X^U)$ for every $t \in T_{\phi^{-1}}$, and this happens if and only if $t+s \in U$ for every $t \in T_{\phi^{-1}}$, if and only if $s + T_{\phi^{-1}} \subseteq U$, that is, if and only if $s \in (U - T_{\phi^{-1}})$. Thus $\mathrm{vMon}(\phi(X^U)) = (U - T_{\phi^{-1}})$, and $\mathrm{Mon}(\phi(X^U)) = (X^U : X^{T_{\phi^{-1}}})$ by Proposition 1.139(c). □

When I is not monomial, we cannot, in general, individuate $\mathrm{Mon}(\phi I)$. However, we have a sort of "global" version.

Proposition 1.143. *Let $R = K[[S]]$, S a numerical semigroup, $I \in \mathcal{F}(R)$. Then*

$$\sum_{\substack{\phi \in K((X)) \\ \mathbf{v}(\phi)=0}} \mathrm{Mon}(\phi I) = X^{\mathbf{v}(I)}.$$

Proof. By Proposition 1.139(a) it is enough to show that $\bigcup_{\mathbf{v}(\phi)=0} \mathrm{vMon}(\phi I) = \mathbf{v}(I)$.

Let $k \in \bigcup_{\mathbf{v}(\phi)=0} \mathrm{vMon}(\phi I)$. Then $k \in \mathrm{vMon}(\phi I)$, that is, $X^k \in \phi I$ for some ϕ such that $\mathbf{v}(\phi) = 0$. Therefore, $\phi^{-1} X^k \in I$, and thus $\mathbf{v}(\phi^{-1} X^k) \in \mathbf{v}(I)$, and $-\mathbf{v}(\phi) + k \in \mathbf{v}(I)$. Hence, $k \in \mathbf{v}(I)$.

Conversely, if $k \in \mathbf{v}(I)$, then there is a $\psi \in I$ such that $\psi = X^k \psi_1$, with $\mathbf{v}(\psi_1) = 0$; however, this implies that $X^k \in \psi_1^{-1} I$, and $k \in \mathrm{vMon}(\psi_1^{-1} I)$. □

Definition 1.144. *Let $R = K[[S]]$, S a numerical semigroup. A star operation $*$ on R is* monomial *if I^* is a monomial ideal for each monomial ideal $I \in \mathcal{F}(R)$.*

It is easily seen that the identity operation d is monomial, and the same is true for the divisorial closure v, since, using Proposition 1.139(c),

$$(X^U)^v = (R : (R : X^U)) = (X^S : (X^S : X^U)) =$$

$$= (X^S : X^{(S-U)}) = X^{(S-(S-U))} = X^{(U^v)}.$$

Let $\sharp$ be a star operation on S. We can define on R the star operation $*$ generated by the X^U, where U ranges among the $\sharp$-closed ideals, that is, $*$ is the maximum star operation that closes all the X^U. We proceed to show that $*$ is monomial, and that we can reobtain $\sharp$ from $*$. To do this, we need to characterize monomial operations through the set of closed ideals.

Proposition 1.145. *Let $R = K[[S]]$, S a numerical semigroup. Let $\Delta \subseteq \mathcal{F}(R)$, suppose $R \in \Delta$, and let $* = *_\Delta$ be the star operation generated by Δ. The following are equivalent:*

(i) $*$ *is monomial;*
(ii) *for every $I \in \mathcal{F}^*$, $\mathrm{Mon}(I) \in \mathcal{F}^*$;*
(iii) *for every $I \in \Delta$ and each $\phi \in K((X)) \setminus (0)$, $\mathrm{Mon}(\phi I) \in \mathcal{F}^*$.*

In this case, $(X^U)^* = \bigcap\{\mathrm{Mon}(\alpha^{-1} I) : I \in \Delta, X^U \subseteq \alpha^{-1} I\}$.

Proof. (i $\Longrightarrow$ ii). Let $I \in \mathcal{F}^*$: $\mathrm{Mon}(I)$ is a monomial ideal, and thus also $(\mathrm{Mon}\, I)^*$ is monomial. Since star operations are extensive and I is $*$-closed, we have $\mathrm{Mon}(I) \subseteq (\mathrm{Mon}\, I)^* \subseteq I^* = I$. By definition, $\mathrm{Mon}\, I$ is the biggest monomial ideal contained in I, and thus $(\mathrm{Mon}\, I)^* = \mathrm{Mon}\, I$.

(ii $\Longrightarrow$ iii) is trivial, since $\Delta \subseteq \mathcal{F}^*$ and $\phi I \in \mathcal{F}^*$ whenever $I \in \mathcal{F}^*$.

(iii $\Longrightarrow$ i). Let $J = (X^U)$ for some ideal U of S. Then $J^* = \bigcap_{\alpha\in A} L_\alpha$, where each L_α is of the form ϕI for some $\phi \in K((X)) \setminus \{0\}$, $I \in \Delta$. On the set $\mathcal{L} := \{L_\alpha : \alpha \in A\}$, define the equivalence relation $\sim$ such that $L_\alpha \sim L_\beta$ if and only if $\mathrm{Mon}(L_\alpha) = \mathrm{Mon}(L_\beta)$: it determines a partition $\{\mathcal{B}_\beta\}$ on $\mathcal{L}$. Let $B_\beta := \mathrm{Mon}(L)$ for some $L \in \mathcal{B}_\beta$. From the definition, it follows that $B_\beta \subseteq L$ for every $L \in \mathcal{B}_\beta$; since $J^* \subseteq L$, we have also $J \subseteq L$ and $J \subseteq B_\beta$ since J is monomial. Moreover, by hypothesis, each B_β is $*$-closed. Thus $J^* = \bigcap B_\beta$ is an intersection of monomial ideals, and hence it is monomial.

The last statement follow directly from the proof of (iii $\Longrightarrow$ i). $\square$

Proposition 1.146. *Let $\{*_\lambda\}_{\lambda\in\Lambda}$ be a set of monomial star operations on $R = K[[S]]$. Then $\inf_{\lambda\in\Lambda} *_\lambda$ and $\sup_{\lambda\in\Lambda} *_\lambda$ are monomial.*

Proof. Let $*_1 := \inf_{\lambda\in\Lambda} *_\lambda$ and $*_2 := \sup_{\lambda\in\Lambda} *_\lambda$.

For every ideal $I \in \mathcal{F}(R)$, $I^{*_1} = \bigcap_{\lambda\in\Lambda} I^{*_\lambda}$. In particular, if I is a monomial ideal, so is I^{*_λ}, for every $\lambda \in \Lambda$. Therefore, by Proposition 1.139(b), I^{*_1} is monomial, and $*_1$ is a monomial star operation.

By definition, $\mathcal{F}^{*_2} = \bigcap_{\lambda\in\Lambda} \mathcal{F}^{*_\lambda}$. Therefore, if $I \in \mathcal{F}^{*_2}$, we have $I \in \mathcal{F}^{*_\lambda}$ for every $\lambda \in \Lambda$, and thus $\mathrm{Mon}(I) \in \mathcal{F}^{*_\lambda}$ for every $\lambda \in \Lambda$. Therefore, $\mathrm{Mon}(I) \in \mathcal{F}^{*_2}$, and $*_2$ is monomial by Proposition 1.145. $\square$

Theorem 1.147. *Let $R = K[[S]]$, S a numerical semigroup. Let $\sharp$ be a star operation on S, and $*$ the associated star operation on R. Then $*$ is monomial and, for each $U \in \mathcal{F}(S)$, $(X^U)^* = (X^{U^\sharp})$.*

Proof. Let $\phi \in K((X)) \setminus \{0\}$. In view of Proposition 1.145, we need to show that $\mathrm{Mon}(\phi(X^U))$ is $*$-closed for each $\sharp$-closed ideal U. By Proposition 1.142 and Proposition 1.139(c), $\mathrm{vMon}(\phi(X^U)) = (U - T_{\phi^{-1}})$.

For every ideal T of S, $(U - T) = \bigcap_{t\in T}(-t + U)$. Since U is $\sharp$-closed, so is each $-t+U$, and thus, by Proposition 1.2(b), also $(U-T)$ is $\sharp$-closed. In particular, the ideal $(U - T_{\phi^{-1}}) = \mathrm{vMon}(\phi(X^U))$ is $\sharp$-closed. Hence, by the definition of $*$, $\mathrm{Mon}(\phi(X^U)) = X^{\mathrm{vMon}(\phi(X^U))}$ is $*$-closed, and $*$ is monomial.

Let $\Delta = \{\phi(X^T) : T^\sharp = T\}$. From Proposition 1.145, we have

$$(X^U)^* = \bigcap\{(X^T)|T^\sharp = T, (X^U) \subseteq (X^T)\} = (X^{\bigcap\{T|T^\sharp=T, U\subseteq T\}}) = (X^{(U^\sharp)}),$$

and the last claim is proved. $\square$

The previous result shows that each star operation on the numerical semigroup S induces a star operation on $R = K[[S]]$, and different star operations on S generate different operations on R. Thus, we have an injective map $\iota_S : \mathrm{Star}(S) \longrightarrow \mathrm{Star}(R)$ and, in particular, $|\mathrm{Star}(R)| \geq |\mathrm{Star}(S)|$. Therefore, we have an analogue of Theorem 1.26:

Theorem 1.148. *Let K be a field, n an integer strictly greater than 1. The set $A_n := \{R = K[[S]] : S$ is a nonsymmetric numerical semigroup, $|\mathrm{Star}(R)| = n\}$ is finite.*

Proof. Since there is a one-to-one correspondence between numerical semigroups and rings in the form $K[[S]]$, A_n has the same cardinality of the set B_n of nonsymmetric numerical semigroups S such that $|\mathrm{Star}(K[[S]])| = n$. For each $S \in B_n$, $|\mathrm{Star}(S)| \leq |\mathrm{Star}(R)| = n$, and thus $B_n \subseteq \{S : S$ is nonsymmetric, $|\mathrm{Star}(S)| \leq n\} = \{S : S$ is a numerical semigroup, $1 < |\mathrm{Star}(S)| \leq n\}$. By Theorem 1.26, the right hand side is finite, and thus B_n and A_n are finite. □

The condition that S is not symmetric is, however, not necessary: in fact, it is true that $|\mathrm{Star}(R)| = 1$ if and only if $|\mathrm{Star}(S)| = 1$ (we will quote the result needed, in greater generality, as Theorem 1.157). Thus, Theorem 1.26 implies also that $\{R = K[[S]] : S$ is a numerical semigroup, $|\mathrm{Star}(R)| = n\}$ is finite.

We could ask if there is an analogous natural map $\mathrm{Star}(R) \longrightarrow \mathrm{Star}(S)$. On monomial operations, this is easy to construct:

Proposition 1.149. *Let S be a numerical semigroup, $R = K[[S]]$, and let $*$ be a monomial star operation on R. For every ideal U of S, define $U^\sharp$ to be the ideal such that $(X^U)^* = (X^{U^\sharp})$. Then $\sharp$ is a star operation on S. In particular, the set of ideals U of S such that X^U is $*$-closed coincides with the set of $\sharp$-closed ideals.*

Proof. Since $*$ is monomial, $\sharp$ is well-defined; by the properties of $*$ it follows easily that $\sharp$ is extensive, order-preserving and idempotent, and that $S^\sharp = S$. Finally, Proposition 1.139(d) implies $\sharp$ is a star operation. □

Let ϵ_S be the map that associates to every monomial star operation on $K[[S]]$ the star operation on S defined in Proposition 1.149: using Theorem 1.147, we see that $\epsilon_S \circ \iota_S$ is the identity on $\mathrm{Star}(S)$. However, $\iota_S \circ \epsilon_S$ is not the identity on the set of monomial star operations: in fact, ϵ_S is not even injective, as the following example shows.

Example 1.150. Let $S := \langle 4, 5, 6, 7\rangle$, $R := K[[S]]$, $T := K[[X^2, X^5]]$, $J := R + (X^2 + X^3)R$. Let $*_1$ be the map $I \mapsto I^{*_1} := I^v \cap IT$ and $*_2$ be the operation generated by $*_1$ and J (that is, generated by the $*_1$-closed ideals and by J). The two operations are different since $J^{*_1} = K[[X]] \cap JT = JT = K[[X^2, X^3]] \neq J$.

We observe that $*_1$ is monomial since I^v and IT are monomial for every monomial ideal I. In order to show that $*_2$ is monomial it is enough to show that $\mathrm{Mon}(\phi J)$ is $*_1$-closed (and hence $*_2$-closed) for every ϕ such that $\mathbf{v}(\phi) = 0$. This will prove also that the corresponding star operations $\sharp_1$ and $\sharp_2$ on S are equal, and that at least one of the two is not induced by an operation on S.

By Proposition 1.143, $\sum\{\mathrm{Mon}(\phi J) : \mathbf{v}(\phi) = 0\} = X^{\mathbf{v}(J)} = R + X^2R$. If M is the maximal ideal of R, then $\phi M = M$ for every ϕ of valuation 0, and thus $\mathrm{Mon}(\phi J)$ must contain both 1 and M. It follows that it has to be either R or $R + X^2R$. By direct calculation, they are both $*_1$-closed.

To extend the correspondence to the whole set of star operations, we need a way to associate to each star operation a monomial one. There are two canonical ways of doing this.

Definition 1.151. *Let $*$ be a star operation on R. Define*

- $*_m^{(i)} := \sup\{*' : *' \leq * \text{ and } *' \text{ is monomial}\}$ *and*
- $*_m^{(s)} := \inf\{*' : *' \geq * \text{ and } *' \text{ is monomial}\}$.

In other words, $*_m^{(i)}$ is the biggest monomial star operation smaller than $*$, while $*_m^{(s)}$ is the smallest monomial star operation bigger than $*$.

Since d and v are monomial star operations (see the remark after Definition 1.144), the sets on the right hand sides are nonempty. Moreover, since the infimum and the supremum of a set of monomial star operations are monomial, $*_m^{(i)}$ and $*_m^{(s)}$ are monomial operations. Clearly, if $*$ is a monomial star operation then $* = *_m^{(i)} = *_m^{(s)}$, while if $* = *_m^{(i)}$ or $* = *_m^{(s)}$ then $*$ is monomial.

The next two propositions characterize $*_m^{(i)}$ and $*_m^{(s)}$ in a more explicit way.

Proposition 1.152. *$*_m^{(i)}$ is the operation generated by $\mathcal{F}^*(R) \cup \{\mathrm{Mon}(I) : I \in \mathcal{F}^*\}$.*

Proof. Let $\overline{*}$ be the star operation generated by $\mathcal{F}^* \cup \{\mathrm{Mon}(I) : I \in \mathcal{F}^*\}$. We claim that $\overline{*}$ is monomial and, to show this, it is enough to prove that $\mathrm{Mon}(\phi I)$ is $\overline{*}$-closed for every $\phi \in K((X))$, $I = \mathrm{Mon}(A)$ for some $A \in \mathcal{F}^*(R)$. By Proposition 1.142, $\mathrm{vMon}(\phi I) = (\mathrm{vMon}(I) - T_{\phi^{-1}})$, and thus

$$\mathrm{Mon}(\phi I) = X^{(\mathrm{vMon}(I) - T_{\phi^{-1}})} = (X^{\mathrm{vMon}(I)} : X^{T_{\phi^{-1}}}) = (\mathrm{Mon}(I) : X^{T_{\phi^{-1}}}).$$

However, $\mathrm{Mon}\, I$ is $\overline{*}$-closed, and thus also $(\mathrm{Mon}(I) : X^{T_{\phi^{-1}}})$ is $\overline{*}$-closed. Therefore, $\overline{*}$ is monomial.

If now $*' \leq *$ is monomial, then each $\mathrm{Mon}\, I$, with $I \in \mathcal{F}^{*'}$, is $*'$-closed. On the other hand, $\mathcal{F}^{*'}(R) \supseteq \mathcal{F}^*(R)$, and thus $\mathrm{Mon}\, I$ is $*'$-closed for every $I \in \mathcal{F}^*(R)$. In particular, since $\overline{*}$ is generated by these, $*' \leq \overline{*}$, and $\overline{*} = *_m^{(i)}$. $\square$

Proposition 1.153. *$*_m^{(s)}$ is the operation generated by $\{\phi(X^U) : (X^U)^* = (X^U)\}$.*

Proof. Let $\Delta = \{\phi(X^U) : (X^U)^* = (X^U)\}$. Then, for each $X^U \in \Delta$, $\mathrm{Mon}(\phi(X^U)) = (X^U : X^{T_{\phi^{-1}}})$ is monomial and $*$-closed, and thus is in Δ. Therefore $\overline{*} = *_\Delta$ is monomial.

Suppose that $*' \geq *$ and that $*'$ is monomial. Then if (X^U) is $*'$-closed it is also $*$-closed, and thus it is also $\overline{*}$-closed. Therefore, $*' \geq \overline{*}$, and $\overline{*} = *_m^{(s)}$. $\square$

Therefore, we have two maps

$$\begin{aligned} \epsilon_S^{(i)} \colon \mathrm{Star}(K[[S]]) &\longrightarrow \mathrm{Star}(S), \\ * &\longmapsto *_m^{(i)}. \end{aligned}$$

and

$$\begin{aligned} \epsilon_S^{(s)} \colon \mathrm{Star}(K[[S]]) &\longrightarrow \mathrm{Star}(S), \\ * &\longmapsto *_m^{(s)}. \end{aligned}$$

and the remarks after Definition 1.151 shows that a star operation $*$ on $K[[S]]$ is monomial if and only if $\epsilon_S^{(i)}(*) = \epsilon_S^{(s)}(*)$.

We end this section with an example of a star operation on R that is not monomial.

Example 1.154. Let $S := \langle 5, 6, 7, 8, 9\rangle$, $R := K[[S]]$, $T := R[X^3 + X^4] = K[[X^3 + X^4, X^5, \ldots]]$, and let $*$ be the operation defined by $I^* := I^v \cap IT$ for every $I \in \mathcal{F}(R)$. Let $J := R + X^2R$: then $J^v = K[[X]]$, so that $J^* = JT = T + X^2T = R + X^2R + (X^3 + X^4)R$. If JT were monomial, it should contain also X^3 and X^4, but this does not happen. Therefore, $*$ is not monomial.

1.8.3. Residually rational rings

Let K be a field. There are many subrings R of the discrete valuation domain $V := K[[X]]$ whose integral closure is $K[[X]]$, but that are not of the form $K[[S]]$, nor are isomorphic to one of this form: for instance, those of the form $F + XK[[X]]$, where F is a subfield of K. To any such R (with the same quotient field of V), however, we can associate a numerical semigroup S by considering $\mathbf{v}(R) := \{\mathbf{v}(r) : r \in R, r \neq 0\}$, where $\mathbf{v}$ is the valuation associated to V: $\mathbf{v}(R)$ is called the *value semigroup* of R, and it brings some information about the structure of R. This situation can be greatly generalized.

Let thus (V, M_V) be a discrete valuation ring and $\mathbf{v}$ the corresponding valuation. As in the previous section, for every subset A of the quotient field of V, let $\mathbf{v}(A) := \{\mathbf{v}(a) : a \in A, a \neq 0\}$. Let $\mathfrak{C}(V)$ be the set of all subrings R of V such that:

- R and V have the same quotient field;
- V is the integral closure of R;
- R is Noetherian;
- the conductor ideal $(R : V)$ is nonzero.

Equivalently, $\mathfrak{C}(V)$ is the set of the analytically irreducible Noetherian one-dimensional domains whose integral closure is V [13, Chapter II] (where a local ring is *analytically irreducible* if its completion, with respect to its maximal ideal, is an integral domain – see e.g. [119, Chapter VIII, §13]). Note that, if $R \in \mathfrak{C}(V)$, then R is local and of dimension 1, $\mathbf{v}(R)$ is a numerical semigroup and $\mathbf{v}(I)$ is an ideal of $\mathbf{v}(R)$ for every ideal I of R. In $\mathfrak{C}(V)$, we consider the set $\mathfrak{V}(V)$ of rings $T \in \mathfrak{C}(V)$ such that the inclusion map $i : T \longrightarrow V$ induces an isomorphism $T/M_T \xrightarrow{\simeq} V/M_V$ (where M_T is the maximal ideal of T). Such rings are said to be *residually rational.*

The last hypothesis, intuitively, guarantees that the value semigroup captures as much information about R as possible: for example, if $R = F + XK[[X]]$, then any relationship between two ideals comprised between R and V is undetectable under the passage to S. Technically, the condition implies the following pivotal result.

Theorem 1.155 [87]. *Let V be a discrete valuation ring, $\mathbf{v}$ its valuation, $R \in \mathfrak{V}(V)$, and $S := \mathbf{v}(R)$. Let $I \subseteq J$ be ideals of R, ℓ_R and ℓ_S be the length of a R-module and of a S-module, respectively. Then,*

$$\ell_R\left(\frac{J}{I}\right) = |\mathbf{v}(J) \setminus \mathbf{v}(I)|.$$

Let R be a local ring and M be its maximal ideal. The *type* of R is $t(R) := \dim_{R/M} \frac{(R:M)}{R}$, and $t(R) > 0$ if and only if M is divisorial, and in particular if R is Noetherian and one-dimensional. If $R \in \mathfrak{V}(V)$, then we have always $t(R) \leq t(\mathbf{v}(R))$, but the inequality can be strict, as the next example shows.

Example 1.156. Let $R := K[[X^4, X^6+X^7, X^{10}]]$, where K is a field whose characteristic is different from 2 [13, Example II.1.19]. Then $S := \mathbf{v}(R) = \langle 4, 6, 11, 13\rangle$, so that $T(S) = \{2, 7, 9\}$ and $t(S) = 3$. On the other hand, there are no elements in $(R : M)$ of valuation 2, and thus $t(R) = 2$. In particular, $S \cup \{2\}$ is a fractional ideal of S, but $\mathbf{v}(I) \neq S \cup \{2\}$ for every ideal I of R.

On the other hand, if $T = K[[U]]$ for some numerical semigroup U, the correspondence becomes much nicer: in this case, $(T : M_T)$ is the ideal generated by the X^u, for $u \in (U - M_U)$, and if J is an ideal of S, then the ideal $I = (X^j : j \in J)$ is such that $\mathbf{v}(I) = J$. In particular, $t(T) = t(U)$. Hence, the ring R defined in Example 1.156 is not isomorphic to $K[[S]]$. Given a numerical semigroup S, if, for any ring R such that $K \subseteq R \subseteq K[[X]]$, $K[[X]] = \overline{R}$ and $\mathbf{v}(R) = S$, we have $R \simeq K[[S]]$, then S is said to be a *monomial semigroup*; see [102] for the original definition and [88, Theorem 3.12] for a complete characterization.

For every $R \in \mathfrak{V}(V)$, we define $g(R) := \min\{n \in \mathbb{N} : M_V^n \subseteq R\}$, where M_V is the maximal ideal of V. The condition $(R : V) \neq 0$ guarantees that $g(R)$ exists, and the equality of the residue fields that $g(R) = g(\mathbf{v}(R))$ [81]. Note also that $(R : V) = M_V^{g+1}$; in the same way, $(S - \mathbb{N}) = \{g + 1, \ldots\} = (g + 1)M_{\mathbb{N}}$, where $M_{\mathbb{N}} = \{1, 2, \ldots\}$ is the maximal ideal of the semigroup $\mathbb{N}$. For this reason, $g(S) + 1$ is called the *conductor* of S.

Using the theory of the Hilbert-Samuel functions, it is also possible to define a notion of *multiplicity* $\mu(R)$ of R, and the hypothesis $R \in \mathfrak{V}(V)$ guarantees that $\mu(R) = \mu(\mathbf{v}(R))$ (see [84] and [13, Section II.2]). However, we will not need it, since we will use directly the multiplicity of the semigroup.

In the basic case, we have the following result.

Theorem 1.157 [14, 81]. *Let R be a Noetherian one-dimensional local domain. Then the v-operation on R is the identity if and only if $t(R) = 1$, if and only if R is a Gorenstein domain. If $R \in \mathfrak{V}(V)$, then this happens if and only if $t(\mathbf{v}(R)) = 1$, that is, if and only if the numerical semigroup $\mathbf{v}(R)$ is symmetric.*

Let $\mathcal{F}_0(R)$ be the set of nonzero fractional ideals between R and V. For every ideal I of R and every $i \in I$ of minimum valuation, $i^{-1}I \in \mathcal{F}_0(R)$, but there could be a $j \in I$ such that $j^{-1}I \in \mathcal{F}_0(R)$ and $i^{-1}I \neq j^{-1}I$. For example, if $I \in \mathcal{F}_0(R)$, $u \in I$, $u^2 \notin I$ and $\mathbf{v}(u) = 0$, then $u^{-1}I \neq I$, but $u^{-1}I$ is still contained between R and V. This means that, even if we suppose $I, J \in \mathcal{F}_0(R)$, it is possible that $I \neq J$ and $*_I = *_J$. The next proposition shows that this is the unique possibility (compare the remark after Proposition 1.110).

Proposition 1.158. *Preserve the notation of Theorem 1.155, let L be the quotient field of V and let I, J be non-divisorial ideals of R. Then $*_I = *_J$ if and only if $I = uJ$ for some $u \in L \setminus \{0\}$. In particular, if $I, J \in \mathcal{F}_0(R)$, this can happen only if $\mathbf{v}(I) = \mathbf{v}(J)$.*

Proof. We can suppose that $I, J \in \mathcal{F}_0(R)$. In this case, $(I : J)$ and $(J : I)$ are both contained in V.

Suppose $0 \notin \mathbf{v}((I : J)(J : I))$. Since I is not divisorial, $\ell_R(I^v/I) \geq 1$, and thus, by Theorem 1.155, $\mathbf{v}(I^v) \neq \mathbf{v}(I)$. Let $\phi \in I^v$ be an element of L such that $\mathbf{v}(\phi) = \sup(\mathbf{v}(I^v) \setminus \mathbf{v}(I))$. Since $(I : J)(J : I) \subseteq (I : I)$, we have $\gamma I^v \subseteq I^v$ and thus $\gamma\phi \in I^v$ for every $\gamma \in (I : J)(J : I)$. But, since $0 \notin \mathbf{v}((I : J)(J : I))$, we have $\mathbf{v}(\gamma) > 0$, and thus $\mathbf{v}(\gamma\phi) = \mathbf{v}(\gamma) + \mathbf{v}(\phi) > \mathbf{v}(\phi)$, and $\gamma\phi \in I$ by the definition of ϕ.

If $*_I = *_J$, then by Proposition 1.110

$$I = I^v \cap \bigcap_{\gamma \in (I:J)(J:I)\setminus\{0\}} (\gamma^{-1} I).$$

By definition, ϕ is not contained in I; however, for the above reasoning, $\phi \in \gamma^{-1}I$ for every $\gamma \in (I : J)(J : I) \setminus \{0\}$, and thus ϕ is contained in the right hand side. This is a contradiction, and thus $*_I \neq *_J$.

If $0 \in \mathbf{v}((I : J)(J : I))$, then (since $(I : J), (J : I) \subseteq V$) there is a $x \in (I : J)$ such that $\mathbf{v}(x) = 0$. Hence, $\mathbf{v}(I) = \mathbf{v}(xI) \subseteq \mathbf{v}(J)$, and symmetrically $\mathbf{v}(J) \subseteq \mathbf{v}(I)$. Therefore, $\mathbf{v}(xI) = \mathbf{v}(J)$ and, since $xI \subseteq J$, Theorem 1.155 implies that $xI = J$. □

When $R = K[[S]]$, this proposition implies that the number of principal star operations on R is at least equal to the number of principal star operations on S, since (X^I) and (X^J) generate different star operations when $I, J \in \mathcal{G}_0(S)$. In particular, we get another version of the proof of Theorem 1.148.

On the other hand, when $R \neq K[[S]]$, we cannot apply the same method of the above corollary, since it is not possible in general to find an ideal of R corresponding to an arbitrary ideal of S. Therefore, to prove an analogue of Theorem 1.26, we will follow the method of its proof by finding analogous ways to count nondivisorial ideals. A similar extension can also be done with regard to the theory of antichains and to the study of the sets $\mathcal{Q}_a$; we prefer to limit ourselves to the counting of principal star operations, highlighting the main technical differences but avoiding a lengthy in-depth "translation" of Section 1.3.2.

The following is an analogue of Lemma 1.19.

Lemma 1.159. *Preserve the notation of Theorem 1.155. Let $I \in \mathcal{F}_0(R)$ and $a := \sup(\mathbb{N} \setminus \mathbf{v}(I))$. If $g - a \notin \mathbf{v}(R)$, then $a \in \mathbf{v}(I^v)$, and in particular I is not divisorial.*

Proof. Let $I \subseteq \gamma^{-1}R$ for some $\gamma \neq 0$ in the quotient field of R. Since $\mathbf{v}(I)$ contains all the integers bigger than a, so does $\mathbf{v}(\gamma^{-1}R) = -\mathbf{v}(\gamma) + \mathbf{v}(R)$, and hence $\mathbf{v}(\gamma) \geq g - a$. However, if $\mathbf{v}(\gamma) = g - a$, then $0 \notin \mathbf{v}(\gamma^{-1}R)$ (since, by hypothesis, $g - a \notin \mathbf{v}(R)$), and this would imply that $I \nsubseteq \gamma^{-1}R$, against the hypothesis. Hence $\mathbf{v}(\gamma) > g - a$. However, R contains all the elements of valuation bigger than g, and thus $\gamma^{-1}R$ contains all the x such that $\mathbf{v}(x) > g - \mathbf{v}(\gamma)$, and in particular all the elements of valuation a. Finally, $I^v = \bigcap \gamma^{-1}R$, where the intersection ranges among the γ such that $I \subseteq \gamma^{-1}R$. In particular, each of these contains all the elements of valuation a, and so does I^v. □

The definition of M_a is, however, more problematic. The point is that, while the union of two ideals of a semigroup is still an ideal, the same does not hold for rings. As a consequence, we may have two ideals I and J such that nor I nor J contains elements of valuation a, but their sum $I+J$ does.

Recall that a *m-canonical* (or simply *canonical*) *ideal* of a domain R is a fractional ideal ω such that, for every fractional ideal F, we have $F=(\omega:(\omega:F))$; equivalently, ω is canonical if $(\omega:\omega)=R$ and the star operation generated by ω is the identity. Not all rings have canonical ideals: see Proposition 3.82 or [60, Theorem 6.7] for a characterization based on localization.

Proposition 1.160. *Let V be a discrete valuation ring with valuation $\mathbf{v}$.*

(a) If R is a local domain, and ω_1,ω_2 are canonical ideals of R, then $\omega_1=\alpha\omega_2$ for some α in the quotient field of R.
(b) If $R\in\mathfrak{C}(V)$, then R has a canonical ideal ω.
(c) Suppose $R\in\mathfrak{V}(V)$. Then, an ideal $\omega\in\mathcal{F}_0(S)$ is a canonical ideal of R if and only if $\mathbf{v}(\omega)$ is the canonical ideal of the numerical semigroup $\mathbf{v}(R)$.

Proof. (a) follows from [60, Proposition 4.2], while (b) is proved in [62, Satz 6.21] and (c) is [74, Satz 5]. □

Fix now a ring $R\in\mathfrak{V}(V)$. For every $a\in\mathbb{N}\setminus\mathbf{v}(R)$, let T_a denote the set $R\cup\{x\in V\mid\mathbf{v}(x)>a\}$; clearly, T_a is a residually rational ring such that $\mathbf{v}(T_a)=\mathbf{v}(R)\cup\{x\in\mathbb{N}\mid x>a\}$, and in particular such that $g(T_a)=a$. Moreover, every T_a-fractional ideal is also a R-fractional ideal. The following can be seen as a ring version of Lemma 1.14(c).

Lemma 1.161. *Preserve the notation introduced above; let $a,b\in\mathbb{N}\setminus\mathbf{v}(R)$ such that $a\geq b$. If $M_a\in\mathcal{F}_0(T_a)$ (respectively, $M_b\in\mathcal{F}_0(T_b)$) is a canonical ideal of T_a (resp., T_b), then $M_b=\gamma^{-1}M_a\cap V$ for some $\gamma\in M_a$.*

Proof. If $a=b$ the claim follows from Proposition 1.160(a) applied on T_a. Suppose $a>b$. The numerical semigroup $\mathbf{v}(T_a)$ does not contain b, and thus its canonical ideal contains $a-b$. Hence, $\mathbf{v}(M_a)$ contains $a-b$, that is, there is a $\gamma_0\in M_a$ such that $\mathbf{v}(\gamma_0)=a-b$. Consider the ideal $I:=\gamma_0^{-1}M_a\cap V$: then, $\mathbf{v}(I)=(b-a+\mathbf{v}(M_a))\cap\mathbb{N}$, and it is not hard to see (proceeding as in the proof of Lemma 1.14(c)) that $\mathbf{v}(I)$ is exactly the canonical ideal of $\mathbf{v}(T_b)$. By Proposition 1.160(c), it follows that I is a canonical ideal of T_b; hence, $M_b=\alpha I$ for some α in the quotient field of R. Therefore, $M_b=\alpha\gamma_0^{-1}M_a\cap\alpha V$. However, $\mathbf{v}(\alpha)=0$ since M_b and I are both in $\mathcal{F}_0(T_b)$, and thus $\alpha V=V$; defining $\gamma:=\alpha^{-1}\gamma_0$ we have our claim. □

Theorem 1.162. *Let R be a residually rational Noetherian one-dimensional domain with integral closure V, and let $\mathbf{v}$ be the valuation relative to V. If R is not Gorenstein, then $|\mathrm{Star}(R)|\geq\delta(\mathbf{v}(R))+1$.*

Proof. We follow a strategy similar to the proof of Theorem 1.25; let $S:=\mathbf{v}(R)$. Let M be the maximal ideal of R, and take a $\tau\in(R:M)$ such that $\mathbf{v}(\tau)\neq g$; let $\lambda:=\min\{\mathbf{v}(\tau),g-\mathbf{v}(\tau)\}$. Consider the three sets

$$A:=\{x\in\mathbb{N}\setminus S\mid x<\lambda,\lambda-x\notin S\},$$

$$B := \{x \in \mathbb{N} \setminus S \mid x < \lambda, \lambda - x \in S\},$$

$$C := \{x \in \mathbb{N} \setminus S \mid x \geq \lambda\}.$$

Note that $\mathbb{N} \setminus S = A \cup B \cup C$ and the three sets are disjoint, and thus $\delta(S) = |A|+|B|+|C|$; we will define for every $x \in \mathbb{N} \setminus S$ a different non-divisorial ideal I_x, whose definition depends on whether $x \in A$, $x \in B$ or $x \in C$.

If $x \in C$, then define I_x as a canonical ideal of the ring T_x. By Lemma 1.161, there is a $\gamma \in I_x$ such that $I_\lambda = \gamma^{-1} I_x \cap V$; hence, if I_x were divisorial (as an ideal of R), so would be I_λ. But this is impossible by Lemma 1.159, and so the I_x are not divisorial.

If $x \in A$, then x belongs to M_λ (in the terminology of numerical semigroups; with respect to S), which by Proposition 1.160(c) is equal to $\mathbf{v}(I_\lambda)$ (where I_λ is defined as above). We define $I_x := S \cup \{\phi \in I_\lambda \mid \mathbf{v}(\phi) > x\}$; then, $\sup(\mathbf{v}(I_\lambda \setminus I_x)) = x$, and thus $I_x \neq I_y$ if $x \neq y$ are in A. Moreover, the I_x are not divisorial by Lemma 1.159.

If $x \in B$, consider $y := g - \lambda + x = g - (\lambda - x)$. Since $\lambda - x \in S$, we have $y \notin S$; moreover, $g - \lambda < y < g$. Let $I_x := R \cup \{\phi \in V \mid \mathbf{v}(\phi) > y\}$; then, g belongs to $\mathbf{v}(I_x)$ while τ does not (since $\mathbf{v}(\tau) < g - \lambda$), and thus I_x is not divisorial (this can be proved with an argument analogous to Proposition 1.12). Moreover, $\sup(\mathbb{N} \setminus I_x) = y$ (so that $I_x \neq I_y$ if $x \neq y$ are in B) and I_y contains $g - \lambda$ (since $x \notin S$); hence, $I_x \neq I_y$.

It is straightforward to see that $\mathbf{v}(I_x) \neq \mathbf{v}(I_y)$ if x and y belongs to different subsets; therefore, $\{I_x \mid x \in \mathbb{N} \setminus S\}$ is a set of $\delta(S)$ non-divisorial ideals of R, each of which generates a different star operation (by Proposition 1.158). Hence, considering also the v-operation, $|\mathrm{Star}(R)| \geq \delta(S) + 1$. □

The previous result shows that, if we want a $R \in \mathfrak{V}(V)$ with n or less star operations, we have only a finite number of choices on its value semigroup $\mathbf{v}(R)$. To get an analogue of Theorem 1.26, we need to show that each semigroup corresponds to only a finite number of rings in $\mathfrak{V}(V)$. Moreover, we have not yet shown that $\mathrm{Star}(R)$ is finite, so that it could be that the analogue of Theorem 1.26 holds, but for trivial reasons. Both problems can be solved with the same hypothesis.

Lemma 1.163. *Let V be a discrete valuation ring with residue field K and quotient field L; suppose that K is finite. Let $\mathcal{S}$ be the class of numerical semigroups.*

(a) The map $\mathbf{v} : \mathfrak{C}(V) \longrightarrow \mathcal{S}$, $R \mapsto \mathbf{v}(R)$ has finite fibres, that is, for every $S \in \mathcal{S}$, $\mathbf{v}^{-1}(S)$ is finite.
(b) For every $R \in \mathfrak{C}(V)$, the cardinality of $\mathcal{F}_0(R)$ is finite, and thus $\mathrm{Star}(R)$ *is finite.*

Proof. (a) Let S be a semigroup. If $\mathbf{v}(R) = S$, then R contains all the elements x such that $\mathbf{v}(x) > g(S)$. Let $H = M_V^{g+1}$ be this set. Since a ring is an additive group, every ring belonging to $\mathbf{v}^{-1}(S)$ defines uniquely a subgroup of V/H, which is finite since its cardinality is bounded by $|K|^{g+1}$. Hence also $\mathbf{v}^{-1}(S)$ is finite.
(b) See the proof of [68, Theorem 2.5]. □

Theorem 1.164. *Let V be a discrete valuation ring with finite residue field. For any $n > 1$, the set $\{R \in \mathfrak{V}(V) : |\mathrm{Star}(R)| = n\}$ is finite.*

Proof. By Theorem 1.157, we can suppose that no $\mathbf{v}(R)$ is symmetric.

Like in the proof of Theorem 1.26, by Theorem 1.162 there are a finite number of possible semigroups for a given n. By Lemma 1.163, each semigroup gives rise to only a finite number of possible rings in $\mathfrak{V}(V)$, and thus the number of rings in $\mathfrak{V}(V)$ with n star operations is finite. $\square$

Theorem 1.164 fails when the residue field of V is infinite. Indeed, suppose $V := K[[X]]$, and let $R := K[[X^3, X^4, X^5]]$. Then, by [68, Theorem 3.8], $|\mathrm{Star}(R)| = 3$. For any $t \in K$, consider the ring isomorphisms $\phi_t : V \longrightarrow V$ such that $\phi_t(X) = X+t$. Then, if $t \neq 0$, $\phi_t(R)$ is a ring isomorphic to R but different from R, since it contains elements in the form $a_0 + a_1X + a_2X^2 + a_3X^3 + \cdots$ with $a_1 \neq 0$. Similarly, $\phi_t(R)$ and $\phi_s(R)$ are isomorphic but different if $t \neq s$, since otherwise $R = \phi_{-s}(\phi_s(R)) = \phi_{-s}(\phi_t(R)) = \phi_{t-s}(R)$. Therefore, if K is infinite, $\{\phi_t(R) : t \in K\}$ is an infinite set of rings in $\mathfrak{V}(V)$, and each one has exactly three star operations.

It is also worth noting that, if $R \in \mathfrak{V}(V)$ and the residue field K of V is infinite, then $\mathrm{Star}(R)$ may be infinite: for example, if $t(R) \geq 3$, then by [68, Theorem 2.7] we have $|\mathrm{Star}(R)| \geq \frac{1}{2}|K| + 3$, and in particular it is infinite if K is.

Remark 1.165. The proof above relies on the fact that every $R \in \mathfrak{V}(V)$ admits a canonical ideal, but this result isn't actually needed; we sketch an alternative proof, referring the reader to [110, Section 5] for the details.

Let $R \in \mathfrak{V}(V)$ and M be its maximal ideal. For every $a \in \mathbb{N} \setminus \mathbf{v}(R)$, define $\mathcal{M}_a$ as the set of ideals I of R that contains every element of valuation $b > a$, but does not contain any element of valuation a; this set is nonempty, since it contains $R+\{x \in V \mid \mathbf{v}(x) > a\}$. Proceeding similarly to the proof of Proposition 1.15, we can see that a maximal element M_a of $\mathcal{M}_a$ is divisorial if and only if $M_a = \gamma^{-1}R \cap V$ for some $\gamma \in R$.

If R is not Gorenstein, then there is a $\tau \in (R : M)$ such that $\mathbf{v}(\tau) \neq g := g(R)$; take any $a \in \mathbb{N} \setminus \mathbf{v}(R)$ such that $a > g/2$. If $g > a > \mathbf{v}(\tau)$, then we associate to a the ideal $R + \{x \in V \mid \mathbf{v}(x) > a\}$, which is not divisorial since it contains elements of valuation g but not τ; if $\mathbf{v}(\tau) > a > g - \mathbf{v}(\tau)$, then M_a is not divisorial, since $M_a = \gamma^{-1}R \cap V$ leads to a contradiction, obtained considering the element $\gamma^{-1}\tau$. Hence,

$$|\mathrm{Star}(R)| \geq |\{a \in \mathbb{N} \setminus S \mid g/2 < a < g\}| \geq \frac{g}{2\mu} - 1.$$

If now both $\mathbf{v}(\tau)$ and $g - \mathbf{v}(\tau)$ are bigger than $\mu := \mu(S)$, let $\lambda := \min\{\mathbf{v}(\tau), g - \mathbf{v}(\tau)\}$; we can find $\nu := \frac{\mu-1}{2}$ ideals of the type $R + \{x \in V \mid \mathbf{v}(x) > \lambda\} + \beta_s R$, where the $\beta_s \notin R$ are elements whose valuation lies between $\lambda - \mu$ and λ (there exist ν elements of different valuation by Lemma 1.63), so that $|\mathrm{Star}(R)| \geq \nu$. On the other hand, if $\lambda < \mu$, we can mix this type of ideals with the one obtained in the previous reasoning to have $|\mathrm{Star}(R)| \geq \mu(S)$.

Therefore, if $|\mathrm{Star}(S)| \leq n$, we have a bound on $\mu(S)$ and, thus, a bound on $g(S)$; since there a finite number of semigroups with $g(S) \leq g$, we have a finite number of possibilities for $\mathbf{v}(R)$, and thus a finite number of possible rings.

2. Semistar operations and topology

2.1. The topology on the set of closure operations

Definition 2.1. *Let $(\mathcal{P},\leq)$ be a partially ordered set. A* closure operation *on $\mathcal{P}$ is a map $c:\mathcal{P}\longrightarrow\mathcal{P}$ such that, for every $x,y\in\mathcal{P}$,*

1. *$x\leq c(x)$ (c is* extensive*);*
2. *$x\leq y$ implies $c(x)\leq c(y)$ (c is* order-preserving*);*
3. *$c(c(x))=c(x)$ (c is* idempotent*).*

An element x of $\mathcal{P}$ is c-closed if $x=c(x)$; we denote by $\mathcal{P}^c:=\{y\in\mathcal{P}\mid y=c(y)\}$ the set of the c-closed elements of $\mathcal{P}$, and by $\mathrm{Clos}(\mathcal{P})$ the set of closure operations on $\mathcal{P}$.

The concept of closure operation was introduced in its generality in [90]; we refer the reader to [16, Chapter IV] for a purely lattice-theoretic study of them.

We start with proving some general properties of closure operations, already present in [16, Chapter IV].

Lemma 2.2. *Let $(\mathcal{P},\leq)$ be a complete lattice.*

(a) If $c\in\mathrm{Clos}(\mathcal{P})$, then $\mathcal{P}^c$ is closed by infima.
(b) If $Y\subseteq\mathcal{P}$ is a subset closed by infima that contains the maximal element of $\mathcal{P}$, then the map $c:x\mapsto\inf\{y\in Y\mid y\geq x\}$ is a closure operation on $\mathcal{P}$.
(c) For every $c\in\mathrm{Clos}(\mathcal{P})$ and for every $x\in\mathcal{P}$ we have

$$c(x)=\inf\{y\in\mathcal{P}^c\mid y\geq x\}.$$

Proof. (a) Let $\Delta\subseteq\mathcal{P}^c$; then, there exist a $y:=\inf\Delta\in\mathcal{P}$. For every $z\in\Delta$, by extensivity we have $c(y)\leq c(z)=z$; hence, $y\leq c(y)\leq\inf\Delta=y$, and thus $y=c(y)$, and $y\in\mathcal{P}^c$.

(b) Note that, since Y contains the maximal element of $\mathcal{P}$, the set $\{y\in Y\mid y\geq x\}$ is nonempty, so c is well-defined. c is clearly extensive and order-preserving. Suppose $y\in Y$ and $y\geq x$. Then, $y\geq c(x)$; hence, $\{y\in Y\mid y\geq x\}=\{y\in Y\mid y\geq c(x)\}$ and so $c(c(x))=c(x)$. Therefore, c is a closure operation.

(c) By part (a), $\mathcal{P}^c$ is a subset of $\mathcal{P}$ closed by infima containing the maximal element of $\mathcal{P}$ and thus, by (b), the right-hand side defines a closure operation d. Then, $d(x)\leq c(x)$ since $c(x)\geq x$ and $c(x)\in\mathcal{P}^c$; on the other hand, $x\leq d(x)$ implies $c(x)\leq c(d(x))$, and $c(d(x))=d(x)$ since $d(x)\in\mathcal{P}^c$ (being $\mathcal{P}^c$ closed by infima, by (a)). Therefore, $c(x)\leq d(x)$ and $c(x)=d(x)$. □

The set $\mathrm{Clos}(\mathcal{P})$ has a natural order such that $c \leq d$ if and only if $c(x) \leq d(x)$ for all $x \in \mathcal{P}$, or equivalently if and only if $\mathcal{P}^d \subseteq \mathcal{P}^c$. This order has a good behaviour:

Proposition 2.3. *If $(\mathcal{P}, \leq)$ is a complete lattice, so is* $\mathrm{Clos}(\mathcal{P})$.

Proof. Let $\Delta \subseteq \mathrm{Clos}(\mathcal{P})$; we have to show that Δ has an infimum and a supremum. Define a map

$$\begin{aligned} i\colon \mathcal{P} &\longrightarrow \mathcal{P} \\ x &\longmapsto \inf\{c(x) \mid c \in \Delta\}. \end{aligned}$$

Proceeding as in the proof of Lemma 2.2(b), we see that i is a closure operation on $\mathcal{P}$. Moreover, $i(x) \leq c(x)$ for every $c \in \Delta$. But if $j \leq c$ for every $c \in \Delta$, then $j(x) \leq c(x)$ for every $c \in \Delta$ and $x \in \mathcal{P}$, i.e., $j(x) \leq \inf\{c(x) \mid c \in \Delta\} = i(x)$ and $j \leq i$. It follows that i is the infimum of Δ.

Let now $Y_s := \bigcap_{c \in \Delta} \mathcal{P}^c = \{y \in \mathcal{P} \mid y = c(y)$ for all $c \in \Delta\}$. Since the maximum of $\mathcal{P}$ is always closed, Y_s is not empty; moreover, Y_s is closed by infima (since so is every $\mathcal{P}^c$, by Lemma 2.2(a)) and thus by Lemma 2.2(b) it defines a closure operation $s : x \mapsto \inf\{y \in Y_s \mid y \geq x\}$. If now $c \in \Delta$ and $x \in \mathcal{P}$, then no element of $\mathcal{P}$ between x and $c(x)$ is in $\mathcal{P}^c$, and thus it can't be in Y_s; it follows that $s(x) \geq c(x)$, and thus $s \geq c$. Moreover, if $t \geq c$ for all $c \in \Delta$ then $\mathcal{P}^t \subseteq \mathcal{P}^c$, so that $\mathcal{P}^t \subseteq Y_s$, i.e., $t \geq s$. Hence, s is the supremum of Δ. □

We can use this order structure to build a topology on $\mathrm{Clos}(\mathcal{P})$.

Definition 2.4. *Let $(\mathcal{P}, \leq)$ be a partially ordered set, and let $\mathscr{F}, \mathscr{P} \subseteq \mathcal{P}$. The $(\mathscr{F}, \mathscr{P})$-*Zariski topology *on $\mathrm{Clos}(\mathcal{P})$ is the topology having, as a subbasis of open sets, the sets of the form*

$$V_{x,y} := \{c \in \mathrm{Clos}(\mathcal{P}) \mid y \leq c(x)\}$$

as x ranges in $\mathscr{F}$ and y ranges in $\mathscr{P}$.

We are mainly interested in this topology when $\mathcal{P}$ is either the set of ideals of a ring R or the set of submodules of the quotient rings of a domain D. In this case, $\mathscr{F}$ will be the whole $\mathcal{P}$, and we will be interested in subspaces of $\mathrm{Clos}(\mathcal{P})$ where some distinguished subsets of $\mathcal{P}$ induce the same topology. On the other hand, $\mathscr{P}$ will usually be the set of principal ideals of R (or D), so that the relation $(a) \subseteq I^c$ will be read as $a \in I^c$; we will also sometimes reduce $\mathscr{P}$ to the singleton $\{1\}$. This also explains why we are restricting ourselves to complete lattices: both the set of ideals a ring and the set of submodules are, in fact, complete lattices.

Definition 2.5. *Let $(\mathcal{P}, \leq)$ be a partially ordered set. A subset $\Delta \subseteq \mathcal{P}$ is* sup-generating *if, for every $z \in \mathcal{P}$,*

$$z = \sup\{y \in \mathscr{P} \mid y \leq z\}.$$

Equivalently, Δ is sup-generating if, for every $z \in \mathcal{P}$, there is a $\Lambda \subseteq \Delta$ such that $z = \sup \Lambda$.

Proposition 2.6. *Let $(\mathcal{P}, \leq)$ be a complete lattice, and let $\mathscr{P} \subseteq \mathcal{P}$. Then, the $(\mathcal{P}, \mathscr{P})$-Zariski topology on $\mathrm{Clos}(\mathcal{P})$ is T_0 if and only if $\mathscr{P}$ is sup-generating.*

Proof. Suppose first that $\mathscr{P}$ is sup-generating, and let $c \neq d$ be closure operations; then, there is an x such that $c(x) \neq d(x)$. For any $z \in \mathcal{P}$, let $\Delta_z := \{y \in \mathscr{P} \mid y \leq z\}$, and consider $\Delta_{c(x)}$ and $\Delta_{d(x)}$; since $\mathscr{P}$ is sup-generating, $c(x) = \sup \Delta_{c(x)}$ and $d(x) = \sup \Delta_{d(x)}$. In particular, $\Delta_{c(x)} \neq \Delta_{d(x)}$ and, without loss of generality, there is a $y \in \Delta_{d(x)} \setminus \Delta_{c(x)}$. Then, $V_{x,y}$ contains d but not c: indeed, if $c \in V_{x,y}$, then $y \leq c(x)$, and so $y \in \Delta_{c(x)}$, against the hypothesis. Therefore, the topology is T_0.

On the other hand, suppose $\mathscr{P}$ is not sup-generating; then, there is an z such that (in the notation of the previous part of the proof) $z \neq \sup \Delta_z =: \hat{z}$. Let 1 be the maximal element of $\mathcal{P}$ and consider the two sets $Y_c := \{z, 1\}$ and $Y_d := \{\hat{z}, 1\}$. Both sets contain 1 and are closed by infima, so by Lemma 2.2(b) they define two star operations, say c and d, and by Lemma 2.2(c) we have $\mathcal{P}^c = Y_c$ and $\mathcal{P}^d = Y_d$. We claim that the $(\mathcal{P}, \mathscr{P})$-Zariski topology does not distinguish c and d.

Consider a $V_{x,y}$. If $x \nleq z$, then both $c(x)$ and $d(x)$ are equal to the maximal element of $\mathcal{P}$, and thus $c, d \in V_{x,y}$. If $x \leq z$, then $c(x) = z$ and $d(x) = \hat{z}$; but now, if $y \in \mathscr{P}$, then $y \leq z$ if and only if $y \leq \sup \Delta_z = \hat{z}$, and thus $c \in V_{x,y}$ if and only if $d \in V_{x,y}$. Therefore, the $(\mathcal{P}, \mathscr{P})$-Zariski topology is not T_0. □

Proposition 2.7. *Let $(\mathcal{P}, \leq)$ be a complete lattice, and denote by 0 and 1, respectively, its minimal and its maximal element. Endow $\mathrm{Clos}(\mathcal{P})$ with the $(\mathcal{P}, \mathscr{P})$-Zariski topology, with $\mathscr{P}$ a sup-generating subset of $\mathcal{P}$.*

(a) The closure of $c \in \mathrm{Clos}(\mathcal{P})$ is $\{d \in \mathrm{Clos}(\mathcal{P}) \mid d \leq c\}$.
(b) The only closed point of $\mathrm{Clos}(\mathcal{P})$ is the identity.
(c) If c is the closure such that $c(x) = 1$ for all $x \in \mathcal{P}$, then c is the generic point of $\mathrm{Clos}(\mathcal{P})$ (i.e., it belongs to every nonempty open set).

Proof. (a) If $d \leq c$ and $d \in V_{x,y}$, then $y \leq d(x) \leq c(x)$ and thus $c \in V_{x,y}$. Hence, c belongs to every open set containing d; therefore, every closed set containing c contains d, or equivalently d is in the closure of c.

Conversely, if $d \nleq c$, there is an $x \in \mathcal{P}$ such that $d(x) \nleq c(x)$; write $d(x) = \sup \Delta_{d(x)}$ where (as in the proof of Proposition 2.6), $\Delta_{d(x)} := \{y \in \mathscr{P} \mid d(x) \leq y\}$. In particular, $w \nleq c(x)$ for some $w \in \Delta_{d(x)}$. We claim that $V_{x,w}$ is an open set containing d but not c. Indeed, $d(w) \leq d(d(x)) = d(x)$, and thus $d \in V_{x,w}$. On the other hand, if $c \in V_{x,w}$, then $w \leq c(x)$, against our choice of w. Therefore, d is not in the closure of c.

(b) follows directly from the point above, since the identity is clearly the smallest closure operation.

(c) is clear, since if c sends everything to 1 then $y \leq c(x)$ for every x, y, that is, $c \in V_{x,y}$ for every c. Moreover, since the space is T_0, the generic point must be unique. □

Proposition 2.8. *Let $\mathcal{P}$ be a complete lattice, let $\mathscr{F}, \mathscr{P} \subseteq \mathcal{P}$ and let $y \in \mathscr{P}$.*

(a) For every $x \in \mathscr{F}$, $V_{x,y}$ is compact in the $(\mathcal{P}, \mathscr{P})$-Zariski topology of $\mathrm{Clos}(\mathcal{P})$.
(b) $\mathrm{Clos}(\mathcal{P})$ is compact.

(c) *If $\Delta \subseteq \mathrm{Clos}(\mathcal{P})$ is closed by infima, then $V_{x,y} \cap \Delta$ (for all $x \in \mathscr{F}$) and Δ are compact.*

Proof. Since $\mathrm{Clos}(\mathcal{P})$ is a complete lattice by Proposition 2.3, it is closed by infima, so it is enough to prove the last statement.

By hypothesis, it exists $c := \inf(V_{x,y} \cap \Delta)$. Moreover, $y \leq d(x)$ for all $d \in V_{x,y} \cap \Delta$, so that $c(x) = \inf\{d(x) \mid d \in V_{x,y} \cap \Delta\} \geq y$, and thus $c \in V_{x,y} \cap \Delta$.

Consider now an open cover $\mathcal{U}$ of $V_{x,y} \cap \Delta$. There must be a $U \in \mathcal{U}$ such that $c \in U$; but then, U contains the whole $V_{x,y} \cap \Delta$, so that $\mathcal{U}$ admits a finite subcover. Hence, $V_{x,y} \cap \Delta$ is compact.

The last claim follows from the fact that, if 1 is the maximal element of $\mathcal{P}$, then $V_{1,x} \cap \Delta = \Delta$ for every $x \in \mathcal{P}$. $\square$

We are almost ready to prove the main result of this section, but, before, we need the following definition.

Definition 2.9. *Let $\mathcal{P}$ be a complete lattice, let $\Delta \subseteq \mathrm{Clos}(\mathcal{P})$. We say that Δ is* sup-normal *if, whenever $\Lambda \subseteq \Delta$ and $y \leq (\sup \Lambda)(x)$, there are $c_1, \ldots, c_n \in \Lambda$ (non necessarily distinct) such that $y \leq (c_1 \circ \cdots \circ c_n)(x)$.*

Note that the definition is inherited by smaller subsets: that is, if $\Delta_1 \subseteq \Delta_2$ and Δ_2 is sup-normal, so is Δ_1. This is not an empty definition, that is, $\mathrm{Clos}(\mathcal{P})$ may not be sup-normal. For example, let $\mathcal{P}$ be the real interval $[0,1]$, endowed with the usual ordering. For every $x \in [0,1]$, let c_x be the closure operation such that $c_x(y) := \max\{x, y\}$, and let $\Lambda := \{c_x \mid x \in [0,1)\}$, $c := \sup \Lambda$. Then, $c_x(y) = y$ if and only if $y \geq x$, and thus $\mathcal{P}^{c_x} = [x,1]$; therefore, $\mathcal{P}^c = \bigcap_{x\in[0,1)}[x,1] = \{1\}$, i.e., $c = c_1$. In particular, $c(0) = 1$; however,

$$(c_{x_1} \circ \cdots c_{x_n})(0) = \max\{x_1, \ldots, x_n\} \neq 1$$

if $x_1, \ldots, x_n \neq 1$. Hence, $\mathrm{Clos}([0,1])$ is not sup-normal.

Theorem 2.10. *Let $\mathcal{P}$ be a complete lattice and let $\mathscr{P}$ be a sup-generating set for $\mathcal{P}$. Let Δ be a subset of $\mathrm{Clos}(\mathcal{P})$ such that:*

(a) *there is a family $\mathcal{A} \subseteq \mathcal{P}$ such that the $(\mathcal{A}, \mathscr{P})$-topology on Δ coincides with the $(\mathcal{P}, \mathscr{P})$-topology and, for every $x \in \mathcal{A}$ and every $y \in \mathscr{P}$, $\inf(V_{x,y} \cap \Delta) \in \Delta$;*
(b) *Δ is sup-normal and $\sup \Lambda \in \Delta$ for every $\Lambda \subseteq \Delta$.*

Then Δ, endowed with the $(\mathcal{P}, \mathscr{P})$-Zariski topology, is a spectral space.

Proof. Since $\mathscr{P}$ is sup-generating, $\mathrm{Clos}(\mathcal{P})$ is T_0 by Proposition 2.6, and thus so is Δ with the subspace topology.

By the first hypothesis, the set

$$\mathcal{S} := \{V_{x,y} \cap \Delta \mid x \in \mathcal{P}, y \in \mathscr{P}\}$$

is a subbasis for the $(\mathcal{P}, \mathscr{P})$-Zariski topology on Δ. Let $\mathscr{U}$ be an ultrafilter on Δ. By [36, Corollary 3.3] (see also Theorem A.2), to show that Δ is spectral it is enough to show that

$$\Delta_{\mathcal{S}}(\mathscr{U}) := \{x \in X \mid \text{ for all } B \in \mathcal{S}, B \in \mathscr{U} \iff x \in B\}$$

is nonempty.

Let c be the closure operation

$$c := \sup\{\inf(B) \mid B \in \mathcal{S} \cap \mathscr{U}\}.$$

Note that $\inf(B) \in \Delta$ by hypothesis, for every such B, and thus also $c \in \Delta$. We want to show that $c \in \Delta_{\mathcal{S}}(\mathscr{U})$. Let $C \in \mathcal{S}$.

Suppose $C = V_{x,y} \cap \Delta \in \mathscr{U}$. Then, $c \geq \inf(C)$ and so clearly $c \in C$.

Conversely, suppose $c \in B = V_{x,y} \cap \Delta$, i.e., $y \leq c(x)$. Then, by sup-normality, there are $B_1, \ldots, B_n \in \Lambda$ such that, defining $c_i := \inf(B_i)$, we have $y \leq (c_1 \circ \cdots \circ c_n)(x)$. If $d \in B_1 \cap \cdots \cap B_n$, then $d \geq c_i$ for every i, and so $y \leq d(x)$; therefore, $d \in V_{x,y}$. It follows that $B_1 \cap \cdots \cap B_n \cap \Delta \subseteq V_{x,y} \cap \Delta$; however, each $B_i \cap \Delta$ in in $\mathscr{U}$, and thus so do $B_1 \cap \cdots \cap B_n \cap \Delta$ and $V_{x,y} \cap \Delta = B$. Therefore, $c \in \Delta_{\mathcal{S}}(\mathscr{U})$ and Δ is a spectral space. $\square$

2.2. The set of semistar operations

Let D be an integral domain with quotient field K. We will denote by:

- $\mathcal{I}(D)$ the set of ideals of D;
- $\mathcal{F}(D)$ the set of fractional ideals of D, i.e., of D-submodules I of K such that $dI \subseteq D$ for some $d \neq 0$;
- $\mathbf{F}(D)$ the set of D-submodules of K.

We will also denote by $\mathcal{I}_f(D)$ and $\mathcal{F}_f(D)$ the set of ideals (respectively, fractional ideals) that are finitely generated. Note that we do not need to define the analogous set $\mathbf{F}_f(D)$ since every finitely generated D-submodule of K is a fractional ideal. An ideal of D is sometimes called, to emphasize the distinction with fractional ideals, an *integral* ideal.

Clearly, we have $\mathcal{I}(D) \subseteq \mathcal{F}(D) \subseteq \mathbf{F}(D)$.

We define $(F : G) := \{x \in K \mid xG \subseteq F\}$ and $(F :_D G) := \{x \in D \mid xD \subseteq F\} = (F : G) \cap D$.

When dealing with closure operations on any of $\mathcal{I}(D)$, $\mathcal{F}(D)$ or $\mathbf{F}(D)$, we will denote, as it is customary, the c-closure of I as I^c instead of $c(I)$.

Definition 2.11. *Let D be an integral domain. Then:*

- *[50, Chapter 32] A closure operation $*$ on $\mathcal{F}(D)$ is called a* star operation *if $D = D^*$ and $x \cdot I^* = (xI)^*$ for every $x \in K$, $I \in \mathcal{F}(D)$. We denote by* Star(D) *the set of star operations on D.*

- *[91] A closure operation $*$ on* $\mathbf{F}(D)$ *is called a* semistar operation *if* $x \cdot I^* = (xI)^*$ *for every* $x \in K$, $I \in \mathbf{F}(D)$. *We denote by* $\mathrm{SStar}(D)$ *the set of star operations on* D.
- *A semistar operation $*$ is called a* (semi)star operation *if* $D^* = D$. *We denote by* $\mathrm{(S)Star}(D)$ *the set of star operations on* D.

We refer the reader to [50, Chapter 32], [91] or [54] for general (non-topological) properties of star and semistar operations.

Example 2.12.

(1) The identity $d : \mathbf{F}(D) \longrightarrow \mathbf{F}(D)$, $I \mapsto I$, is a semistar operation. Likewise, the identity on $\mathcal{F}(D)$ (which we still denote by d) is a star operation.

(2) If T is an overring of D, then the map $\wedge_{\{T\}} : \mathbf{F}(D) \longrightarrow \mathbf{F}(D)$ such that $I^{\wedge_{\{T\}}} = IT$ is a semistar operation, called the *extension* to T.

(3) In particular, if $T = K$ is the quotient field of D, we get the map

$$I^{\wedge_{\{K\}}} = IK = \begin{cases} K & \text{if } I \neq (0) \\ (0) & \text{if } I = (0), \end{cases}$$

that is the maximum of $\mathrm{SStar}(D)$.

(4) If Δ is a family of overrings of D, then we define $\wedge_\Delta := \inf\{\wedge_{\{T\}} \mid T \in \Delta\}$, or more explicitly

$$I^{\wedge_\Delta} := \bigcap_{T \in \Delta} IT.$$

If $\bigcap\{T \mid T \in \Delta\} = D$, then the restriction of $\wedge_\Delta$ to $\mathcal{F}(D)$ is also a star operation.

(5) The v-operation is the map $I \mapsto (D : (D : I))$, as I ranges among D-fractional ideals or D-submodules of K (respectively, if we want to define a star or a semistar operation). Moreover, v (when considered as a star operation) is the maximum of $\mathrm{Star}(D)$.

(6) For any D-submodule I of K, let I^b be the set of elements x of K such that there is a $n \in \mathbb{N}$ and there are $a_1, \ldots, a_n \in K$, $a_i \in I^i$, such that

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$

Then, the map $I \mapsto I^b$ defines a semistar operation, called the *b-operation.*

While star and semistar operations seem very similar, the appearance of the two sets $\mathrm{Star}(D)$ and $\mathrm{SStar}(D)$ is very sensitive to the two differences between the definitions. Firstly, the set $\mathbf{F}(D)$ is (usually) much bigger than the set $\mathcal{F}(D)$ of fractional ideals, and thus the set of semistar operation tends to be bigger than the set of star operations. Secondly, the axiom $D = D^*$ confers to star operations a certain "rigidity", while the absence of this property makes semistar operations a much more "flexible" way to look at a ring and at its ideals. We shall see an example of this phenomenon when dealing with closures induced by sets of overrings (see Remark 2.61 at the beginning of Section 2.3).

The latter is the main reason why, at least in this chapter, we will mostly work with semistar operation. Note, however, that (semi)star operations provide a link between the two worlds:

Proposition 2.13. *Let D be an integral domain.*

(a) If $*$ is a (semi)star operation on D, then $*|_{\mathcal{F}(D)}$ is a star operation on D.
(b) If $*$ is a star operation on D, there is (at least) one (semi)star operation $\sharp$ such that $\sharp|_{\mathcal{F}(D)} = *$.

Proof. The first claim is obvious. For the second, we can define $\sharp$ as

$$I^\sharp := \begin{cases} I^* & \text{if } I \in \mathcal{F}(D) \\ K & \text{otherwise} \end{cases}$$

and it is immediate to see that $\sharp$ verifies the claimed properties. □

The semistar operation $\sharp$ defined in the proof above is called the *trivial extension* of $*$, and it is often denoted by $*_e$.

The order structure on $\mathrm{Clos}(\mathbf{F}(D))$ carries over with no problem to the set of semistar operation; moreover, the infimum and the supremum of any subset of $\mathrm{SStar}(D)$ is still a semistar operation. Indeed, the proof of Proposition 2.3 shows that the infimum of Δ is the map $\flat : I \mapsto \bigcap\{I^* \mid * \in \Delta\}$, and clearly

$$x \cdot I^\flat = x \cdot \bigcap_{*\in\Delta} I^* = \bigcap_{*\in\Delta} x \cdot I^* = \bigcap_{*\in\Delta} (xI)^* = (xI)^\flat.$$

On the other hand, we can modify Lemma 2.2(b) by requiring that the set Y is not only closed by intersection, but also by multiplication by elements, that is, that $xI \in Y$ whenever $I \in Y$ and $x \in K$; this property is inherited by the intersection of sets having it, and thus the same proof of Proposition 2.3 shows that the supremum of a family of semistar operation is a semistar operation.

Definition 2.14. *Let D be an integral domain. We define the* Zariski topology *(without specifications) on* $\mathrm{SStar}(D)$ *to be the topology induced by the* $(\mathbf{F}(D), \{D\})$*-Zariski topology on* $\mathrm{Clos}(\mathbf{F}(D))$*; that is, the topology generated by the subsets*

$$V_I := \{* \in \mathrm{SStar}(D) \mid D \subseteq I^*\} = \{* \in \mathrm{SStar}(D) \mid 1 \in I^*\}$$

as I ranges among the D-submodules of K.

We can also make a parallel definition for the Zariski topology on $\mathrm{Star}(D)$, by restricting I to range only between the fractional ideals of D. (We must, however, be a little careful in applying Section 2.1, since $\mathcal{F}(D)$ is not a complete lattice.) We shall investigate the link between the topologies on these two sets in Section 2.2.5.

An equivalent definition of the Zariski topology uses, instead of only D, any principal ideal xD, or, if repetition is not a problem, the whole array of principal fractional ideals of D. (The equivalence follows from the fact that $x \in I^*$ if and only if $1 \in (x^{-1}I^*)$, i.e., in the notation of Section 2.1, $V_{I,xD} = V_{x^{-1}I,D}$.) While this is unnecessary in the case of semistar operation, it will be useful when dealing with semiprime operations in Section 2.2.5.1.

Being a special case of the $(\mathscr{F}, \mathscr{P})$-Zariski topology, we can apply the results of Section 2.1: keeping in mind the previous observation, and since the set of principal ideals is sup-generating for $\mathbf{F}(D)$ (because $I = \sum\{xD \mid x \in I\}$ for every D-submodule I), we have that $\mathrm{SStar}(D)$ is a T_0 space, and by Proposition 2.8 every V_I, and in particular $\mathrm{SStar}(D)$ itself, is compact. By Proposition 2.7, moreover, $\mathrm{SStar}(D)$ has a unique closed point (the identity d); it also has a generic point, the closure $\wedge_{\{K\}}$ (see Example 2.12(3)). Thus, the only thing needed to apply Theorem 2.10 and show that $\mathrm{SStar}(D)$ is a spectral space is to show that it is sup-normal; that is, that if Λ is a family of semistar operation then

$$I^{\sup \Lambda} = \bigcup\{I^{*_1 \circ \cdots \circ *_n} \mid *_1, \ldots, *_n \in \Lambda\}.$$

Unfortunately, this is not true, as the next example shows.

Example 2.15. Let D be a ring with quotient field K, and suppose that the complete integral closure T of D is different from K and is not a fractional ideal of D. (This holds, for example, if D is a Noetherian domain whose integral closure is not finitely generated over D; see [96] for explicit examples.) Let Δ be the set of overrings of D that are fractional ideals over D; note that every member of Δ is almost integral over D, and thus it is contained in T.

For each $A \in \Delta$, define $*_A$ as the map such that, for every $I \in \mathbf{F}(D)$,

$$I^{*_A} := \begin{cases} IA & \text{if } I \in \mathcal{F}(D) \\ K & \text{otherwise.} \end{cases}$$

Since A is a fractional ideal of D, so is IA if $I \in \mathcal{F}(D)$; hence, we see that $*_A$ is idempotent, and thus a semistar operation on D. (Extensivity and order-preservation are clear.)

We first claim that $*_{A_1} \circ *_{A_2} = *_{A_1A_2}$. (Note that A_1A_2 is still a fractional ideal of D.) Indeed, if $I \in \mathcal{F}(D)$ then

$$I^{*_{A_1} \circ *_{A_2}} = (I^{*_{A_1}})^{*_{A_2}} = (IA_1)^{*_{A_2}} = IA_1A_2 = I^{*_{A_1A_2}},$$

since $IA_1 \in \mathcal{F}(D)$; on the other hand, if $I \notin \mathcal{F}(D)$, then $I^{*_{A_1} \circ *_{A_2}} = K = I^{*_{A_1A_2}}$. Therefore, by induction, $*_{A_1} \circ \cdots \circ *_{A_n} = *_{A_1 \cdots A_n}$ for every $A_1, \ldots, A_n \in \Delta$, and

$$\bigcup\{D^{*_{A_1} \circ \cdots \circ *_{A_n}}\} = \bigcup\{D^{*_A} \mid A \in \Delta\} = \bigcup_{A \in \Delta} DA = \bigcup_{A \in \Delta} A = T.$$

Let now $* := \sup\{*_A \mid A \in \Delta\}$. If an overring B of D is $*$-closed, then it has to be $*_A$-closed for every A; therefore, it must be either K or a fractional ideal of D. In the latter case, we must also have $BA = B$ for every $A \in \Delta$; in particular, $A \subseteq B$ for every $A \in \Delta$. But this would imply that $T \subseteq B$, against the fact that T is not a D-fractional ideal while B is. Hence, the only $*$-closed overring of D is K, and $* = \wedge_{\{K\}}$ is the trivial extension of D. In particular, $D^* = K$, which is not contained in T.

Therefore, $\{*_A \mid A \in \Delta\}$ is not a sup-normal family.

We are not able to prove that $\mathrm{SStar}(D)$ is a spectral space, nor we have examples where it is not. Therefore, our goal becomes to find subsets of $\mathrm{SStar}(D)$ that *are* sup-normal and thus spectral.

2.2.1. Finite-type operations

Definition 2.16. *Let $*$ be a semistar operation on an integral domain D. Define $*_f : \mathbf{F}(D) \longrightarrow \mathbf{F}(D)$ as the map*

$$*_f : I \mapsto \bigcup\{F^* \mid F \subseteq I, F \in \mathcal{F}_f(D)\}.$$

If $* = *_f$, we say that $*$ is a semistar operation of finite type*; we denote by* $\mathrm{SStar}_f(D)$ *the set of semistar operation of finite type on D.*

It is easy to see that, even if $*$ is not of finite type, $*_f$ is still a semistar operation, and it is of finite type; we call it the semistar operation of finite type *associated* to $*$. In particular, we can build a map

$$\begin{aligned} \Psi_f \colon \mathrm{SStar}(D) &\longrightarrow \mathrm{SStar}_f(D) \\ * &\longmapsto *_f. \end{aligned}$$

Remark 2.17.

(1) Semistar operation of finite type are determined by their action on finitely-generated ideals: that is, once we know the closure F^* for every $F \in \mathcal{F}_f(D)$, and that $*$ is a semistar operation of finite type, we can reconstruct the whole $*$.
(2) If F is finitely generated, then $F^* = F^{*_f}$. Indeed, the inclusion $I^{*_f} \subseteq I^*$ holds for every I; on the other hand, F^{*_f} contains F^* since F is finitely generated and contained in itself. Therefore, $F^* = F^{*_f}$.
(3) By the point above, and by the fact that $*_f$ is a semistar operation, we have $*_f \leq *$. Indeed, $*_f$ is the biggest semistar operation of finite type that is smaller than $*$.

The map Ψ_f and the topology induced on $\mathrm{SStar}_f(D)$ are closely interlinked, as the next proposition shows. See Section A.1 for the definition of the canonical T_0 quotient.

Proposition 2.18. *Let D be an integral domain.*

(a) The $(\mathcal{F}_f(D), \{D\})$-Zariski topology on $\mathrm{SStar}_f(D)$ *coincides with the Zariski topology.*
(b) The map Ψ_f, defined above, is a topological retraction.

(c) *If also* $\mathrm{SStar}(D)$ *is endowed with the* $(\mathcal{F}_f(D),\{D\})$*-topology, then* Ψ_f *is the canonical* T_0 *quotient of* $\mathrm{SStar}(D)$.

Proof. (a) The topology induced on $\mathrm{SStar}_f(D)$ by $(\mathcal{F}_f(D),\{D\})$-Zariski topology coincides with the topology whose subbasic open sets are those in the form $U_F := V_F \cap \mathrm{SStar}_f(D)$, as F ranges among the finitely generated fractional ideals of D; since every such U_F is also the restriction of an open set of $\mathrm{SStar}(D)$, it is enough to show that $V_I \cap \mathrm{SStar}_f(D)$ can be generated by the U_F. But, if $* \in V_I \cap \mathrm{SStar}_f(D)$, then $1 \in I^*$, and since $*$ is of finite type $1 \in F^*$ for some finitely generated $F \subseteq I$, that is, $* \in V_F$. It follows that

$$V_I \cap \mathrm{SStar}_f(D) = \bigcup\{U_F \mid F \subseteq I, F \in \mathcal{F}_f(D)\}.$$

(b) By definition, Ψ_f is the identity on $\mathrm{SStar}_f(D)$, and in particular Ψ_f is surjective. The continuity of Ψ_f follows from the fact that $\Psi_f^{-1}(U_F) = V_F$ for every $F \in \mathcal{F}_f(D)$; hence, since the family of the U_F is a subbase (point (a)), Ψ_f is a topological retraction.

(c) It is enough to note that, if $\mathrm{SStar}(D)$ is endowed with the $(\mathcal{F}_f(D),\{D\})$-Zariski topology, then two semistar operations $*_1, *_2$ are topologically indistinguishable if and only if $(*_1)_f = (*_2)_f$. $\square$

The next step is showing that the infimum of every U_F is still of finite type.

Proposition 2.19. *Let* D *be an integral domain and let* F *be a finitely generated fractional ideal of* D*. Then, the infimum of* $U_F = V_F \cap \mathrm{SStar}_f(D)$ *is of finite type; in particular, every* U_F *is compact.*

Proof. Let $*$ be the infimum of U_F in $\mathrm{SStar}(D)$ (it exists since $\mathrm{SStar}(D)$ is a complete lattice). Then, $1 \in F^*$; but since F is finitely generated, $1 \in F^{*_f}$, and $*_f \in U_F$. However, $*_f \leq *$; since $*$ was defined as an infimum, $* = *_f$, and thus $* \in \mathrm{SStar}_f(D)$. The last claim follows with the same proof of Proposition 2.8. $\square$

Hence, the family $\mathcal{S}_f := \{U_F \mid F \in \mathcal{F}_f(D)\}$ satisfies the first hypothesis of Theorem 2.10; to show that $\mathrm{SStar}_f(D)$ is a spectral space, we only need to show that it is supnormal and that it contains the supremum of every subset. To do this, we merely translate to the semistar setting a proof originally given for star operations in [5, p.1628].

Lemma 2.20. *Let* D *be an integral domain and let* $\Lambda \subseteq \mathrm{SStar}_f(D)$*. Then,* $\sup\Lambda$ *is of finite type and, for any* $I \in \mathbf{F}(D)$,

$$I^{\sup\Lambda} = \bigcup\{I^{*_1\circ\cdots\circ *_n} \mid *_1, \ldots *_n \in \Lambda\}.$$

Proof. Let $*$ be the map

$$* : I \mapsto \bigcup\{I^{*_1\circ\cdots\circ *_n} \mid *_1, \ldots *_n \in \Lambda\}.$$

Clearly, $*$ is an extensive and order-preserving map (if $I \subseteq J$, then $I^{*_1\circ\cdots\circ *_n} \subseteq J^{*_1\circ\cdots\circ *_n}$); moreover, $x \cdot I^* = (xI)^*$. We claim that $*$ is also idempotent.

Indeed, let $x \in (I^*)^*$. Then, there are $*_1, \ldots, *_n \in \Lambda$ such that

$$x \in (I^*)^{*_1 \circ \cdots \circ *_n} = (I^{* \circ *_1 \circ \cdots \circ *_{n-1}})^{*_n}.$$

Since $*_n$ is of finite type, there is a finitely generated ideal $F_n \subseteq I^{* \circ *_1 \circ \cdots \circ *_{n-1}}$ such that $x \in F_n^{*_n}$. Suppose $F_n := f_1 D + \cdots + f_k D$. For every i, the element f_i is in $I^{* \circ *_1 \circ \cdots \circ *_{n-1}}$; hence, there is a finitely generated $G_{n-1,i} \subseteq I^{* \circ *_1 \circ \cdots \circ *_{n-2}}$ such that $f_i \in G_{n-1,i}^{*_{n-1}}$; thus, if $F_{n-1} := G_{n-1,1} + \cdots + G_{n-1,k}$, we have $F_n \subseteq F_{n-1}^{*_{n-1}}$, and thus $x \in F_{n-1}^{*_{n-1} \circ *_n}$. repeating the process, we can find a finitely generated $F_1 \subseteq I^*$ such that $x \in F_1^{*_1 \circ \cdots \circ *_n}$.

If now $F_1 := a_1 D + \cdots + a_t D$, with the same reasoning we can find for each i some semistar operations $\sharp_1, \ldots, \sharp_n \in \Lambda$ (dependent on i) such that $a_i \in H_i^{\sharp_1 \circ \cdots \circ \sharp_n}$ for some finitely generated $H_i \subseteq I$; putting this sets of closures one after another, we get a family $\flat_1, \ldots, \flat_N$ such that $a_i \in H^{\flat_1 \circ \cdots \circ \flat_N}$ for every i, where $H := H_1 + \ldots + H_t$; that is, $F_1 \subseteq H^{\flat_1 \circ \cdots \circ \flat_N}$, and so

$$x \in H^{\flat_1 \circ \cdots \circ \flat_N \circ *_1 \circ \cdots \circ *_n} \subseteq I^*.$$

Since x was arbitrarily chosen in $(I^*)^*$, we have $I^* = (I^*)^*$, that is, $*$ is idempotent and thus a semistar operation; moreover, the fact that H is finitely generated and $H \subseteq I$ implies that $*$ is of finite type.

Finally, we show that $* = \sup \Lambda$. Clearly $I^\sharp \subseteq I^*$ for every $\sharp \in \Lambda$, and thus $* \geq \sup \Lambda$. On the other hand, if $I \in \mathbf{F}(D)$ and $*_1, \ldots, *_n \in \Lambda$, then, by definition, $I^{*_1} \subseteq I^{\sup \Lambda}$. Hence,

$$I^{*_1 \circ *_2} = (I^{*_1})^{*_2} \subseteq (I^{\sup \Lambda})^{*_2}.$$

However, $I^{\sup \Lambda}$ is $(\sup \Lambda)$-closed, and thus, by the characterization of the supremum, it is $*$-closed for every $* \in \Lambda$; in particular, it is $*_2$-closed. Therefore, $(I^{\sup \Lambda})^{*_2} = I^{\sup \Lambda}$; repeating this process we get $I^{*_1 \circ \cdots \circ *_n} \subseteq I^{\sup \Lambda}$, and so $I^* \subseteq I^{\sup \Lambda}$. It follows that $* = \sup \Lambda$. $\square$

Theorem 2.21. *For any domain D, the set* $\mathrm{SStar}_f(D)$*, endowed with the Zariski topology, is a spectral space.*

Proof. It is enough to put together Propositions 2.18 and 2.19 (to find $\mathcal{A} = \mathcal{S}_f$) and Lemma 2.20, and then apply Theorem 2.10. $\square$

Corollary 2.22. *For any domain D, the set* $\mathrm{(S)Star}_f(D)$*, endowed with the Zariski topology, is a spectral space.*

Proof. It is enough to note that the set $\mathrm{(S)Star}(D)$ is closed by supremum (since D is closed by every (semi)star operation) and so the proof of Theorem 2.21 apply to this case as well. $\square$

2.2.2. Stable semistar operations

Definition 2.23. *A semistar operation $*$ on an integral domain D is called* stable *if it distributes over finite intersections, that is, if, for every $I, J \in \mathbf{F}(D)$, $(I \cap J)^* = I^* \cap J^*$. We denote by* $\mathrm{SStar}_{st}(D)$ *the set of stable semistar operations on D.*

The stronger property that $*$ distributes over arbitrary intersections, i.e., that $\bigcap_{\alpha\in A} I_\alpha^* = (\bigcap_{\alpha\in A} I_\alpha)^*$ is very rare; indeed, a finite-type star operation with this property has to be the identity [3, Theorem 7]. In view of the following Proposition 2.25, the same happens for semistar operations.

Just like in the case of finite-type operations, we can associate to an arbitrary semistar operation $*$ a stable semistar operation $\overline{*}$ by defining (following [6, Definition 2.2])

$$E^{\overline{*}} := \bigcup\{(E : I) \mid I \in \mathcal{I}(D), I^* = D^*\}.$$

The map $\overline{*}$ is always a stable semistar operation such that $\overline{*} \leq *$, and it is the biggest stable semistar operation smaller than $*$; in particular, $* = \overline{*}$ if and only if $*$ is stable. Thus, we have a map

$$\begin{aligned} \Psi_{st} \colon \mathrm{SStar}(D) &\longrightarrow \mathrm{SStar}_{st}(D) \\ * &\longmapsto \overline{*} \end{aligned}$$

and we can show that is behaves very much like Ψ_f; the next proposition is an analogue of Proposition 2.18.

Proposition 2.24. *Let D be an integral domain.*

(a) The $(\mathcal{I}(D), \{D\})$-Zariski topology on $\mathrm{SStar}_{st}(D)$ coincides with the Zariski topology.
(b) The map Ψ_{st}, defined above, is a topological retraction.
(c) If also $\mathrm{SStar}(D)$ is endowed with the $(\mathcal{I}(D), \{D\})$-topology, then Ψ_{st} is the canonical T_0 quotient of $\mathrm{SStar}(D)$.

Proof. (a) For any nonzero D-submodule E of K, and any stable semistar operation $*$, we have $1 \in E^*$ if and only if $1 \in E^* \cap D^* = (E \cap D)^*$. Therefore, $* \in V_E$ if and only if $* \in V_{E\cap D}$. Hence, $V_E \cap \mathrm{SStar}_{st}(D) = V_{E\cap D} \cap \mathrm{SStar}_{st}(D)$, and the claim follows.

(b) By definition, Ψ_{st} is the identity on $\mathrm{SStar}_{st}(D)$, and in particular Ψ_{st} is surjective. By the previous point, if E is an integral ideal of D then $\Psi_{st}^{-1}(V_E \cap \mathrm{SStar}_{st}(D)) = V_E$, and so Ψ_{st} is continuous, and thus a topological retraction.

(c) It is enough to note that, if $\mathrm{SStar}(D)$ is endowed with the $(\mathcal{I}(D), \{D\})$-Zariski topology, then two semistar operations $*_1, *_2$ are topologically indistinguishable if and only if $1 \in I^{*_1}$ is equivalent to $1 \in I^{*_2}$ for every integral ideal I. We claim that this holds if and only if $\overline{*_1} = \overline{*_2}$.

Indeed, if $\overline{*_1} = \overline{*_2}$ and $1 \in I^{*_1}$, then

$$1 \in (I : I) \subseteq I^{\overline{*_1}} = I^{\overline{*_2}} \subseteq I^{*_2}$$

and so $1 \in I^{*_2}$; by symmetry, $1 \in I^{*_2}$ implies $1 \in I^{*_1}$.

Conversely, if $1 \in I^{*_1}$ and $1 \in I^{*_2}$ are equivalent, then $I^{*_1} = D^{*_1}$ if and only if $I^{*_2} = D^{*_2}$, and thus the set of ideals used in the definition of $E^{\overline{*_1}}$ and $E^{\overline{*_2}}$ is the same; therefore, $\overline{*_1} = \overline{*_2}$. □

The proposition above can be summarized by saying that a stable semistar operation on a domain D is determined by its action on the integral ideals of D. It is therefore natural to think that stable semistar operation and stable star operation are much more linked than general semistar and star operations.

Proposition 2.25. *Let D be an integral domain. For any stable star operation $*$, there is exactly one stable semistar operation $\sharp$ on D such that $\sharp|_{\mathcal{F}(D)} = *$.*

Proof. Suppose there exist two stable semistar extensions $\sharp_1$ and $\sharp_2$ of the star operation $*$, i.e., $\sharp_1|_{\mathcal{F}(D)} = \sharp_2|_{\mathcal{F}(D)} = *$. Since $\mathrm{SStar}_{st}(D)$ is T_0, there is a subbasic open set $\overline{U}_I := V_I \cap \mathrm{SStar}_{st}(D)$, where I is a proper ideal of D, such that $\sharp_1 \in \overline{U}_I$ but $\sharp_2 \notin \overline{U}_I$ (or conversely). But this would imply $I^* = I^{\sharp_1} \neq I^{\sharp_2} = I^*$, a contradiction.

For the existence, consider the semistar operation $\sharp := \overline{*_e}$, where $*_e$ is the trivial semistar extension of $*$ defined in the proof of Proposition 2.13. By definition, $\sharp$ is a stable semistar operation. On the other hand, since $D^\sharp = D$, if I is a nonzero D-submodule of K such that $I^\sharp = D^\sharp$, then I is an ideal in D. It follows that $\sharp|_{\mathcal{F}(D)}$ is the stable closure of $*$ as a *star* operation, as defined in [6, Definition 2.2]. However, since $*$ is already stable, we have $\sharp|_{\mathcal{F}(D)} = *$, i.e., $\sharp$ is an extension of $*$. □

Note the trivial extension $*_e$ of a stable star operation $*$ is almost never stable: indeed, if there are two D-submodules of K, say I and J, that are not fractional ideals but such that $I \cap J$ is a fractional ideal, then $*_e$ is not stable, since $(I \cap J)^{*_e} \neq K = I^{*_e} \cap J^{*_e}$. This happens fairly often: for example, it is enough to have a domain D such that $\bigcap_{n \geq 0} s^n D = (0)$ for every nonunit $s \in D$ and two localizations $S^{-1}D$ and $T^{-1}D$ (both different from K) such that $S^{-1}D \cap T^{-1}D = D$ (the condition on the intersection guarantees that no nontrivial localization is a fractional ideal). An explicit example is a Noetherian domain D with exactly two maximal ideals, M and N, where we consider D_M and D_N.

2.2.3. Spectral semistar operations

Definition 2.26. *Let D be an integral domain and $\Delta \subseteq \mathrm{Spec}(D)$, $\Delta \neq \emptyset$. Then, we define s_Δ as the map*

$$s_\Delta : I \mapsto \bigcap_{P \in \Delta} ID_P.$$

We say that a semistar operation $*$ on D is spectral *if $* = s_\Delta$ for some $\Delta \subseteq \mathrm{Spec}(D)$. We denote by $\mathrm{SStar}_{sp}(D)$ (respectively, $\mathrm{SStar}_{f,sp}(D)$) the set of spectral (respectively, spectral of finite type) semistar operation.*

Remark 2.27.

(1) s_Δ is always a semistar operation.
(2) If $\Delta = \{P\}$ is a singleton we will often denote $s_{\{P\}}$ as s_P.
(3) The condition $\Delta \neq \emptyset$ is not really restrictive: indeed, if $\Delta = \emptyset$, we can think of the intersection $\bigcap\{ID_P \mid P \in \Delta\}$ as the empty intersection, and thus defining $I^{s_\Delta} = K$ for all I. Under our notation (with closure operations defined on complete lattices), this is *not* a semistar operation, since in particular $(0)^{s_\Delta} = K$, while, if $*$ is a semistar operation, we must have

$$(0)^* = (0 \cdot I)^* = 0 \cdot I^* = (0)$$

where I is an arbitrary D-submodule of K. However, semistar operations are usually defined only on *nonzero* D-submodule of K; in this case, s_Δ becomes a full-fledged semistar operation, but it also becomes equal to $s_{(0)}$, the spectral semistar association associated to the zero ideal.

There is obviously a relation between a set Δ of prime ideals of D and the spectral operation s_Δ generated by Δ. To analyze this relationship, we introduce the following definitions.

Definition 2.28. *Let D be an integral domain and $*$ be a semistar operation on D.*

- *An integral ideal I of D is a* quasi-$*$-ideal *if $I = I^* \cap D$.*
- *A maximal element in the set of proper quasi-$*$-ideals if called a* $*$-maximal ideal*; we denote by* $\mathrm{QMax}^*(D)$ *the set of $*$-maximal ideals.*
- *A prime ideal of D that is also a quasi-$*$-ideal is called a* quasi-$*$-prime*; we denote by* $\mathrm{QSpec}^*(D)$ *the set of quasi-$*$-primes, and call it the* quasi-$*$-spectrum *of D.*
- *$*$ is* semifinite *(also called* quasi-spectral *in [45]) if every proper quasi-$*$-ideal is contained in a quasi-$*$-prime.*

Every finite-type semistar operation is semifinite; more strongly, if $*$ is of finite type then every proper quasi-$*$-ideal is contained in a $*$-maximal ideal. This is not true for general semifinite operations; indeed, $*$ may be semifinite without having $*$-maximal ideals. For example, if $D := K[X_1, \ldots, X_n, \ldots]$ is a polynomial ring in infinitely many indeterminates, and Δ is the set of finitely generated prime ideals of D, then $* := s_\Delta$ is a semifinite operation without $*$-maximal ideals (see [40, Remark 5.6] for more details).

Proposition 2.29 [42, Section 4]. *Let D be an integral domain and let $\Delta, \Lambda \subseteq \mathrm{Spec}(D)$, $\Delta, \Lambda \neq \emptyset$. Then:*

(a) $\mathrm{QSpec}^{s_\Delta}(D) = \Delta^\downarrow = \{Q \mid Q \subseteq P \text{ for some } P \in \Delta\}$
(b) $s_\Delta = s_\Lambda$ *if and only if* $\Delta^\downarrow = \Lambda^\downarrow$*;*
(c) $s_\Delta \leq s_\Lambda$ *if and only if* $\Delta^\downarrow \supseteq \Lambda^\downarrow$*.*

Proof. (a) Suppose that $Q \in \Delta^\downarrow$; then, $Q \subseteq P$ for some $P \in \Delta$, and thus $Q^* \cap D \subseteq QD_P \cap P = Q$, i.e., Q is a quasi-$*$-ideal. On the other hand, if $Q \notin \Delta^\downarrow$, then $QD_P = D_P$ for every $P \in \Delta$, and thus $Q^* \cap D = D$, i.e., Q is not a quasi-$*$-ideal. Therefore, $\mathrm{QSpec}^*(D) = \Delta^\downarrow$.

(b) The proof above shows that if $s_\Delta = s_\Lambda$ then $\Delta^\downarrow = \Lambda^\downarrow$. To show the converse, it is enough to show that $s_\Delta = s_{\Delta^\downarrow}$; but, for every $Q \in \Delta^\downarrow \setminus \Delta$, there is a $P \in \Delta$ such that $Q \subseteq P$, and so $D_P \subseteq D_Q$ and $ID_P \subseteq ID_Q$ for every $I \in \mathbf{F}(D)$; it follows that, in the intersection $\bigcap\{ID_A \mid A \in \mathrm{Spec}(D)\}$, the set ID_Q is superfluous. Hence $s_\Delta = s_{\Delta^\downarrow}$.

(c) By point (b), we can suppose $\Delta = \Delta^\downarrow$ and $\Lambda = \Lambda^\downarrow$. If $\Delta \supseteq \Lambda$, then the intersection defining s_Λ is only a part of the intersection defining s_Δ; hence, $I^{s_\Delta} \subseteq I^{s_\Lambda}$ and $s_\Delta \leq s_\Lambda$. Conversely, if $s_\Delta \leq s_\Lambda$, then every quasi-s_Λ-ideal is quasi-s_Δ-ideal; in particular, $\Delta = \mathrm{QSpec}^{s_\Delta}(D) \supseteq \mathrm{QSpec}^{s_\Lambda}(D) = \Lambda$. $\square$

Spectral semistar operations are also closely linked to stable operations; we premise a more general lemma, which generalizes [86, Theorem 7.4] and [2, Theorem 2]. In turn, it will be generalized from a topological point of view in Theorem 2.76.

Recall that a map $\phi : A \longrightarrow B$ is an *epimorphism* in the category of rings if, for every $\psi_1, \psi_2 : B \longrightarrow C$, the equality $\psi_1 \circ \phi = \psi_2 \circ \phi$ implies that $\psi_1 = \psi_2$. If the inclusion map $A \hookrightarrow B$ is an epimorphisms, we will call the ring extension $A \subseteq B$ an *epimorphic extension*. Examples of epimorphisms are surjective maps and localizations; our main example will be the extension $D \subseteq K$, where D is a domain and K its quotient field.

Lemma 2.30. *Let $A \subseteq B$ be an epimorphic extension, and let $I, G_1, \ldots, G_n$ be A-submodules of B. If I is flat over A, then*

$$I(G_1 \cap \ldots \cap G_n) = IG_1 \cap \ldots \cap IG_n.$$

Proof. By induction, it suffices to show the statement for $n = 2$. Consider the map

$$\begin{aligned} \lambda \colon B \otimes_A B &\longrightarrow B \\ b_1 \otimes b_2 &\longmapsto b_1 b_2, \end{aligned}$$

and, for any A-submodules I, G of B, denote by $I \otimes G$ the subset of $B \otimes B$ generated by the elements $i \otimes g$, as i varies in I and g varies in G.

Then, $\lambda(I \otimes G) = IG$; therefore, by [86, Theorem 7.4]

$$I(G_1 \cap G_2) = \lambda(I \otimes (G_1 \cap G_2)) = \lambda((I \otimes G_1) \cap (I \otimes G_2)).$$

Since $A \subseteq B$ is an epimorphic extension, λ is an isomorphism (indeed, this property actually characterizes epimorphisms [83, Lemma 1.0]); in particular, λ is a bijection, and thus

$$\lambda((I \otimes G_1) \cap (I \otimes G_2)) = \lambda(I \otimes G_1) \cap \lambda(I \otimes G_2) = IG_1 \cap IG_2.$$

This completes the proof. $\square$

Note that this proposition does not hold for a generic extension $A \subseteq B$: for example, if X is an indeterminate over A, $B = A[X] = I$, $G_1 = A$, $G_2 = XA[X]$, then $G_1 \cap G_2 = (0)$ and so $I(G_1 \cap G_2) = (0)$, while $IG_1 \cap IG_2 = XA[X]$.

Proposition 2.31. *Let D be an integral domain and let $*$ be a semistar operation on D.*

- *(a) If $*$ is spectral, then it is stable.*
- *(b) If $*$ is stable, then it is spectral if and only if it is semifinite.*
- *(c) If $*$ is of finite type, then it is stable if and only if it is spectral.*

Proof. Part (a) follows from the fact that any localization is flat and from Lemma 2.30; part (b) and (c) follow from [91, Theorem 22] or [42, Theorem 4.12(3) and Proposition 4.23(2)]. $\square$

A different way to prove the previous proposition is to start from the fact that it holds in the setting of star operations (see [3]) and then use the extension results (Proposition 2.25 and the following 2.36).

An important information about s_Δ that we can extract from Δ is whether s_Δ is of finite type or not; the following is an improvement of [42, Corollary 4.6].

Proposition 2.32. *Let D be an integral domain and let $\Delta \subseteq \mathrm{Spec}(D)$, $\Delta \neq \emptyset$. Then, the semistar operation s_Δ is of finite type if and only if Δ is compact (in the Zariski topology).*

Proof. Suppose s_Δ is of finite type, and let $\{\mathcal{D}(I_\alpha) \mid \alpha \in A\}$ be an open cover of Δ, where each I_α is an integral ideal of D. Then, for each $P \in \Delta$ there is an α such that $I_\alpha \nsubseteq P$; it follows that $I := \sum\{I_\alpha \mid \alpha \in A\}$ is not contained in any $P \in \Delta$, and thus $I^{s_\Delta} = D^{s_\Delta}$. In particular, $1 \in I^{s_\Delta}$, and thus there is a finitely generated $F \subseteq I$ such that $1 \in F^{s_\Delta}$. There are $\alpha_1, \ldots, \alpha_n \in A$ such that $F \subseteq I_{\alpha_1} + \cdots + I_{\alpha_n}$; in particular, for each $P \in \Delta$ there is an i such that $I_{\alpha_i} \nsubseteq P$, and so $\{\mathcal{D}(I_{\alpha_1}), \ldots, \mathcal{D}(I_{\alpha_n})\}$ is a subcover of Δ. Hence, Δ is compact.

Conversely, suppose Δ is compact, and let $x \in I^{s_\Delta}$. Consider the set

$$\mathcal{U} := \{\mathcal{D}((H :_D x)) \mid H \subseteq I, H \in \mathcal{I}_f(D)\};$$

we claim that it is a cover of Δ. Otherwise, there is a $P \in \Delta$ such that $(H :_D x) = x^{-1}H \cap D \subseteq P$ for every finitely generated $H \subseteq I$. However, since a spectral operation is stable (Proposition 2.31) $(x^{-1}I \cap D)^{s_\Delta} = x^{-1}I^{s_\Delta} \cap D = D$ (since $x \in I^{s_\Delta}$) and thus there is an $y \in (x^{-1}I \cap D) \setminus P$; hence, $xy \in I$, which means that $\mathcal{D}((xyD :_D x)) \in \mathcal{U}$. But $(xyD :_D x) = x^{-1}xyD \cap D = yD \cap D$ is not contained in P, against the choice of P. Therefore, $\mathcal{U}$ is an open cover of Δ, and by compactness there is a finite subcover $\{\mathcal{D}((H_1 :_D x)), \ldots, \mathcal{D}((H_n :_D x))\}$.

We claim that $x \in H^{s_\Delta}$, where $H := H_1 + \cdots + H_n$. Otherwise, $1 \notin (x^{-1}H)^{s_\Delta}$, and so $1 \notin (x^{-1}H)^* \cap D^* = (x^{-1}H \cap D)^* = (H :_D x)^{s_\Delta}$. It follows that $(H :_D x)$ is contained in a $P \in \Delta$; but $(H_i :_D x) \subseteq (H :_D x)$ for every i, and thus P would not be contained in any $\mathcal{D}((H_i :_D x))$, against the previous paragraph. It follows that $x \in H^{s_\Delta}$, and so s_Δ is of finite type. $\square$

Keeping in mind the examples in Sections 2.2.1 and 2.2.2, we can ask if we can build, in the case of spectral operations, a map analogous to Ψ_f and Ψ_{st}, that is, a topological retraction $\Psi_{sp} : \mathrm{SStar}(D) \longrightarrow \mathrm{SStar}_{sp}(D)$. A natural try would be to define $\Psi_{sp}(*) := s_{\mathrm{QSpec}^*(D)}$; this is actually a good definition when $*$ is a semifinite operation, but in the general case it breaks down, since the quasi-$*$-spectrum need not to be very faithful to $*$. For example, if V is a non-discrete valuation ring of dimension 1, and $v : I \mapsto (V : (V : I))$ is the v-operation, then $\mathrm{QSpec}^v(V) = \{(0)\}$, and so $\Psi_{sp}(*)$ would just be the trivial extension $\wedge_{\{K\}}$ that sends every nonzero submodule I to K.

However, it is possible to construct such a map if we restrict to finite-type spectral operations; this is somewhat natural, since by Proposition 2.31 $\mathrm{SStar}_{f,sp}(D) = \mathrm{SStar}_f(D) \cap \mathrm{SStar}_{st}(D)$, and the two sets on the right-hand side are both topological retracts of $\mathrm{SStar}(D)$. Given a semistar operation $*$, following [34, Definition 3] and [6, Definition 2.2], we define $\widetilde{*}$ as the map such that, for every $E \in \mathbf{F}(D)$,

$$E^{\widetilde{*}} := \bigcup\{(E : I) \mid I \in \mathcal{I}_f(D), I^* = D^*\},$$

and define Ψ_w as the map

$$\begin{aligned}\Psi_w \colon \mathrm{SStar}(D) &\longrightarrow \mathrm{SStar}_{f,sp}(D)\\ * &\longmapsto \widetilde{*}.\end{aligned}$$

The map Ψ_w follows the same pattern of Ψ_f and Ψ_{st}:

Proposition 2.33. *Let D be an integral domain.*

(a) The $(\mathcal{I}_f(D), \{D\})$-Zariski topology on $\mathrm{SStar}_{f,sp}(D)$ coincides with the Zariski topology.
(b) The map Ψ_w, defined above, is a topological retraction.
(c) If also $\mathrm{SStar}(D)$ is endowed with the $(\mathcal{I}_f(D), \{D\})$-topology, then Ψ_w is the canonical T_0 quotient of $\mathrm{SStar}(D)$.

Proof. Repeat the proof of Proposition 2.24. □

We could have also defined Ψ_w as a composition:

Proposition 2.34. *Let D be an integral domain and $*$ be a semistar operation on D. Then, $\Psi_{st} \circ \Psi_f(*) = \Psi_w(*)$.*

Proof. We start by expliciting $\Psi_{st} \circ \Psi_f(*)$. We have

$$E^{\Psi_{st}\circ\Psi_f(*)} = E^{\Psi_{st}(\Psi_f(*))} = \bigcup\{(E : I) \mid I \in \mathcal{I}(D), I^{*_f} = D^{*_f}\}.$$

If $x \in E^{\widetilde{*}}$, then $x \in (E : I)$ for some $I \in \mathcal{I}_f(D)$ such that $I^* = D^*$; hence, $I^{*_f} = D^{*_f}$, so that $(E : I)$ appear also in the union defining $\Psi_{st} \circ \Psi_f(*)$, and $x \in E^{\Psi_{st}\circ\Psi_f(*)}$.

Conversely, if $x \in E^{\Psi_{st}\circ\Psi_f(*)}$, then $x \in (E : I)$ for some $I \in \mathcal{I}(D)$ such that $I^{*_f} = D^{*_f}$. In particular, $1 \in I^{*_f}$, and thus there is a finitely generated $J \subseteq I$ such that $1 \in J^*$; since I is an integral ideal of D, so is J, and thus $J^* = D^*$. But $x \in (E : I)$ means that $xI \subseteq E$, and thus $xJ \subseteq E$; hence, $x \in (E : J)$ and $x \in E^{\widetilde{*}}$. Therefore, $\Psi_{st}\circ\Psi_f = \Psi_w$. □

Remark 2.35. The order under which we compose Ψ_{st} and Ψ_f is important; indeed, $\Psi_f \circ \Psi_{st}$ is not always equal to Ψ_w, since the finite-type operation associated to a stable semistar operation may not be stable (or, equivalently, spectral) [6, p. 2466]. For example, let D be an essential ring that is not a PvMD; that is, suppose that D is the intersection of a family of valuation rings, each of which is a localization of D, but suppose that there is a t-prime ideal P such that D_P is not a valuation ring (where t is the finite-type star operation associated to the v-operation $v : I \mapsto (D : (D : I))$). Let Δ be the set of prime ideals P of D such that D_P is a valuation ring; then, $(s_\Delta)_f = t$ [50, Proposition 44.13] but t is not stable since D is not a PvMD [3, Theorem 6].

An explicit example of an appropriate ring D was given in [59]; we will describe it briefly, deferring the reader to [59] for the proof that D has the requested properties. Let L be a field, and take independent indeterminates $y, z, x_0, \ldots, x_n, \ldots$ Let $R := L(x_0, \ldots, x_n, \ldots)[y, z]_{(y,z)}$; for each i, let V_i be the valuation ring whose valuation $\mathbf{v}_i$ is such that:

- $\mathbf{v}_i(x_i) = \mathbf{v}_i(y) = \mathbf{v}_i(z) = 1$;
- $\mathbf{v}_i(x_j) = 0$ if $j \neq i$ (in particular, V_i contains the field $L(\{x_j\}_{j\neq i})$);
- if f is a polynomial in $y, z, x_0, \ldots, x_n, \ldots$, then $\mathbf{v}_i(f)$ is the minimum of $\mathbf{v}_i(\mu)$, as μ ranges among the monomial of f;
- if $h := f/g$ is a rational function, then $\mathbf{v}_i(h) = \mathbf{v}_i(f) - \mathbf{v}_i(g)$.

(The last two points can be summarized by saying that $\mathbf{v}_i$ is a monomial valuation – see e.g. [72, Section 6.1].) The requested ring D is now defined as $D := R \cap \bigcap\{V_i \mid i \in \mathbb{N}\}$.

The existence of Ψ_w allows to prove quickly an analogue of Proposition 2.25, and to generalize Proposition 2.32 to star operations.

Proposition 2.36. *Let D be an integral domain. For any spectral star operation of finite type $\sharp$, there is exactly one spectral semistar operation of finite type $*$ on D such that $*|_{\mathcal{F}(D)} = \sharp$.*

Proof. If $\sharp$ is of finite type, then both $\widetilde{\sharp}$ and $\overline{\sharp}$ are stable extensions of $\sharp$. Since the extension is unique, it must be $\widetilde{\sharp} = \overline{\sharp}$. □

Corollary 2.37. *Let D be an integral domain and let $\Delta \subseteq \mathrm{Spec}(D)$, $\Delta \neq \emptyset$ such that $D = \bigcap\{D_P \mid P \in \Delta\}$. Then, the star operation $s_\Delta|_{\mathcal{F}(D)}$ is of finite type if and only if Δ is compact (in the Zariski topology).*

Proof. By Propositions 2.25 and 2.36, spectral star operations of finite type correspond bijectively to spectral (semi)star operations of finite type. But, for a (semi)star operation s_Δ, being of finite type is equivalent (by Proposition 2.32) to Δ being compact; hence the same holds for star operations. □

Another consequence of the existence of Ψ_w is that we get a new way to compare different spectral operations. The correct way of expliciting this composition is topological but, instead of the usual Zariski topology on $\mathrm{Spec}(D)$, we shall use the *inverse topology* (see Section A.3 for the definitions). We denote by $\mathrm{Cl}^{\mathrm{inv}}(Y)$ the closure of $Y \subseteq \mathrm{Spec}(D)$ in the inverse topology. We start with a more general result.

Proposition 2.38. *Let D be an integral domain and let $*$ be a semifinite semistar operation on D. Then, $\widetilde{*} = s_{\mathrm{Cl}^{\mathrm{inv}}(\mathrm{QSpec}^*(D))}$.*

Proof. Let E be any D-submodule of K, and let $\sharp := s_{\mathrm{Cl}^{\mathrm{inv}}(\mathrm{QSpec}^*(D))}$.

If $x \in E^{\widetilde{*}}$, then $x \in (E : I)$ for some $I \in \mathcal{I}_f(D)$ such that $I^* = D^*$. In particular, I is not contained in any quasi-$*$-prime, and so $\mathrm{QSpec}^*(D) \subseteq \mathcal{D}(I)$; but since I is finitely generated, $\mathcal{D}(I)$ is closed in the inverse topology, and thus $\mathrm{Cl}^{\mathrm{inv}}(\mathrm{QSpec}^*(D)) \subseteq \mathcal{D}(I)$.

Hence, for any $P \in \mathrm{Cl}^{\mathrm{inv}}(\mathrm{QSpec}^*(D))$, we have $ID_P = D_P$, and using the flatness of D_P over D we obtain

$$x \in (E : I)D_P = (ED_P : ID_P) = (ED_P : D_P) = ED_P$$

so that $x \in E^\sharp$, and $\widetilde{*} \leq \sharp$.

Conversely, if $x \in E^{s_{\mathrm{QSpec}^*(D)}}$, then $x \in ED_P$ for every $P \in \mathrm{QSpec}^*(D)$. But now, since $*$ is semifinite, $E^* = \bigcap\{ED_P \mid P \in \mathrm{QSpec}^*(D)\}$ [42, Proposition 4.8], and so $x \in E^*$. Therefore, $\sharp \leq s_{\mathrm{QSpec}^*(D)} \leq *$, and applying Ψ_w we get

$$\widetilde{\sharp} = \Psi_w(s_{\mathrm{Cl}^{\mathrm{inv}}(\mathrm{QSpec}^*(D))}) \leq \Psi_w(s_{\mathrm{QSpec}^*(D)}) \leq \widetilde{*}. \tag{2.1}$$

By Proposition 2.34, $\Psi_w(\sharp) = \Psi_{st}(\Psi_f(\sharp))$; however, $\mathrm{Cl}^{\mathrm{inv}}(\mathrm{QSpec}^*(D))$ is compact in the Zariski topology (see [37, Remark 2.2 and Proposition 2.6]), and thus by Proposition 2.32 $\sharp$ is of finite type; it follows that the leftmost side in (2.1) is equal to $\sharp$. Applying the inequality obtained in the previous paragraph we have

$$s_{\mathrm{Cl}^{\mathrm{inv}}(\mathrm{QSpec}^*(D))} \leq \widetilde{*} \leq s_{\mathrm{Cl}^{\mathrm{inv}}(\mathrm{QSpec}^*(D))}$$

and thus the two semistar operations are equal, as claimed. □

This gives immediately the following corollaries.

Corollary 2.39. *Let D be an integral domain and let Δ and Λ be nonempty subsets of* $\mathrm{Spec}(D)$*. Then, $\widetilde{s_\Delta} = \widetilde{s_\Lambda}$ if and only if* $\mathrm{Cl}^{\mathrm{inv}}(\Delta) = \mathrm{Cl}^{\mathrm{inv}}(\Lambda)$.

Corollary 2.40. *Let D be an integral domain. Then, a subset $\Delta \subseteq$* $\mathrm{Spec}(D)$ *is dense in* $\mathrm{Spec}(D)$*, with respect to the inverse topology, if and only if $\widetilde{s_\Delta}$ is the identity.*

Define the w-operation as the spectral operation of finite type associated to the v-operation, and say that a domain is a *DW-domain* if w is the identity.

Corollary 2.41. *Let D be an integral domain. The following are equivalent.*

(i) D is a DW-domain.
(ii) Every $\Delta \subseteq$ $\mathrm{Spec}(D)$ *such that $\bigcap\{D_P \mid P \in \Delta\} = D$ is dense in* $\mathrm{Spec}(D)$*, with respect to the inverse topology.*
(iii) $\mathrm{QMax}^t(D)$ *is dense in* $\mathrm{Spec}(D)$*, with respect to the inverse topology.*

Proof. (i $\Longrightarrow$ ii) If D is a DW domain and $\bigcap\{D_P \mid P \in \Delta\} = D$, then $s_Y|_{\mathcal{F}_f(D)}$ is a star operation and therefore $\widetilde{s_Y} \leq \widetilde{v} \leq w = d$, and thus Δ is dense in $\mathrm{Spec}(D)$.

(ii $\Longrightarrow$ iii) It is enough to note that $\bigcap\{D_P \mid P \in \mathrm{QMax}^t(D)\} = D$.

(iii $\Longrightarrow$ i) If $\mathrm{QMax}^t(D)$ is dense, then $w = s_{\mathrm{QMax}^t(D)} = s_{\mathrm{Spec}(D)} = d$. □

Corollary 2.39, in particular, allows to view spectral semistar operation in a new light:

Proposition 2.42. *Let D be an integral domain. Then, there is a natural bijection between the set of spectral semistar operation of finite type on D and the set $\boldsymbol{\mathcal{X}}(D)$ of nonempty subsets of* $\mathrm{Spec}(D)$ *that are closed with respect to the inverse topology.*

In particular, we can see this result as a way to endow the latter set with a topology; since this set is defined not also in the case of domains, but for general rings, it is natural to ask if there is a generalization of this topology, and if it can be described using only the spectrum. We shall follow this point of view in Section 2.4.

We now resume the study of whether $\mathrm{SStar}_{f,sp}(D)$ is a spectral space. Since $\mathrm{SStar}_f(D)$ is sup-normal, so is $\mathrm{SStar}_{f,sp}(D)$. Hence, to show that the latter is a spectral space, we only need to show that it is closed by taking suprema and that $V_I \cap \mathrm{SStar}_{f,sp}(D)$ has an infimum for every $I \in \mathcal{I}_f(D)$.

Lemma 2.43. *Let D be an integral domain, and let $\mathscr{D}$ be a nonempty set of spectral semistar operations on D. For each spectral semistar operation $*$, set $\Delta(*) := \mathrm{QSpec}^*(D)$.*

(a) $\inf \mathscr{D}$ *is spectral, and* $\Delta(\inf \mathscr{D}) = \bigcup\{\Delta(*) \mid * \in \mathscr{D}\}$.
(b) If $\sup \mathscr{D}$ *is semifinite, then it is spectral, and* $\Delta(\sup \mathscr{D}) = \bigcap\{\Delta(*) \mid * \in \mathscr{D}\}$.

Proof. (a) Set $\mathbf{\Delta} := \bigcup\{\Delta(*) \mid * \in \mathscr{D}\}$. For each $E \in \mathbf{F}(D)$,

$$E^{\inf \mathscr{D}} = \bigcap\{ED_P \mid P \in \Delta(*), * \in \mathscr{D}\} = \bigcap\{ED_P \mid P \in \mathbf{\Delta}\}.$$

In particular, $\inf \mathscr{D}$ is spectral, and $\mathbf{\Delta} \subseteq \mathrm{QSpec}^{\inf \mathscr{D}}(D)$. On the other hand, if $Q \in \mathrm{QSpec}^{\inf \mathscr{D}}(D)$, then $Q^* \neq D^*$ for some $* \in \mathscr{D}$, and hence $Q \in \mathrm{QSpec}^*(D)$. Therefore, $\mathbf{\Delta} = \mathrm{QSpec}^{\inf \mathscr{D}}(D)$.

(b) Let $P \in \mathrm{QSpec}^{\sup \mathscr{D}}(D)$. Then, P belongs to $\mathrm{QSpec}^*(D)$ for each $* \in \mathscr{D}$, i.e., $\mathrm{QSpec}^{\sup \mathscr{D}}(D) \subseteq \bigcap\{\Delta(*) \mid * \in \mathscr{D}\}$. Since each $* \in \mathscr{D}$ is spectral, then, for each $E \in \mathbf{F}(D)$, $E^* \subseteq ED_P$ for all $P \in \Delta(*)$, and in particular for all $P \in \bigcap\{\Delta(*) \mid * \in \mathscr{D}\}$. Hence,

$$E^{\sup \mathscr{D}} \subseteq \bigcap\left\{ED_P \mid P \in \bigcap\{\Delta(*) \mid * \in \mathscr{D}\}\right\} \subseteq \bigcap\{ED_P \mid P \in \mathrm{QSpec}^{\sup \mathscr{D}}(D)\}.$$

However, if $\sup \mathscr{D}$ is semifinite, then the right hand side is contained in $E^{\sup \mathscr{D}}$ [42, Proposition 4.8]; therefore, they are equal, and hence $\sup \mathscr{D}$ is spectral, with $\Delta(\sup \mathscr{D}) = \bigcap\{\Delta(*) \mid * \in \mathscr{D}\}$. □

Example 2.44. Lemma 2.43(b) does not hold if we drop the assumption that $\sup \mathscr{D}$ is semifinite. For example, let $\mathbb{A}$ be the ring of algebraic integers, i.e., the integral closure of $\mathbb{Z}$ in the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. Recall that $\mathbb{A}$ is a one-dimensional Bézout domain [77, p. 72]. If P is a maximal ideal of $\mathbb{A}$, we will denote by $\Delta(P)$ the set $\mathrm{Max}(\mathbb{A}) \setminus \{P\}$. Our proof will go through three steps:

Step 1: $\Delta(P)$ is not compact;

Step 2: $\bigcap\{\mathbb{A}_Q \mid Q \in \Delta(P)\} = \mathbb{A}$;

Step 3: $* := \sup\{s_{\Delta(P)} \mid P \in \mathrm{Max}(\mathbb{A})\}$ is not spectral.

Proof of Step 1. Suppose $\Delta(P)$ is compact. Then, since $\Delta(P)$ is open in $\mathrm{Max}(\mathbb{A})$, we should have $\Delta(P) = \mathcal{D}(J) \cap \mathrm{Max}(\mathbb{A})$ for some finitely generated J, or equivalently $\mathcal{V}(J) = \{P\}$, i.e., J is a P-primary ideal. Since $\mathbb{A}$ is a Bézout domain, J would be principal, i.e., $J = \alpha\mathbb{A}$ for some $\alpha \in \mathbb{A}$. Let K be the Galois closure of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$, and consider the prime ideal $P_K := P \cap \mathcal{O}_K$, where $\mathcal{O}_L$ the ring of integers of the field L. Let $P \cap \mathbb{Z} = p\mathbb{Z}$ for some prime integer p, and let F be a Galois extension of $\mathbb{Q}$ where p splits and such that $F \cap K = \mathbb{Q}$ (since p splits in infinitely many quadratic extension of $\mathbb{Q}$ and $[K : \mathbb{Q}] < \infty$, F has to exist). We claim that P_K splits in the compositum FK: if this is true, then α would be contained in more than a single prime ideal of $\mathbb{A}$, against the hypothesis.

Set $P_{FK} := P \cap \mathcal{O}_{FK}$ and $P_F := P \cap \mathcal{O}_F = P_{FK} \cap \mathcal{O}_F$. Suppose P_K does not split in $\mathcal{O}_{FK}$: then $P_K\mathcal{O}_{FK}$ would be primary to P_{FK}. On the other hand, $P_F \cap \mathbb{Z} = p\mathbb{Z}$; since p splits in $\mathcal{O}_F$, and the Galois group of F over $\mathbb{Q}$ acts transitively over the primes of $\mathcal{O}_F$ lying over p, there is an automorphism σ of F such that $\sigma(P_F) \neq P_F$. Since $K \cap F = \mathbb{Q}$, there is an automorphism τ of FK such that $\tau|_F = \sigma$ and $\tau|_K$ is the identity. Therefore, $\tau(P_K) = P_K$ and $\tau(P_{FK})$ must contain P_K, i.e., $\tau(P_{FK}) = P_{FK}$. However, P_{FK} contains P_F, and $\tau(P_{FK})$ contains $\sigma(P_F)$; therefore, P_{FK} must contain both P_F and $\sigma(P_F)$, which is impossible. Therefore, P_K splits in $\mathcal{O}_{FK}$, and $\alpha\mathbb{A}$ cannot be P-primary.

Proof of Step 2. Since $\mathbb{A}$ is a Bézout domain, it is a Prüfer domain; it follows that the spectrum of $B := \bigcap\{\mathbb{A}_Q \mid Q \in \Delta(P)\}$ is homeomorphic to the set of prime ideals Q of $\mathbb{A}$ such that $QB \neq B$. However, each prime in $\Delta(P)$ survives in B; therefore, $\mathrm{Spec}(B)$ is homeomorphic either to $\mathrm{Spec}(\mathbb{A})$ or to $\Delta(P) \cup \{(0)\}$. By Step 1, the latter set is not compact, and so $\mathrm{Spec}(B) \simeq \mathrm{Spec}(\mathbb{A})$; it follows that $PB \neq B$ and so $B = \mathbb{A}$.

Proof of Step 3. By Step 2, $\mathbb{A}^{s_{\Delta(P)}} = \mathbb{A}$; hence, $\mathbb{A}$ is closed by $*$. However, $P^{s_{\Delta(P)}} = \mathbb{A}$ for every P, and thus $P^* = \mathbb{A}$ for every P. If $*$ were spectral, it would follow that $* = *_{(0)} = \wedge_{\{K\}}$ (where K is the quotient field of $\mathbb{A}$); but then $\mathbb{A}^* = K$, a contradiction. Hence, $*$ is not spectral.

Theorem 2.45. *Let D be an integral domain. Then, the set $\mathrm{SStar}_{f,sp}(D)$, endowed with the Zariski topology, is a spectral space.*

Proof. By Lemma 2.20, $\mathrm{SStar}_{f,sp}(D)$ is sup-normal, and by Lemma 2.43 it is closed by taking suprema. In the notation of Theorem 2.10, we take $\mathcal{A} = \mathcal{I}_f(D)$; then, the $(\mathcal{A}, \mathscr{P})$-Zariski topology on $\mathrm{SStar}_{f,sp}(D)$ coincides with the Zariski topology (where $\mathscr{P}$ is the set of principal fractional ideals). We only need to show that $V_I \cap \mathrm{SStar}_{f,sp}(D)$ has an infimum in $\mathrm{SStar}_{f,sp}(D)$ whenever $I \in \mathcal{I}_f(D)$; however, as in the proof of Proposition 2.19, if $* := \inf(V_I \cap \mathrm{SStar}_{f,sp}(D))$ (where the infimum is taken in $\mathrm{SStar}(D)$) then $\widetilde{*}$ is also in V_I, since, for every finitely generated integral ideal I, $1 \in I^*$ if and only if $1 \in I^{\widetilde{*}}$. Hence, $\widetilde{*} \leq *$ and $\widetilde{*} \in V_I \cap \mathrm{SStar}_{f,sp}(D)$, and the two facts together imply $* = \widetilde{*}$. □

2.2.3.1. Localizing systems

Definition 2.46. *Let D be an integral domain. A* localizing system *on D is a subset $\mathcal{F} \subseteq \mathcal{I}(D)$ such that:*

- *if $I \in \mathcal{F}$ and J is an ideal of D such that $I \subseteq J$, then $J \in \mathcal{F}$;*
- *if $I \in \mathcal{F}$ and J is an ideal of D such that, for each $i \in I$, $(J :_D iD) \in \mathcal{F}$, then $J \in \mathcal{F}$.*

A localizing system $\mathcal{F}$ is of finite type *if, for each $I \in \mathcal{F}$, there exists a nonzero finitely generated ideal $J \in \mathcal{F}$ such that $J \subseteq I$.*

An example of localizing system on D is, given an overring T, the family $\mathcal{F}(T) := \{I \in \mathcal{I}(D) \mid IT = T\}$; every $\mathcal{F}(T)$ is of finite type. An example of localizing system which is not of finite type is, given a valuation domain V and a nonzero idempotent prime ideal P, the family $\widehat{\mathcal{F}}(P) := \{I \in \mathcal{I}(V) \mid I \supsetneq P\}$. To every localizing system $\mathcal{F}$ we can associate a localizing system of finite type $\mathcal{F}_f$ defined by

$$\mathcal{F}_f := \{I \in \mathcal{F} \mid I \supseteq J \text{ for some } J \in \mathcal{F}\}.$$

We denote by $\mathrm{LS}(D)$ (respectively, $\mathrm{LS_f}(D)$) the set of all localizing systems (respectively, localizing systems of finite type) on D. We can endow these two sets with a topology (which we call the *Zariski topology*) by declaring open the sets of the form

$$W_I := \{\mathcal{F} \in \mathrm{LS}(D) \mid I \in \mathcal{F}\},$$

as I ranges among the integral ideals of D.

Every localizing system $\mathcal{F}$ induces a stable semistar operation $*_\mathcal{F}$ defined by

$$E^{*_\mathcal{F}} := \bigcup\{(E : H) \mid H \in \mathcal{F}\}.$$

Hence, there is a map

$$\begin{aligned} \mathrm{sl}\colon \mathrm{LS}(D) &\longrightarrow \mathrm{SStar}_{st}(D) \\ \mathcal{F} &\longmapsto *_\mathcal{F} \end{aligned}$$

which is a bijection such that $\mathrm{sl}(\mathrm{LS_f}(D)) = \mathrm{SStar}_{f,sp}(D)$ [42, Theorem 2.10, Corollary 2.11, and Proposition 3.2]. This correspondence carry over to the topological level.

Proposition 2.47. *Let D be an integral domain, and endow $\mathrm{SStar}_{st}(D)$ and $\mathrm{LS}(D)$ with the respective Zariski topologies.*

(a) sl *is a homeomorphism between* $\mathrm{LS}(D)$ *and* $\mathrm{SStar}_{st}(D)$.
(b) $\mathrm{sl}|_{\mathrm{LS_f}(D)}$ *is a homeomorphism between* $\mathrm{LS_f}(D)$ *and* $\mathrm{SStar}_{f,sp}(D)$.
(c) $\mathrm{LS_f}(D)$ *is a spectral space.*

Proof. By the previous remarks, we only need to show that sl is continuous and open. If $U_I := V_I \cap \mathrm{SStar}_{st}(D)$ is a subbasic open set of $\mathrm{SStar}_{st}(D)$ (with I an integral ideal of D) then

$$\begin{aligned} \mathrm{sl}^{-1}(U_I) &= \{\mathcal{F} \in \mathrm{LS}(D) \mid 1 \in I^{*_\mathcal{F}}\} = \\ &= \{\mathcal{F} \in \mathrm{LS}(D) \mid 1 \in (I : J) \text{ for some } J \in \mathcal{F}\} = \\ &= \{\mathcal{F} \in \mathrm{LS}(D) \mid H \subseteq I \text{ for some } H \in \mathcal{F}\} = \textstyle\bigcup_{H \subseteq I} W_H, \end{aligned}$$

which is open in $\mathrm{LS}(D)$. Moreover, this union is equal to W_I: clearly $W_I \subseteq \bigcup_{H\subseteq I} W_H$, while if $\mathcal{F} \in W_H$ for some $H \subseteq I$, then $H \in \mathcal{F}$ and thus $I \in \mathcal{F}$ by the definition of localizing system; that is, $\mathcal{F} \in W_I$. Thus, $\mathrm{sl}(W_I) = U_I$, and, since $\{W_H \mid H \in \mathcal{I}(D)\}$ is a basis for $\mathrm{LS}(D)$, sl is open, and hence a homeomorphism.

Point (b) follows since $\mathrm{sl}(\mathrm{LS_f}(D)) = \mathrm{SStar}_{f,sp}(D)$, and point (c) follows by using Theorem 2.45. □

2.2.4. Functorial properties

Let $A \subseteq B$ be an extension of integral domains, and let K be the quotient field of A. For every semistar operation $*$ on B, we can define a map $\sigma(*) : \mathbf{F}(A) \longrightarrow \mathbf{F}(A)$ by

$$I^{\sigma(*)} := (IB)^* \cap K$$

for all $I \in \mathbf{F}(A)$. It is straightforward to see that $\sigma(*)$ is a semistar operation; hence, we have a map

$$\begin{aligned} \sigma\colon \mathrm{SStar}(B) &\longrightarrow \mathrm{SStar}(A) \\ * &\longmapsto \sigma(*). \end{aligned} \tag{2.2}$$

Proposition 2.48. *Preserve the notation above. Then:*

(a) σ is a continuous map (in the Zariski topology);
(b) $\sigma(\mathrm{SStar}_f(B)) \subseteq \mathrm{SStar}_f(A)$.

Proof. (a) Denote by $V_F^{(A)}$ and $V_G^{(B)}$, respectively, the subbasic open sets of $\mathrm{SStar}(A)$ and $\mathrm{SStar}(B)$. For every $F \in \mathbf{F}(A)$, we have

$$\begin{aligned} \sigma^{-1}(V_F^{(A)}) &= \{* \in \mathrm{SStar}(B) : \sigma(*) \in V_F\} = \{* \in \mathrm{SStar}(B) : 1 \in F^{\sigma(*)}\} = \\ &= \{* \in \mathrm{SStar}(B) : 1 \in (FB)^*\} = V_{FB}^{(B)}. \end{aligned}$$

Hence, σ is continuous.

(b) Let $I \in \mathbf{F}(A)$ and $x \in I^{\sigma(*)}$; then $x \in (IB)^*$, and thus there are $y_1, \dots, y_n \in IB$ such that $x \in (y_1B + \cdots + y_nB)^*$. For every y_i, there is a finitely generated A-module $F_i \subseteq I$ such that $y_i \in F_iB$; let $F := F_1 + \cdots + F_n$. Then $F \subseteq I$ is finitely generated (as an A-module), and $y_1B + \cdots + y_nB \subseteq FB$; therefore, $x \in (FB)^*$ and $x \in F^{\sigma(*)}$. Thus $\sigma(*)$ is of finite type. □

If we also assume B to be an overring of A, then σ has even better properties.

Proposition 2.49. *Let A be an integral domain, and let $B \in \mathrm{Over}(A)$; let $\sigma : \mathrm{SStar}(B) \longrightarrow \mathrm{SStar}(A)$ be the map defined above. Then:*

(a) for every $* \in \mathrm{SStar}(B)$, $\sigma(*)|_{\mathcal{F}(B)} = *$;
(b) σ is a topological embedding;
(c) if $\sigma(*)$ is of finite type, so is $*$.

Proof. The first point is straightforward, since if I is a B-module, then $I^{\sigma(*)} = (IB)^* = I^*$. In particular, σ is injective; to show that it is an embedding, we need to show that $\sigma(V_F^{(B)})$ is open in $\sigma(\mathrm{SStar}(B))$.

Since B is an overring of A, then F is an A-module and thus is defined the open set $V_F^{(A)}$. But, since $F^* = F^{\sigma(*)}$ for every $* \in \mathrm{SStar}(B)$, by the above paragraph, we have $\sigma(V_F^{(B)}) = V_F^{(A)} \cap \sigma(\mathrm{SStar}(B))$, which is an open set in $\sigma(\mathrm{SStar}(B))$. Hence, σ is an embedding.

If now $\sigma(*)$ is of finite type, suppose $x \in I^*$. Then $x \in I^{\sigma(*)}$, and thus there is a finitely generated A-module $F \subseteq I$ such that $x \in F^{\sigma(*)}$. Hence, $x \in (FB)^*$, and $*$ is of finite type. □

We observe that, if B is an overring of A, then there is an obvious canonical inclusion $\mathrm{Over}(B) \hookrightarrow \mathrm{Over}(A)$, which is a topological embedding when the two sets are endowed with an "appropriate", natural, topology. We shall show in Section 2.3 that the result on semistar operations is actually an extension of this fact about overrings.

Remark 2.50. Points (b) and (c) can fail if the quotient fields of A and B are different. (Point (a) is not even well-defined.)

For the failure of injectivity, the simplest example is obtained when B is not a field but contains the quotient field K of A, since for every $* \in \mathrm{SStar}(B)$ and every $I \in \mathbf{F}(A)$, $I^{\sigma(*)} = (IB)^* \cap K = K^* \cap K = K$. The injectivity is not preserved even if we suppose $B \cap K = A$: for example, let A be a rank one discrete valuation ring, L a field containing K and define B as the integral closure of A in L. If B is not local (e.g., if $A = \mathbb{Z}_{(p)}$ and L is an algebraic extension of $\mathbb{Q}$ where p splits), then σ is not injective, since $|\mathrm{SStar}(A)| = 2$ (see e.g. [103, Proposition 4.2]) while B admits at least 3 semistar operation of finite type: the identity, $\wedge_{\{L\}}$ and $\wedge_{\{B_P\}}$, where P is a maximal ideal of B.

In the same way, $\sigma(*)$ could be of finite type even if $*$ is not and $B \cap K = A$: for example, let Z be an indeterminate over $\mathbb{C}$, set $A := \mathbb{C}[Z]$ and let B be the ring of all entire functions. Then the map $*$ defined by, for any $F \in \mathbf{F}(B)$,

$$F \mapsto F^* := \bigcap_{\alpha \in \mathbb{C}} FB_{(Z-\alpha)},$$

is a (semi)star operation on B which is not of finite type. Indeed, since B is a Bézout domain [61], all finitely generated ideals are quasi-$*$-ideals but, if $\mathfrak{b} \subsetneq B$ is *free ideal* (i.e., the functions belonging to $\mathfrak{b}$ have no common zeros), then clearly $\mathfrak{b}^* = B$, while for any finite subset $\{f_1, \ldots, f_n\}$ of $\mathfrak{b}$, we have $((f_1, \ldots, f_n)B)^* = (f_1, \ldots, f_n)B \subseteq \mathfrak{b} \subsetneq B$. This shows that $*$ is not of finite type. Since $B \cap \mathbb{C}(Z) = A$, $\sigma(*)$ is a (semi)star operation on A, and it is not hard to see that it is the identity, and thus of finite type.

Proposition 2.51. *Let $A \subseteq B$ be a ring extension, and let σ be the map defined in* (2.2)*. Suppose also that B is flat over A. Then:*

(a) if $*$ is stable, so is $\sigma(*)$;
(b) if $*$ is spectral, so is $\sigma(*)$.

Proof. We first claim that, if B is flat over A, then $(I \cap J)B = IB \cap JB$ for every $I, J \in \mathbf{F}(A)$. Note that the setup is different from Lemma 2.30: we are not supposing that I and J are submodules of B, but rather submodules of the quotient field K of A.

If I and J are integral ideals of A, then the claim is exactly [86, Theorem 7.4(ii)]. If they are fractional ideals, then we can find a $z \in A$ such that $zI, zJ \subseteq A$; then, by the previous case,

$$\begin{aligned}(I \cap J)B &= z^{-1}(z(I \cap J)B) = z^{-1}(zI \cap zJ)B = \\ &= z^{-1}(zIB \cap zJB) = z^{-1}zIB \cap z^{-1}zJB = IB \cap JB,\end{aligned}$$

as claimed. Suppose I and J are merely submodules of K. Clearly, $(I \cap J)B \subseteq IB \cap JB$. Conversely, suppose $x \in IB \cap JB$. Then, $x = i_1b_1 + \cdots + i_nb_n = j_1c_1 + \cdots + j_mc_m$ for some $i_1, \ldots, i_n \in I$, $j_1, \ldots, j_m \in J$, $b_1, \ldots, b_n, c_1, \ldots, c_m \in B$; therefore, $x \in I'B \cap J'B$, where $I' := (i_1, \ldots, i_n)A \subseteq I$ and $J' := (j_1, \ldots, j_m)A \subseteq J$ are fractional ideals of A. Therefore,

$$x \in I'B \cap J'B = (I' \cap J')B \subseteq (I \cap J)B.$$

(a) Let $*$ be a stable semistar operation on B. Then, using the previous paragraph,

$$(I \cap J)^{\sigma(*)} = ((I \cap J)B)^* \cap K = (IB \cap JB)^* = (IB)^* \cap (JB)^* = I^{\sigma(*)} \cap J^{\sigma(*)},$$

and so $\sigma(*)$ is stable.

(b) Suppose now $*$ is spectral; by point (a) and Proposition 2.31, it is enough to show that $\sigma(*)$ is semifinite. Suppose thus that I is a proper quasi-$\sigma(*)$-ideal of A; by definition, $I = I^{\sigma(*)} \cap A = (IB)^* \cap A$. In particular, $1 \notin (IB)^*$; therefore, $J := (IB)^* \cap B$ is a proper quasi-$*$-ideal of B. Hence, there is a prime ideal P of B that is quasi-$*$-closed; if $Q := P \cap A$, then

$$Q^{\sigma(*)} \cap A = (QB)^* \cap A \subseteq (P^* \cap B) \cap A = P \cap A = Q,$$

and so Q is a quasi-$\sigma(*)$-prime. But $I = J \cap A \subseteq P \cap A = Q$; hence, $*$ is semifinite. $\square$

2.2.5. The relation between semistar and star operations

In this section we explore the relationship between the topological spaces $\mathrm{Star}(D)$ and $\mathrm{SStar}(D)$. We start by a topological version of Proposition 2.13:

Proposition 2.52. *Let D be an integral domain. Then, the maps*

$$\begin{aligned}\rho\colon \mathrm{(S)Star}(D) &\longrightarrow \mathrm{Star}(D) \\ * &\longmapsto *|_{\mathcal{F}(D)}\end{aligned} \qquad \textit{and} \qquad \begin{aligned}\iota\colon \mathrm{Star}(D) &\longrightarrow \mathrm{(S)Star}(D) \\ \sharp &\longmapsto \sharp_e\end{aligned}$$

are continuous and $\iota \circ \rho$ is the identity on $\mathrm{Star}(D)$*; in particular, ι is injective and ρ is surjective.*

Proof. To avoid confusion, denote by V_I the subbasic open sets of (S)Star(D) and by W_I the subbasic open sets of Star(D).

If I is a fractional ideal of D, then $I^* = I^{*|_{\mathcal{F}(D)}}$; therefore, $\rho^{-1}(W_I) = V_I$ and ρ is continuous. Similarly, $\iota^{-1}(V_I) = W_I$. Moreover, if I is a D-submodule of K that is not a fractional ideal, then $I^{\sharp_e} = K$, and thus $\iota^{-1}(V_I) = \mathrm{Star}(D)$. Thus, $\iota^{-1}(V_I)$ is always open, and so ι is continuous.

The other claims are obvious. □

When we restrict to finite-type operations, the relation becomes a homeomorphism; recall that a domain D is said to be *conductive* if every overring of D, different from its quotient field K, is a fractional ideal of D, or, equivalently, if $\mathbf{F}(D) = \mathcal{F}(D) \cup \{K\}$.

Proposition 2.53. *Let D be an integral domain, and let $\rho : \mathrm{(S)Star}(D) \longrightarrow \mathrm{Star}(D)$ and $\iota : \mathrm{Star}(D) \longrightarrow \mathrm{(S)Star}(D)$ be the maps defined in Proposition 2.52. Then:*

(a) The restriction $\rho_f = \rho|_{\mathrm{(S)Star}_f(D)}$ is a homeomorphism between $\mathrm{(S)Star}_f(D)$ and $\mathrm{Star}_f(D)$, whose inverse is the map $\iota_f = \Psi_f \circ \iota : \mathrm{Star}_f(D) \longrightarrow \mathrm{(S)Star}_f(D)$;
(b) $\iota(\mathrm{Star}_f(D)) \subseteq \mathrm{(S)Star}_f(D)$ if and only if D is conductive.

Proof. Preserve the notation of the proof of Proposition 2.52.

(a) Suppose $*$ is a (semi)star operation of finite type. By definition, for every fractional ideal I, $I^* = \bigcup\{F^* \mid F \in \mathcal{F}_f(D), F \subseteq I\}$. This equality continues to hold if $*$ is substituted by $\rho(*) = *|_{\mathcal{F}(D)}$, and thus $\rho(*)$ is still of finite type.

For every finitely-generated fractional ideal F, $\iota_f^{-1}(V_F) = W_F$; therefore, ι_f is continuous. Moreover, for every $\sharp \in \mathrm{Star}_f(D)$,

$$\rho_f \circ \iota_f(\sharp) = \rho_f((\sharp_e)_f) = \rho(\sharp_e) = \sharp,$$

while, if $* \in \mathrm{SStar}_f(D)$,

$$\iota_f \circ \rho_f(*) = \Psi_f \circ \iota \circ \rho(*) = \Psi_f(*) = *;$$

hence, ρ_f and ι_f are inverses, and thus are homeomorphisms.

(b) If D is conductive, then

$$I^{\iota(\sharp)} = \begin{cases} I^\sharp & \text{if } I \neq K \\ K & \text{if } I = K, \end{cases}$$

and so $\iota(\sharp)$ is of finite type if $\sharp$ is. On the other hand, if D is not conductive, then the trivial extension d_e of the identity is not of finite type, since $F^{d_e} = F$ if F is finitely generated (since every finitely generated D-submodule of K is a fractional ideal) while there is at least one D-submodule I of K, $I \notin \mathcal{F}(D)$, that is different from K itself; in particular, $I^{*_f} = I \neq K = I^*$. □

The same holds for stable and spectral operations; the following can be seen as the topological version of Propositions 2.25 and 2.36.

Proposition 2.54. *Let D be an integral domain, and let $\rho : \mathrm{(S)Star}(D) \longrightarrow \mathrm{Star}(D)$ and $\iota : \mathrm{Star}(D) \longrightarrow \mathrm{(S)Star}(D)$ be the maps defined in Proposition 2.52. Then:*

1. *$\rho|_{\mathrm{(S)Star}_{st}(D)}$ is a homeomorphism between $\mathrm{(S)Star}_{st}(D)$ and $\mathrm{Star}_{st}(D)$, whose inverse is $\iota_{st} := \Psi_{st} \circ \iota$;*
2. *$\rho|_{\mathrm{(S)Star}_{sp}(D)}$ is a homeomorphism between $\mathrm{(S)Star}_{sp}(D)$ and $\mathrm{Star}_{sp}(D)$, whose inverse is $\iota_{sp} := \Psi_{st}|_{\mathrm{(S)Star}_{sp}(D)} \circ \iota$;*
3. *$\rho|_{\mathrm{(S)Star}_{f,sp}(D)}$ is a homeomorphism between $\mathrm{(S)Star}_{f,sp}(D)$ and $\mathrm{Star}_{f,sp}(D)$, whose inverse is $\iota_w := \Psi_w \circ \iota$.*

Proof. The first point follows with the same proof of Proposition 2.53, using the bijection proved in Proposition 2.25. The other two points follow by restricting the homeomorphism $\rho|_{\mathrm{(S)Star}_{st}(D)}$, and noting that $*$ is semifinite if and only if so is $\rho(*)$. $\square$

Corollary 2.55. *Let D be an integral domain. Then, $\mathrm{Star}_f(D)$ and $\mathrm{Star}_{f,sp}(D)$, endowed with the Zariski topology, are spectral spaces.*

As a side note, observe that the previous corollary can also be proved without using semistar and (semi)star operation, but by applying Theorem 2.10 with the same method used in the proofs of Theorems 2.21 and 2.45. The only difficulty lies in the fact that $\mathcal{F}(D)$ is not a complete lattice; however, we can augment the definition of star operation by requiring that $*$ is a map from $\mathcal{F}(D) \cup \{K\}$ (that *is* a complete lattice) to itself. With this modification, the proof runs as smoothly as in the semistar case.

2.2.5.1. Standard and semiprime operations

If $*$ is a semistar but not a (semi)star operation, that is, if $D \neq D^*$, then we can't define a closure operation on the ideals of D just by restricting $*$. However, if I is an integral ideal of D, then we can define

$$I^{\kappa(*)} := I^* \cap D.$$

It is not hard to see that $\kappa(*)$ is a closure operation on the set $\mathcal{I}(D)$ of ideals of D, and that the $\kappa(*)$-closed ideals are exactly the quasi-$*$-ideals of D. Hence, we have a map

$$\begin{aligned} \kappa\colon \mathrm{SStar}(D) &\longrightarrow \mathrm{Clos}(\mathcal{I}(D)) \\ * &\longmapsto \kappa(*). \end{aligned}$$

Note that, if $D = D^*$, then $I^* \subseteq D$, so that $I^{\kappa(*)} = I^*$; it follows that, when restricted to (semi)star operations, κ acts exactly like the map ρ defined in Proposition 2.52.

To study the range of κ, we introduce two new definition.

Definition 2.56. *Let R be a ring (not necessarily a domain). A closure operation c on $\mathcal{I}(R)$ is:*

- *[33, Definition 2.2]* standard *if, for all $I \in \mathcal{I}(R)$ and every non-zerodivisor w, $((wI)^c :_R w) = I^c$;*
- semiprime *if, for all $I \in \mathcal{I}(R)$ and every non-zerodivisor w, $w \cdot I^c \subseteq (wI)^c$.*

We denote by $\mathrm{Std}(R)$ *and by* $\mathrm{Sp}(R)$*, respectively, the set of standard and of semiprime closure operations on* R*.*

In both the standard and the semiprime case, we can define finite-type operations like in the star and semistar case.

The topology we work with on $\mathrm{Std}(R)$ and on $\mathrm{Sp}(R)$ (and that we call simply the *Zariski topology*) is the $(\mathcal{I}(\mathcal{R}),\mathscr{P})$-Zariski topology (where $\mathscr{P}$ is the set of principal ideals), that is, the topology generated by the subbasic open sets

$$V_{I,x} := \{c \in \Delta \mid x \in I^c\},$$

where x ranges in R and I in $\mathcal{I}(R)$. (Here Δ denotes alternatively $\mathrm{Std}(R)$ or $\mathrm{Sp}(R)$; since we will not have to use both sets concurrently, this should not cause any confusion.) Since every standard operation is semiprime, and every star operation is standard, if D is a domain we have inclusions

$$\mathrm{Star}(D) \subseteq \mathrm{Std}(D) \subseteq \mathrm{Sp}(D)$$

that, in the Zariski topology, are topological embeddings; if R is not a domain, the inclusion $\mathrm{Std}(R) \subseteq \mathrm{Sp}(R)$ is still an embedding.

Endowing the whole $\mathrm{Clos}(\mathcal{I}(D))$ with the $(\mathcal{I}(D),\mathscr{P})$-Zariski topology, it is easy to see that κ is continuous: indeed, if $x \in D \setminus \{0\}$ and $I \in \mathcal{I}(D)$, then

$$\begin{aligned}\kappa_f^{-1}(V_{I,x}) &= \{* \in \mathrm{SStar}(D) \mid x \in I^{\kappa_f}\} = \\ &= \{* \in \mathrm{SStar}(D) \mid x \in I^* \cap R\} = V_{x^{-1}I}\end{aligned}$$

which is an open set of $\mathrm{SStar}(D)$, while if $x = 0$ then $V_{I,x} = \mathrm{Clos}(\mathcal{I}(D))$, and so $\kappa_f^{-1}(V_{I,x}) = \mathrm{SStar}(D)$.

In the case of finite-type operations, standard closures allow to find the range of κ:

Proposition 2.57. *Let* D *be an integral domain. Then, the restriction* $\kappa_f := \kappa|_{\mathrm{SStar}_f(D)}$ *is a homeomorphism between* $\mathrm{SStar}_f(D)$ *and the set* $\mathrm{Std}_f(D)$ *of standard closure operations of finite type.*

Proof. By [33, Theorem 3.1], κ_f is a bijection; by the previous remark, it is continuous. By [33], the inverse of κ_f is the map σ_f such that, if $c \in \mathrm{Std}(D)$ and F is finitely generated, then

$$\sigma_f(c) : F \mapsto \sum_{\substack{x\in K\setminus\{0\}\\ xF\subseteq D}} x^{-1}(xF)^c,$$

while if I is not finitely generated then $I^{\sigma_f(c)} := \bigcup\{F^{\sigma_f(c)} \mid F \in \mathcal{I}_f(D), F \subseteq I\}$.

Fix now a finitely generated fractional ideal E of D. Then,

$$\begin{aligned}\kappa_f(V_E) &= \{c \in \mathrm{Std}(D) \mid c = \kappa_f(*) \text{ for some } * \in V_E\} = \{c \in \mathrm{Std}(D) \mid \sigma_f(c) \in V_E\} = \\ &= \{c \in \mathrm{Std}(D) \mid 1 \in E^c\} = \left\{c \in \mathrm{Std}(D) \mid 1 \in \sum_{\substack{x\in K\setminus\{0\}\\ xE\subseteq D}} x^{-1}(xE)^c\right\} = \\ &= \Big\{c \in \mathrm{Std}(D) \mid 1 \in x^{-1}(xE)^c \text{ for some } x \text{ s.t. } xE \subseteq D\Big\} = \bigcup_{\substack{x\in K\setminus\{0\}\\ xE\subseteq D}} V_{xE,x}\end{aligned}$$

which is open. Since $\{V_E \mid E \in \mathcal{I}_f(D)\}$ is a subbasis for $\mathrm{SStar}_f(D)$, it follows that κ_f is open, and thus κ_f is a homeomorphism. □

Corollary 2.58. *Let D be a domain. Then,* $\mathrm{Std}_f(D)$*, endowed with the Zariski topology, is a spectral space.*

What happens when D is not a domain? [33, Theorem 3.1] works on arbitrary rings R, provided that we define semistar operations as closure operations c on the set of R-submodule of the total quotient ring Q such that $w \cdot I^c = (wI)^c$ for all I and all *non-zerodivisors* w (this is done, for example, in [33]). It is possible to generalize to this case the results of Section 2.2.1, showing that $\mathrm{SStar}_f(D)$ is a spectral space; [33, Theorem 3.1] shows that κ_f is still a bijection, and the obvious modifications allows to prove that Proposition 2.57 continues to hold. Hence, $\mathrm{Std}_f(R)$ is a spectral space for any ring R.

However, it is not possible, in general, to prove that $\mathrm{Std}_f(D)$ is a spectral space by using Theorem 2.10, since the supremum of a family of standard operations may not be standard, as the next example shows.

Example 2.59. Let $(D, \mathfrak{m})$ be a one-dimensional local domain, and suppose that there are two different one-dimensional valuation domains V, W over $\mathfrak{m}$. (This is possible, for example, if D is Noetherian with non-local integral closure.) Consider the semistar operations $*_1 : I \mapsto IV$ and $*_2 : I \mapsto IW$, and let $* := *_1 \vee *_2$. Then,

$$\mathfrak{m}^* \supseteq (\mathfrak{m}^{*_1})^{*_2} = \mathfrak{m}VW = K;$$

thus, $*$ must be the trivial extension $\wedge_{\{K\}}$, and $\kappa_f(*)$ sends every nonzero ideal to R.

However, $\mathfrak{m}$ is fixed by both $\kappa_f(*_1)$ and $\kappa_f(*_2)$, since

$$\mathfrak{m}^{\kappa_f(*_i)} = \mathfrak{m}V \cap D = \mathfrak{m}.$$

(Note that $*_1$ and $*_2$ are both of finite type.) Hence, $\sup\{\kappa_f(*_1), \kappa_f(*_2)\}$ fixes $\mathfrak{m}$, and so $\sup\{\kappa_f(*_1), \kappa_f(*_2)\} \neq \kappa_f(\sup\{*_1, *_2\})$.

However, note that $\mathrm{Std}(D)$ is a complete lattice, since the homeomorphism with $\mathrm{SStar}(D)$ is, in particular, an isomorphism of partially ordered set. The difference is that the supremum of a family of standard operations on D in $\mathrm{Std}(D)$ need not to coincide with its supremum in $\mathrm{Clos}(D)$, which would be needed to apply Theorem 2.10.

When dealing with semiprime operations, this difficulty vanishes:

Proposition 2.60. *Let R be a ring. Then, the set* $\mathrm{Sp}_f(R)$ *of semiprime operations of finite type, endowed with the Zariski topology, is a spectral space.*

Proof. The proof can be obtained following, *mutatis mutandis*, the proof of Theorem 2.21. □

2.3. Overrings

We have seen in Example 2.12 (specifically, points (2) and (4)) that overrings are a rich source of examples of semistar operations; any subset of Over(D) can be used to build a semistar operation. In particular, there is a natural map

$$\begin{aligned} \iota\colon \mathrm{Over}(D) &\longrightarrow \mathrm{SStar}(D) \\ T &\longmapsto \wedge_{\{T\}} \end{aligned} \tag{2.3}$$

(not to be confused with the map ι defined in Proposition 2.52; we will have no occasion to use both simultaneously). Note that each $\wedge_{\{T\}}$ is of finite type, so ι can also be seen as a map $\mathrm{Over}(D) \longrightarrow \mathrm{SStar}_f(D)$.

Remark 2.61. The existence of this embedding shows a difference between star and semistar operations. Indeed, we have seen that both star and semistar operations can be defined by sets of overrings: however, while a semistar operation $\wedge_\Delta$ can be thought of the infimum of the family $\{\iota(T) = \wedge_{\{T\}} \mid T \in \Delta\}$, obtaining a sort of "factorization" of the semistar operation, the same cannot be done in the star case, since $\wedge_{\{T\}}|_{\mathcal{F}(D)}$ is not a star operation unless $T = D$ (for the simple reason that $D^{\wedge_{\{T\}}} = T \neq D$).

If we want to study ι topologically, we need to put a topology on Over(D); the following is the most common and natural choice, generalizing the topology on spaces of valuation domains defined in [119, Chapter 6, §17].

Definition 2.62. *Let D be an integral domain with quotient field K. The* Zariski topology *on* Over(D) *is the topology whose subbasic open sets are those in the form*

$$B_F := \mathrm{Over}(D[F]) = \{T \in \mathrm{Over}(D) \mid F \subseteq T\}$$

as F ranges among the finite subsets of K (or, equivalently, among the finitely generated D-submodules of K).

Note that to generate the Zariski topology it is enough to declare open the sets in the form B_x, as x ranges in K, since $B_{f_1,\ldots,f_n} = B_{f_1} \cap \cdots \cap B_{f_n}$.

Proposition 2.63. *Let D be an integral domain and endow* Over(D) *and* SStar(D) *with the respective Zariski topologies. Then, the map ι defined in* (2.3) *is a topological embedding.*

Proof. The injectivity of ι follows from the fact that $D^{\wedge_{\{T\}}} = T$ for all $T \in \mathrm{Over}(D)$. To prove that ι is continuous, consider a D-submodule I of K. Then,

$$\iota^{-1}(V_I) = \{T \in \mathrm{Over}(D) \mid \wedge_{\{T\}} \in V_I\} = \{T \in \mathrm{Over}(D) \mid 1 \in IT\}.$$

Fix a ring $A \in \iota^{-1}(V_I)$. Then there are $a_1, \ldots, a_n \in A$, $i_1, \ldots, i_n \in I$, such that $1 = a_1 i_1 + \cdots + a_n i_n$. Hence, $1 \in FC$ for each $C \in B_{\{a_1,\ldots,a_n\}}$, and thus $C \in \iota^{-1}(V_I)$. Therefore, $B_{\{a_1,\ldots,a_n\}}$ is an open neighborhood of A contained in $\iota^{-1}(V_I)$, which is thus open.

We now prove that ι is a homeomorphism on its image; it is enough to prove that $\iota(B_F)$ is open in $\iota(\mathrm{Over}(D))$ for every finite set F. Let thus $F := \{f_1, \dots, f_n\}$; without loss of generality we can suppose that $f_i \neq 0$ for all i. We claim that $\iota(B_F) = U \cap \iota(\mathrm{Over}(D))$, where $U := V_{f_i^{-1}D} \cap \cdots \cap V_{f_n^{-1}D}$.

Indeed, if $\wedge_{\{T\}} \in U$, then $1 \in (f_i^{-1}D)^{\wedge_{\{T\}}} = f_i^{-1}T$ for every i, that is, $f_i \in T$ and $F \subseteq T$, or equivalently $T \in B_F$ and $\wedge_{\{T\}} \in \iota(B_F)$. Conversely, if $F \subseteq T$, then $f_i \in T$ for all T, and thus $1 \in f_i^{-1}f_iD \subseteq f_i^{-1}T = (f_i^{-1}D)^{\wedge_{\{T\}}}$, that is, $\wedge_{\{T\}} \in V_{f_i^{-1}D}$, and so $\wedge_{\{T\}} \in U$. Hence, $\iota(B_F) = U \cap \iota(\mathrm{Over}(D))$ is open in $\iota(\mathrm{Over}(D))$, and ι is a topological embedding. $\square$

Conversely, we have also a canonical map from $\mathrm{SStar}(D)$ to $\mathrm{Over}(D)$:

Proposition 2.64. *Let D be an integral domain, and let π be the map*

$$\begin{aligned} \pi\colon \mathrm{SStar}(D) &\longrightarrow \mathrm{Over}(D) \\ * &\longmapsto D^*. \end{aligned}$$

Then, π is a topological retraction, and $\pi \circ \iota$ is the identity.

Proof. The fact that $\pi \circ \iota$ is the identity is clear (and was observed in the proof of Proposition 2.63); in particular, π is surjective. To show that it is continuous, it is enough to note that $\pi^{-1}(B_x) = V_{x^{-1}D}$. Coupling the two results we see that π is a retraction. $\square$

Note that, since $\pi \circ \iota$ is the identity, the map $\pi|_{\mathrm{SStar}_f(D)}$ is also surjective.

We next investigate what happens to ι and π when we restrict to spectral semistar operations.

Proposition 2.65. *Let D be an integral domain and let $\iota : \mathrm{Over}(D) \longrightarrow \mathrm{SStar}_f(D)$ be the canonical topological embedding. If $T \in \mathrm{Over}(D)$, then $\iota(T) \in \mathrm{SStar}_{sp}(D)$ if and only if T is flat over D.*

Proof. If T is flat, $(I \cap J)T = IT \cap JT$ holds by Lemma 2.30; if it is not, then there are ideals where the equality fails [2, Theorem 2]. $\square$

However, in this case the behaviour of π does not match the behaviour of ι. Indeed, by definition, an overring T of D is in $\pi(\mathrm{SStar}_{sp}(D))$ if and only if there is a subset $\Delta \subseteq \mathrm{Spec}(D)$ such that $T = \bigcap\{D_P \mid P \in \Delta\}$; such overrings are said to be *sublocalizations* of D. While every D-flat overring is a sublocalization of D [105, p. 795], the converse is not true: for example, let L be a field, define $D := L[[X^2, X^3, Y, XY]]$ and let $T := D[X] = L[[X, Y]]$. Then, T is a sublocalization of D (it is the intersection of localization of D at its height-1 primes) but it is not D-flat, since T is the integral closure of D and $T \neq D$. Therefore, in the case of spectral operations, the symmetry that ι and π exhibited in the case of general semistar operations breaks down. We shall come back to sublocalization and flat overrings of D (and in particular, to the problem of if and when the set of all such overrings is a spectral space) in Section 2.5.2.

Remark 2.66. The Zariski topology can be defined in much greater generality. Indeed, if $A \subseteq B$ is any extension of rings and $\mathrm{Over}(A|B)$ denotes the set of rings between A and B, we can define a topology on $\mathcal{F}(A|B)$ by declaring open the sets of the form

$$B_F := \mathrm{Over}(A[F]|B) = \{T \in \mathrm{Over}(A|B) \mid F \subseteq T\}$$

as F ranges among the finite subsets of B. Even more generally, if B is any A-module, we can define a similar topology on the set $\mathbf{F}_A(B)$ of A-submodules of B, declaring open the sets of the form

$$B_F := \{T \in \mathbf{F}_A(B) \mid F \subseteq T\}$$

as F ranges among the finite subsets of B. For analogy, and in no danger of confusion, both are called the *Zariski topology*.

2.3.1. The rôle of compactness

An equivalent way to study spectral semistar operations is to see them as a peculiar kind of operation generated by a family of localizations of the base ring D; indeed, if $\Delta \subseteq \mathrm{Spec}(R)$, then $s_\Delta = \wedge_{\{\lambda(\Delta)\}}$, where $\lambda(\Delta) = \{D_P \mid P \in \Delta\}$. In view of Proposition 2.32, it is natural to ask what is the relationship between the Zariski topology on $\mathrm{Spec}(D)$ and the Zariski topology on $\mathrm{Over}(D)$. The answer is the following result.

Proposition 2.67 [28, Lemma 2.4]. *Let D be an integral domain, and let λ be the map*

$$\begin{aligned} \lambda\colon \mathrm{Spec}(D) &\longrightarrow \mathrm{Over}(D)\\ P &\longmapsto D_P. \end{aligned}$$

Then, λ is a topological embedding.

Proof. Clearly, λ is injective. Let $x \in K$; then,

$$\begin{aligned} \lambda^{-1}(B_x) &= \{P \in \mathrm{Spec}(D) \mid x \in D_P\}\\ &= \{P \in \mathrm{Spec}(D) \mid 1 \in (D_P : x) \cap D\} =\\ &= \{P \in \mathrm{Spec}(D) \mid 1 \in (D :_D\colon x)D_P\} =\\ &= \{P \in \mathrm{Spec}(D) \mid (D :_D\colon x) \subsetneq P\} = \mathcal{D}((D :_D x)) \end{aligned}$$

and thus λ is continuous. Conversely, if $\mathcal{D}(aD)$ is a subbasic open set of $\mathrm{Spec}(D)$, then

$$\begin{aligned} \lambda(\mathcal{D}(aD)) &= \lambda(\{P \in \mathrm{Spec}(D) \mid a \notin P\}) =\\ &= \{D_P \mid a^{-1} \in D_P\} = B_{a^{-1}} \cap \lambda(\mathrm{Spec}(D)) \end{aligned}$$

which is open in $\lambda(\mathrm{Spec}(D))$. Therefore, λ is a topological embedding. □

Note that the open set $\mathcal{D}((D :_D x))$ need not to be compact; that is, the map λ is not necessarily a spectral map. This observation will be exploited again in Section 2.5.2.

Hence, Proposition 2.32 can be rewritten as follows:

Proposition 2.68. *Let D be an integral domain, and let Λ be a set of localizations of D. (We assume that each $T \in \Lambda$ is local.) Then, $\wedge_\Lambda$ is of finite type if and only if Λ is compact.*

Is this a general property of families of overrings or is this something special about localizations? As it stands, it is the latter, as the next example shows (disproving a conjecture made in [40]).

Example 2.69. Let L be a field, X be an indeterminate over L, and define $D := L[[X^4, X^5, X^6, X^7]] = L + X^4L[[X]]$; let $K := L((X))$ be the quotient field of D. Then, D is Noetherian and conductive, and thus each D-submodule of K, except K itself, is finitely generated over D; it follows that every semistar operation on D is of finite type.

For every $\alpha \in L$, let $T_\alpha := D[X^2 + \alpha X^3] = D + (X^2 + \alpha X^3)L$; if $A \subseteq L$, define $\mathcal{A} := \{T_\alpha \mid \alpha \in A\}$. Then, for each α, we have $B_{X^2+\alpha X^3} \cap \mathcal{A} = \{T_\alpha\}$; it follows that $\mathcal{A}$ is noncompact whenever A is infinite, since $\{B_{X^2+\alpha X^3} \mid \alpha \in A\}$ is an infinite cover without proper subcovers (and so without finite subcovers). However, by the previous paragraph, $\wedge_{\mathcal{A}}$ is always of finite type; hence, there are noncompact families of overrings that generate finite-type operations.

However, things are not so bleak. The other implication, in fact, holds; we can prove it in greater generality.

Proposition 2.70. *Let D be an integral domain and let Δ be a compact subset of* $\mathrm{SStar}_f(D)$. *Then,* $\inf \Delta$ *is a semistar operation of finite type.*

Proof. Set $\Delta := \{*_\alpha \mid \alpha \in A\}$, $* := \inf \Delta$; fix a D-submodule F of K and let $x \in F^*$. Since $F^* = \bigcap_{\alpha \in A} F^{*_\alpha}$, and each $*_\alpha$ is of finite type, there are finitely generated ideals $G_\alpha \subseteq F$ such that $x \in G_\alpha^{*_\alpha}$; thus, for any α,

$$1 \in x^{-1}G_\alpha^{*_\alpha} = (x^{-1}G_\alpha)^{*_\alpha},$$

that is, $*_\alpha \in U_{x^{-1}G_\alpha} := \Omega_\alpha$. Therefore, $\{\Omega_\alpha \mid \alpha \in A\}$ is an open cover of Δ; by compactness, it admits a finite subcover $\{\Omega_{\alpha_1}, \ldots, \Omega_{\alpha_n}\}$. Set $G := G_{\alpha_1} + \cdots + G_{\alpha_n} \subseteq F$; we claim that $x \in G^*$.

Indeed, take a $*_\alpha \in \Delta$. Then, $*_\alpha \in \Omega_\beta$ for some $\beta \in \{\alpha_1, \ldots, \alpha_n\}$, that is, $*_\alpha \in U_{x^{-1}G_\beta}$. This means that $1 \in x^{-1}G_\beta^{*_\alpha}$, or equivalently $x \in G_\beta^{*_\alpha} \subseteq G^{*_\alpha}$. Hence, $x \in \bigcap\{G^{*_\alpha} \mid \alpha \in A\} = G^*$, and $*$ is of finite type. □

Corollary 2.71. *Let D be an integral domain and let $\Delta \subseteq \mathrm{Over}(D)$. If Δ is compact, then the semistar operation $\wedge_\Delta$ is of finite type.*

Proof. By Proposition 2.63, $\iota(\Delta)$ is compact. By definition, $\wedge_\Delta = \inf \iota(\Delta)$, and thus by Proposition 2.70 $\wedge_\Delta$ is of finite type. □

Recall that a family Δ of overrings of D is said to be *locally finite* if, for every $x \in D$, $x \neq 0$, is noninvertible in only finitely many elements of Δ. The following result generalizes [3, Theorem 2(4)].

Proposition 2.72. *Let D be an integral domain, let $\{T_\alpha \mid \alpha \in A\}$ be a locally finite family of overrings of D and, for each $\alpha \in A$, let $*_\alpha$ be a semistar operation of finite type on T_α. Then, the map*

$$\begin{aligned} *\colon \mathbf{F}(D) &\longrightarrow \mathbf{F}(D) \\ I &\longmapsto \bigcap_{\alpha \in A} (IT_\alpha)^{*_\alpha} \end{aligned}$$

is a semistar operation of finite type on D.

Proof. Let $\sharp_\alpha$ be the map $\sharp_\alpha : F \mapsto (FB_i)^{*_\alpha}$; by Proposition 2.48(b), every $\sharp_\alpha$ is a semistar operation of finite type on D. Moreover, $* = \inf \Lambda$, where $\Lambda := \{\sharp_\alpha \mid \alpha \in A\}$; in view of Proposition 2.70, we only need to show that Λ is compact.

Let $\mathcal{U}$ be an open cover of Λ; by Alexander Subbase Theorem (see e.g. [55, d-5] or [114, Problem 17S]) we can assume, without loss of generality, that each member of $\mathcal{U}$ is a subbasic open set of $\mathrm{SStar}_f(D)$. Choose a finitely generated fractional ideal F of A such that $U_F \in \mathcal{U}$, and pick a $x_0 \in F \setminus \{0\}$. Since Δ is locally finite, there are only a finite number of overrings, say $T_{\alpha_1}, \ldots, T_{\alpha_n}$, that does not contain x_0^{-1}. Hence, if $T \in \Lambda \setminus \{T_1, \ldots, T_n\}$, $x_0 T = T$, and so $1 \in FT$; it follows that U_F contains $\Delta \setminus \{T_{\alpha_1}, \ldots, T_{\alpha_n}\}$. But now, for every $\alpha \in A$, $\sharp_\alpha \geq \wedge_{\{T_\alpha\}}$, and thus $\sharp_\alpha \in U_F$ if $T_\alpha \in \Delta \setminus \{T_1, \ldots, T_n\}$. Therefore, picking open sets $\Omega_i \in \mathcal{U}$ such that $\sharp_{\alpha_i} \in \Omega_i$, the subset $\{U_F, \Omega_1, \ldots, \Omega_n\}$ of $\mathcal{U}$ is an open cover of Λ. Hence, Λ is compact and $*$ is of finite type. □

Corollary 2.73. *Let D be an integral domain and $\Delta \subseteq \mathrm{Over}(D)$ be locally finite. Then, $\wedge_\Delta$ is a semistar operation of finite type.*

Proof. Apply the previous Proposition by taking every $*_i$ to be the identity. □

Proposition 2.72 does not hold if we replace locally finite families with compact families. Indeed, suppose that $\Delta \subseteq \mathrm{Over}(D)$ is a set such that $\wedge_\Delta$ is not of finite type (for example, if Δ is a non-compact set of localizations of D – see Proposition 2.68). If $T \in \Delta$, define $*_T := \wedge_{\{T\}}$; if $T \in \mathrm{Over}(D) \setminus \Delta$, define $*_T := \wedge_{\{K\}}|_{\mathbf{F}(T)}$. Then, every $*_T$ is a semistar operation of finite type on T. Let $* := \inf\{*_T \mid T \in \mathrm{Over}(D)\}$, i.e.,

$$* : I \mapsto \bigcap_{T \in \mathrm{Over}(D)} (IT)^{*_T}.$$

It is clear that $*$ is just equal to $\wedge_\Delta$, and thus it is not of finite type; on the other hand, $\mathrm{Over}(D)$ is a compact space. A subtler question is what happens if we also impose that $D^{*_T} = T$ for every $T \in \Delta$; we do not know if this is enough to recover the fact that $*$ is of finite type.

We have seen in Example 2.69 that a family of overrings can define a finite-type semistar operation even without being compact. Proposition 2.68, on the other hand, shows that this is possible if we restrict ourselves to some “good” overrings (namely, localizations). Therefore, we would like to find some other cases where such property hold; we present now what can be seen as a generalization of the spectral case, while in the next section we shall study a different, but surprisingly similar, case.

Let $\mathrm{LocOver}(D)$ be the set of local overrings of D; for all $T \in \mathrm{LocOver}(D)$, let $\mathfrak{m}_T$ be the maximal ideal of T. We endow $\mathrm{LocOver}(D)$ with the Zariski topology inherited from $\mathrm{Over}(D)$. Let γ denote the *center map*

$$\begin{aligned}\gamma\colon \mathrm{LocOver}(D) &\longrightarrow \mathrm{Spec}(D)\\ T &\longmapsto \mathfrak{m}_T.\end{aligned}$$

Then, γ is a topological retraction: indeed, it is continuous since

$$\gamma^{-1}(\mathcal{D}(f)) = \mathrm{LocOver}(D[f^{-1}]) = B_{f^{-1}} \cap \mathrm{LocOver}(D),$$

and it is surjective since $\gamma \circ \lambda(P) = P$ (where λ is the localization map defined in Proposition 2.67).

Proposition 2.74. *Let D be an integral domain and let $\Delta \subseteq \mathrm{LocOver}(D)$.*

(a) If $\wedge_\Delta$ is of finite type, $\gamma(\Delta)$ is compact.
(b) If $\gamma|_\Delta$ is a topological embedding, then $\wedge_\Delta$ is of finite type if and only if Δ is compact.

Proof. (a) Let $\mathcal{U} := \{\mathcal{D}(f_\alpha) \mid \alpha \in A\}$ be an open cover of $\gamma(\Delta)$. For any $T \in \Delta$, there is a f_α such that $f_\alpha \notin \mathfrak{m}_T$; that is, $f_\alpha^{-1} \in T$. Let I be the ideal of D generated by all the f_α; then, $IT = T$ for all $T \in \Delta$, and thus $1 \in I^{\wedge_\Delta}$. Since $\wedge_\Delta$ is of finite type, there is a finitely generated ideal $J \subseteq I$ such that $1 \in J^{\wedge_\Delta}$; in particular, $J \subseteq f_{\alpha_1}D + \cdots + f_{\alpha_n}D$ for some $\alpha_1, \dots, \alpha_n \in A$, and so we can suppose $J = f_{\alpha_1}D + \cdots + f_{\alpha_n}D$. In particular, J is not contained in any $P \in \gamma(\Delta)$; that is, $\gamma(\Delta) \subseteq \mathcal{D}(J) = \mathcal{D}(f_{\alpha_1}D) \cup \cdots \cup \mathcal{D}(f_{\alpha_n}D)$, and so $\{\mathcal{D}(f_{\alpha_1}D), \dots, \mathcal{D}(f_{\alpha_n}D)\}$ is an open subcover of $\gamma(\Delta)$.

(b) If Δ is compact, $\wedge_\Delta$ is of finite type by Proposition 2.70. Conversely, if $\wedge_\Delta$ is of finite type, then $\gamma(\Delta)$ is compact by the previous part of the proof; but, under the hypothesis that $\gamma|_\Delta$ is a topological embedding, this is equivalent to the fact that Δ itself is compact. □

This allows to re-prove Proposition 2.68:

Corollary 2.75. *Let D be an integral domain, and let Δ be a set of localizations of D. Then, $\wedge_\Delta$ is of finite type if and only if Δ is compact.*

Proof. When restricted to the set of localizations of D, γ is a homeomorphism, since it is the inverse of the embedding λ. Hence, we can apply Proposition 2.74(b). □

We end this section with a topological generalization of Lemma 2.30.

Theorem 2.76. *Let $A \subseteq B$ be an epimorphic extension; endow the set $\mathbf{F}_A(B)$ of A-submodules of B with the Zariski topology (see Remark 2.66). Let I be a flat A-submodule of B and let Δ be a (nonempty) compact subspace of $\mathbf{F}_A(B)$. Then, the following equality holds:*

$$\left(\bigcap_{U\in\Delta} IU\right)T = \bigcap_{U\in\Delta}(IUT).$$

In particular, $\left(\bigcap_{U\in\Delta} U\right)T = \bigcap_{U\in\Delta}(UT)$.

Proof. Let $I' := \bigcap_{U\in\Delta} IU$. Clearly $I' \subseteq IU$ for every $U \in \Delta$, and thus $I'T \subseteq IUT$ for every $U \in \Delta$; hence $I'T$ is contained in the right hand side.

Let $x \in \bigcap_{U\in\Delta}(IUT)$. For every $U \in \Delta$, there are $i_1,\ldots,i_n \in I$, $u_1,\ldots,u_n \in U$, $t_1,\ldots,t_n \in T$ such that $x = i_1u_1t_1 + \cdots + i_nu_nt_n$, and thus $x \in IF_UT$, where $F_U = (u_1,\ldots,u_n)A$ is a finitely generated A-module. Hence, the set $\{B_{F_U} : U \in \Delta\}$ is an open cover of Δ, and thus it admits a finite subcover $\{B_{F_{U_1}},\ldots,B_{F_{U_m}}\}$. Let $\Delta_k := B_{F_{U_k}} \cap \Delta$. Since multiplication by a flat module commutes with finite intersections in an epimorphic extension (Lemma 2.30), we have

$$\left(\bigcap_{U\in\Delta} IU\right)T = \left(\bigcap_{i=1}^{n}\bigcap_{U\in\Delta_i} IU\right)T = \bigcap_{i=1}^{n}\left(\left(\bigcap_{U\in\Delta_i} IU\right)T\right).$$

However, $F_{U_i} \subseteq \bigcap_{U\in\Delta_i} U$, and thus $x \in \left(\bigcap_{U\in\Delta_i} IU\right)T$. Hence, $x \in I'T$. □

Note that the previous theorem holds, in particular, when $\Delta \subseteq \mathrm{Over}(D)$.

2.3.2. Valutative semistar operations

The good behaviour described in Propositions 2.68 and 2.74 – namely, the equivalence between compactness of a family of overrings and the fact that the semistar operation it generates is of finite type – ultimately depends on the fact that we can link the overrings with the base ring; in one case through localization, in the other through the center map. However, Proposition 2.70 (or rather its overring version, Corollary 2.71) shows that compact sets generate finite-type operations without assuming any link between the rings in the family and the base ring (with the obvious restriction that the rings in the family are overrings); similarly, Propositions 2.48(b) and 2.49 show that finite-type operations on an overring can be considered also as finite-type operations on the base ring. Hence, the next step is trying to found some subset of $\mathrm{Over}(D)$, defined (as much as possible) *independently from* D, such that the behaviour of the semistar operations generated by its subsets can be controlled better than in the case of general overrings.

Under this point of view, the most natural try is the set of valuation overrings, since valuation domains are, in some way, the simplest rings (fields excluded). Recall that the set of valuation overrings of a domain D, endowed with the topology induced by $\mathrm{Over}(D)$, is called the *Riemann-Zariski space* of D (or the *Zariski space*, or *abstract Riemann surface*), and is denoted by $\mathrm{Zar}(D)$. It is well known that, endowed with this topology, $\mathrm{Zar}(D)$ is a spectral space [29, 28].

Definition 2.77. *Let D be an integral domain. A* valutative semistar operation *on D is a semistar operation in the form $\wedge_\Delta$ for some $\Delta \subseteq \mathrm{Zar}(D)$.*

In particular, the b-operation (see Example 2.12(6)) is a valutative semistar operation, since it can be shown that it is equal to $\wedge_{\mathrm{Zar}(D)}$.

The following is an analogue of Proposition 2.29.

Proposition 2.78. *Let D be an integral domain and let $\Delta,\Lambda\subseteq\mathrm{Zar}(D)$, $\Delta,\Lambda\neq\emptyset$. For any $\Gamma\subseteq\mathrm{Zar}(D)$, let $\Gamma^\uparrow:=\{W\in\mathrm{Zar}(D)\mid W\supseteq V$ for some $V\in\Gamma\}$. Then:*

(a) $\wedge_\Delta\leq\wedge_\Lambda$ if and only if $\Delta^\uparrow\supseteq\Lambda^\uparrow$;
(b) $\wedge_\Delta=\wedge_\Lambda$ if and only if $\Delta^\uparrow=\Lambda^\uparrow$.

Proof. We begin by showing that $\wedge_\Delta=\wedge_{\Delta^\uparrow}$ for any $\Delta\subseteq\mathrm{Over}(D)$. Indeed, clearly $\wedge_\Delta\leq\wedge_\Lambda$ if $\Delta\supseteq\Lambda$ and thus $\wedge_\Delta\geq\wedge_{\Delta^\uparrow}$. Conversely, suppose $x\in I^{\wedge_\Delta}$; then, $x\in IV$ for every $V\in\Delta$. If $W\in\Delta^\uparrow$, then $W\supseteq V$ for some $V\in\Delta$, and so $x\in IV\subseteq IW$; hence $x\in I^{\wedge_{\Delta^\uparrow}}$ and $\wedge_\Delta\leq\wedge_{\Delta^\uparrow}$; hence the two semistar operations are equal.

(a) If $\Delta^\uparrow\supseteq\Lambda^\uparrow$ then $\wedge_\Delta\leq\wedge_\Lambda$ by the first part of the proof. Conversely, if $\Delta^\uparrow\not\supseteq\Lambda^\uparrow$, there is a $V\in\Delta^\uparrow\setminus\Lambda^\uparrow$. Then, V does not contain any $W\in\Lambda$, and thus for each such W there is a $x_W\in W\setminus V$; let $I:=\sum_{W\in\Lambda}x_W^{-1}D$. Then, I is contained in the maximal ideal of V, and thus $1\notin I^{\wedge_\Delta}$; however, $1=x_W^{-1}x_W\in IW$ for every $W\in\Lambda$, and thus $1\in I^{\wedge_\Lambda}$. Hence, $I^{\wedge_\Delta}\nsubseteq I^{\wedge_\Lambda}$ and $\wedge_\Delta\nleq\wedge_\Lambda$.

(b) It is enough to apply the previous part to the couples (Δ,Λ) and (Λ,Δ). □

Remark 2.79. To obtain a full analogue of Proposition 2.29, we would need to have also an analogue of its point (a); that is, we would have to characterize the valuation domains that are closed by a valutative semistar operation. However, the obvious analogue, namely

$$\{V\in\mathrm{Zar}(D)\mid V=V^{*_\Delta}\}=\Delta^\uparrow$$

in *not* true: for example, if D is a valuation domain whose maximal ideal M is not branched (i.e., such that M is the union of the prime ideals properly contained in M) and $\Delta:=\mathrm{Spec}(D)\setminus\{M\}$, then $\Delta^\uparrow=\Delta$, while $D^{*_\Delta}=D$.

Our next result is an analogue of Proposition 2.32.

Proposition 2.80. *Let D be an integral domain and let $\Delta\subseteq\mathrm{Zar}(D)$ be nonempty. Then, $\wedge_\Delta$ is of finite type if and only if Δ is a compact subspace of $\mathrm{Zar}(D)$.*

Proof. If Δ is compact, then $\wedge_\Delta$ is of finite type by Corollary 2.71.

Conversely, suppose $\wedge_\Delta$ is of finite type, and let $\mathcal{U}$ be an open cover of Δ; we can suppose that $\mathcal{U}:=\{B_{f_\alpha}\mid\alpha\in A\}$, where each f_α is in $K\setminus\{0\}$. Let $I:=\sum_{\alpha\in A}f_\alpha^{-1}D$; then, 1 belongs to IV for every $V\in\Delta$ (since $1=f_\alpha f_\alpha^{-1}$) and thus there is a finitely generated ideal $J\subseteq I$ such that $1\in J^{\wedge_\Delta}$; in particular, we can suppose $J=f_{\alpha_1}^{-1}D+\cdots+f_{\alpha_n}^{-1}D$ for a finite set $\{\alpha_1,\ldots\alpha_n\}\subseteq A$. For any $V\in\Delta$, $1\in JV$; it follows that $V\subseteq JV$, and in particular $V\subseteq f_{\alpha_i}^{-1}V$ for some i. Being V a valuation domain, this implies that f_{α_i} belongs to V, i.e., that $V\in B_{f_{\alpha_i}}$. Therefore, $\{B_{f_{\alpha_1}},\ldots,B_{f_{\alpha_n}}\}$ is an open cover of Δ, and Δ is compact. □

This result implies two corollaries; the first is a new way to prove a well-known fact (see e.g. [119, Theorem 40, page 113]; note that this proof is completely independent from the fact that $\mathrm{Zar}(D)$ is spectral), while the second can be seen as an analogue of Proposition 2.42.

Corollary 2.81. *Let D be an integral domain. Then,* $\mathrm{Zar}(D)$ *is compact.*

Proof. The semistar operation $\wedge_{\mathrm{Zar}(D)}$ is equal to the b-operation defined in Example 2.12(6) (see e.g. [72, Proposition 6.8.2], with the *caveat* that the definition used therein is slightly different, but the proof continue to hold with our notation). Through the use of the definition, it is not hard to see that the b-operation is of finite type, and by Proposition 2.80 it follows that $\mathrm{Zar}(D)$ is compact. □

Corollary 2.82. *Let D be an integral domain. Then, there is an order-preserving bijection between the set* $\mathrm{SStar}_{f,val}(D)$ *of valutative semistar operation of finite type and the set* $\boldsymbol{\mathcal{X}}(\mathrm{Zar}(D))$ *of nonempty subset of* $\mathrm{Zar}(D)$ *that are closed in the inverse topology.*

Proof. By Proposition 2.80, a subset $\Delta \subseteq \mathrm{Zar}(D)$ generates a finite-type operation if and only if it is compact; moreover, by Proposition 2.78, $\wedge_\Delta = \wedge_\Lambda$ if and only if $\Delta^\uparrow = \Lambda^\uparrow$, and thus the finite-type valutative operations on D are in bijective correspondence with the nonempty subsets Δ of $\mathrm{Zar}(D)$ that are compact and such that $\Delta = \Delta^\uparrow$. However, these subsets are exactly the nonempty closed subsets of $\mathrm{Zar}(D)$ in the inverse topology, since the order induced by the Zariski topology on $\mathrm{Zar}(D)$ is the inverse of the topology given by the set-theoretic containment. □

Again, we are using the notation $\boldsymbol{\mathcal{X}}$ in a sense which will become clear in Section 2.4.

Thus, one of the central results of Section 2.2.3, namely the equivalence between compactness and being of finite type, carries over to valutative operations. Therefore, we could ask how much of the remaining theory of spectral semistar operations can be extended to the case of valuation rings; in the next sections, we shall explore how this analogy plays out in three different (but closely related) areas.

2.3.2.1. $\mathrm{SStar}_{f,val}(D)$ as a spectral space

The proof that $\mathrm{SStar}_{f,sp}(D)$ is a spectral space was based on a few properties:

- the fact that $\mathrm{SStar}_{f,sp}(D)$ is sup-normal, which in turn was based on the sup-normality on $\mathrm{SStar}_f(D)$ (Lemma 2.20);
- the fact that $\mathrm{SStar}_{f,sp}(D)$ is closed by taking suprema (Lemma 2.43);
- the existence of a "good" set of D-submodules of K, namely finitely-generated integral ideals, that generates the Zariski topology (Proposition 2.33(a));
- the existence of a topological retraction Ψ_w (Proposition 2.33(b)).

The first of these properties is shared by $\mathrm{SStar}_{f,val}(D)$, being a subset of $\mathrm{SStar}_f(D)$, and we will see in Section 2.3.2.3 that we can actually find a topological retraction $\Psi_a : \mathrm{SStar}(D) \longrightarrow \mathrm{SStar}_{f,val}(D)$. On the other hand, the second property is quite natural, but it is not clear if it is true; and it is quite hard to come up with a distinguished set of D-submodules of K linked to valuation rings (the set of the modules I we used in the proofs of Propositions 2.78 and 2.80 does not seem to have any distinctive feature).

However, Corollary 2.82, when confronted with Proposition 2.42, seems to suggest that $\mathrm{SStar}_{f,val}(D)$ is not very different from $\mathrm{SStar}_{f,sp}(D)$, especially knowing that $\mathrm{Zar}(D)$

itself is a spectral space: both spaces of semistar operations are represented by closed sets of the inverse topology. We proceed in this section to prove that $\mathrm{SStar}_{f,val}(D)$ is actually a spectral space, and we do so in an indirect way, by using what we have proved in the spectral case.

Definition 2.83. *Let D be an integral domain. The* Kronecker function ring $\mathrm{Kr}(D)$ *of D is the domain*

$$\mathrm{Kr}(D) := \left\{ \frac{f}{g} \mid f, g \in D[X], \boldsymbol{c}(f) \subseteq \boldsymbol{c}(g)^b \right\}$$

where X is an indeterminate, $\boldsymbol{c}(f)$ indicates the content *of the polynomial f (that is, the ideal of D generated by the coefficients of f), and b denotes the b-operation on D.*

The theory of Kronecker function rings is quite rich; in particular, the very definition of Kronecker function ring can be generalized by considering, instead of the b-operation, any valutative semistar operation $*$, obtaining the so-called *$*$-Kronecker function ring* $\mathrm{Kr}(D,*)$. Since we will not be using this generalization, however, we restrict to the notationally simpler case of the “classical” Kronecker function ring, deferring the reader interested in this generalization to [45].

The next lemma summarizes the properties of $\mathrm{Kr}(D)$ that we will be using.

Lemma 2.84. *Let D be an integral domain with quotient field K, and let $\mathrm{Kr}(D)$ be its Kronecker function ring.*

(a) $\mathrm{Kr}(D)$ *is a Bézout domain.*
(b) For each polynomial $f \in D[X]$, $\boldsymbol{c}(f)\mathrm{Kr}(D) = f\mathrm{Kr}(D)$.
(c) For each $E \in \mathbf{F}(D)$, $E^b = E\mathrm{Kr}(D) \cap K$; in particular, $\mathrm{Kr}(D) \cap K$ is equal to the integral closure of D.
(d) The maps

$$\begin{aligned} \theta\colon \mathrm{Spec}(\mathrm{Kr}(D)) &\longrightarrow \mathrm{Zar}(D) \\ P &\longmapsto \mathrm{Kr}(D)_P \cap K \end{aligned} \quad \text{and} \quad \begin{aligned} \phi\colon \mathrm{Zar}(D) &\longrightarrow \mathrm{Spec}(\mathrm{Kr}(D)) \\ V &\longmapsto \mathfrak{m}_{V(X)} \cap \mathrm{Kr}(D) \end{aligned}$$

(where $\mathfrak{m}_T$ indicates the maximal ideal of the local ring T) are well-defined and homeomorphisms (when $\mathrm{Spec}(\mathrm{Kr}(D))$ and $\mathrm{Zar}(D)$ are endowed with the respective Zariski topologies), inverse one of the other.

Proof. Point (a) is [50, Theorem 32.7(b)], while point (b) follows from the proof of the latter; (c) is proved in [50, Theorem 32.7(c)] for finitely generated ideals, and can be immediately generalized to arbitrary submodules of K. The fact that θ and ϕ are bijections is proved in [50, Theorems 3.10 and 3.15], and the fact that they are homeomorphisms in [63, Proposition 2.7]. □

Given these properties, we are ready to extend our result to semistar operations.

Theorem 2.85. *Let D be an integral domain with quotient field K, and let $\mathrm{Kr}(D)$ denote its Kronecker function ring; let θ be the homeomorphism defined in Lemma 2.84(d). Endow all spaces of semistar operation with the Zariski topology.*

(a) There is a continuous bijection

$$\begin{aligned}\Theta\colon \mathrm{SStar}_{sp}(\mathrm{Kr}(D)) &\longrightarrow \mathrm{SStar}_{val}(D)\\ s_\Delta &\longmapsto \wedge_{\theta(\Delta)}.\end{aligned}$$

(b) Θ *restricts to a homeomorphism* $\Theta_f : \mathrm{SStar}_{f,sp}(\mathrm{Kr}(D)) \longrightarrow \mathrm{SStar}_{f,val}(D)$*.*
(c) $\mathrm{SStar}_{f,val}(D)$ *is a spectral space.*

Proof. (a) Since θ is a homeomorphism, it is in particular an order-isomorphism in the order induced by the topology (see Section A.1 for the definition of the latter concept); this order coincides with the set-theoretic inclusion on $\mathrm{Spec}(\mathrm{Kr}(D))$, while it is the opposite on $\mathrm{Zar}(D)$. Hence, θ induces a bijective correspondence Θ_0 between the family of subsets Δ of $\mathrm{Spec}(\mathrm{Kr}(D))$ such that $\Delta = \Delta^\downarrow$ and the family of subsets Λ of $\mathrm{Zar}(D)$ such that $\Lambda = \Lambda^\uparrow$. Moreover, by Proposition 2.29, $s_{\Delta_1} = s_{\Delta_2}$ if and only if $\Delta_1^\downarrow = \Delta_2^\downarrow$, while by Proposition 2.78 $\wedge_{\Lambda_1} = \wedge_{\Lambda_2}$ if and only if $\Lambda_1^\uparrow = \Lambda_2^\uparrow$; therefore, Θ_0 induces Θ, which in particular is well-defined and bijective.

We need to prove that Θ is continuous; to do so, we claim that, for every $E \in \mathbf{F}(D)$,

$$E^{\Theta(*)} = (E\mathrm{Kr}(D))^* \cap K;$$

that is, we claim that Θ is actually the restriction of the functorial map σ (defined in Section 2.2.4) when $A = D$ and $B = \mathrm{Kr}(D)$, which is continuous by Proposition 2.48(a). Indeed, fix an $E \in \mathbf{F}(D)$ and let $\Delta \subseteq \mathrm{Spec}(\mathrm{Kr}(D))$. Then,

$$(E\mathrm{Kr}(D))^{s_\Delta} \cap K = \bigcap_{P\in\Delta} E\mathrm{Kr}(D)_P \cap K = \bigcap_{W\in\lambda(\Delta)} EW \cap K,$$

where λ is the localization map (relative to $\mathrm{Kr}(D)$) defined in Proposition 2.67. Since each W is a valuation ring (being $\mathrm{Kr}(D)$ a Bézout domain by Lemma 2.84(a)), $EW\cap K = E(W\cap K)$, and thus $(E\mathrm{Kr}(D))^{s_\Delta} \cap K = \bigcap\{EV \mid V \in \mathcal{L}\}$, where $\mathcal{L}$ is the set of $W\cap K$, as W ranges in $\lambda(\Delta)$. But this is exactly $\theta(\Delta)$, and so

$$(E\mathrm{Kr}(D))^{s_\Delta} \cap K = \bigcap_{V\in\theta(\Delta)} EV = E^{\wedge_{\theta(\Delta)}} = E^{\Theta(s_\Delta)},$$

as claimed. Hence, Θ is a continuous bijection.

(b) Since θ is a homeomorphism, $\Delta \subseteq \mathrm{Spec}(D)$ is compact if and only if $\theta(\Delta)$ is; since compactness corresponds to finite-type operations in both the spectral (Proposition 2.68) and valutative case (Proposition 2.80), Θ restricts to a continuous bijection Θ_f : $\mathrm{SStar}_{f,sp}(\mathrm{Kr}(D)) \longrightarrow \mathrm{SStar}_{f,val}(D)$. Thus, we only need to show that Θ_f is open.

Consider a subbasic open set $U_F := V_F \cap \mathrm{SStar}_{f,sp}(\mathrm{Kr}(D))$; since $\mathrm{Kr}(D)$ is a Bézout domain, F is principal over $\mathrm{Kr}(D)$, and thus we can write $F = (\alpha/\beta)\mathrm{Kr}(D)$ for some $\alpha, \beta \in D[X]$; let $\beta := b_0 + b_1X + \cdots + b_nX^n$. We claim that

$$\Theta_f(U_F) = V_{b_0^{-1}\boldsymbol{c}(\alpha)} \cap \cdots V_{b_n^{-1}\boldsymbol{c}(\alpha)} \cap \mathrm{SStar}_{f,val}(D). \tag{2.4}$$

Indeed, if $* \in U_F$, then for every i we have

$$(b_i^{-1}\boldsymbol{c}(\alpha))^{\Theta_f(*)} = (b_i^{-1}\boldsymbol{c}(\alpha)\mathrm{Kr}(D))^* = b_i^{-1}(\alpha\mathrm{Kr}(D))^*;$$

however, $1 \in ((\alpha/\beta)\mathrm{Kr}(D))^*$, and thus $\beta \in (\alpha\mathrm{Kr}(D))^*$; since $\boldsymbol{c}(\beta)\mathrm{Kr}(D) = \beta\mathrm{Kr}(D)$, each b_i is in $\beta\mathrm{Kr}(D)$ and thus $b_i \in (\alpha\mathrm{Kr}(D))^*$ for each i, and thus $\Theta_f(*)$ is in the intersection given in (2.4).

Conversely, if $\Theta_f(*)$ is in the intersection (recall that Θ_f is surjective), then $b_i \in (\alpha\mathrm{Kr}(D))^*$ for every i, which means (with the same reasoning) that $\beta \in (\alpha\mathrm{Kr}(D))^*$, and thus $1 \in ((\alpha/\beta)\mathrm{Kr}(D))^*$, i.e., $* \in U_F$ and $\Theta_f(*) \in \Theta_f(U_F)$.

Now the intersection on the right hand side of (2.4) is open in $\mathrm{SStar}_{f,val}(D)$, and thus Θ_f is open. Hence, it is a homeomorphism.

(c) follows directly from the previous point, since $\mathrm{SStar}_{f,sp}(D)$ is a spectral space by Theorem 2.45. □

2.3.2.2. eab and ab semistar operations

If valutative operations correspond to spectral operations, we could ask if there is an analogue of stable operations; that is, a class of closures that works as an "even less finite" version of valutative operations. There are two classical concepts that can fill this place, although neither of them is a complete analogue.

Definition 2.86. *Let D be an integral domain, and let $*$ be a semistar operation on D. We say that $*$ is:*

- eab *(short for "**e**ndlich **a**rithmetisch **b**rauchbar", literally "finitely arithmetically useful") if, for every $F, G, H \in \mathcal{F}_f(D)$, the inclusion $(FG)^* \subseteq (FH)^*$ implies $G^* \subseteq H^*$;*
- ab *(short for "**a**rithmetisch **b**rauchbar", literally "arithmetically useful") if, for every $F \in \mathcal{F}_f(D)$, $G, H \in \mathbf{F}(D)$, the inclusion $(FG)^* \subseteq (FH)^*$ implies $G^* \subseteq H^*$.*

The concept of ab (star) operation was first considered by Krull [80]; subsequently, Gilmer showed that the theory flowed without problems from the weaker concept of eab (star) operation [50]. However, the first example of semistar operation that is eab but non ab was given much later, in [47, Example 16]. We summarize the relationship between the two concepts in the following proposition.

Proposition 2.87. *Let D be an integral domain.*

(a) An ab semistar operation is eab.
(b) If $*$ is of finite type, then it is eab if and only if it is ab.
(c) If $*$ is eab, so is $*_f$.

Proof. Straightforward from the definitions. □

Corollary 2.88. *Let D be an integral domain, $* \in \mathrm{SStar}(D)$. If $*$ is valutative, so is $*_f$.*

Proof. Since $*$ is valutative, it is eab. Therefore, $*_f$ is eab and of finite type, and thus valutative. □

Corollary 2.89. *Let D be an integral domain, $\Delta,\Lambda\subseteq\mathrm{Zar}(D)$. Then:*

1. *[37, Corollary 4.17]* $(\wedge_\Delta)_f=\wedge_{\mathrm{Cl}^{\mathrm{inv}}(\Delta)}$*;*
2. *[37, Theorem 4.9 and Corollary 4.17]* $(\wedge_\Delta)_f=(\wedge_\Lambda)_f$ *if and only if* $\mathrm{Cl}^{\mathrm{inv}}(\Delta)=\mathrm{Cl}^{\mathrm{inv}}(\Lambda)$.

Proof. By the previous corollary, $(\wedge_\Delta)_f$ is a valutative semistar operation, say $(\wedge_\Delta)_f=\wedge_\Gamma$; by Proposition 2.80, Γ has to be compact, and by Corollary 2.82 we can suppose $\Gamma=\Gamma^\uparrow=\mathrm{Cl}^{\mathrm{inv}}(\Gamma)$. In particular, $\Delta\subseteq\mathrm{Cl}^{\mathrm{inv}}(\Delta)\subseteq\Gamma$, and if the containment is strict then (again by Corollary 2.82) we would have $*\geq\wedge_{\mathrm{Cl}^{\mathrm{inv}}(\Delta)}>\wedge_\Gamma=*_f$. However, $\wedge_{\mathrm{Cl}^{\mathrm{inv}}(\Delta)}$ is still of finite type by Proposition 2.80, so this would contradict the fact that $*_f$ is the biggest semistar operation of finite type smaller than $*$. Hence, $\Gamma=\mathrm{Cl}^{\mathrm{inv}}(\Delta)$.

The second claim follows with the same reasoning. □

The following proposition clarifies the relationship between eab and valutative semistar operations.

Proposition 2.90. *Let D be an integral domain and $*$ be a semistar operation on D.*

(a) If $*$ is valutative, then $*$ is eab.
(b) If $*$ is eab and of finite type, then it is valutative.

Proof. The claims follow from [47, Lemma 3 and p.2097-2098]; they were also proved, in the case of star operations, in [50, Theorems 32.5 and 32.12]. □

As a corollary, we find that the property of being eab is probably the simpler property that is not always shared by the identity, as the next proposition shows.

Corollary 2.91. *Let D be an integral domain.*

(a) If the identity is eab, then D is a Prüfer domain.
(b) If D is a Prüfer domain, every semistar operation is ab.

Proof. If the identity semistar operation d is eab, than it must be equal (by Proposition 2.90(b)) to $\wedge_\Delta$ for some Δ, and thus $d=\wedge_{\mathrm{Zar}(D)}=b$. However, the b-operation is equal to the identity if and only if D is Prüfer [50, Theorem 24.7].

Conversely, if D is a Prüfer domain, then every finitely generated fractional ideal is invertible. Hence, if $F\in\mathcal{F}_f(D)$, $G,H\in\mathbf{F}(D)$, $*$ be a semistar operation on D, and $(FG)^*\subseteq(FH)^*$, then, applying Lemma 1.111 (that holds, with the same proof, for semistar operations), we have $FG^*\subseteq FH^*$ and so $G^*\subseteq H^*$ (again, since F is invertible). Hence, $*$ is ab. □

Proposition 2.90 should be seen as an analogue of Proposition 2.31, that was about the relationship between stable and spectral semistar operations; more precisely, the two claims correspond to points (a) and (c) of Proposition 2.31. The third result of that proposition was a characterization of spectral operation among the stable closures; that is, we showed that a stable operation is spectral if and only if it is semifinite. This equivalence does not carry over to this case, as the next example shows.

Example 2.92 [47, Example 15]. Let D be an almost Dedekind domain (that is, suppose that D_N is a discrete valuation ring for every $N \in \mathrm{Max}(D)$), and suppose that D has a unique maximal ideal that is not finitely generated, say M_∞. Such a ring does indeed exist: see [85] for an explicit construction.

Let $*$ be the semistar operation generated (in the terminology of Chapter 1) by M_∞; that is, $I^* := (M_\infty : (M_\infty : I))$. Note that $D^* = (M_\infty : (M_\infty : D)) = (M_\infty : M_\infty) = D$ since D is completely integrally closed, being a one-dimensional Prüfer domain (and so the intersection of one-dimensional valuation domains, which are completely integrally closed). In particular, every invertible (i.e., finitely generated, since D is Prüfer) fractional ideal of D is $*$-closed, and so every maximal ideal different from M_∞ is $*$-closed. Likewise, $M_\infty^* = (M_\infty : (M_\infty : M_\infty)) = (M_\infty : D) = M_\infty$; therefore, every maximal ideal of D is $*$-closed, and hence $*$ is semifinite.

We now claim that $*$ is not valutative. Indeed, $\mathrm{Zar}(D) = \{D_N \mid N \in \mathrm{Max}(D)\} \cup \{K\}$; hence, if $\Delta = \{D_N \mid N \in \Lambda\}$ for some $\Lambda \subseteq \mathrm{Max}(D)$, then $\mathrm{QSpec}^*(D) = \Lambda \cup \{(0)\}$. Therefore, the unique possibility is $\Lambda = \mathrm{Max}(D)$, and thus $*$ should be the identity. However, we claim that $(M_\infty^2)^* = M_\infty \neq M_\infty^2$. Indeed, if $x \in (M_\infty : M_\infty^2)$, then $xM_\infty^2 \subseteq M_\infty$, and thus $xM_\infty^2 D_N \subseteq M_\infty D_N$ for every $N \in \mathrm{Max}(D) \setminus \{M_\infty\}$; since M_∞^2 is M_∞-primary, it follows that $x \in D_N$ for every $N \in \mathrm{Max}(D)$, and so $x \in \bigcap\{D_N \mid N \in \mathrm{Max}(D) \setminus \{M_\infty\}\}$. If the latter intersection is different from D (say, it is equal to T), then T would be an almost Dedekind domain with all prime ideals finitely generated (see Example 2.44 for a similar situation), so a Dedekind domain, and thus $D = T \cap D_N$ would be a finite intersection of Dedekind domains. This would imply that D itself is Dedekind, against our assumption. Therefore, $T = D$, and thus $D \subseteq (M_\infty : M_\infty^2) \subseteq D$; therefore,

$$(M_\infty^2)^* = (M_\infty : (M_\infty : M_\infty^2)) = (M_\infty : D) = M_\infty$$

which is different from M_∞^2 since M_∞^k extends to $(M_\infty D_{M_\infty})^k$ and D_{M_∞} is a discrete valuation ring. Hence, $*$ is not valutative.

Indeed, as far as I know, there is no known characterization of valutative semistar operations among the ab or the eab operations that does not involve valuation overrings but only "internal" properties of the semistar operation.

2.3.2.3. The map Ψ_a and the b-topology

Although the sets of eab and ab semistar operations seem to be similar to the set of stable operations, there is no known analogue of the topological retraction Ψ_{st}. However, we can define an analogue of Ψ_w: let $*$ be a semistar operation and let F be a finitely generated fractional ideal of D. Then, we define, following [44],

$$F^{*_a} := \bigcup\{((FG)^* : G^*) \mid G \in \mathcal{F}_f(D)\}$$

and, if I is an arbitrary D-submodule of K, we define $I^{*_a} := \bigcup\{F^{*_a} \mid F \subseteq I, F \in \mathcal{F}_f(D)\}$; then, we would wish to define the map

$$\begin{aligned}\Psi_a \colon \mathrm{SStar}(D) &\longrightarrow \mathrm{SStar}_{f,val}(D)\\ * &\longmapsto *_a.\end{aligned}$$

The next proposition shows that this map does actually what we would like it to do.

Proposition 2.93. *Let D be an integral domain and let $*$ be a semistar operation on D.*

(a) $*_a$ is an eab semistar operation of finite type; in particular, it is valutative.
(b) $*_a \geq *$, and $*_a$ is the smallest eab operation bigger than $*$.
(c) If $*$ is eab, then $*_a = *_f$.
(d) Ψ_a is a well-defined topological retraction, when $\mathrm{SStar}(D)$ and $\mathrm{SStar}_{f,val}(D)$ are endowed with the Zariski topology.

Proof. The proof of (a), (b) and (c) can be found in [44, Proposition 4.5] (note that we have to use Proposition 2.90(b) to show that $*_a$ is valutative).

For (d), we note that Ψ_a is well-defined by point (a); moreover, if $*$ is of eab of finite type, then $* = *_a$ by (c), so to show that Ψ_a is a retraction we only need to show that it is continuous. Take a finitely-generated fractional ideal F. Then,

$$\begin{aligned}\Phi_a^{-1}(V_F \cap \mathrm{SStar}_{f,val}(D)) &= \{* \in \mathrm{SStar}(D) \mid 1 \in F^{*_a}\} = \\ &= \{* \in \mathrm{SStar}(D) \mid F^* \subseteq (FH)^* \text{ for some } H \in \mathcal{F}_f(D)\} = \\ &= \bigcup \{\{* \in \mathrm{SStar}(D) \mid F^* \subseteq (FH)^*\} \mid H \in \mathcal{F}_f(D)\}\,.\end{aligned}$$

However, if $F = f_1D + \cdots + f_nD$, then $F^* \subseteq (FH)^*$ if and only if $f_i \in (FH)^*$ for all i, that is, if and only if $* \in V_{f_i^{-1}FH}$. Hence,

$$\{* \in \mathrm{SStar}(D) \mid F^* \subseteq (FH)^*\} = \bigcap_{i=1}^{n} V_{f_i^{-1}FH}$$

is open in $\mathrm{SStar}(D)$, and thus also their union is open. Therefore, $\Phi_a^{-1}(V_F \cap \mathrm{SStar}_{f,val}(D))$ is open and Ψ_a is continuous and a topological retraction. □

Another natural question – in the spirit of Propositions 2.18, 2.24 and 2.33 – would be if we can endow $\mathrm{SStar}(D)$ with a topology such that Ψ_a is the canonical T_0 quotient or, better, if this topology can be thought of as a $(\mathcal{A}, \mathscr{P})$-Zariski topology, for some class $\mathcal{A}$ of D-submodules of K. The former question has a positive, but trivial, answer: we can take the coarsest topology such that Ψ_a is continuous, or more explicitly the topology generated by the subbasic open sets $V_I^{(a)} := \{* \mid 1 \in I^{*_a}\}$, as I ranges among the D-submodules of K. On the other hand, there does not seem to be a family of D-submodules that can become an appropriate $\mathcal{A}$.

However, if we restrict to the space of overrings, we can use Ψ_a to find an alternative description of the Zariski topology. For any $T \in \mathrm{Over}(D)$, define $b(T) := \wedge_{\mathrm{Zar}(T)}$; then, $b(T)$ can also be viewed as $\sigma_T(b_T)$, there b_T is the b-operation on T and $\sigma_T : \mathrm{SStar}(T) \longrightarrow \mathrm{SStar}(D)$ is the canonical map associated to the inclusion $D \hookrightarrow T$. Note that $b(T) = b(\overline{T})$, where $\overline{T}$ is the integral closure of T, and that $D^{b(T)} = T^{b(T)} = \overline{T}$, since every integrally closed ring is the intersection of its valuation overrings (see e.g. [12, Corollary 5.22]).

Let $\mathrm{Over}_{\mathrm{ic}}(D)$ denote the space of integrally closed overrings of D. The *b-topology* was defined in [95] as the topology generated by the subbasic open sets

$$\mathscr{U}_{ic}(I,J) := \{T \in \mathrm{Over}_{\mathrm{ic}}(D) \mid I \subseteq J^{b(T)}\},$$

as I and J ranges among the finitely-generated fractional ideals of D. We also extend this topology to the whole $\mathrm{Over}(D)$ by taking as subbasic open sets the sets

$$\mathscr{U}(I,J) := \{T \in \mathrm{Over}(D) \mid I \subseteq J^{b(T)}\},$$

again with $I, J \in \mathcal{F}_f(D)$, so that $\mathscr{U}_{ic}(I,J) = \mathscr{U}(I,J) \cap \mathrm{Over}_{\mathrm{ic}}(D)$. Since $b(T) = b(\overline{T})$, as observed above, it is clear that the b-topology on $\mathrm{Over}(D)$ can't be T_0; we shall be more precise in Proposition 2.98.

Lemma 2.94. *Let D be an integral domain. Then, the b-topology on $\mathrm{Over}_{\mathrm{ic}}(D)$ is finer (or equal) than the Zariski topology.*

Proof. Let B_F be a subbasic open set of the Zariski topology of $\mathrm{Over}_{\mathrm{ic}}(D)$. Then, using the fact that $T^{b(T)} = T$ if T is integrally closed,

$$\begin{aligned} B_F &= \{T \in \mathrm{Over}_{\mathrm{ic}}(D) \mid F \subseteq T\} = \{T \in \mathrm{Over}_{\mathrm{ic}}(D) \mid F \subseteq T^{b(T)}\} = \\ &= \{T \in \mathrm{Over}_{\mathrm{ic}}(D) \mid F \subseteq D^{b(T)}\} = \mathscr{U}_{ic}(F,D) \end{aligned}$$

which is open in the b-topology. □

Proposition 2.95. *Let D be an integral domain, and define $\iota_{ic,a}$ as the map*

$$\begin{aligned} \iota_{ic,a} \colon \mathrm{Over}_{\mathrm{ic}}(D) &\longrightarrow \mathrm{SStar}(D) \\ T &\longmapsto b(T). \end{aligned}$$

Endow $\mathrm{SStar}(D)$ with the Zariski topology. Then:

(a) If $\mathrm{Over}_{\mathrm{ic}}(D)$ is endowed with the Zariski topology, then $\iota_{ic,a}$ is continuous and injective.
(b) If $\mathrm{Over}_{\mathrm{ic}}(D)$ is endowed with the b-topology, then $\iota_{ic,a}$ is topological embedding.

Proof. (a) We first show that

$$\sigma \circ \Psi_a^{(T)} = \Psi_a^{(D)} \circ \sigma, \tag{2.5}$$

where $\Psi_a^{(T)}$ and $\Psi_a^{(D)}$ are the maps Ψ_a in $\mathrm{SStar}(T)$ and $\mathrm{SStar}(D)$, respectively, and σ is the canonical map $\mathrm{SStar}(T) \longrightarrow \mathrm{SStar}(D)$. Indeed, let $* \in \mathrm{SStar}(T)$; since both sides of (2.5) are of finite type, it is enough to check equality at finitely-generated ideals, and so we take a $I \in \mathcal{F}_f(D)$. Then,

$$I^{\sigma\circ\Psi_a^{(T)}(*)} = I^{\sigma(\Psi_a^{(T)}(*))} = (IT)^{\Psi_a^{(T)}(*)} = \bigcup_{F \in \mathcal{F}_f(T)} ((IT)^* : F^*),$$

while

$$I^{\Psi_a^{(D)}\circ\sigma(*)}=\bigcup_{G\in\mathcal{F}_f(D)}(I^{\sigma(*)}:G^{\sigma(*)})=\bigcup_{G\in\mathcal{F}_f(D)}((IT)^*:(GT)^*).$$

However, as G ranges among all the finitely-generated fractional ideals of D, GT ranges among all the finitely-generated fractional ideals of T; it follows that $I^{\sigma\circ\Psi_a^{(T)}(*)}=I^{\Psi_a^{(D)}\circ\sigma(*)}$ and thus (2.5) holds.

For any domain A, the semistar operation b_A is equal to $(d_A)_a$, where d_A is the identity on A; therefore,

$$\iota_{ic,a}(T)=b(T)=\sigma(b_T)=\sigma(\Psi_a^{(T)}(d_T))=\Psi_a^{(D)}(\sigma(d_T))=\Psi_a^{(D)}(\wedge_{\{T\}})=\Psi_a^{(D)}\circ\iota(T),$$

where $\iota:\mathrm{Over}(D)\longrightarrow\mathrm{SStar}(D)$ is the natural inclusion defined in (2.3). Hence, $\iota_{ic,a}$ is the composition of the two continuous maps $\Psi_a^{(D)}$ and $\iota|_{\mathrm{Over}_{\mathrm{ic}}(D)}$, and so it is continuous itself.

Moreover, $\iota_{ic,a}$ is injective since, if $T_1,T_2\in\mathrm{Over}_{\mathrm{ic}}(D)$, then $\mathrm{Zar}(T_1)^\uparrow=\mathrm{Zar}(T_1)\neq\mathrm{Zar}(T_2)=\mathrm{Zar}(T_2)^\uparrow$, and so $b(T_1)\neq b(T_2)$ by Proposition 2.78(b).

(b) Since the b-topology is finer than the Zariski topology (Lemma 2.94), $\iota_{ic,a}$ is continuous even when $\mathrm{Over}_{\mathrm{ic}}(D)$ is endowed with the b-topology, by the previous part of the proof. Let now $\mathrm{SStar}_b(D)$ be the range of $\iota_{ic,a}$, and take a subbasic open set $\mathscr{U}_{ic}(I,J)$ of the b-topology. If $I=x_1D+\cdots+x_nD$, then $\mathscr{U}_{ic}(I,J)=\mathscr{U}_{ic}(x_1D,J)\cap\cdots\cap\mathscr{U}_{ic}(x_nD,J)$; therefore, we can suppose $I=xD$ for some $x\in K\setminus\{0\}$. Then,

$$\iota_{ic,a}(\mathscr{U}_{ic}(xD,J))=\{*\in\mathrm{SStar}_b(D)\mid x\in J^*\}=V_{x^{-1}J}\cap\mathrm{SStar}_b(D),$$

which is open in $\mathrm{SStar}_b(D)$. Hence, $\iota_{ic,a}$ is a topological embedding. □

Corollary 2.96. *Let D be an integral domain. The b-topology and the Zariski topology coincide on* $\mathrm{Over}_{\mathrm{ic}}(D)$.

Proof. By Lemma 2.94, the b-topology is finer than the Zariski topology. Moreover, any subbasic open set $\mathscr{U}_{ic}(D)$ of the b-topology is equal to $\iota_{ic,a}^{-1}(V_{x^{-1}G})$, which is open in the Zariski topology since $\iota_{ic,a}$ is continuous when $\mathrm{Over}_{\mathrm{ic}}(D)$ is endowed with the Zariski topology. Hence, the Zariski topology is finer than the b-topology, and the two topologies are equal. □

In particular, the previous corollary generalizes [95, Corollary 2.8], which proved that the two topology coincide on $\mathrm{Zar}(D)$.

Corollary 2.97. *Let D be an integral domain and $T\in\mathrm{Over}(D)$. Then, $\wedge_{\{T\}}$ is eab if and only if T is a Prüfer domain.*

Proof. By the proof of Proposition 2.95, $\wedge_{\{T\}}=\sigma(d_T)$ is eab if and only if it coincides with $\Psi_a(\wedge_{\{T\}})=\Psi_a^{(D)}\circ\sigma(d_T)=\sigma\circ\Psi_a^{(T)}(d_T)=\sigma(b_T)$; since σ is injective, being T an overring (Proposition 2.49), this happens if and only if $d_T=b_T$. By Corollary 2.91, this is equivalent to T being Prüfer. □

To end this section, we concentrate on the relationship between $\mathrm{Over}(D)$ and $\mathrm{Over}_{\mathrm{ic}}(D)$. There is a natural map

$$\begin{aligned}\beta\colon \mathrm{Over}(D) &\longrightarrow \mathrm{Over}_{\mathrm{ic}}(D)\\ T &\longmapsto \overline{T};\end{aligned}$$

moreover, if we extend $\iota_{ic,a}$ to a map $\iota_a : \mathrm{Over}(D) \longrightarrow \mathrm{SStar}(D)$, $T \mapsto b(T)$, we see that $\iota_a = \iota_{ic,a} \circ \beta$ and thus β can be seen as a way to generalize $\iota_{ic,a}$.

Proposition 2.98. *Let D be an integral domain; let β be the map defined above*

- *(a) If $\mathrm{Over}_{\mathrm{ic}}(D)$ and $\mathrm{Over}(D)$ are endowed with the Zariski topology, then β is continuous and, hence, it is a topological retraction of $\mathrm{Over}(D)$ onto $\mathrm{Over}_{\mathrm{ic}}(D)$.*
- *(b) If $\mathrm{Over}_{\mathrm{ic}}(D)$ is endowed with the Zariski topology and $\mathrm{Over}(D)$ is endowed with the b-topology, then β is the canonical T_0 quotient of $\mathrm{Over}(D)$.*

Proof. (a) Let x be a nonzero element of K and let B_x be a subbasic open set of the Zariski topology on $\mathrm{Over}_{\mathrm{ic}}(T)$. Then,

$$\begin{aligned}\beta^{-1}(B_x) &= \{T \in \mathrm{Over}(D) \mid x \in \overline{T}\} = \{T \in \mathrm{Over}(D) \mid x \text{ is integral over } T\} =\\ &= \bigcup\Big\{B_{\{\alpha_{k-1},\alpha_{k-2},\ldots,\alpha_0\}} \mid x^k + \alpha_{k-1}x^{k-1} + \cdots + \alpha_1 x + \alpha_0 = 0\Big\}\end{aligned}$$

which is open since it is a union of open sets. Hence, β is continuous; moreover, $\beta|_{\mathrm{Over}_{\mathrm{ic}}(D)}$ is the identity, so β is a topological retraction.

(b) Since the Zariski topology on $\mathrm{Over}_{\mathrm{ic}}(D)$ coincide with the b-topology, we can consider the subbasic open sets in the form $\mathscr{U}_{ic}(I,J)$, for $I, J \in \mathcal{F}_f(D)$. Since $b(T) = b(\overline{T})$, we have $\beta^{-1}(\mathscr{U}_{ic}(I,J)) = \mathscr{U}(I,J)$, and thus β is continuous. Analogously, $\beta(\mathscr{U}(I,J)) = \mathscr{U}_{ic}(I,J)$, and thus β is open. Finally, T_1 and T_2 can be distinguished by the b-topology if and only if their respective integral closures are different; the claim is completely proved. ☐

2.3.3. Subsets of $\mathrm{Zar}(D)$ that are not compact

Proposition 2.70 (or, more precisely, Corollary 2.71) can be thought of as a test for compactness of families of overrings: if Δ is compact, then $\wedge_\Delta$ must be of finite type. Equivalently, if we can somehow prove that $\wedge_\Delta$ is not of finite type, then we can deduce that Δ is not compact.

However, this is usually not quite useful, since it is not clear how to prove that a given semistar operation is or is not of finite type, especially if it is given simply in the form $\wedge_\Delta$. In this section, we show how this can be done in a particular case for valutative operations; our main tool will be the dual definition of the b-operation, either as the set of elements that satisfy an equation of integral dependence over an ideal (Example 2.12(6)) or as the semistar operation $\wedge_{\mathrm{Zar}(D)}$. In particular, we will use the fact that, against what happens with spectral operations (Propositions 2.25, 2.36 and 2.54), valutative (semi)star and star operations does not correspond bijectively; we start with an example that contains the main idea of our method.

Example 2.99. Let D be a Noetherian domain, $x \in D$, I an ideal of D; suppose also that D is integrally closed (this is not strictly needed, but makes $b_D|_{\mathcal{F}(D)}$ a star operation). Then, $x \in I^b$ if and only if, for every *discrete* valuation overring V of D, $x \in IV$ [72, Proposition 6.8.2]. Therefore, $I^b = \bigcap\{IV \mid V \in \Delta\} = I^{\wedge_\Delta}$, where Δ is the set of discrete valuation overrings of D. However, if $\dim(D) \geq 2$, then $b \neq \wedge_\Delta$, since $\Delta = \Delta^\uparrow \neq \mathrm{Zar}(D)$, as the latter contains valuation rings of dimension 2 or more.

An immediate consequence of this example is the following proposition:

Proposition 2.100. *Let D be a Noetherian domain such that $\dim(D) \geq 2$. Then, the set Δ of discrete valuation overrings of D is not compact.*

Proof. We have seen in Example 2.99 that $b|_{\mathcal{F}(D)} = \wedge_\Delta|_{\mathcal{F}(D)}$; since every finitely generated D-submodule of K is a fractional ideal, it follows that $b = b_f = (\wedge_\Delta)_f$. In particular, since $b \neq \wedge_\Delta$, it follows that $\wedge_\Delta$ is not of finite type, and thus, by Proposition 2.80, Δ is not compact. □

This section will be essentially devoted to generalizations of this example; our idea is to find condition on some $\Delta \subseteq \mathrm{Zar}(D)$ to have $I^b = \bigcap\{IV \mid V \in \Delta\}$ for every ideal I. To ease the notation, we introduce the following definitions.

Definition 2.101. *Let D be an integral domain and let $\Delta, \Lambda \subseteq \mathrm{Over}(D)$. We say that Λ dominates Δ if, for every $T \in \Delta$ and every $M \in \mathrm{Max}(T)$ there is a $A \in \Lambda$ such that $MA \neq A$.*

Definition 2.102. *Let D be an integral domain domain. We denote by $D[\mathcal{F}_f]$ the subset of $\mathrm{Over}(D)$ defined by*

$$D[\mathcal{F}_f] := \{D[I] : I \in \mathcal{F}_f(R)\} = \left\{D\left[\frac{I}{x}\right] : I \in \mathcal{I}_f, x \in D \setminus \{0\}\right\}.$$

We denote by $\mathrm{Zar}_{\min}(D)$ the set of minimal valuation overrings of D, that is, the set of valuation overrings of D that do not contain any other valuation overring. Note that, since elements of $\mathrm{Zar}_{\min}(D)$ corresponds to maximal ideals of $\mathrm{Kr}(D)$ (see Lemma 2.84), every valuation overring of D contains a minimal valuation overring.

Proposition 2.103. *Let D be an integral domain, and let $\Delta \subseteq \mathrm{Zar}(D)$ be a set that dominates $D[\mathcal{F}_f]$. Then, Δ is compact if and only if it contains $\mathrm{Zar}_{\min}(D)$.*

Proof. Clearly, if Δ contains $\mathrm{Zar}_{\min}(D)$ then a family of open sets is a cover of Δ if and only if it is a cover of $\mathrm{Zar}(D)$, and thus Δ is compact since $\mathrm{Zar}(D)$ is.

To show the converse, we first show that, if Δ dominates $D[\mathcal{F}_f]$, then $(\wedge_\Delta)_f = b$; we proceed following the proof of [72, Proposition 6.8.2]. Clearly, $b \leq \wedge_\Delta$ and so $b \leq (\wedge_\Delta)_f$.

Suppose I is a finitely generated integral ideal of D and $x \in I^{\wedge_\Delta}$; let $J := x^{-1}I \in \mathcal{F}_f(D)$. Define $A := D[J]$, and suppose $JA \neq A$. Then (note that $J \subseteq A$), there is a maximal ideal M of A containing J, and thus – by domination – there is a $V \in \Delta$, $A \subseteq V$ such that $JV \subseteq \mathfrak{m}_V$. But then, $x^{-1}IV \subseteq \mathfrak{m}_V$, and thus $IV \subseteq x\mathfrak{m}_V$. However,

$x \in I^b \subseteq IV$; then $x \in x\mathfrak{m}_V$, a contradiction. Hence, $JA = A$, i.e., $1 = j_1a_1 + \cdots + j_na_n$ for some $j_t \in J$, $a_t \in A$; expliciting the elements of A as elements of $D[J]$ and using $J = x^{-1}I$, we find that there must be an $N \in \mathbb{N}$ and elements $i_t \in I^t$ such that $x^N = i_1x^{N-1} + \cdots + i_{N-1}x + i_N$, which gives an equation of integral dependence of x over I. Therefore, $x \in I^b$ and $(\wedge_\Delta)_f \leq b$, so that $(\wedge_\Delta)_f = b$.

If now Δ does not contain $\mathrm{Zar}_{\min}(D)$, then $\Delta^\uparrow \neq \mathrm{Zar}(D)$, and thus by Proposition 2.78 $\wedge_\Delta \neq \wedge_{\mathrm{Zar}(D)} = b$. In particular, $\wedge_\Delta$ is not of finite type and thus, by Proposition 2.80, Δ is not compact. $\square$

Proposition 2.104. *Let D be an integral domain and let $V \in \mathrm{Zar}_{\min}(D)$. If $\mathrm{Zar}(D) \setminus \{V\}$ is compact, then V is the integral closure of $D[x_1, \ldots, x_n]_M$ for some $x_1, \ldots, x_n \in K$ and some $M \in \mathrm{Max}(D[x_1, \ldots, x_n])$.*

Proof. If $\Delta := \mathrm{Zar}(D) \setminus \{V\}$ is compact, then by Proposition 2.103 it cannot dominate $D[\mathcal{F}_f]$ (for otherwise it would contain $\mathrm{Zar}_{\min}(D)$, against our choice of V). Hence, there is a finitely generated fractional ideal I such that Δ does not dominate $A := D[I]$, and so a maximal ideal M of A such that $MW = W$ for every $W \in \Delta \cap \mathrm{Zar}(A)$. In particular, $A \neq K$.

However, there must be a valuation ring containing A_M whose center (on A_M) is MA_M, and the unique possibility for this valuation ring is V: it follows that V is the unique valuation ring centered on MA_M. However, the integral closure of A_M is the intersection of the valuation rings with center MA_M (since every valuation ring containing A_M contains a valuation ring centered on MA_M [50, Corollary 19.7]); thus, $\overline{A_M} = V$. $\square$

This result actually generalizes Proposition 2.100: indeed, if D is Noetherian, then $A := D[I]$ and A_M must be Noetherian as well, and thus $\overline{A_M}$ has to be a Krull domain. But a Krull domain that is also a valuation ring is discrete, and thus it cannot have dimension strictly greater than 1.

For the following result, if D is fixed and $V \in \mathrm{Zar}(D)$, denote by $\iota_V : \mathrm{Spec}(V) \longrightarrow \mathrm{Spec}(D)$ the canonical map associated to the inclusion $D \hookrightarrow V$.

Proposition 2.105. *Let D be an integral domain, let $V \in \mathrm{Zar}_{\min}(D)$ and suppose that $\mathrm{Zar}(D) \setminus \{V\}$ is compact. For every $P \in \mathrm{Spec}(D)$, $|\iota_V^{-1}(P)| \leq 2$; in particular, $\dim(V) \leq 2\dim(D)$.*

Proof. Suppose $|\iota_V^{-1}(P)| > 2$: then, there are prime ideals $Q_1 \subsetneq Q_2 \subsetneq Q_3$ of V such that $\iota_V(Q_1) = \iota_V(Q_2) = \iota_V(Q_3)$. If $\mathrm{Zar}(D) \setminus \{V\}$ is compact, by Proposition 2.104 there is a finitely-generated D-algebra $A := D[a_1, \ldots, a_n]$ such that V is the integral closure of A_M, for some maximal ideal M of A. We can write A_M as a quotient $\frac{D[X_1,\ldots,X_n]_\mathfrak{a}}{\mathfrak{b}}$, where $X_1, \ldots, X_n$ are independent indeterminates and $\mathfrak{a}, \mathfrak{b} \in \mathrm{Spec}(D[X_1, \ldots, X_n])$; since $A_M \subseteq V$ is an integral extension, $Q_i \cap A \neq Q_j \cap A$ if $i \neq j$.

For $i \in \{1, 2, 3\}$, let $\mathfrak{q}_i$ be the prime ideal of $D[X_1, \ldots, X_n]$ whose image in A is Q_i; then, $\mathfrak{q}_1$, $\mathfrak{q}_2$ and $\mathfrak{q}_3$ are distinct, $\mathfrak{q}_i \cap D = P$ for each i, and the set of ideals between $\mathfrak{q}_1$ and $\mathfrak{q}_3$ is linearly ordered. However, the prime ideals of $D[X_1, \ldots, X_n]$ contracting

$$\begin{tikzcd}
 & D[X_1,\ldots,X_n] \arrow[r, hook] \arrow[d, two heads] & D[X_1,\ldots,X_n]_{\mathfrak{a}} \arrow[d, two heads] & \\
D \arrow[r, hook] \arrow[ur, hook] & A=D[a_1,\ldots,a_n] \arrow[r, hook] & A_M=\frac{D[X_1,\ldots,X_n]_{\mathfrak{a}}}{\mathfrak{b}} \arrow[r, hook] & V
\end{tikzcd}$$

Figure 2.1: Rings involved in the proof of Proposition 2.105.

to P are in a bijective and order-preserving correspondence with the prime ideals of $Q(D/P)[X_1,\ldots,X_n]$; since the latter is a Noetherian ring, there are an infinite number of prime ideals between the ideals corresponding to $\mathfrak{q}_1$ and $\mathfrak{q}_3$. This is a contradiction, and $|\iota_V^{-1}(P)|\leq 2$.

For the "in particular" statement, take a chain $(0)\subsetneq Q_1\subsetneq\cdots\subsetneq Q_k$ in $\mathrm{Spec}(V)$. Then, the corresponding chain of the $P_i:=Q_i\cap D$ has length at most $\dim(D)$, and moreover $\iota^{-1}((0))=\{(0)\}$. Hence, $k+1\leq 2\dim(D)+1$ and $\dim(V)\leq 2\dim(D)$. □

For the next corollary, recall that the *valutative dimension* of D, indicated by $\dim_v(D)$, is defined as the supremum of the dimensions of the valuation overrings of D; we have always $\dim(D)\leq\dim_v(D)$, and when the two dimensions coincide we say that D is a *Jaffard domain* [10] (for example, Noetherian and Prüfer domains are Jaffard domains – see e.g. [50, Theorem 3.8 and Corollary 3.10]). Moreover, recall that a topological space X is said to be *Noetherian* if the family of the open sets satisfies the ascending chain condition, or equivalently if every subset of X is compact.

Corollary 2.106. *Let D be an integral domain such that* $\mathrm{Zar}(D)$ *is Noetherian. Then,* $\dim_v(D)\leq 2\dim(D)$.

Proof. If $\mathrm{Zar}(D)$ is Noetherian, then in particular $\mathrm{Zar}(D)\setminus\{V\}$ is compact for every $V\in\mathrm{Zar}_{\min}(D)$. Hence, $\dim(V)\leq 2\dim(D)$ for every $V\in\mathrm{Zar}_{\min}(D)$, by Proposition 2.105; since, if $W\supseteq V$ are valuation domain, $\dim(W)\leq\dim(V)$, the claim follows. □

Example 2.107. The inequality given in Corollary 2.106 is sharp, that is, we may have $\mathrm{Zar}(D)$ Noetherian and $\dim_v(D)=2\dim(D)$. For example, let L be an algebraically closed field, and consider the ring $A:=L+YL(X)[[Y]]$, where X and Y are independent indeterminates. Then, the valuation overrings of A are the rings in the form $V+YL(X)[[Y]]$, as V ranges among the valuation rings containing L and having quotient field $L(X)$. Any of these must contain X or X^{-1}; thus, $\mathrm{Zar}(A)$ can be written as $Z_1\cup Z_2$, where Z_1 is homeomorphic to $\mathrm{Zar}(L[X])$ and Z_2 is homeomorphic to $\mathrm{Zar}(L[X^{-1}])$. The spaces Z_1 and Z_2 are both Noetherian, since are homeomorphic to the spectrum of a principal ideal domain, and so $\mathrm{Zar}(A)$ is Noetherian, being the union of a finite number of Noetherian spaces. However, $\dim_v(A)=2$, while $\dim(A)=1$ (the unique nonzero prime ideal of A is $YL(X)[[Y]]$).

Despite this example, Proposition 2.105 can be strengthened if we add some hypotheses. Recall that a PvMD is an integral domain such that the localization at every t-prime ideal is a valuation domain.

Proposition 2.108. *Let D be an integral domain, and let $V \in \mathrm{Zar}_{\min}(R)$ be such that $\mathrm{Zar}(D) \setminus \{V\}$ is compact; let $(0) \subsetneq P_1 \subsetneq \cdots \subsetneq P_k$ be the chain of prime ideals of V and let $Q_i := P_i \cap R$. Then:*

(a) for every $0 \leq t \leq \dim(D)$, we have

$$\dim(V) \leq \dim_v(D_{Q_t}) + 2(\dim(D) - h(Q_t));$$

(b) if D_{Q_t} is a valuation ring, then $\dim(V) \leq 2\dim(D) - h(Q_t)$;
(c) if D is a PvMD, then $\dim(V) \leq 2\dim(D) - 1$.

Proof. (a) Let $(0) \subsetneq Q^{(1)} \subsetneq Q^{(2)} \subsetneq \cdots \subsetneq Q^{(s)}$ be the chain of the Q_i without the repetitions, and let a be the index such that $Q^{(a)} = Q_t$. For every $b > a$, by the proof of Proposition 2.105 there can be at most two prime ideals of V over $Q^{(b)}$; on the other hand, V_{P_t} is a valuation overring of D_{Q_t}, and thus $t = \dim(V_{P_t}) \leq \dim_v(D_{Q_t})$. Therefore,

$$\dim(V) \leq t + 2(s-a) \leq \dim_v(D_{Q_t}) + 2(\dim(D) - h(Q_t))$$

since each ascending chain of prime ideals starting from Q_t has length at most $\dim(D) - h(Q_t)$.

Point (b) follows from the above point, since if D_{Q_t} is a valuation ring then $\dim_v(D_{Q_t}) = \dim(D_{Q_t}) = h(Q_t)$.

(c) Suppose now D is a PvMD; we can suppose, without loss of generality, that the dimension of D is finite. If the chain of the Q_i does not contain any height-1 prime, then the same proof of Proposition 2.105 shows that $\dim(V) \leq 2\dim(D) - 2$; otherwise, the height-1 prime Q_1 is such that D_{Q_1} is a valuation ring (since every height-1 prime is a t-prime) and thus we can apply (b). Hence, (c) is proved. □

Corollary 2.109. *Let D be a PvMD. If $\mathrm{Zar}(D)$ is a Noetherian space, then $\dim_v(D) \leq 2\dim(D) - 1$.*

Proof. It is enough to proceed like in the proof of Corollary 2.106, using Proposition 2.108(c). □

2.3.4. Proconstructible spaces of overrings

In this section, we focus on the problem of determining subsets of the space $\mathrm{Over}(D)$ that are spectral. The first result involves the whole space.

Proposition 2.110 [36, Proposition 3.5]. *Let D be an integral domain. Then, $\mathrm{Over}(D)$ is a spectral space.*

Proof. Let $\mathscr{U}$ be an ultrafilter on $\mathrm{Over}(D)$, and define

$$A_{\mathscr{U}} := \{x \in K \mid B_x \in \mathscr{U}\}.$$

Then, it is easy to see that $A_{\mathscr{U}}$ is a ring, and that $A_{\mathscr{U}}$ belongs to the set

$$\mathrm{Over}(D)(\mathscr{U}) = \{A \in \mathrm{Over}(D) \mid [\forall B_x, A \in B_x \iff B_x \in \mathscr{U}]\}.$$

By [36, Corollary 3.3] (see also Theorem A.2), it follows that $\mathrm{Over}(D)$ is a spectral space. □

This method can be generalized:

Proposition 2.111. *Let D be an integral domain, and let $X \subseteq \mathrm{Over}(D)$. If, for any ultrafilter $\mathscr{U}$ on X, the set*

$$A_{\mathscr{U}} := \{x \in K \mid B_x \cap X \in \mathscr{U}\}$$

is a ring belonging to X, then X is a spectral space.

The proof follows essentially the same lines of the proof of Proposition 2.110. The sets X that satisfy the hypothesis of the proposition are said to be *proconstructible subsets* of $\mathrm{Over}(D)$; they are the closed sets of a topology, called the *constructible topology*, and $\mathrm{Over}(D)$, endowed with this topology, is a spectral space. (See Section A.4 for more details.) Note that the same reasoning applies if, instead of the space $\mathrm{Over}(D)$, we start with the space $\mathrm{Over}(A|B)$ of rings contained between the two rings A and B (where $A \subseteq B$ is a ring extension).

Quite a few subspaces of $\mathrm{Over}(D)$ can be proven to be proconstructible; note that a similar reasoning is presented in [98, Example 2.2].

Corollary 2.112. *Let $D \subseteq K$ be an extension of domains. The following sets are proconstructible in* $\mathrm{Over}(D|K)$*:*

(a) [39, Example 2.11] $\{T \mid T$ is local$\}$.
(b) $\{T \mid T$ is local and its residue field has cardinality at most $n\}$ (with n fixed).
(c) $\{T \mid T$ is a valuation domain$\}$.
(d) $\{T \mid T$ is a pseudo-valuation domain$\}$.
(e) $\{T \mid T$ is seminormal$\}$ [i.e., if x is in the quotient field of T and $x^2, x^3 \in T$ then $x \in T$].
(f) [36, Proposition 3.6] $\{T \mid T$ is integrally closed in $K\}$.
(g) $\{T \mid T$ is integrally closed in its quotient field$\}$.

Proof. In all the cases, let $\mathscr{U}$ be an ultrafilter on X. To ease the notation, whenever it does not lead to confusion, we denote the subbasic open sets $B_x \cap X$ of X simply as B_x.

(a) Let $a, b \in A_{\mathscr{U}}$, and suppose that neither a nor b are invertible in $A_{\mathscr{U}}$; to show that $A_{\mathscr{U}}$ is local we need to show that $a + b$ is not invertible in $A_{\mathscr{U}}$. By definition of $A_{\mathscr{U}}$, we have $B_a, B_b \in \mathscr{U}$ and $B_{a^{-1}}, B_{b^{-1}} \notin \mathscr{U}$; hence, $Z := B_a \cap B_b \cap (B_{a^{-1}})^c \cap (B_{b^{-1}})^c \in \mathscr{U}$.

If T is a local ring in Z, then both a and b are non-invertible in T, and so $a+b$ is not-invertible; it follows that $Z \subseteq (B_{(a+b)^{-1}})^c$, and thus the latter set is in $\mathscr{U}$. Hence, $(a+b)^{-1} \notin A_{\mathscr{U}}$, and $A_{\mathscr{U}}$ is local.

(b) By the previous point, $A_{\mathscr{U}}$ is local; denote by $\kappa(T)$ the residue field of the local ring T and by $\mathfrak{m}_T$ its maximal ideal. If $|\kappa(A_{\mathscr{U}})| \geq n+1$, then there are $x_1, \ldots, x_{n+1} \in A_{\mathscr{U}}$ such that $x_i + \mathfrak{m}_{A_{\mathscr{U}}} \neq x_j + \mathfrak{m}_{A_{\mathscr{U}}}$ if $i \neq j$, i.e., such that $x_i - x_j \notin \mathfrak{m}_{A_{\mathscr{U}}}$ whenever $i \neq j$. This means that

$$Z := \bigcap_{i=1}^{n+1} B_{x_i} \cap \bigcap_{i \neq j} B_{(x_i-x_j)^{-1}} \in \mathscr{U};$$

but if we take $T \in Z$, then this would mean that $|\kappa(T)| \geq n+1$, against the hypothesis. Therefore, $|\kappa(A_{\mathscr{U}})| \leq n$, as claimed.

(c) Take $a, b \in A_{\mathscr{U}}$. Then, $Z := B_a \cap B_b \in \mathscr{U}$, and if $T \in Z$ then either $ab^{-1} \in T$ or $a^{-1}b \in T$. Hence, $Z \subseteq B_{ab^{-1}} \cap B_{a^{-1}b} \cap X$, and thus at least one between $B_{ab^{-1}}$ and $B_{a^{-1}b}$ is in $\mathscr{U}$. But this means that $ab^{-1} \in A_{\mathscr{U}}$ or $ba^{-1} \in A_{\mathscr{U}}$, and thus $A_{\mathscr{U}}$ is a valuation ring.

(d) A local ring T is a PVD if and only if, whenever x, y are in the quotient field of T and $xy \in \mathfrak{m}_T$, then $x \in \mathfrak{m}_T$ or $y \in \mathfrak{m}_T$ [56, Theorem 1.4]. Thus, suppose x, y are in the quotient field of $A_{\mathscr{U}}$ and $xy \in \mathfrak{m}_{A_{\mathscr{U}}}$; then, we can find $a, b, c, d \in A_{\mathscr{U}}$ such that $x = ab^{-1}$ and $y \in cd^{-1}$. Hence, we have

$$Z := B_a \cap B_b \cap B_c \cap B_d \cap B_{xy} \cap (B_{(xy)^{-1}})^c \in \mathscr{U};$$

if $T \in Z$, then x and y are in the quotient field of T, and thus $x \in \mathfrak{m}_T$ or $y \in \mathfrak{m}_T$. Hence,

$$Z \subseteq [B_x \cap (B_{x^{-1}})^c] \cup [B_y \cap (B_{y^{-1}})^c],$$

and thus one of the two sets in in $\mathscr{U}$; if it is $B_x \cap (B_{x^{-1}})^c$, then $x \in \mathfrak{m}_{A_{\mathscr{U}}}$, while if it is $B_y \cap (B_{y^{-1}})^c$ then $y \in \mathfrak{m}_{A_{\mathscr{U}}}$. Therefore, $A_{\mathscr{U}}$ is a pseudo-valuation domain.

(e) Suppose $x^2, x^3 \in A_{\mathscr{U}}$. Then, $Z := B_{x^2} \cap B_{x^3} \in \mathscr{U}$, and if $T \in Z$ then $x \in T$; i.e., $Z \subseteq B_x$ and so $B_x \in \mathscr{U}$, i.e., $x \in A_{\mathscr{U}}$.

(f) For each sequence $\mathbf{a} := a_0, \ldots, a_{n-1}$, let $Z(\mathbf{a})$ be the elements x of K such that $a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + x^n = 0$, and let $X(\mathbf{a})$ be the rings T such that either $\mathbf{a} \subsetneq T$ or $\mathbf{a} \subseteq T$ and $x \in T$. Every $X(\mathbf{a})$ is proconstructible: indeed, if $A_{\mathscr{U}}(\mathbf{a})$ denotes the corresponding ring, then either $\mathbf{a} \nsubseteq A_{\mathscr{U}}(\mathbf{a})$ (and so $A_{\mathscr{U}}(\mathbf{a}) \in X(\mathbf{a})$) or $\mathbf{a} \subseteq A_{\mathscr{U}}(\mathbf{a})$, and so $B_{\mathbf{a}} \in \mathscr{U}$. But if $T \in B_{\mathbf{a}} \cap X(\mathbf{a})$ then $x \in T$, and thus $B_{\mathbf{a}} \cap X(\mathbf{a}) \subseteq B_x \cap X(\mathbf{a})$, so that the latter in in $\mathscr{U}$ and $x \in A_{\mathscr{U}}(\mathbf{a})$.

Now $X = \bigcap_{\mathbf{a}} X(\mathbf{a})$, and since the proconstructible subsets are closed set in a topology also X is proconstructible.

(g) It is enough to repeat the previous proof, but using instead of x the couples (x, y) such that $x^n + a_{n-1}x^{n-1}y + \cdots + a_1xy^{n-1} + a_0y^n = 0$, and restrict $\mathbf{a}$ to vary in the quotient field of $A_{\mathscr{U}}$. $\square$

Note that, when K is a field but not the quotient field of D, the set of valuation rings between D and K is *not* the Zariski space of $D|K$, which instead comprises only the valuation domains whose quotient field is K.

We can also tie proconstructibility and semistar operations.

Proposition 2.113. *Let D be an integral domain and let $*$ be a semistar operation of finite type on D. Then, the set of* $\mathrm{Over}^*(D) := \{T \in \mathrm{Over}(D) \mid T = T^*\}$ *is proconstructible in* $\mathrm{Over}(D)$.

Proof. Take an ultrafilter $\mathscr{U}$ on $X := \mathrm{Over}^*(D)$. Let $x \in A_{\mathscr{U}}{}^*$: since $*$ is of finite type there are $f_1, \ldots, f_n \in A_{\mathscr{U}}$ such that $x \in (f_1, \ldots, f_n)^*$. For all $T \in B_{f_1,\ldots,f_n} \cap X$, $x \in T^*$; therefore, $B_{f_1,\ldots,f_n} \cap X \subseteq B_x \cap X$. However, $B_{f_1,\ldots,f_n} \cap X \in \mathscr{U}$, and thus so does $B_x \cap X$; by definition, $x \in A_{\mathscr{U}}$, and $A_{\mathscr{U}} = A_{\mathscr{U}}{}^* \in X$, as requested. □

2.3.4.1. The proconstructibility of compact subspaces

Every proconstructible subspace is compact in the Zariski topology, since it is compact in the constructible topology, and the latter is finer than the Zariski topology. On the other hand, sometimes we can prove that a subspace X is proconstructible, but only with the additional hypothesis that X is compact. The following is the simplest case.

Proposition 2.114. *Let $D \subseteq K$ be a ring extension, and let $X \subseteq \mathrm{Over}(D|K)$ be a subspace such that $X = X^{\uparrow}$ (i.e., if $T \in X$ and $T \subseteq T'$, then $T' \in X$). Then, X is proconstructible if and only if it is compact.*

Proof. The necessity of compactness is clear. Suppose that X is compact: since X is, by hypothesis, closed by generizations (in the order induced by the Zariski topology, which is the opposite of the containment), it follows that X is closed in the inverse topology. But the inverse topology is weaker than the constructible topology, and thus X is proconstructible. □

Corollary 2.115. *Let D be an integral domain. The following subsets are proconstructible in* $\mathrm{Over}(D)$ *if and only if they are compact:*

(a) $\{T \mid T$ *is a Prüfer domain*$\}$.
(b) $\{T \mid T$ *is a valuation domain such that* $\dim(T) \leq n\}$ *(for any fixed n).*
(c) $\{T \mid T$ *is a Prüfer domain such that* $\dim(T) \leq n\}$ *(for any fixed n).*
(d) $\{T \mid \dim_v(T) \leq n\}$ *(for any fixed n).*
(e) $\{T \mid$ *every finitely generated ideal of T is generated by at most n elements*$\}$ *(for any fixed n).*
(f) $\{T \mid T$ *is a Bézout domain*$\}$.
(g) $\{T \mid T$ *is a discrete valuation ring*$\}$.
(h) $\{T \mid T$ *is a principal ideal domain*$\}$.
(i) $\{T \mid T$ *is a Dedekind domain*$\}$.
(j) $\{T \mid T$ *is a Noetherian domain of dimension at most* $1\}$.

Proof. We have to check that each of the sets verifies the hypothesis of Proposition 2.114. (a) and (c) follow from [50, Theorems 26.1 and 26.2], while (b) follows from [50, Theorem 17.6]; (d) is a direct consequence of the definition of valutative dimension (see the paragraph before Corollary 2.106).

For (e), suppose every finitely generated ideal of D is generated by n elements, and let $T \in \mathrm{Over}(D)$. If I is finitely generated over D, say $I = i_1T + \cdots + i_kT$, then

$J := i_1D + \cdots + i_kD$ is a fractional ideal of D, and thus it is generated by n elements: say $J = j_1D + \cdots + j_nD$. Then, $I = j_1T + \cdots + j_nT$ is generated over T by n elements. (f) is simply the previous point for $n = 1$.

(g) follows from [50, Theorem 17.6] and the definition of discrete valuation ring; likewise, [50, Theorems 26.1 and 26.2] and point (f) imply (h) and (i). (j) is implied by [77, Theorem 93]. □

Note that some of these subsets can indeed be noncompact – for example, as we saw in Proposition 2.100, if D is Noetherian and $\dim(D) \geq 2$, then the set of discrete valuation overrings of D is not compact.

With the same hypotheses of Proposition 2.114, we get another spectral space.

Proposition 2.116. *Let $D \subseteq K$ be an extension of domains, and let $X \subseteq \mathrm{Over}(D|K)$ be a compact set such that $X = X^{\uparrow}$. Then, the set*

$$X^{\cap} := \left\{ \bigcap \{T \mid T \in Z\} \mid Z \subseteq X \right\}$$

is proconstructible in $\mathrm{Over}(D|K)$.

Proof. Let $\mathbf{F}(D|K)$ be the set of D-submodules of K. Let $*$ be the map

$$\begin{aligned} * : \mathbf{F}(D|K) &\longrightarrow \mathbf{F}(D|K) \\ I &\mapsto \bigcap_{S \in X} IS. \end{aligned}$$

(We are defining a generalization of the concept of semistar operation.) Note that $*$ is well-defined since the fact that K is a ring implies $K \cdot K = K$, and thus $IS \subseteq K$ for all $I \in \mathbf{F}(D|K)$, $S \in X$.

We proceed in three steps:

Step 1: for every $I \in \mathbf{F}(D|K)$, and each $x \in I^*$, there is a finitely generated module $J \subseteq I$ such that $x \in J^*$.

Step 2: $\mathrm{Over}^*(D|K) := \{T \in \mathrm{Over}(D|K) \mid T = T^*\}$ is proconstructible.

Step 3: $X^{\cap} = \mathrm{Over}^*(D|K)$.

Step 1 can be proved in a manner completely analogous to Proposition 2.70, while to prove Step 2 it is enough to repeat the proof of Proposition 2.113.

For Step 3, suppose $T \in X^{\cap}$. Then, $T = \cap_{S \in Z} S$ for some $Z \subseteq X$; for all $S \in Z$, $T \subseteq S$, and thus $TS = S$. Hence,

$$T^* = \bigcap_{S \in X} TS \subseteq \bigcap_{S \in Z} TS = \bigcap_{S \in Z} S = T;$$

therefore, $T \in \mathrm{Over}^*(D|K)$.

Conversely, if $T \in \mathrm{Over}^*(D|K)$, then $T = \bigcap_{S \in X} TS$; however, each TS is a ring contained in K, and $S \subseteq TS$, together with $X = X^{\uparrow}$, implies that $TS \in X$. Hence, $T \in X^{\cap}$. □

The following can be seen as a companion of Corollary 2.112(b). Denote by $\kappa(T)$ the residue field of the local domain T. Note also that, until the end of this section, for the sake of clarity, we will use $B(F)$ to denote the open set B_F.

Proposition 2.117. *Let $D \subseteq K$ be a ring extension, and denote by* $\mathrm{LocOver}(D|K)$ *the set of local rings in* $\mathrm{Over}(D|K)$*. Each of the following subspaces is spectral whenever it is compact:*

(a) $\{T \in \mathrm{LocOver}(D|K) : |\kappa(T)| \geq n\}$*;*
(b) $\{T \in \mathrm{LocOver}(D|K) : |\kappa(T)| = n\}$*;*
(c) $\{T \in \mathrm{LocOver}(D|K) : |\kappa(T)| = \infty\}$*.*

Proof. Let $X := \{T \in \mathrm{LocOver}(D|K) : |\kappa(T)| \geq n\}$ and denote by $\mathscr{U}$ an ultrafilter on X. For every $T \in X$, there are elements $x_1, \dots, x_n$ of T such that $x_1 + \mathfrak{m}_T, \dots, x_n + \mathfrak{m}_T$ are different in $\kappa(T)$ ($\mathfrak{m}_T$ denotes the maximal ideal of T); in particular, $(x_i - x_j)^{-1} \in T$ for every $i \neq j$. Let $\Omega(T) := B(x_1, \dots, x_n, \{(x_i - x_j)^{-1}\}_{i \neq j})$; then, $\{\Omega(T) \mid T \in X\}$ is an open cover of X, and by compactness there is a $\widehat{T} \in X$ such that $\Omega(\widehat{T}) \in \mathscr{U}$, that is, $x_1, \dots, x_n, (x_i - x_j)^{-1} \in A_{\mathscr{U}}$ for every $i \neq j$. But since $A_{\mathscr{U}}$ is local (being $\mathrm{LocOver}(D|K)$ a proconstructible subspace of $\mathrm{Over}(D|K)$), this means that $x_1 + \mathfrak{m}_{A_{\mathscr{U}}}, \dots, x_n + \mathfrak{m}_{A_{\mathscr{U}}}$ are different in $\kappa(A_{\mathscr{U}})$, and thus $|\kappa(A_{\mathscr{U}})| \geq n$, i.e., $A_{\mathscr{U}} \in X$.

The case $|\kappa(T)| = n$ follows with the same proof, but using $\{T \in \mathrm{LocOver}(D|K) : |\kappa(T)| \leq n\}$ instead of $\mathrm{LocOver}(D|K)$ (note that the former is proconstructible by Corollary 2.112(b)). Analogously, the case $|\kappa(T)| = \infty$ follows with the same reasoning, but using an arbitrary N instead of n. $\square$

Remark 2.118. (1) In the third case of the above proposition, we cannot use the fact that

$$\{T \in \mathrm{LocOver}(D|K) : |\kappa(T)| = \infty\} = \bigcap_n \{T \in \mathrm{LocOver}(D|K) : |\kappa(T)| \geq n\},$$

since we are supposing that the left hand side is compact, but we do not know whether the subspaces on the right hand side are compact or not.

(2) Unlike the previous cases, the set $X := \{T \in \mathrm{LocOver}(D|K) : |\kappa(T)| < \infty\}$ may be compact but not spectral. Indeed, if $T = \mathbb{Z}$, $K = \mathbb{Q}$, then $X = \{D_M \mid M \in \mathrm{Max}(\mathbb{Z})\} \simeq \mathrm{Max}(\mathbb{Z})$, which is compact but not spectral (being 0-dimensional but not Hausdorff).

Sometimes, not even assuming compactness suffices to show that a subspace is proconstructible, but we can still prove it if we add more hypoteses. We premise a lemma that will be of use also in Section 2.5.

Lemma 2.119. *Let $Y \subseteq X$ be spectral spaces. Suppose that there is a subbasis $\mathcal{B}$ of X such that, for every $B \in \mathcal{B}$, both B and $B \cap Y$ are compact. Then, Y is a proconstructible subset of X.*

Proof. The hypothesis on $\mathcal{B}$ implies that the inclusion map $Y \hookrightarrow X$ is a spectral map; by [25, 1.9.5(vii)], it follows that Y is a proconstructible subset of X. $\square$

Compare the following result with Proposition 2.100 and Corollary 2.115(j).

Proposition 2.120. *Let D be a Noetherian domain. The spaces*

- $\mathrm{NoethOver}(D) := \{T \in \mathrm{Over}(D) \mid T \text{ is Noetherian}\}$*, and*
- $\mathrm{KrullOver}(D) := \{T \in \mathrm{Over}(D) \mid T \text{ is a Krull domain}\}$

are spectral if and only if $\dim(D) = 1$.

Proof. If $\dim(D) = 1$, then $\mathrm{NoethOver}(D)$ coincides with $\mathrm{Over}(D)$ [77, Theorem 93], while $\mathrm{KrullOver}(D)$ coincides with $\mathrm{Over}(\overline{D})$ (where $\overline{D}$ is the integral closure of D), and both these spaces are spectral.

Suppose now $\dim(D) \geq 2$, let X be $\mathrm{NoethOver}(D)$ or $\mathrm{KrullOver}(D)$, and let $\mathcal{B}$ be the canonical subbasis of $\mathrm{Over}(D)$. For every $B \in \mathcal{B}$, we claim that $X \cap B$ is compact. Indeed, $X \cap B$ has always a minimum: explicitly, the minimum of $\mathrm{NoethOver}(D) \cap B$ is $D[x_1, \dots, x_n]$ (which is Noetherian since so is D) while the minimum of $\mathrm{KrullOver}(D) \cap B$ is the integral closure of $D[x_1, \dots, x_n]$.

Hence, if X were spectral, then by Lemma 2.119 it would be a proconstructible subspace of $\mathrm{Over}(D)$; hence, so would be $X \cap \mathrm{Zar}(D)$. However, for both choices of X, $X \cap \mathrm{Zar}(D)$ is the set of the discrete valuation overrings of D, that by Proposition 2.100 is not compact (since D is Noetherian with $\dim(D) \geq 2$), and thus not proconstructible. Therefore, $\mathrm{NoethOver}(D)$ and $\mathrm{KrullOver}(D)$ cannot be spectral, as claimed. $\square$

Recall that the *Picard group* of a domain is the quotient between the group of invertible ideals and the subgroup of principal ideals.

Proposition 2.121. *Let $D \subseteq K$ be an extension of domains, and let* $\mathrm{Pic}(R)$ *denote the Picard group of a domain R. In each of the following cases, X is proconstructible whenever $X \cap B(x_1, \dots, x_n)$ is compact for every $x_1, \dots, x_n \in K$.*

(a) $X = \{T \mid \mathrm{Pic}(T) \text{ is torsion}\}$.
(b) $X = \{T \mid n \cdot \mathrm{Pic}(T) = 0\}$ *(for any fixed n).*
(c) $X = \{T \mid \mathrm{Pic}(T) = 0\}$.
(d) $X = \{T \mid \cdot n : \mathrm{Pic}(T) \longrightarrow \mathrm{Pic}(T) \text{ is injective}\}$ *(for any fixed n).*

Proof. (a) Let $\mathscr{U}$ be an ultrafilter on $X := \{T \mid \mathrm{Pic}(T) \text{ is torsion}\}$. Let $(x_1, \dots, x_n)A_{\mathscr{U}}$ be an invertible ideal of $A_{\mathscr{U}}$; we can suppose, without loss of generality, that $x_1, \dots, x_n \in A_{\mathscr{U}}$. There are $y_1, \dots, y_m \in Q(A_{\mathscr{U}})$ such that $(y_1, \dots, y_m)A_{\mathscr{U}}$ is the inverse of $(x_1, \dots, x_n)A_{\mathscr{U}}$; that is, there are $\lambda_{ij} \in A_{\mathscr{U}}$ such that

$$\begin{cases} x_iy_j \in A_{\mathscr{U}} & \text{for all } i, j \\ 1 = \sum_{i,j} \lambda_{ij}x_iy_j. \end{cases}$$

Let $Z := B(\{x_iy_j, \lambda_{ij} : i, j\}) \cap X$; then, $Z \in \mathscr{U}$, and $[(x_1, \dots, x_n)T] \cdot [(y_1, \dots, y_m)T] = T$ for all $T \in Z$, and in particular $(x_1, \dots, x_n)T$ is invertible in T. Thus, for all

$T \in Z$ there is an integer N_T and a μ_T such that $(x_1,\dots,x_n)^{N_T}T = \mu_T T$, i.e., $T = \mu_T^{-1}(x_1,\dots,x_n)^{N_T}T$. It follows that

$$\begin{cases} y_T^{-1}\xi \in T & \text{for all monomial } \xi \text{ in the } x_i \text{ of degree } N_T \\ 1 = \sum_\xi \nu_\xi^{(T)}\xi y_T^{-1} & \text{for some } \nu_\xi^{(T)} \in T. \end{cases}$$

Let $\Omega(T) := B(\{y_T^{-1}\xi, \nu_\xi^{(T)} : \xi\})$; then, $\{\Omega(T) \mid T \in Z\}$ is an open cover of Z. By hypothesis, Z is compact, and so there is a finite subcover; thus, there is a $\widehat{T}$ such that $\Omega(\widehat{T}) \in \mathscr{U}$. In particular, $(x_1,\dots,x_n)^{N_{\widehat{T}}}A_\mathscr{U} = y_{\widehat{T}}A_\mathscr{U}$. Since $(x_1,\dots,x_n)A_\mathscr{U}$ was arbitrary, it follows that $\mathrm{Pic}(A_\mathscr{U})$ is torsion, i.e., $A_\mathscr{U} \in X$ and X is proconstructible.

(b) follows using n instead of the N_T, and (c) follows by taking $n = 1$.

(d) Take $X, \mathscr{U}$ as usual. Let $I = (i_1,\dots,i_k)A_\mathscr{U}$ be an invertible ideal of $A_\mathscr{U}$ such that $I^n = \mu A_\mathscr{U}$ is principal. Then, $\mu = \sum_a \lambda_a \mathbf{i}_a$, as $\mathbf{i}_a$ ranges between the monomial of degree n in $i_1,\dots,i_k$; let $Z := B(\lambda_a, \mu^{-1}\mathbf{i}) \cap X$. For all $T \in Z$, $(IT)^n = \mu T$, and in particular IT is invertible; by injectivity, $I = \mu_T T$ for some μ_T. Therefore, there are $\tau_l^{(T)}$ such that $\mu_T = \sum_l \tau_l^{(T)} i_l$; taking $\Omega(T) := B(\tau_l^{(T)}, \mu_T^{-1}i_l) \cap X$, we get an open cover of Z, and compactness implies, as before, that $\Omega(\widehat{T}) \in \mathscr{U}$ for some $\widehat{T}$. In particular, $\tau_l^{(\widehat{T})}, \mu_{\widehat{T}}^{-1}i_l \in A_\mathscr{U}$ and $I = \mu_{\widehat{T}}A_\mathscr{U}$ is principal. □

The previous proposition can in particular be applied to rings of algebraic integers.

Corollary 2.122. *The set of integral closures of $\mathbb{Z}$ in algebraic extensions of $\mathbb{Q}$ with torsion class group is a spectral space.*

Proof. Let X be the set under consideration. Let Y be the subspace of $\mathrm{Over}(\mathbb{Z}|\mathbb{A})$ (where $\mathbb{A}$ is the set of all algebraic integers) that are integrally closed in their quotient field; Y is proconstructible by Corollary 2.112. Then, X is the subspace of Y composed of the rings of torsion class group; reasoning as in the above proposition we have that X is proconstructible in $\mathrm{Over}(\mathbb{Z}|\mathbb{A})$ provided that $B(x_1,\dots,x_n) \cap X$ is compact for every $x_1,\dots,x_n \in \mathbb{A}$. However, $B(x_1,\dots,x_n) \cap X$ contains the integral closure R of $\mathbb{Z}$ in $\mathbb{Q}(x_1,\dots,x_n)$ (since, by well-known results, $\mathrm{Pic}(R)$ is finite, and in particular a torsion group – see e.g. [82, Chapter V, §1 and Chapter VI, §1] or [111, Theorem 9.7]); R is the minimum of $B(x_1,\dots,x_n) \cap X$, that hence is compact. Therefore, X is proconstructible and thus spectral. □

Proposition 2.123. *Let $D \subseteq K$ be an extension of rings, and let X be the set of elements of* $\mathrm{Over}(D|K)$ *whose prime spectrum is totally ordered. If $X \cap B(x,y)$ is compact for every $x, y \in K$, then X is proconstructible.*

Proof. For any ring R, $\mathrm{Spec}(R)$ is totally ordered if and only if $\mathrm{rad}(xR)$ and $\mathrm{rad}(yR)$ are comparable for every $x, y \in R$ (if P and Q are not comparable, $x \in P \setminus Q$, $y \in Q \setminus P$ then $\mathrm{rad}(xR)$ and $\mathrm{rad}(yR)$ are not comparable). Consider thus an ultrafilter $\mathscr{U}$ on X and let $x, y \in A_\mathscr{U}$, $Z := X \cap B(x,y)$. If $x = 0$ or $y = 0$ the claim is obvious. Otherwise, let $T \in Z$, and denote by $U(T)$ the set of invertible elements of T. We distinguish four cases:

1. $x \in U(T)$: then, $T \in B(x^{-1}) =: \Omega(T)$;
2. $y \in U(T)$: then, $T \in B(y^{-1}) =: \Omega(T)$;
3. $x, y \notin U(T)$ and $\mathrm{rad}(xR) \subseteq \mathrm{rad}(yR)$: there is a $n_T \in \mathbb{N}$ such that $x^{n_T} \in yR$, i.e., $T \in B(y^{-1}x^{n_T}) =: \Omega(T)$;
4. $x, y \notin U(T)$ and $\mathrm{rad}(yR) \subseteq \mathrm{rad}(xR)$: as above, there is a $m_T \in \mathbb{N}$ such that $T \in B(x^{-1}y^{m_T}) =: \Omega(T)$.

Then, $\{\Omega(T) \mid T \in Z\}$ is an open cover of Z, and thus admits a finite subcover; hence, at least one among $B(x^{-1})$, $B(y^{-1})$, $B(y^{-1}x^n)$ and $B(x^{-1}y^m)$ is in $\mathscr{U}$ (for some $n, m \in \mathbb{N}$). In the first two cases, $x \in U(A_{\mathscr{U}})$ or $y \in U(A_{\mathscr{U}})$; in the latter two, $x \in y^m R$ or $y \in x^n R$, that is, $x \in \mathrm{rad}(yA_{\mathscr{U}})$ or $y \in \mathrm{rad}(xA_{\mathscr{U}})$. Hence, $A_{\mathscr{U}} \in X$ and X is proconstructible. $\square$

Proposition 2.124. *Let $D \subseteq K$ be an extension of rings, and let $X_n := \{T \in \mathrm{Over}(D|K) \mid \dim(T) \leq n\}$. If, for every $x_0, x_1, \ldots, x_n \in K$, the subspace $X_n \cap B(x_0, x_1, \ldots, x_n)$ is compact, then X_n is proconstructible.*

Proof. We will use the elementary characterization proved in [22, 21]: given a ring T, the dimension of T is at most n if and only if there are $a_0, \ldots, a_n \in T$, $m_0, \ldots, m_n \in \mathbb{N}$ such that

$$x_0^{m_0}\left(x_1^{m_1}\left(\cdots\left(x_n^{m_n}(1 + a_n x_n)\right)\cdots + a_1 x_1\right) + a_0 x_0\right) = 0. \tag{2.6}$$

Suppose the hypothesis hold, take an ultrafilter $\mathscr{U}$ on X_n and let $x_0, \ldots, x_n \in A_{\mathscr{U}}$. For every $T \in Z := X_n \cap B(x_0, \ldots, x_n)$, we can find appropriate $a_0^{(T)}, \ldots, a_n^{(T)} \in T$, $m_0^{(T)}, \ldots, m_n^{(T)} \in \mathbb{N}$; let $\Omega(T) := B(a_0^{(T)}, \ldots, a_n^{(T)}) \cap X_n$. Then, $\{\Omega(T) \mid T \in Z\}$ is an open cover of Z, that by compactness admits a finite subcover, and thus $\Omega(\widehat{T}) \in \mathscr{U}$ for some $\widehat{T}$. If $a_i := a_i^{(\widehat{T})}$, it follows that $a_0, \ldots, a_n \in A_{\mathscr{U}}$; therefore, condition (2.6) holds for $A_{\mathscr{U}}$, and $A_{\mathscr{U}} \in X_n$, i.e., X_n is proconstructible. $\square$

2.4. The space $\boldsymbol{\mathcal{X}}(X)$

The starting point of this section is the topological characterization of the set $\mathrm{SStar}_{f,sp}(D)$ given in Proposition 2.42: $\mathrm{SStar}_{f,sp}(D)$ is in bijective correspondence with the set $\boldsymbol{\mathcal{X}}(D)$ of the nonempty subsets of $\mathrm{Spec}(D)$ that are closed in the inverse topology. The latter construction can be considered also when D is not a domain, but an arbitrary ring. Moreover, since this is a purely topological construction, instead of working with $\mathrm{Spec}(R)$ we will work directly with a spectral space X.

Let thus X be a spectral space, and denote by $\boldsymbol{\mathcal{X}}(X)$ the set of nonempty subsets of X that are closed in the inverse topology. Endow $\boldsymbol{\mathcal{X}}(X)$ with the topology (which we call the *Zariski topology*) whose subbasic open subsets are those in the form

$$\boldsymbol{\mathcal{U}}(\Omega) := \{Y \in \boldsymbol{\mathcal{X}}(X) \mid Y \subseteq \Omega\},$$

as Ω ranges among the open and compact subsets of X (in the given spectral topology). If $X = \mathrm{Spec}(R)$, then we denote $\boldsymbol{\mathcal{X}}(\mathrm{Spec}(R))$ by $\boldsymbol{\mathcal{X}}(R)$; in this case, the subbasic open sets have the form

$$\boldsymbol{\mathcal{U}}(\mathcal{D}(I)) := \{Y \in \boldsymbol{\mathcal{X}}(R) \mid Y \subseteq \mathcal{D}(I)\}$$

as I ranges among the finitely generated ideals of R.

The following proposition contains the basic properties of the topological space $\boldsymbol{\mathcal{X}}(X)$.

Proposition 2.125. *Let X be a spectral space, and endow $\boldsymbol{\mathcal{X}}(X)$ with the Zariski topology.*

(a) For any open and compact subset Ω of X, $\Omega \in \boldsymbol{\mathcal{U}}(\Omega)$.
(b) $\{\boldsymbol{\mathcal{U}}(\Omega) \mid \Omega$ is open and compact in $X\}$ is a basis of $\boldsymbol{\mathcal{X}}(X)$.
(c) Let $Y_1, Y_2 \in \boldsymbol{\mathcal{X}}(X)$. Then, $Y_2 \in \mathrm{Cl}_{\boldsymbol{\mathcal{X}}(X)}(Y_1)$ if and only if $Y_1 \subseteq Y_2$.
(d) $\boldsymbol{\mathcal{X}}(X)$ is a T_0 space.
(e) $\boldsymbol{\mathcal{X}}(X)$ has a unique closed point, namely X itself.

Proof. (a) follows directly from the fact that an open and compact subset of X is closed in the inverse topology; to prove (b) it is enough to note that

$$\begin{aligned}\boldsymbol{\mathcal{U}}(\Omega_1) \cap \boldsymbol{\mathcal{U}}(\Omega_2) &= \{Y \in \boldsymbol{\mathcal{X}}(X) \mid Y \subseteq \Omega_1\} \cap \{Y \in \boldsymbol{\mathcal{X}}(X) \mid Y \subseteq \Omega_2\} = \\ &= \{Y \in \boldsymbol{\mathcal{X}}(X) \mid Y \subseteq \Omega_1 \cap \Omega_2\} = \boldsymbol{\mathcal{U}}(\Omega_1 \cap \Omega_2),\end{aligned}$$

which is open in $\boldsymbol{\mathcal{X}}(X)$ since, being X a spectral space, $\Omega_1 \cap \Omega_2$ is still an open and compact subset of X.

(c) Let $Y_1, Y_2 \in \boldsymbol{\mathcal{X}}(X)$. If $Y_1 \subseteq Y_2$, then Y_1 belongs to every basic open set $\boldsymbol{\mathcal{U}}(\Omega)$ that contains Y_2; therefore, if $Y_1 \in \boldsymbol{\mathcal{X}}(X) \setminus \boldsymbol{\mathcal{U}}(\Omega)$ then $Y_2 \in \boldsymbol{\mathcal{X}}(X) \setminus \boldsymbol{\mathcal{U}}(\Omega)$, and $Y_2 \in \mathrm{Cl}_{\boldsymbol{\mathcal{X}}(X)}(Y_1)$.

Conversely, if $Y_1 \nsubseteq Y_2$, there is an $x \in Y_1 \setminus Y_2$; consider the open and compact set $\mathcal{D}(x) := X \setminus \mathrm{Cl}_X(x)$. Then, $Y \in \mathcal{D}(x)$ if and only if $x \notin Y$, and thus $\boldsymbol{\mathcal{U}}(\mathcal{D}(x)) = \{Z \in \boldsymbol{\mathcal{X}}(X) \mid x \notin Z\}$. In particular, $Y_1 \notin \boldsymbol{\mathcal{U}}(\mathcal{D}(x))$ while $Y_2 \in \boldsymbol{\mathcal{U}}(\mathcal{D}(x))$; equivalently, Y_1 is contained in the closed set $\boldsymbol{\mathcal{X}}(X) \setminus \boldsymbol{\mathcal{U}}(\mathcal{D}(x))$ while Y_2 is not. It follows that $Y_2 \notin \mathrm{Cl}_{\boldsymbol{\mathcal{X}}(X)}(Y_1)$.

From (c) the next points follow directly: if $Y_2 \in \mathrm{Cl}_{\boldsymbol{\mathcal{X}}(X)}(Y_1)$ and $Y_1 \in \mathrm{Cl}_{\boldsymbol{\mathcal{X}}(X)}(Y_2)$ then we should have $Y_1 \subseteq Y_2$ and $Y_2 \subseteq Y_1$, i.e., $Y_1 = Y_2$, and thus (d) holds. At the same time, every set in the form $\mathrm{Cl}_{\boldsymbol{\mathcal{X}}(X)}(Y)$ contains X, because $Y \subseteq X$, and $\mathrm{Cl}_{\boldsymbol{\mathcal{X}}(X)}(X) = \{X\}$ because no other Y contains X; hence (e) follows. $\square$

A different way to see point (c) of the above proposition is by saying that the order induced by the Zariski topology on $\boldsymbol{\mathcal{X}}(X)$ coincides with the set-theoretic containment.

Theorem 2.126. *Let X be a spectral space. Then, $\boldsymbol{\mathcal{X}}(X)$, endowed with the Zariski topology, is a spectral space.*

Proof. For shortness, let $\boldsymbol{\mathcal{X}} := \boldsymbol{\mathcal{X}}(X)$. By Proposition 2.125(d), $\boldsymbol{\mathcal{X}}$ is T_0. Let

$$\boldsymbol{\mathcal{T}} := \{\boldsymbol{\mathcal{U}}(\Omega) \mid \Omega \text{ is a compact open subspace of } X\}$$

denote the canonical basis of the open sets of $\boldsymbol{\mathcal{X}}$. As in Theorem 2.10, by [36, Corollary 3.3] we have to show that, if $\mathscr{U}$ is an ultrafilter on $\boldsymbol{\mathcal{X}}$, the set

$$\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{T}}}(\mathscr{U}) := \{Y \in \boldsymbol{\mathcal{X}} \mid [\forall \boldsymbol{\mathcal{U}}(\Omega) \in \boldsymbol{\mathcal{T}}, Y \in \boldsymbol{\mathcal{U}}(\Omega) \iff \boldsymbol{\mathcal{U}}(\Omega) \in \mathscr{U}]\}$$

is nonempty.

Denote by $\mathscr{F}(\mathscr{U})$ the set of open and compact subsets Ω of X such that $\boldsymbol{\mathcal{U}}(\Omega) \in \mathscr{U}$; then, $\emptyset \notin \mathscr{F}(\mathscr{U})$ (since $\boldsymbol{\mathcal{U}}(\emptyset) = \emptyset$ and $\mathscr{U}$ is an ultrafilter) and, by the proof of Proposition 2.125(b), if $\Omega_1, \dots, \Omega_n \in \mathscr{F}(\mathscr{U})$ then $\Omega_1 \cap \dots \cap \Omega_n$ is still in $\mathscr{F}(\mathscr{U})$, since

$$\boldsymbol{\mathcal{U}}(\Omega_1 \cap \dots \cap \Omega_n) = \boldsymbol{\mathcal{U}}(\Omega_1) \cap \cdots \boldsymbol{\mathcal{U}}(\Omega_n) \in \mathscr{U},$$

again by properties of ultrafilters. Therefore, $\mathscr{F}(\mathscr{U})$ is a family of open and compact sets in the compact topological space X with the finite intersection property; it follows that the intersection Y_0 of all the elements of $\mathscr{F}(\mathscr{U})$ is nonempty. At the same time, each $\Omega \in \mathscr{F}(\mathscr{U})$ is closed in X^{inv}, and thus Y_0 is closed too. It follows that Y_0 is in fact a member of $\boldsymbol{\mathcal{X}}$, and we claim that $Y_0 \in \boldsymbol{\mathcal{X}}_{\mathcal{T}}(\mathscr{U})$.

Indeed, let $\overline{\Omega}$ be an open and compact subset of X. If $\boldsymbol{\mathcal{U}}(\overline{\Omega}) \in \mathscr{U}$, then $\overline{\Omega} \in \mathscr{F}(\mathscr{U})$ and so $Y_0 \subseteq \overline{\Omega}$, i.e., $Y_0 \in \boldsymbol{\mathcal{U}}(\overline{\Omega})$. Conversely, suppose $Y_0 \in \boldsymbol{\mathcal{U}}(\overline{\Omega})$ and $\boldsymbol{\mathcal{U}}(\overline{\Omega}) \notin \mathscr{U}$. Consider the set

$$\mathscr{C} := \{\Omega \cap (X \setminus \overline{\Omega}) \mid \Omega \in \mathscr{F}(\mathscr{U})\}.$$

Since $\boldsymbol{\mathcal{U}}(\overline{\Omega}) \notin \mathscr{U}$, $\boldsymbol{\mathcal{U}}(\Omega) \nsubseteq \boldsymbol{\mathcal{U}}(\overline{\Omega})$ for every $\Omega \in \mathscr{F}(\mathscr{U})$, that is, $\Omega \nsubseteq \overline{\Omega}$ for every such Ω; hence, every member of $\mathscr{C}$ is nonempty.

Since, for every compact $\Omega_1, \dots, \Omega_n$, we have

$$\bigcap_{i=1}^{n} [\Omega_i \cap (X \setminus \overline{\Omega})] = \left[\bigcap_{i=1}^{n} \Omega_i\right] \cap (X \setminus \overline{\Omega})$$

and $\Omega_1 \cap \dots \cap \Omega_n$ is still open and compact, $\mathscr{C}$ is closed by finite intersection; moreover, every member of $\mathscr{C}$ is closed in the constructible topology; since X, endowed with the constructible topology, is a compact space, we have $Z := \bigcap\{C \mid C \in \mathscr{C}\} \neq \emptyset$. However,

$$Z = \bigcap_{\Omega \in \mathscr{F}(\mathscr{U})} [\Omega \cap (X \setminus \overline{\Omega})] = \left[\bigcap_{\Omega \in \mathscr{F}(\mathscr{U})} \Omega\right] \cap (X \setminus \overline{\Omega}) = Y_0 \cap (X \setminus \overline{\Omega}) = \emptyset$$

since $Y_0 \in \boldsymbol{\mathcal{U}}(\overline{\Omega})$ is equivalent to $Y \subseteq \overline{\Omega}$. This is a contradiction, and thus $\boldsymbol{\mathcal{U}}(\overline{\Omega}) \in \mathscr{U}$. Therefore, $Y_0 \in \boldsymbol{\mathcal{X}}_{\mathcal{T}}(\mathscr{U})$, and $\boldsymbol{\mathcal{X}}(X)$ is a spectral space. □

A fundamental property of $\mathrm{SStar}_f(D)$ is that it contains a homeomorphic copy of the space $\mathrm{Over}(D)$; likewise, $\mathrm{SStar}_{f,sp}(D)$ contains a copy of $\mathrm{Spec}(D)$. In the same way, $\boldsymbol{\mathcal{X}}(X)$ can be thought of as an extension of X, as the following proposition shows. Recall that the *generization* of a point x is the set of point y such that $x \in \mathrm{Cl}(y)$ (see also Section A.1).

Proposition 2.127. *Let X be a spectral space. Let χ be the map*

$$\begin{aligned} \chi\colon X &\longrightarrow \boldsymbol{\mathcal{X}}(X) \\ x &\longmapsto \{x\}^{\mathrm{gen}} = \mathrm{Cl}_X^{\mathrm{inv}}(x). \end{aligned}$$

Then, χ is a topological embedding such that $\chi(X)$ is a dense and proconstructible subset of $\boldsymbol{\mathcal{X}}(X)$.

Proof. Since X^{inv} is a spectral space, it is T_0, and thus χ is injective. Moreover, for every open and compact subset Ω of X,

$$\chi^{-1}(\boldsymbol{\mathcal{U}}(\Omega)) = \{x \in X \mid \mathrm{Cl}_X^{\mathrm{inv}}(x) \subseteq \Omega\} = \Omega$$

so that χ is continuous, and

$$\boldsymbol{\mathcal{U}}(\Omega) \cap \chi(X) = \{\chi(x) \mid \chi(x) \in \boldsymbol{\mathcal{U}}(\Omega)\} = \{\chi(x) \mid x \in \Omega\} = \chi(\Omega),$$

so that χ is a topological embedding. Moreover, the last equation shows that $\chi(X)$ is proconstructible in $\boldsymbol{\mathcal{X}}(X)$ (by Lemma 2.119, since $\chi(X) \simeq X$ is spectral and $\chi(\Omega) \simeq \Omega$ is compact). Finally, the fact that $\chi(X)$ meets every basic open subset $\boldsymbol{\mathcal{U}}(\Omega)$ of $\boldsymbol{\mathcal{X}}(X)$ implies that $\chi(X)$ is dense in $\boldsymbol{\mathcal{X}}(X)$. □

It is possible to characterize the range of χ.

Proposition 2.128. *Let X be a spectral space, and let $\chi : X \longrightarrow \boldsymbol{\mathcal{X}}(X)$ be the topological embedding defined in Proposition 2.127; let $C \in \boldsymbol{\mathcal{X}}(X)$.*

(a) $C \in \chi(X)$ if and only if C is an irreducible set of X^{inv}.
(b) χ is surjective if and only if X is linearly ordered (in the order induced by the topology).

Proof. (a) If $C \in \chi(X)$, then $C = \chi(x) = \{x\}^{\mathrm{gen}}$ is irreducible in the inverse topology. Conversely, if C is irreducible in X^{inv}, then (since X^{inv} is spectral) $C = \{x\}^{\mathrm{gen}}$ for some $x \in X$, i.e., $C = \chi(x)$.

(b) By the point above, χ is surjective if and only if every closed subspace of X^{inv} is the closure of a single point. If X is linearly ordered, this is obvious; if X is not linearly ordered, then there are $x_1, x_2 \in X$ that are not comparable, and thus $\{x_1\}^{\mathrm{gen}} \cup \{x_2\}^{\mathrm{gen}} = \mathrm{Cl}^{\mathrm{inv}}(\{x_1, x_2\})$ is a non-irreducible closed subset of X^{inv}. □

We next find the dimension of $\boldsymbol{\mathcal{X}}(X)$.

Proposition 2.129. *Let X be a spectral space. Then,* $\dim \boldsymbol{\mathcal{X}}(X) = |X| - 1 \geq \dim(X)$.

Proof. Let $x_1, \ldots, x_n$ be different elements of X; we can order them in such a way that $x_i \not\geq x_j$ if $i < j$. For each k, consider the subset

$$Y_k := \{x_1\}^{\mathrm{gen}} \cup \cdots \cup \{x_k\}^{\mathrm{gen}} = \mathrm{Cl}^{\mathrm{inv}}(\{x_1, \ldots, x_k\}).$$

(Notice that the finiteness of $\{x_1, \ldots, x_k\}$ guarantees that the last equality holds.) Clearly, $Y_i \subseteq Y_j$ if $i \leq j$; we claim that, if $i \neq j$, then $Y_i \neq Y_j$. Indeed, suppose $i < j$, and consider x_j. If $Y_i = Y_j$, then $x_j \in Y_i$, that is, $x_j \in \{x_t\}^{\mathrm{gen}}$ for some $t \in \{1, \ldots, j\}$, and so $x_j \leq x_t$. However, this contradicts our ordering of the x_i, and thus $x_j \in Y_j \setminus Y_i$. Hence, $Y_1 \subsetneq Y_2 \subsetneq \cdots \subsetneq Y_n$ is a chain in $\boldsymbol{\mathcal{X}}(X)$, and thus $\dim \boldsymbol{\mathcal{X}}(X) \geq n - 1$. If X is finite, it follows that $\dim \boldsymbol{\mathcal{X}}(X) \geq |X| - 1$; if X is infinite, it follows that $\dim \boldsymbol{\mathcal{X}}(X) \geq n - 1$ for every $n \in \mathbb{N}$, and so $\dim \boldsymbol{\mathcal{X}}(X) = \infty = |X| - 1$.

Suppose X is finite and consider a chain $Y_0 \subsetneq \cdots \subsetneq Y_n$ in $\boldsymbol{\mathcal{X}}(X)$. Then, $|Y_i| > |Y_j|$ if $i > j$; in particular, $|Y_n| \geq |Y_0| + n \geq n + 1$. Now $\boldsymbol{\mathcal{X}}(X)$ is finite (being a subset of the power set of X, which is finite) and thus admits a maximal chain of length $\dim \boldsymbol{\mathcal{X}}(X)$, whose maximal element is X; hence, $|X| \geq \dim \boldsymbol{\mathcal{X}}(X) + 1$, i.e., $\dim \boldsymbol{\mathcal{X}}(X) \leq |X| - 1$. Coupling this with the result in the previous paragraph, we have $\dim \boldsymbol{\mathcal{X}}(X) = |X| - 1$ in every case.

The inequality $|X| - 1 \geq \dim(X)$ is obvious. □

Before turning to the study of the functorial properties of $\boldsymbol{\mathcal{X}}$, we give a topological version of Proposition 2.42, checking that the Zariski topology we put on $\boldsymbol{\mathcal{X}}(X)$ is, in fact, a generalization of the space of spectral semistar operations of finite type.

Proposition 2.130. *Let D be an integral domain. Then, the maps*

$$\begin{aligned} s\colon \boldsymbol{\mathcal{X}}(D) &\longrightarrow \mathrm{SStar}_{f,sp}(D) \\ \Delta &\longmapsto s_\Delta \end{aligned} \quad \textit{and} \quad \begin{aligned} \mathrm{QSpec}\colon \mathrm{SStar}_{f,sp}(D) &\longrightarrow \boldsymbol{\mathcal{X}}(D) \\ * &\longmapsto \mathrm{QSpec}^*(D) \end{aligned}$$

are homeomorphisms, inverse one of each other.

Proof. The fact that s and QSpec are bijective and inverses one of each other follows from Proposition 2.42 and the results before it. For any subbasic open set $U_J := V_J \cap \mathrm{SStar}_{f,sp}(D)$, where J is a finitely generated ideal, we have

$$s^{-1}(U_J) = \{\Delta \mid 1 \in J^{s_\Delta}\} = \{\Delta \mid \Delta \subseteq \mathcal{D}(J)\} = \boldsymbol{\mathcal{U}}(\mathcal{D}(J)),$$

which is an open set of $\boldsymbol{\mathcal{X}}(D)$; moreover, for every basic open set $\boldsymbol{\mathcal{U}}(\mathcal{D}(J))$ of $\boldsymbol{\mathcal{X}}(D)$, $s(\boldsymbol{\mathcal{U}}(\mathcal{D}(J))) = U_J$. Hence, s is continuous and open, and thus a homeomorphism. □

2.4.1. $\boldsymbol{\mathcal{X}}$ as a functor

Recall that a map $\psi : X_1 \longrightarrow X_2$ between spectral space is a *spectral map* if $\psi^{-1}(\Omega)$ is open and compact for every open and compact subset Ω of X_2. Note that a spectral map is always continuous.

Proposition 2.131. *Let X_1, X_2 be spectral spaces, and let $\psi : X_1 \longrightarrow X_2$ be a spectral map. Define $\boldsymbol{\mathcal{X}}(\psi)$ as the map*

$$\begin{aligned} \boldsymbol{\mathcal{X}}(\psi)\colon \boldsymbol{\mathcal{X}}(X_1) &\longrightarrow \boldsymbol{\mathcal{X}}(X_2) \\ C &\longmapsto \psi(C)^{\mathrm{gen}}. \end{aligned}$$

Then, the following properties hold.

(a) $\boldsymbol{\mathcal{X}}(\psi)$ is a well-defined spectral map.
(b) If $\chi_1 : X_1 \longrightarrow \boldsymbol{\mathcal{X}}(X_1)$ and $\chi_2 : X_2 \longrightarrow \boldsymbol{\mathcal{X}}(X_2)$ are the embeddings defined in Proposition 2.127, then $\chi_2 \circ \psi = \boldsymbol{\mathcal{X}}(\psi) \circ \chi_1$.
(c) If ψ is a topological embedding (respectively, a homeomorphism) so is $\boldsymbol{\mathcal{X}}(\psi)$.

Proof. (a) If $C \in \boldsymbol{\mathcal{X}}(X_1)$, then it is compact (in the given topology of X_1), and thus so is $\psi(C)$. Hence, $\psi(C)^{\text{gen}}$ is the closure of $\psi(C)$ in X_2^{inv}, and thus $\boldsymbol{\mathcal{X}}(\psi)(C)$ is a member of $\boldsymbol{\mathcal{X}}(X_2)$.

For spectrality, it is enough to consider a basic subset $\boldsymbol{\mathcal{U}}(\Omega)$ of $\boldsymbol{\mathcal{X}}(X_2)$, for some open and compact $\Omega \subseteq X_2$. We have (since Ω is itself closed in the inverse topology of X_2)

$$\begin{aligned}\boldsymbol{\mathcal{X}}(\psi)^{-1}(\Omega) &= \{C \in \boldsymbol{\mathcal{X}}(X_1) \mid \psi(C)^{\text{gen}} \subseteq \Omega\} = \\ &= \{C \in \boldsymbol{\mathcal{X}}(X_1) \mid \psi(C) \subseteq \Omega\} = \\ &= \{C \in \boldsymbol{\mathcal{X}}(X_1) \mid C \subseteq \psi^{-1}(\Omega)\} = \boldsymbol{\mathcal{U}}(\psi^{-1}(\Omega)),\end{aligned}$$

which is open and compact in $\boldsymbol{\mathcal{X}}(X_1)$ since $\psi^{-1}(\Omega)$ is open and compact in X_1, being ψ spectral.

(b) If $x \in X_1$, then $\chi_2 \circ \psi(x) = \chi_2(\psi(x)) = \psi(x)^{\text{gen}}$, while

$$\boldsymbol{\mathcal{X}}(\psi) \circ \chi_1(x) = \boldsymbol{\mathcal{X}}(\psi)(\{x\}^{\text{gen}}) = (\psi(\{x\}^{\text{gen}}))^{\text{gen}} = \psi(x)^{\text{gen}}.$$

(c) Both χ_1 and χ_2 are embeddings; if so is ψ, then $\chi_2 \circ \psi$ is an embedding, and thus so is $\boldsymbol{\mathcal{X}}(\psi) \circ \chi_1$. Hence, also $\boldsymbol{\mathcal{X}}(\psi)$ must be an embedding.

If ψ is a homeomorphism, then $C = \boldsymbol{\mathcal{X}}(\psi)(\psi^{-1}(C))$ for every $C \in \boldsymbol{\mathcal{X}}(X_2)$, and thus $\boldsymbol{\mathcal{X}}(\psi)$ is surjective. Moreover, any homeomorphism is an embedding, so that $\boldsymbol{\mathcal{X}}(\psi)$ turns out to be a surjective topological embedding, and hence aa homeomorphism. $\square$

To complete the previous proposition, we show that $\boldsymbol{\mathcal{X}}$ is a functor.

Proposition 2.132. *The assignment $X \mapsto \boldsymbol{\mathcal{X}}(X)$, $\psi \mapsto \boldsymbol{\mathcal{X}}(\psi)$ is a covariant functor from the category of spectral spaces and spectral maps to itself.*

Proof. The only missing point is that, if $X_1 \xrightarrow{\psi_1} X_2 \xrightarrow{\psi_2} X_3$ is a chain of spectral maps, then the spectral map $\boldsymbol{\mathcal{X}}(\psi_2 \circ \psi_1) : \boldsymbol{\mathcal{X}}(X_1) \longrightarrow \boldsymbol{\mathcal{X}}(X_3)$ induced by $\psi_2 \circ \psi_1$ is equal to the composition $\boldsymbol{\mathcal{X}}(\psi_2) \circ \boldsymbol{\mathcal{X}}(\psi_1)$. But this follows directly by the definitions. $\square$

Corollary 2.133. *Let D be an integral domain. Then,* $\text{SStar}_{f,val}(D)$ *is homeomorphic to* $\boldsymbol{\mathcal{X}}(\text{Zar}(D))$.

Proof. By Theorem 2.85(b), $\text{SStar}_{f,val}(D)$ is homeomorphic to $\text{SStar}_{f,sp}(\text{Kr}(D))$; by Proposition 2.130, the latter set is homeomorphic to $\boldsymbol{\mathcal{X}}(\text{Kr}(D))$. However, $\text{Spec}(\text{Kr}(D)) \simeq \text{Zar}(D)$ (see Lemma 2.84(d)) and thus $\boldsymbol{\mathcal{X}}(\text{Kr}(D)) \simeq \boldsymbol{\mathcal{X}}(\text{Zar}(D))$; hence, $\text{SStar}_{f,val}(D) \simeq \boldsymbol{\mathcal{X}}(\text{Zar}(D))$. $\square$

Suppose now that X_1 and X_2 are spectral spaces. If X_1 and X_2 are not homeomorphic, the same happens to the spaces X_1^{inv} and X_2^{inv}, since otherwise $(X_1^{\text{inv}})^{\text{inv}}$ and $(X_2^{\text{inv}})^{\text{inv}}$ would be homeomorphic, against the fact that, for an arbitrary spectral space, $(X^{\text{inv}})^{\text{inv}} \simeq X$ (see [30, Proposition 3.1(c)] or [38, Corollary 4.8(4)]). In particular, we cannot obtain a "perfect" correspondence between the closed sets of X_1^{inv} and X_2^{inv}; that is, we do not expect that $\boldsymbol{\mathcal{X}}(X_1)$ and $\boldsymbol{\mathcal{X}}(X_2)$ can be homeomorphic if X_1 and X_2 are not. This is actually the case; before proving it, we study the maps between $\boldsymbol{\mathcal{X}}(X_1)$ and $\boldsymbol{\mathcal{X}}(X_2)$ in relation with the maps between X_1 and X_2.

$$
\begin{tikzcd}
 & & & \mathrm{SStar}(D)^{(\blacklozenge)} & & & \\
 & & \mathrm{SStar}_{eab}(D)^{(\blacklozenge)} \arrow[ur, hook] & & \mathrm{SStar}_{st}(D)^{(\blacklozenge)} \arrow[ul, hook] & & \\
 & & \mathrm{SStar}_{val}(D)^{(\blacklozenge)} \arrow[u, hook] & \mathrm{SStar}_{f}(D) \arrow[uu, hook] & \mathrm{SStar}_{sp}(D)^{(\blacklozenge)} \arrow[u, hook] & & \\
\boldsymbol{\mathcal{X}}(\mathrm{Zar}(D)) & \simeq & \mathrm{SStar}_{f,val}(D) \arrow[u, hook] \arrow[ur, hook] & & \mathrm{SStar}_{f,sp}(D) \arrow[u, hook] \arrow[ul, hook] & \simeq & \boldsymbol{\mathcal{X}}(\mathrm{Spec}(D)) \\
 & & & \mathrm{Over}(D) \arrow[uu, hook] & & & \\
 & & \mathrm{Over}_{\mathrm{ic}}(D) \arrow[ur, hook] & \mathrm{LocOver}(D) \arrow[u, hook] & & & \\
 & & \mathrm{Zar}(D) \arrow[u, hook] \arrow[ur, hook] \arrow[uuull, hook] & \mathrm{Loc}(D) \arrow[u, hook] \arrow[uuur, hook] & \simeq & & \mathrm{Spec}(D) \arrow[uuu, hook]
\end{tikzcd}
$$

Figure 2.2: Diagram of the inclusions between the spaces of overrings and of semistar operations considered. All spaces are spectral, except possibly the spaces denoted with $(\blacklozenge)$.

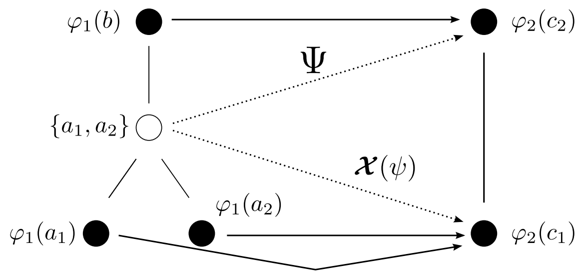


Figure 2.3: The spectral spaces of Example 2.135: black circle represents elements of $\chi_1(X_1)$ and $\chi_2(X_2)$.

Proposition 2.134. *Let X_1, X_2 be spectral spaces, and let $\Psi : \boldsymbol{\mathcal{X}}(X_1) \longrightarrow \boldsymbol{\mathcal{X}}(X_2)$ be a spectral map. If there is a spectral map $\psi : X_1 \longrightarrow X_2$ such that $\chi_2 \circ \psi = \Psi \circ \chi_1$, then $\boldsymbol{\mathcal{X}}(\psi) \leq \Psi$, that is, $\boldsymbol{\mathcal{X}}(\psi)(C) \subseteq \Psi(C)$ for every $C \in \boldsymbol{\mathcal{X}}(X_1)$.*

Proof. Let $C \in \boldsymbol{\mathcal{X}}(X_1)$, and let $c \in C$. Then, $\{c\}^{\text{gen}} \subseteq C$, and thus (since Ψ is continuous, and the containment is the opposite order of the Zariski topology on both $\boldsymbol{\mathcal{X}}(X_1)$ and $\boldsymbol{\mathcal{X}}(X_2)$) it follows that

$$\psi(c)^{\text{gen}} = \chi_2 \circ \psi(c) = \Psi \circ \chi_1(c) = \Psi(\{c\}^{\text{gen}}) \subseteq \Psi(C);$$

hence, $\psi(C) \subseteq \Psi(C)$, and since $\Psi(C)$ is closed in the inverse topology of X_2 we have $\text{Cl}^{\text{inv}}(\psi(C)) \subseteq \Psi(C)$. On the other hand, $\boldsymbol{\mathcal{X}}(\psi)(C) = \psi(C)^{\text{gen}} = \text{Cl}^{\text{inv}}(\psi(C))$; hence $\boldsymbol{\mathcal{X}}(\psi(C)) \subseteq \Psi(C)$, and $\boldsymbol{\mathcal{X}}(\psi) \leq \Psi$. □

A different way to see Proposition 2.134 is by viewing Ψ as an extension of ψ; thus, we can interpret it as saying that $\boldsymbol{\mathcal{X}}(\psi)$ is the minimal extension of ψ. However, it may not be the unique extension, as the next example shows.

Example 2.135. Let $X_1 = \{a_1, a_2, b\}$ and $X_2 := \{c_1, c_2\}$. Suppose that a_1 and a_2 are incomparable but both smaller than b, and suppose also that $c_1 < c_2$. Since X_1 and X_2 are finite, this order structures are compatible with the order of suitable spectral topologies on X_1 and X_2. Direct inspection shows that $\boldsymbol{\mathcal{X}}(X_1) = \{\{a_1\}, \{a_2\}, \{a_1, a_2\}, \{b, a_1, a_2\}\} = \chi_1(X_1) \cup \{\Delta\}$ (where $\Delta := \{a_1, a_2\}$), while $\boldsymbol{\mathcal{X}}(X_2) = \{\{c_1\}, \{c_1, c_2\}\} = \chi(X_2)$. (See Figure 2.3.)

Let $\psi : X_1 \to X_2$ be the spectral map defined by $\psi(a_1) := \psi(a_2) := c_1$ and $\psi(b) := c_2$. Let $\Psi : \boldsymbol{\mathcal{X}}(X_1) \longrightarrow \boldsymbol{\mathcal{X}}(X_2)$ be a map extending ψ; the unique element of $\boldsymbol{\mathcal{X}}(X_1)$ whose image is not determined by ψ is Δ. There are two possible ways, which we call Ψ_1 and Ψ_2, to extend ψ to Δ: the first is by defining $\Psi_1(\Delta) := c_1$, and second by defining $\Psi_2(\Delta) := c_2$. In both cases, Ψ remains an order-preserving map, and thus (since the sets are finite) a spectral map; Ψ_1 is the smallest, and thus is equal to $\boldsymbol{\mathcal{X}}(\psi)$, while Ψ_2 is a different extension.

Proposition 2.136. *Let X_1, X_2 be spectral spaces. If $\Psi : \boldsymbol{\mathcal{X}}(X_1) \longrightarrow \boldsymbol{\mathcal{X}}(X_2)$ is a homeomorphism, then there is a unique homeomorphism $\psi : X_1 \longrightarrow X_2$ such that $\Psi = \boldsymbol{\mathcal{X}}(\psi)$. In particular, X_1 and X_2 are homeomorphic if and only if $\boldsymbol{\mathcal{X}}(X_1)$ and $\boldsymbol{\mathcal{X}}(X_2)$ are homeomorphic.*

Proof. We first show that Ψ restricts to a map $\Psi_0 : \chi_1(X_1) \longrightarrow \chi_2(X_2)$. Indeed, if $C \in \chi_1(X_1)$ then, by Proposition 2.128(a) C is irreducible in X_1^{inv}; we claim that $\Psi(C)$ is irreducible in X_2^{inv}. If not, there are two sets $B_1, B_2 \in \boldsymbol{\mathcal{X}}(X_2)$ such that $\Psi(C) = B_1 \cup B_2$ while $\Psi(C)$ is not contained in both B_1 and B_2. However, Ψ (being a homeomorphism) is an isomorphism of partially ordered set; hence, $C = \Psi^{-1}(\Psi(C)) = \Psi^{-1}(B_1) \cup \Psi^{-1}(B_2)$. Since C is irreducible, it would follow that $C \subseteq \Psi^{-1}(B_1)$ or $C \subseteq \Psi^{-1}(B_2)$, which would imply that $\Psi(C) \subseteq B_1$ or $\Psi(C) \subseteq B_2$. This is a contradiction; therefore, $\Psi(C)$ is irreducible and thus (again by Proposition 2.128(a)) $\Psi(C) \in \chi_2(X_2)$, and Ψ_0 is well-defined.

A symmetric reasoning shows that Ψ^{-1} restricts to a map $\Psi_0^{-1} : \chi_2(X_2) \longrightarrow \chi_1(X_1)$; since clearly Ψ_0 and Ψ_0^{-1} are inverses one of each other, $\psi := \chi_2^{-1} \circ \Psi_0 \circ \chi_1$ is a homeomorphism between X_1 and X_2 (and, in particular, a spectral map).

We need now to show that $\Psi = \boldsymbol{\mathcal{X}}(\psi)$. (Note that the ψ defined this way is the unique map φ that can have this property, since $\boldsymbol{\mathcal{X}}(\varphi)|_{\chi_1(X_1)} = \chi_2 \circ \varphi$.) By construction, they agree on $\chi_1(X_1)$; in particular, $\chi_2 \circ \psi = \Lambda \circ \chi_1$ for both $\Lambda = \Psi$ and $\Lambda = \boldsymbol{\mathcal{X}}(\psi)$. By Proposition 2.134, it follows that $\boldsymbol{\mathcal{X}}(\psi) \leq \Psi$. Note also that ψ can also be thought of as a homeomorphism between X_1^{inv} and X_2^{inv}; in particular, it is a closed map (with respect to the inverse topology) and thus $\boldsymbol{\mathcal{X}}(\psi)(C) = \mathrm{Cl}^{\mathrm{inv}}(\psi(C)) = \psi(C)$ for every $C \in \boldsymbol{\mathcal{X}}(X_1)$.

Suppose $d \in \Psi(C) \setminus \boldsymbol{\mathcal{X}}(\psi)(C) = \Psi(C) \setminus \psi(C)$. Then, $\{d\}^{\mathrm{gen}} \subseteq \Psi(C)$; however,

$$\{d\}^{\mathrm{gen}} = \chi_2(d) = \chi_2 \circ \psi(f) = \Psi \circ \chi_1(f) = \Psi(\{f\}^{\mathrm{gen}})$$

where $f = \psi^{-1}(d)$ does not belong to C (for otherwise $d = \psi(f) \in C$). However, being Ψ a homeomorphism, it is an isomorphism of partially ordered sets; therefore, $\Psi(\{f\}^{\mathrm{gen}}) \subseteq \Psi(C)$ implies $\{f\}^{\mathrm{gen}} \subseteq C$, i.e., $f \in C$. But this is a contradiction; therefore $\Psi(C) = \psi(C)$ and $\Psi = \boldsymbol{\mathcal{X}}(\psi)$.

The last claim follows directly from the rest of the proof. □

Corollary 2.137. *Let D_1, D_2 be two integral domains. Then,* $\mathrm{Spec}(D_1)$ *and* $\mathrm{Spec}(D_2)$ *are homeomorphic if and only if so are* $\mathrm{SStar}_{f,sp}(D_1)$ *and* $\mathrm{SStar}_{f,sp}(D_2)$.

Proof. By Proposition 2.130, $\mathrm{SStar}_{f,sp}(D_i) \simeq \boldsymbol{\mathcal{X}}(\mathrm{Spec}(D_i))$ for $i = 1, 2$. It is then enough to apply Proposition 2.136. □

Note that one implication of Corollary 2.137 could have been proved also without resorting to the terminology of $\boldsymbol{\mathcal{X}}(X)$: indeed, if $\mathrm{Spec}(D_1) \simeq \mathrm{Spec}(D_2)$, there is a bijective correspondence between radical ideals, that preserves whether a radical ideal I can be obtained as a radical of a finitely generated ideal (since this is equivalent to asking if $\mathcal{D}(I)$ is compact). In particular, this preserves the subbasic open sets U_J of $\mathrm{SStar}_{f,sp}(D)$, which implies that $\mathrm{SStar}_{f,sp}(D_1) \simeq \mathrm{SStar}_{f,sp}(D_2)$. However, the use of the construction $\boldsymbol{\mathcal{X}}$ allows to clarify the proof, pushing the technical problems in the background (where they can be dealt with in a topological way).

2.4.2. The space $\boldsymbol{\mathcal{Z}}(X)$

The interpretation of the space $\mathrm{SStar}_{f,sp}(D)$ of finite-type spectral semistar operations as the geometric object $\boldsymbol{\mathcal{X}}(X)$ was based on two facts: firstly, we can represent uniquely finite-type spectral operations by subsets of $\mathrm{Spec}(D)$; secondly, we can interpret the sets obtained in the previous points as closed sets of a topology.

By Proposition 2.29, the first property holds also if we drop the condition of being finitely generated: indeed, given $\Delta, \Lambda \subseteq \mathrm{Spec}(D)$, $s_\Delta = s_\Lambda$ if and only if $\Delta^{\mathrm{gen}} = \Lambda^{\mathrm{gen}}$. Hence, $\mathrm{SStar}_{sp}(D)$ is in bijective correspondence with the subsets of $\mathrm{Spec}(D)$ that are closed by generizations.

Now the family of subsets of $\mathrm{Spec}(D)$ that are closed by generizations is the family of closed sets of a topology, called the *R-topology* (R stands for "right"; the corresponding *L-topology* is the topology whose closed sets are the subsets of $\mathrm{Spec}(D)$ that are closed by specialization) [30]. If X^R (that is, $X := \mathrm{Spec}(D)$ endowed with this topology) were spectral, then $\mathrm{SStar}_{sp}(D)$ would be isomorphic to $\boldsymbol{\mathcal{X}}((X^R)^{\mathrm{inv}})$, which would be a spectral space by Theorem 2.126; however, the spectrality of X^R is equivalent to X being a Noetherian space [30, Corollary 4.3], and in this case, $\mathrm{SStar}_{sp}(D)$ actually coincides with $\mathrm{SStar}_{f,sp}(D)$ (due to the equivalence between compact subsets and finite-type operations – Proposition 2.32), so we don't get anything new.

Still, it may be useful to make explicit a construction similar to $\boldsymbol{\mathcal{X}}(X)$, but corresponding to non-finite type operations. Given a spectral space X, denote by $\boldsymbol{\mathcal{Z}}(X)$ the set of closed sets in the R-topology (that is, the set of subsets of X that are closed by generizations), endowed with the topology (which, again, we call the *Zariski topology*) whose subbasic open sets are those in the form

$$\boldsymbol{\mathcal{U}}_{\boldsymbol{\mathcal{Z}}}(\Omega) := \{Y \in \boldsymbol{\mathcal{Z}}(X) \mid Y \subseteq \Omega\},$$

as Ω ranges among the open (not necessarily compact) subsets of X. If $X = \mathrm{Spec}(R)$ for some ring R, denote $\boldsymbol{\mathcal{Z}}(\mathrm{Spec}(R))$ by $\boldsymbol{\mathcal{Z}}(R)$.

Many of the results proved for $\boldsymbol{\mathcal{X}}(X)$ have an analogue for $\boldsymbol{\mathcal{Z}}(X)$: for example, Proposition 2.125 continues to hold, when the obvious changes are made. Instead of a an embedding $X \longrightarrow \boldsymbol{\mathcal{X}}(X)$, we can embed $\boldsymbol{\mathcal{X}}(X)$ into $\boldsymbol{\mathcal{Z}}(X)$; indeed, we can view $\boldsymbol{\mathcal{X}}(X)$ as a subset of $\boldsymbol{\mathcal{Z}}(X)$ (since a subset closed in the inverse topology is, in particular, closed by generizations); to see that the Zariski topology on $\boldsymbol{\mathcal{X}}(X)$ is the restriction of the Zariski topology on $\boldsymbol{\mathcal{Z}}(X)$, since clearly $\boldsymbol{\mathcal{U}}_{\boldsymbol{\mathcal{X}}}(\Omega) = \boldsymbol{\mathcal{U}}_{\boldsymbol{\mathcal{Z}}}(\Omega) \cap \boldsymbol{\mathcal{X}}(X)$ for every open and compact Ω, we have to show that $\boldsymbol{\mathcal{U}}_{\boldsymbol{\mathcal{Z}}}(\Omega) \cap \boldsymbol{\mathcal{X}}(X)$ is open in $\boldsymbol{\mathcal{X}}(X)$ even if Ω is not compact.

We claim that

$$\boldsymbol{\mathcal{U}}_{\boldsymbol{\mathcal{Z}}}(\Omega) \cap \boldsymbol{\mathcal{X}}(X) = \bigcup\{\boldsymbol{\mathcal{U}}_{\boldsymbol{\mathcal{X}}}(\Omega') \mid \Omega' \subseteq \Omega \text{ and } \Omega' \text{ is open and compact in } X\}.$$

One inclusion is clear; suppose now that $Y \subseteq \Omega$. Since the family of open and compact subsets is a basis for X, in particular there is a family $\mathcal{O} := \{\Omega_\alpha \mid \alpha \in A\}$ such that Ω is the union of the Ω_α; hence, $\mathcal{O}$ is an open cover of Y, and thus $Y \subseteq \Omega_1 \cup \cdots \cup \Omega_n \subseteq \Omega$, where $\Omega_1, \ldots, \Omega_n$ are open and compact. Hence, $\Omega_1 \cup \cdots \cup \Omega_n$ is also open and compact and $Y \in \boldsymbol{\mathcal{U}}(\Omega_1 \cup \cdots \cup \Omega_n)$, as requested.

As we did with $\boldsymbol{\mathcal{X}}$, we can build a functor $\boldsymbol{\mathcal{Z}}$ by defining, for any continuous map $\psi : X_1 \longrightarrow X_2$ between spectral spaces,

$$\begin{aligned} \boldsymbol{\mathcal{Z}}(\psi)\colon X_1 &\longrightarrow X_2 \\ C &\longmapsto \psi(C)^{\mathrm{gen}}; \end{aligned}$$

the proofs of Propositions 2.131 and 2.132 carry over, *mutatis mutandis*, to show that $\boldsymbol{\mathcal{Z}}(\psi)$ is a continuous map that respects embeddings and compositions, and so the assignment $X \mapsto \boldsymbol{\mathcal{Z}}(X)$, $\psi \mapsto \boldsymbol{\mathcal{Z}}(\psi)$ defines a functor from the category of topological spaces to itself. An inspection of the proof of Proposition 2.136 shows also that, if $\Psi : \boldsymbol{\mathcal{Z}}(X_1) \longrightarrow \boldsymbol{\mathcal{Z}}(X_2)$ is a homeomorphism, then $\Psi = \boldsymbol{\mathcal{Z}}(\psi)$ for some continuous map $\psi : X_1 \longrightarrow X_2$.

The last point, in particular, shows that the topology on $\boldsymbol{\mathcal{Z}}(X)$ does not depend exclusively on the space X^R; indeed, if X and Y are spectral spaces whose underlying partial order is isomorphic, then X^R and Y^R are homeomorphic [30, Lemma 2.6], but if X and Y are not homeomorphic then neither are $\boldsymbol{\mathcal{Z}}(X)$ and $\boldsymbol{\mathcal{Z}}(Y)$.

We end this section by showing that the topology on $\boldsymbol{\mathcal{Z}}(X)$ is actually a generalization of the space $\mathrm{SStar}_{sp}(D)$.

Proposition 2.138. *Let D be an integral domain. Then, the maps*

$$\begin{aligned} s\colon \boldsymbol{\mathcal{Z}}(D) &\longrightarrow \mathrm{SStar}_{sp}(D) \\ \Delta &\longmapsto s_\Delta \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \mathrm{QSpec}\colon \mathrm{SStar}_{sp}(D) &\longrightarrow \boldsymbol{\mathcal{Z}}(D) \\ * &\longmapsto \mathrm{QSpec}^*(D) \end{aligned}$$

are homeomorphisms, inverse one of each other.

Proof. The fact that s and QSpec are bijective and inverses one of each other follows from Proposition 2.29. For any subbasic open set $U_J := V_J \cap \mathrm{SStar}_{sp}(D)$, where J is an integral ideal of D, we have

$$s^{-1}(U_J) = \{\Delta \mid 1 \in J^{s_\Delta}\} = \{\Delta \mid \Delta \subseteq \mathcal{D}(J)\} = \boldsymbol{\mathcal{U}}_{\boldsymbol{\mathcal{Z}}}(\mathcal{D}(J)),$$

which is an open set of $\boldsymbol{\mathcal{Z}}(D)$; moreover, for every basic open set $\boldsymbol{\mathcal{U}}_{\boldsymbol{\mathcal{Z}}}(\mathcal{D}(J))$ of $\boldsymbol{\mathcal{Z}}(D)$, $s(\boldsymbol{\mathcal{U}}_{\boldsymbol{\mathcal{Z}}}(\mathcal{D}(J))) = U_J$. Hence, s is continuous and open, and thus a homeomorphism. □

Corollary 2.139. *Let D_1, D_2 be two integral domains. Then,* $\mathrm{Spec}(D_1)$ *and* $\mathrm{Spec}(D_2)$ *are homeomorphic if and only if so are* $\mathrm{SStar}_{sp}(D_1)$ *and* $\mathrm{SStar}_{sp}(D_2)$*.*

Proof. It is enough to repeat, the proof of Corollary 2.137, substituting the results on $\boldsymbol{\mathcal{Z}}$ to the results on $\boldsymbol{\mathcal{X}}$. □

On the other hand, we cannot obtain an analogue of Corollary 2.133, since we do not know whether the continuous bijection $\Theta : \mathrm{SStar}_{sp}(\mathrm{Kr}(D)) \longrightarrow \mathrm{SStar}_{val}(D)$ is a homeomorphism or not.

2.5. Localizations as overrings

We have seen in Proposition 2.67, at the beginning of Section 2.3.1, that there is a topological embedding

$$\begin{aligned} \lambda\colon \mathrm{Spec}(D) &\longrightarrow \mathrm{Over}(D) \\ P &\longmapsto D_P, \end{aligned}$$

so that, in particular, the space $\mathrm{Loc}(D) := \lambda(\mathrm{Spec}(D))$ of the localizations of D at prime ideals is a spectral space.

A natural question is whether we could have reached this conclusion without λ, by methods dealing exclusively with overrings; or, more practically, if $\mathrm{Loc}(D)$ is a proconstructible subset of $\mathrm{Over}(D)$. The answer is, in general, negative, as the next example shows.

Example 2.140. Let D be an essential domain that is not a PvMD (see Remark 2.35), and let $\mathcal{E}$ be the set of prime ideals P of D such that D_P is a valuation domain. Since D is not a PvMD, not all t-primes are in $\mathcal{E}$. Since $\mathcal{E} \subseteq \mathrm{QSpec}^t(D)$ [76, Lemma 3.17], we thus have $\mathcal{E} \subsetneq \mathrm{QSpec}^t(D)$. If $\mathcal{E}$ were compact, $s_{\mathcal{E}}$ would define a semistar operation of finite type on D; however, since D is essential (and thus, by definition, $\bigcap\{D_P \mid P \in \mathcal{E}\} = D$) we have $D^{s_{\mathcal{E}}} = D$, and so $s_{\mathcal{E}} \leq w$, and $\mathrm{QSpec}^{s_{\mathcal{E}}}(D) \supseteq \mathrm{QSpec}^w(D) \supseteq \mathrm{QSpec}^t(D)$. This is a contradiction, and so $\mathcal{E}$ is not compact.

However, $\lambda(\mathcal{E}) = \mathrm{Loc}(D) \cap \mathrm{Zar}(D)$; if $\mathrm{Loc}(D)$ were to be proconstructible in $\mathrm{Over}(D)$, so would be $\lambda(\mathcal{E})$ (since $\mathrm{Zar}(D)$ is always proconstructible). But this would imply that $\lambda(\mathcal{E})$ is, in particular, compact, a contradiction. Hence $\mathrm{Loc}(D)$ is not proconstructible in $\mathrm{Over}(D)$.

Definition 2.141. *Let D be an integral domain. We say that D is* rad-colon coherent *if, for every $x \in K \setminus D$, there is a finitely generated ideal I such that* $\mathrm{rad}(I) = \mathrm{rad}((D :_D x))$*, i.e., if and only if $\mathcal{D}((D :_D x))$ is compact in* $\mathrm{Spec}(D)$ *for every $x \in K$.*

Obvious examples of rad-colon coherent domains are Noetherian domains or, more generally, domains with Noetherian spectrum. Another large class of such domains is the class of *coherent domains*, i.e., domains where the intersection of two finitely generated ideals is still finitely generated; this follows from the fact that $(D :_D x) = D \cap x^{-1}D$. In particular, this class comprises all Prüfer domains [50, Proposition 25.4(1)], or more generally the polynomial rings in finitely many variables over a Prüfer domain [52, Corollary 7.3.4].

Proposition 2.142. *Let D be an integral domain. Then,* $\mathrm{Loc}(D)$ *is a proconstructible subspace of* $\mathrm{Over}(D)$ *if and only if D is rad-colon coherent.*

Proof. By the proof of Proposition 2.67, $B_x \cap \mathrm{Loc}(D) = \lambda(\mathcal{D}((D :_D x)))$.

Suppose $\mathrm{Loc}(D)$ is proconstructible. Since B_x is also a proconstructible subspace of $\mathrm{Over}(D)$, so is $B_x \cap \mathrm{Loc}(D)$, that is closed (and thus compact) is $\mathrm{Over}(D)^{\mathrm{cons}}$. Since the Zariski topology is weaker than the constructible topology, $B_x \cap \mathrm{Loc}(D)$ must be compact is the Zariski topology; and since λ is a homeomorphism between $\mathrm{Spec}(D)$ and $\mathrm{Loc}(D)$, also $\mathcal{D}((D :_D x))$ must be compact. But this means exactly that $\mathrm{rad}((D :_D x)) = \mathrm{rad}(I)$ for some finitely generated I.

Conversely, suppose the latter property hold. Then, each $B_x \cap \mathrm{Loc}(D)$ is compact, and thus $\{B_x \cap \mathrm{Loc}(D) \mid x \in K\}$ is a subbasis of compact subsets for $\mathrm{Loc}(D)$; applying Lemma 2.119 we see that $\mathrm{Loc}(D)$ is a proconstructible subset of $\mathrm{Over}(D)$. ☐

There are at least three natural ways to extend $\mathrm{Loc}(D)$ to non-local overrings of D.

The first is by considering general *localizations* (or *quotient rings*) of D, that is, overrings in the form $S^{-1}D$ for some multiplicatively closed subsets S of D. This is, in some way, the most natural generalization; moreover, it is attractive since we can use multiplicatively closed subset instead of overrings. We denote this set by $\mathrm{Over}_{\mathrm{qr}}(D)$.

The second is through the set of *flat* overrings of D (that is, overrings that are flat when considered as D-modules); we have already linked these overrings to semistar operation in Proposition 2.65, as $\iota(T)$ is a spectral semistar operation if and only if T is flat. (Here $\iota : \mathrm{Over}(D) \longrightarrow \mathrm{SStar}(D)$ is the natural inclusion map.) We denote the set of flat overrings of D by $\mathrm{Over}_{\mathsf{flat}}(D)$.

The third is by considering *sublocalizations* of D, i.e., overrings that are intersection of localizations (or, equivalently, quotient rings) of D. Also sublocalizations are linked with spectral operations, since $T = D^{s_\Delta}$ for some $\Delta \subseteq \mathrm{Spec}(D)$ if and only if T is a sublocalization, or, equivalently, the set of sublocalizations is equal to $\pi(\mathrm{SStar}_{sp}(D))$, where $\pi : \mathrm{SStar}(D) \longrightarrow \mathrm{Over}(D)$ is the map sending $*$ to D^*. We denote this set by $\mathrm{Over}_{\mathrm{sloc}}(D)$.

It is well-known that $\mathrm{Over}_{\mathrm{qr}}(D) \subseteq \mathrm{Over}_{\mathsf{flat}}(D) \subseteq \mathrm{Over}_{\mathrm{sloc}}(D)$, and that both inclusions may be strict. For example, any overring of a Prüfer domain is flat, but it need not be a quotient ring (if D is a Prüfer domain such that $\mathrm{Over}_{\mathrm{qr}}(D) = \mathrm{Over}_{\mathsf{flat}}(D) = \mathrm{Over}(D)$, then D is said to be a *QR-domain* – see [50, Section 27] or [43, Section 3.2]): in the case of Dedekind domains, this happens if and only if the class group of D is torsion [51, Corollary 2.6]. As for sublocalizations that are not flat, we refer the reader to the example given just before Section 2.3.1.

In all three cases, a natural question is to ask if (or when) the spaces are spectral, and if (or when) they are proconstructible in $\mathrm{Over}(D)$; moreover, we could ask if there is some way to represent them in function of D or of $\mathrm{Spec}(D)$. We shall treat the case of quotient rings in Section 2.5.1, while the case of flat overrings and of sublocalizations will be dealt with in Section 2.5.2.

A first result is a relation between their proconstructibility and the proconstructibility of $\mathrm{Loc}(D)$.

Proposition 2.143. *Let D be an integral domain. If $\mathrm{Over}_{\mathrm{qr}}(D)$ or $\mathrm{Over}_{\mathsf{flat}}(D)$ is proconstructible, so is $\mathrm{Loc}(D)$, and thus D is rad-colon coherent.*

Proof. If X designates $\mathrm{Over}_{\mathrm{qr}}(D)$ or $\mathrm{Over}_{\mathsf{flat}}(D)$, then $X \cap \mathrm{LocOver}(D) = \mathrm{Loc}(D)$. Since $\mathrm{LocOver}(D)$ is always proconstructible (Corollary 2.112(a)), if X is proconstructible so is $\mathrm{Loc}(D)$. □

2.5.1. Quotient rings and the space $\boldsymbol{\mathcal{S}}(R)$

Definition 2.144. *Let R be a ring (not necessarily a domain). A* semigroup prime *on R is a nonempty subset $\mathscr{Q} \subseteq R$ such that:*

1. *for each $r \in R$ and for each $\pi \in \mathscr{Q}$, $r\pi \in \mathscr{Q}$;*
2. *for all $\sigma, \tau \in R \setminus \mathscr{Q}$, $\sigma\tau \in R \setminus \mathscr{Q}$;*

3. $\mathscr{Q} \neq R$*.*

Remark 2.145.

(1) Semigroup primes owe their name to the fact that, if we consider the semigroup $(R, \cdot)$, then the semigroup primes $\mathscr{Q}$ are exactly the ideals of the semigroup $(R, \cdot)$ (in particular, proper subsemigroups; condition 1) that are "prime" in the sense that, if $ab \in \mathscr{Q}$, then $a \in \mathscr{Q}$ or $b \in \mathscr{Q}$ (condition 2).
(2) A prime ideal P of R is a semigroup prime: indeed, hypothesis 1 is satisfied by every ideal of R, and the second weeds out everything but prime ideals.
(3) The union of any family of semigroup primes is again a semigroup prime: indeed, the first hypothesis works on a element-to-element basis, so it is automatically fulfilled, while if $\sigma\tau \in R \setminus \bigcup_\alpha \mathscr{Q}_\alpha$ then $\sigma\tau \notin \mathscr{Q}_{\overline{\alpha}}$ for some $\overline{\alpha}$, and so (without loss of generality) $\sigma \notin \mathscr{Q}_{\overline{\alpha}}$ and $\sigma \notin \bigcup_\alpha \mathscr{Q}_\alpha$.
(4) The intersection of two semigroup primes is not, in general, a semigroup prime: for example, if P and Q are prime ideals, each not contained in the other, then $P \cap Q$ is not a semigroup prime, since taking $\sigma \in P \setminus Q$, $\tau \in Q \setminus P$ then both are in $R \setminus (P \cap Q)$ but $\sigma\tau \in P \cap Q$.

Points (2) and (3) of the Remark above can be coupled to give an alternate characterization of semigroup primes:

Proposition 2.146 [95, (2.3)]. *Let R be a ring and $\mathscr{Q} \subseteq R$. The following are equivalent:*

(i) $\mathscr{Q}$ is a semigroup prime;
(ii) $\mathscr{Q}$ is the union of a family of prime ideals;
(iii) $R \setminus \mathscr{Q}$ is a saturated multiplicatively closed subset.

Proof. The equivalence of (i) and (iii) follows directly from the definitions; Remark 2.145(2) shows that (ii) implies (i), while the fact that (iii) implies (ii) is a standard result (see e.g. [77, Theorem 2] or [12, Chapter 3, Exercise 7]). □

Let now $\boldsymbol{\mathcal{S}}(R)$ denote the set of semigroup primes of a ring R; as in [95], we endow it with the topology (which we call the *hull-kernel topology* or the *Zariski topology*) whose subbasic closed sets have the form

$$\mathcal{V}(x_1, \ldots, x_n) := \{\mathscr{Q} \in \boldsymbol{\mathcal{S}}(R) \mid x_1, \ldots, x_n \in \mathscr{Q}\},$$

as $x_1, \ldots, x_n$ ranges in R; equivalently, we can consider the subbasis of open sets

$$\mathcal{D}(x_1, \ldots, x_n) := \boldsymbol{\mathcal{S}}(R) \setminus \mathcal{V}(x_1, \ldots, x_n) = \{\mathscr{Q} \in \boldsymbol{\mathcal{S}}(R) \mid x_i \notin \mathscr{Q} \text{ for some } i\}.$$

This notation is chosen to correspond to the closed and open sets in $\mathrm{Spec}(R)$; indeed, by Remark 2.145(2) there is a set-theoretic inclusion $\mathrm{Spec}(R) \subseteq \boldsymbol{\mathcal{S}}(R)$ (this is the chief reason why we work with $\boldsymbol{\mathcal{S}}(R)$ and not with the set of multiplicatively closed subsets). Whenever confusion may arise, we refer to the sets in $\boldsymbol{\mathcal{S}}(R)$ defined above with

$\mathcal{D}_{\boldsymbol{\mathcal{S}}}(x_1, \ldots, x_n)$ and $\mathcal{V}_{\boldsymbol{\mathcal{S}}}(x_1, \ldots, x_n)$, denoting by $\mathcal{D}_{\mathrm{Spec}}(x_1, \ldots, x_n)$ and $\mathcal{V}_{\mathrm{Spec}}(x_1, \ldots, x_n)$ the sets in $\mathrm{Spec}(R)$. It is straightforward to see that

$$\mathcal{V}_{\mathrm{Spec}}(x_1, \ldots, x_n) = \mathcal{V}_{\boldsymbol{\mathcal{S}}}(x_1, \ldots, x_n) \cap \mathrm{Spec}(R)$$

and

$$\mathcal{D}_{\mathrm{Spec}}(x_1, \ldots, x_n) = \mathcal{D}_{\boldsymbol{\mathcal{S}}}(x_1, \ldots, x_n) \cap \mathrm{Spec}(R),$$

and thus the Zariski topology on $\mathrm{Spec}(R)$ is exactly the restriction of the Zariski topology on $\boldsymbol{\mathcal{S}}(R)$.

Lemma 2.147. *Let R be a ring. Then, the family $\{\mathcal{D}(x) \mid x \in R\}$ is a basis of open subsets of $\boldsymbol{\mathcal{S}}(R)$, which is closed by intersections.*

Proof. Let $\Delta := \{\mathcal{D}(x) \mid x \in R\}$. By the definition of semigroup prime, it follows that $\mathcal{D}(xy) = \mathcal{D}(x) \cap \mathcal{D}(y)$; hence Δ is closed by intersections.

By definition of the Zariski topology, a basis of $\boldsymbol{\mathcal{S}}(R)$ is formed by the finite intersections of sets of the form $\mathcal{D}(x_1, \ldots, x_n)$. However, from the definition of the subbasic open sets, it follows easily that $\mathcal{D}(x_1, \ldots, x_n) = \mathcal{D}(x_1) \cup \cdots \mathcal{D}(x_n)$; we claim that

$$\mathcal{D}(x_1, \ldots, x_n) \cap \mathcal{D}(y_1, \ldots, y_m) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \mathcal{D}(x_i y_j).$$

Indeed, if $\mathscr{Q}$ is in the left hand side then $\mathscr{Q} \in \mathcal{D}(x_i) \cap \mathcal{D}(y_j)$ for some i and j, and thus $\mathscr{Q} \in \mathcal{D}(x_i y_j)$; on the other hand, if $\mathscr{Q} \in \mathcal{D}(x_i y_j)$ for some i, j, then $\mathscr{Q} \in \mathcal{D}(x_i) \cap \mathcal{D}(y_j) \subseteq \mathcal{D}(x_1, \ldots, x_n) \cap \mathcal{D}(y_1, \ldots, y_m)$.

Therefore, Δ is a basis. $\square$

In our context, the two important properties of $\mathrm{Spec}(R)$, and its Zariski topology, are that $\mathrm{Spec}(R)$ is a spectral space and that (if R is a domain) it can be embedded into $\mathrm{Over}(R)$. These properties are shared with $\boldsymbol{\mathcal{S}}(R)$, as the next two propositions show.

Proposition 2.148. *Let R be a ring. Then $\boldsymbol{\mathcal{S}}(R)$, endowed with the Zariski topology, is a spectral space.*

Proof. By Lemma 2.147, the family $\{\mathcal{D}(x) \mid x \in R\}$ is an open basis for $\boldsymbol{\mathcal{S}}(R)$. Let $\mathscr{U}$ be an ultrafilter on $\boldsymbol{\mathcal{S}}(R)$, and define

$$\mathscr{Q}_{\mathscr{U}} := \{r \in R \mid \mathcal{V}(r) \in \mathscr{U}\} = \{r \in R \mid \mathcal{D}(r) \notin \mathscr{U}\}.$$

We claim that $\mathscr{Q}_{\mathscr{U}}$ is a semigroup prime of R: indeed, if $s \in R$ and $\pi \in \mathscr{Q}_{\mathscr{U}}$, then $\boldsymbol{\mathcal{S}}(R) \setminus \mathcal{D}(\pi) \subseteq \boldsymbol{\mathcal{S}}(R) \setminus \mathcal{D}(s\pi)$, and thus the latter is in $\mathscr{U}$, which means $s\pi \in \mathscr{Q}_{\mathscr{U}}$; moreover, if $\sigma, \tau \notin \mathscr{Q}_{\mathscr{U}}$, then $\mathcal{V}(\sigma), \mathcal{V}(\tau) \notin \mathscr{U}$, that is, $\mathcal{D}(\sigma), \mathcal{D}(\tau) \in \mathscr{U}$, and $\mathcal{D}(\sigma\tau) = \mathcal{D}(\sigma) \cap \mathcal{D}(\tau) \notin \mathscr{U}$, so that $\sigma\tau \notin \mathscr{Q}_{\mathscr{U}}$.

Rewriting the definition of $\mathscr{Q}_{\mathscr{U}}$, we see that, for all $x \in R$, $\mathscr{Q}_{\mathscr{U}} \in \mathcal{D}(x)$ if and only if $\mathcal{D}(x) \in \mathscr{U}$; hence, by the usual [36, Corollary 3.3]/Theorem A.2, $\boldsymbol{\mathcal{S}}(R)$ is a spectral space. $\square$

Proposition 2.149. *Let D be an integral domain, and let λ_{qr} be the map*

$$\begin{aligned}\lambda_{qr}\colon \boldsymbol{\mathcal{S}}(D) &\longrightarrow \mathrm{Over}_{\mathrm{qr}}(D)\\ \mathscr{Q} &\longmapsto (R\setminus\mathscr{Q})^{-1}D.\end{aligned}$$

Endow $\boldsymbol{\mathcal{S}}(D)$ and $\mathrm{Over}_{\mathrm{qr}}(D)$ with the respective Zariski topologies. Then, λ_{qr} is a homeomorphism.

Proof. By Proposition 2.146, λ_{qr} is bijective. The remainder of the proof proceeds like the proof of Proposition 2.67.

To prove that λ_{qr} is continuous it is enough to prove that $\lambda_{qr}^{-1}(B_x)$ is open for every $x\in K$ (where K is the quotient field of D). However,

$$\begin{aligned}\lambda_{qr}^{-1}(B_x) &= \{\mathscr{Q}\in\boldsymbol{\mathcal{S}}(D)\mid x\in(D\setminus\mathscr{Q})^{-1}D\} =\\ &= \{\mathscr{Q}\in\boldsymbol{\mathcal{S}}(D)\mid 1\in((D\setminus\mathscr{Q})^{-1}D:_D x)\} =\\ &= \{\mathscr{Q}\in\boldsymbol{\mathcal{S}}(D)\mid 1\in(D:_D x)\cdot(D\setminus\mathscr{Q})^{-1}D\} =\\ &= \{\mathscr{Q}\in\boldsymbol{\mathcal{S}}(D)\mid (D:_D x)\nsubseteq\mathscr{Q}\} = \textstyle\bigcup_{z\in(D:_D x)}\mathcal{D}(z)\end{aligned}$$

which is open in $\boldsymbol{\mathcal{S}}(D)$.

On the other hand, consider a basic open set $\mathcal{D}(x)$ of $\boldsymbol{\mathcal{S}}(D)$. Then,

$$\lambda_{qr}(\mathcal{D}(x)) = \{S^{-1}D\mid x\in S\} = B_{x^{-1}}\cap\mathrm{Over}_{\mathrm{qr}}(D)$$

is open. Hence, λ_{qr} is a homeomorphism. □

Let now $f: R_1\longrightarrow R_2$ be a ring homomorphism. It is well-known that we can associate to f a continuous map $f^a:\mathrm{Spec}(R_2)\longrightarrow\mathrm{Spec}(R_1)$ by defining $f^a(P):=f^{-1}(P)$. We can do the same at the level of prime semigroups: to any homomorphism f, we associate a map

$$\begin{aligned}\boldsymbol{\mathcal{S}}(f)\colon \boldsymbol{\mathcal{S}}(R_2) &\longrightarrow \boldsymbol{\mathcal{S}}(R_1)\\ \mathscr{Q} &\longmapsto f^{-1}(\mathscr{Q}).\end{aligned}$$

We investigate the properties of this map.

Proposition 2.150. *Let $f: R_1\longrightarrow R_2$ be a ring homomorphism, and let $\boldsymbol{\mathcal{S}}(f)$ be the map defined above. Then:*

(a) $\boldsymbol{\mathcal{S}}(f)$ is well-defined, continuous and spectral in the Zariski topology;
(b) if $i_k:\mathrm{Spec}(R_k)\longrightarrow\boldsymbol{\mathcal{S}}(R_k)$ is the set-theoretic inclusion, then $\boldsymbol{\mathcal{S}}(f)\circ i_2 = i_1\circ f^a$;
(c) the assignment $R\mapsto\boldsymbol{\mathcal{S}}(R)$, $f\mapsto\boldsymbol{\mathcal{S}}(f)$, is a functor from the category of rings to the category of spectral spaces.

Proof. To show that $\boldsymbol{\mathcal{S}}(f)$ is well-defined we have to show that $f^{-1}(\mathscr{Q})$ is a semigroup prime when $\mathscr{Q}$ is. Indeed, if $r\in R_1$ and $\pi\in f^{-1}(\mathscr{Q})$ then $f(\pi r)=f(\pi)f(r)\in f(r)\mathscr{Q}\subseteq\mathscr{Q}$, so that $r\pi\in f^{-1}(\mathscr{Q})$; moreover, if $\sigma,\tau\notin f^{-1}(\mathscr{Q})$, then $f(\sigma),f(\tau)\notin\mathscr{Q}$ and thus $f(\sigma)f(\tau)\notin\mathscr{Q}$, that is, $\sigma\tau\notin f^{-1}(\mathscr{Q})$. Hence $\boldsymbol{\mathcal{S}}(f)$ is well-defined.

To show that it is continuous, it is enough to note that $\boldsymbol{\mathcal{S}}(f)^{-1}(\mathcal{D}(x)) = \mathcal{D}(f(x))$; this also implies that $\boldsymbol{\mathcal{S}}(f)$ is a spectral map, since $\{\mathcal{D}(y) \mid y \in A\}$ is a basis of compact subsets of $\boldsymbol{\mathcal{S}}(A)$, for any ring A. Point (b) follows from the equalities

$$\boldsymbol{\mathcal{S}}(f) \circ i_1(P) = \boldsymbol{\mathcal{S}}(f)(P) = f^{-1}(P) = f^a(P) = i_2 \circ f^a(P).$$

The last thing needed to show that $\boldsymbol{\mathcal{S}}$ is a functor is that $\boldsymbol{\mathcal{S}}(f \circ g) = \boldsymbol{\mathcal{S}}(g) \circ \boldsymbol{\mathcal{S}}(f)$. But this is a direct consequence of the definition. $\square$

Corollary 2.151. *Let R be a ring. Each basic open set $\mathcal{D}(x)$ of $\boldsymbol{\mathcal{S}}(R)$ is compact, in the Zariski topology.*

Proof. The set $\mathcal{D}(x)$ is the image of $\boldsymbol{\mathcal{S}}(R_x)$ under the canonical map $f^a : \boldsymbol{\mathcal{S}}(R_x) \longrightarrow \boldsymbol{\mathcal{S}}(R)$, where R_x is the localization of R at the multiplicatively closed set $\{1, x, \dots, x^n, \dots, \}$. However, $\boldsymbol{\mathcal{S}}(R_x)$ is compact (this can be see either because it has a maximum in the order induced by the Zariski topology – namely, $R_x \setminus U(R_x)$ – or because it is a spectral space), and since f^a is continuous it follows that $\mathcal{D}(x)$ is compact too. $\square$

2.5.1.1. The relationship between $\boldsymbol{\mathcal{S}}(R)$ and $\boldsymbol{\mathcal{X}}(R)$

Sections 2.4 and 2.5.1 can be seen as two very different ways to answer the same problem: find a functorial construction $\boldsymbol{\mathcal{A}}$ that associates to any ring R a topological space $\boldsymbol{\mathcal{A}}(R)$ that is both spectral and an extension of $\mathrm{Spec}(R)$. This section explores the relationship between the two construction $\boldsymbol{\mathcal{S}}(R)$ and $\boldsymbol{\mathcal{X}}(R)$, and how they underline different aspects of the ring R.

Proposition 2.152. *Let R be a ring, and denote by ζ and $\mathscr{P}$ the maps*

$$\begin{aligned}\zeta\colon \boldsymbol{\mathcal{S}}(R) &\longrightarrow \boldsymbol{\mathcal{X}}(R)\\ \mathscr{Q} &\longmapsto \{P \in \mathrm{Spec}(R) \mid P \subseteq \mathscr{Q}\}\end{aligned} \quad \textit{and} \quad \begin{aligned}\mathscr{P}\colon \boldsymbol{\mathcal{X}}(R) &\longrightarrow \boldsymbol{\mathcal{S}}(R)\\ \Delta &\longmapsto \bigcup_{P \in \Delta} P.\end{aligned}$$

Then:

- *(a) ζ and $\mathscr{P}$ are continuous, and $\mathscr{P}$ spectral;*
- *(b) $\mathscr{P} \circ \zeta$ is the identity on $\boldsymbol{\mathcal{S}}(R)$;*
- *(c) $\mathscr{P}$ is a topological retraction;*
- *(d) ζ is a topological embedding;*
- *(e) if $i : \mathrm{Spec}(R) \longrightarrow \boldsymbol{\mathcal{S}}(R)$ and $\chi : \mathrm{Spec}(R) \longrightarrow \boldsymbol{\mathcal{X}}(R)$ are the canonical embeddings, then $\zeta \circ i = \chi$;*
- *(f) for every $\Delta \in \boldsymbol{\mathcal{X}}(R)$, $\zeta \circ \mathscr{P}(\Delta) = \bigcap\{\mathcal{D}_{\mathrm{Spec}}(a) \mid \Delta \subseteq \mathcal{D}_{\mathrm{Spec}}(a)\}$.*

Proof. We first note that ζ and $\mathscr{P}$ are well-defined: the latter follows by Remark 2.145 (or by Proposition 2.146). The former is true since $\zeta(\mathscr{Q})$ is equal to the image of $\mathrm{Spec}((R \setminus \mathscr{Q})^{-1}R)$ into $\mathrm{Spec}(R)$ under the map corresponding to the inclusion of rings, and this image is compact and closed by generizations, i.e., closed in the inverse topology.

(a) To prove that ζ is continuous, take a subbasic open set $\boldsymbol{\mathcal{U}}(\Omega)$ of $\boldsymbol{\mathcal{X}}(R)$, where $\Omega = \mathcal{D}_{\mathrm{Spec}}(J)$ for some finitely generated ideal J. Then,

$$\begin{aligned}
\zeta^{-1}(\boldsymbol{\mathcal{U}}(\Omega)) &= \{\mathscr{Q} \in \boldsymbol{\mathcal{S}}(R) \mid \zeta(\mathscr{Q}) \in \boldsymbol{\mathcal{U}}(\Omega)\} = \\
&= \{\mathscr{Q} \in \boldsymbol{\mathcal{S}}(R) \mid \{P : P \subseteq \mathscr{Q}\} \in \boldsymbol{\mathcal{U}}(\Omega)\} = \\
&= \{\mathscr{Q} \in \boldsymbol{\mathcal{S}}(R) \mid P \subseteq \Omega \text{ for all } P \subseteq \mathscr{Q}\} = \\
&= \{\mathscr{Q} \in \boldsymbol{\mathcal{S}}(R) \mid P \in \mathcal{D}_{\mathrm{Spec}}(J) \text{ for all } P \subseteq \mathscr{Q}\} = \\
&= \{\mathscr{Q} \in \boldsymbol{\mathcal{S}}(R) \mid I \subsetneq P \text{ for all } P \subseteq \mathscr{Q}\}.
\end{aligned}$$

We claim that this set is equal to $\{\mathscr{Q} \in \boldsymbol{\mathcal{S}}(R) \mid I \subsetneq \mathscr{Q}\}$. Indeed, if $\mathscr{Q}$ is in the latter set then it is in $\zeta^{-1}(\boldsymbol{\mathcal{U}}(\Omega))$, for the above calculation. On the other hand, if I is not contained in any prime ideal $P \subseteq \mathscr{Q}$, then $(R \setminus \mathscr{Q})^{-1}I = (R \setminus \mathscr{Q})^{-1}R$, and thus $I \cap (R \setminus \mathscr{Q}) \neq \emptyset$, i.e., $I \subsetneq \mathscr{Q}$. However,

$$\{\mathscr{Q} \in \boldsymbol{\mathcal{S}}(R) \mid I \subsetneq \mathscr{Q}\} = \bigcup_{i \in I} \mathcal{D}_{\boldsymbol{\mathcal{S}}}(i),$$

which is an open set of $\boldsymbol{\mathcal{S}}(R)$. Hence, ζ is continuous.

For $\mathscr{P}$, consider a basic open set $\mathcal{D}_{\boldsymbol{\mathcal{S}}}(x)$ of $\boldsymbol{\mathcal{S}}(R)$. Then,

$$\begin{aligned}
\mathscr{P}^{-1}(\mathcal{D}_{\boldsymbol{\mathcal{S}}}(x)) &= \{\Delta \in \boldsymbol{\mathcal{X}}(R) \mid \mathscr{P}(\Delta) \in \mathcal{D}_{\boldsymbol{\mathcal{S}}}(x)\} = \\
&= \{\Delta \in \boldsymbol{\mathcal{X}}(R) \mid x \notin \mathscr{P}(\Delta)\} = \\
&= \{\Delta \in \boldsymbol{\mathcal{X}}(R) \mid x \notin \bigcup\{P \mid P \in \Delta\}\} = \\
&= \{\Delta \in \boldsymbol{\mathcal{X}}(R) \mid x \notin P \text{ for every } P \in \Delta\} = \\
&= \{\Delta \in \boldsymbol{\mathcal{X}}(R) \mid Y \subseteq \mathcal{D}_{\mathrm{Spec}}(x)\} = \boldsymbol{\mathcal{U}}(\mathcal{D}_{\mathrm{Spec}}(x)),
\end{aligned}$$

which is open and compact on $\boldsymbol{\mathcal{X}}(R)$. Hence, $\mathscr{P}$ is continuous and spectral.

(b) Let $\mathscr{Q} \in \boldsymbol{\mathcal{S}}(R)$; then,

$$\mathscr{P} \circ \zeta(\mathscr{Q}) = \bigcup\{P \mid P \in \zeta(\mathscr{Q})\} = \bigcup\{P \mid P \subseteq \mathscr{Q}\} = \mathscr{Q},$$

with the last equality coming from Proposition 2.146.

(c) follows directly from (b); to show (d) (i.e., that ζ is an embedding) it is enough to note that, for every $x \in R$, $\zeta(\mathcal{D}_{\boldsymbol{\mathcal{S}}}(x)) = \zeta(\boldsymbol{\mathcal{S}}(R)) \cap \boldsymbol{\mathcal{U}}(\mathcal{D}_{\mathrm{Spec}}(x))$. (e) is an easy consequence of the definitions, since

$$\zeta \circ i(P) = \zeta(P) = \{Q \in \mathrm{Spec}(R) \mid Q \subseteq P\} = \{P\}^{\mathrm{gen}} = \chi(P).$$

(f) Let $\Delta \in \boldsymbol{\mathcal{X}}(R)$. If $\Delta \subseteq \mathcal{D}_{\mathrm{Spec}}(a)$, then $a \notin P$ for every $P \in \Delta$, and thus $a \notin \bigcup\{P \mid P \in \Delta\} = \mathscr{P}(\Delta)$. Hence, if $Q \in \zeta \circ \mathscr{P}(\Delta)$ then $a \notin Q$ and so $Q \in \mathcal{D}_{\mathrm{Spec}}(a)$. Conversely, suppose Q belongs to the intersection. If $Q \notin \zeta \circ \mathscr{P}(\Delta)$, then there would be an element $q \in Q \setminus \mathscr{P}(\Delta)$; but this would imply $\Delta \subseteq \mathcal{D}_{\mathrm{Spec}}(q)$ while $Q \notin \mathcal{D}_{\mathrm{Spec}}(q)$, a contradiction. □

Remark 2.153. Note that the definition of $\mathscr{P}(\Delta)$ works not only for elements of $\boldsymbol{\mathcal{X}}(X)$, but more generally for arbitrary subsets $\Delta \subseteq \mathrm{Spec}(R)$. However, $\mathscr{P}(\Delta)$ can always be reduced to an image of $\mathscr{P}(\boldsymbol{\mathcal{X}}(R))$; more precisely, we claim that $\mathscr{P}(Y) = \mathscr{P}(\mathrm{Cl}^{\mathrm{inv}}(Y))$ for every $Y \subseteq \mathrm{Spec}(R)$.

One inclusion is obvious; suppose $x \in \mathscr{P}(\mathrm{Cl}^{\mathrm{inv}}(Y)) \setminus \mathscr{P}(Y)$. Then, $x \notin Q$ for every $Q \in Y$, that is, $Y \subseteq \mathcal{D}_{\mathrm{Spec}}(x)$; however, $\mathcal{D}_{\mathrm{Spec}}(x)$ is a closed set, in the inverse topology, and thus $\mathrm{Cl}^{\mathrm{inv}}(Y) \subseteq \mathcal{D}_{\mathrm{Spec}}(x)$. But this would imply that $x \notin P$ for every $P \in \mathrm{Cl}^{\mathrm{inv}}(Y)$, and so $x \notin \mathscr{P}(\mathrm{Cl}^{\mathrm{inv}}(Y))$, against the assumptions. Therefore, $\mathscr{P}(Y) = \mathscr{P}(\mathrm{Cl}^{\mathrm{inv}}(Y))$.

Any ring homomorphism $f : R_1 \longrightarrow R_2$ gives rise to three spectral maps: the classical one $f^a : \mathrm{Spec}(R_2) \longrightarrow \mathrm{Spec}(R_1)$, the semigroup prime map $\boldsymbol{\mathcal{S}}(f) : \boldsymbol{\mathcal{S}}(R_2) \longrightarrow \boldsymbol{\mathcal{S}}(R_1)$, and the $\boldsymbol{\mathcal{X}}$-version $\boldsymbol{\mathcal{X}}(f^a) : \boldsymbol{\mathcal{X}}(R_2) \longrightarrow \boldsymbol{\mathcal{X}}(R_1)$. The three maps are compatible, in a sense made precise by the following proposition.

Proposition 2.154. *Let $f : R_1 \to R_2$ be a ring homomorphism, and define f^a, $\boldsymbol{\mathcal{S}}(f)$ and $\boldsymbol{\mathcal{X}}(f^a)$ as above. Let $i_k : \mathrm{Spec}(R_k) \longrightarrow \boldsymbol{\mathcal{S}}(R_k)$ and $\zeta_k : \boldsymbol{\mathcal{S}}(R_k) \longrightarrow \boldsymbol{\mathcal{X}}(R_k)$ (for $k = 1, 2$) be the canonical inclusions. Then, the diagram*

$$\begin{tikzcd} \mathrm{Spec}(R_2) \arrow[r, "i_2"] \arrow[d, "f^a"'] & \boldsymbol{\mathcal{S}}(R_2) \arrow[r, "\zeta_2"] \arrow[d, "\boldsymbol{\mathcal{S}}(f)"'] & \boldsymbol{\mathcal{X}}(R_2) \arrow[d, "\boldsymbol{\mathcal{X}}(f^a)"'] \\ \mathrm{Spec}(R_1) \arrow[r, "i_1"] & \boldsymbol{\mathcal{S}}(R_1) \arrow[r, "\zeta_1"] & \boldsymbol{\mathcal{X}}(R_1) \end{tikzcd} \tag{2.7}$$

commutes.

Proof. The left square of (2.7) commutes by Proposition 2.150(b). Let now $\mathscr{Q} \in \boldsymbol{\mathcal{S}}(R_2)$; then,

$$\zeta_1 \circ \boldsymbol{\mathcal{S}}(f)(\mathscr{Q}) = \zeta_1(f^{-1}(\mathscr{Q})) = \{P \in \mathrm{Spec}(R) \mid P \subseteq f^{-1}(\mathscr{Q})\},$$

while

$$\begin{aligned} \boldsymbol{\mathcal{X}}(f^a) \circ \zeta_2(\mathscr{Q}) &= \boldsymbol{\mathcal{X}}(f^a)\left(\{P \in \mathrm{Spec}(R) \mid P \subseteq \mathscr{Q}\}\right) = \\ &= \left(f^a\left(\{P \mid P \subseteq \mathscr{Q}\}\right)\right)^{\mathrm{gen}} \\ &= \left(\{f^{-1}(P) \mid P \subseteq \mathscr{Q}\}\right)^{\mathrm{gen}}. \end{aligned}$$

Let $Q \in \mathrm{Spec}(R_1)$. If $Q \in \boldsymbol{\mathcal{X}}(f^a) \circ \zeta_2(\mathscr{Q})$, then $Q \subseteq f^{-1}(P)$ for some $P \subseteq \mathscr{Q}$; hence, $Q \subseteq f^{-1}(\mathscr{Q})$ and $Q \in \zeta_1 \circ \boldsymbol{\mathcal{S}}(f)(\mathscr{Q})$.

Conversely, suppose $Q \in \zeta_1 \circ \boldsymbol{\mathcal{S}}(f)(\mathscr{Q})$. Then, $Q \subseteq f^{-1}(\mathscr{Q})$, and thus $f(Q) \subseteq \mathscr{Q}$; hence, $f(Q)R_2 \cap \Sigma_{\mathscr{Q}} = \emptyset$, where $\Sigma_{\mathscr{Q}} := R_2 \setminus \mathscr{Q}$. It follows that $f(Q)R_2$ extends to a proper ideal of $\Sigma_{\mathscr{Q}}^{-1}R_2$, and in particular there is a prime ideal P of R_2 such that $f(Q) \subseteq P$ and $\Sigma_{\mathscr{Q}}^{-1}P \neq \Sigma_{\mathscr{Q}}^{-1}R_2$. Therefore, $P \subseteq \mathscr{Q}$. It follows that

$$Q \subseteq f^{-1}(f(Q)) \subseteq f^{-1}(P) \subseteq f^{-1}(\mathscr{Q}),$$

and so $Q \in \boldsymbol{\mathcal{X}}(f^a) \circ \zeta_2(\mathscr{Q})$. Hence, also the right square of (2.7) commutes. □

The next corollary can be seen as a "$\boldsymbol{\mathcal{S}}$-version" of Proposition 2.136.

Corollary 2.155. *Let $f : R_1 \to R_2$ be a ring homomorphism, and let $f^a : \mathrm{Spec}(R_2) \to \mathrm{Spec}(R_1)$ be the associated spectral map. If f^a is a spectral embedding (respectively, a homeomorphism) then so is $\boldsymbol{\mathcal{S}}(f)$.*

Proof. If f^a is a topological embedding then, by Proposition 2.131, so is $\boldsymbol{\mathcal{X}}(f^a)$, and thus also $\boldsymbol{\mathcal{X}}(f^a) \circ \zeta_2$ is a topological embedding. By Proposition 2.150, it follows that $\zeta_1 \circ \boldsymbol{\mathcal{S}}(f)$ is an embedding, and thus so is $\boldsymbol{\mathcal{S}}(f)$.

If f^a is an homeomorphism, then by the previous paragraph $\boldsymbol{\mathcal{S}}(f)$ is a topological embedding. Let $\mathscr{Q} \in \boldsymbol{\mathcal{S}}(R_1)$, and let $\mathscr{L} := \bigcup\{\mathrm{rad}(f(P)R_2) \mid P \subseteq \mathcal{Q}\}$. Since f^a is an homeomorphism, $\mathrm{rad}(f(P)R_2)$ is a prime ideal of R_2, and so $\mathscr{L}$ is a prime semigroup. We claim that $\boldsymbol{\mathcal{S}}(f)(\mathscr{L}) = \mathscr{Q}$. Clearly if $q \in \mathscr{Q}$ then $f(q) \in \mathscr{L}$, and $q \in f^{-1}(\mathscr{L}) = \boldsymbol{\mathcal{S}}(f)(\mathscr{L})$. Conversely, if $q \in \boldsymbol{\mathcal{S}}(f)(\mathscr{L})$, then $f(q)^n \in f(P)R_2$ for some $n \geq 1$ and some $P \subseteq \mathscr{Q}$. Hence $q^n \in f^{-1}(f(P)R_2) = P$, the last equality coming from the bijectivity of f^a. Thus, $q \in P \subseteq \mathscr{Q}$. Therefore, $\boldsymbol{\mathcal{S}}(f)$ is surjective, and thus a homeomorphism. □

Despite the similarities, the great difference between $\boldsymbol{\mathcal{S}}(R)$ and $\boldsymbol{\mathcal{X}}(R)$ is that the latter is an entirely geometric construction – depending only on $\mathrm{Spec}(R)$ – while the former is inextricably tied with the algebraic properties of R itself. One example of this phenomenon happens in the set-up of Proposition 2.154: while $\boldsymbol{\mathcal{X}}(f^a)$ is defined in term of the spectral map f^a, $\boldsymbol{\mathcal{S}}(f)$ has to be defined directly from the ring homomorphism f. The next two examples shed more light on this difference.

Example 2.156. The image of $\mathrm{Spec}(R)$ in $\boldsymbol{\mathcal{S}}(R)$ cannot be determined uniquely by topological means; that is, a homeomorphism $\boldsymbol{\mathcal{S}}(R_1) \longrightarrow \boldsymbol{\mathcal{S}}(R_2)$ does not always descend to a homeomorphism $\mathrm{Spec}(R_1) \longrightarrow \mathrm{Spec}(R_2)$.

For example, let R be a unique factorization domain, and let $\boldsymbol{\mathcal{P}}(R)$ be the set of prime elements of R modulo multiplication by units. Any nonzero prime P of R is the union of the principal prime ideals contained in P; therefore, a semigroup prime $\mathscr{Q}$ is uniquely determined by the prime elements it contains, and any set of prime elements determines a different prime semigroup. It follows that there is a bijective correspondence between $\boldsymbol{\mathcal{S}}(R)$ and the power set $\mathscr{B}$ of $\boldsymbol{\mathcal{P}}(R)$, which becomes a homeomorphism if we take, as a subbasis for $\mathscr{B}$, the family of the subsets of the form $\mathcal{V}(p) := \{B \in \mathscr{B} \mid p \notin B\}$, as p runs in $\boldsymbol{\mathcal{P}}(R)$.

In particular, the topology of $\boldsymbol{\mathcal{S}}(R)$ depends uniquely on the cardinality of $\boldsymbol{\mathcal{P}}(R)$ (i.e., of the set of height-1 primes of R), and thus it is independent, for example, from the dimension of R. Thus, if R_1 and R_2 are unique factorization domains (UFD) such that $|\boldsymbol{\mathcal{P}}(R_1)| = |\boldsymbol{\mathcal{P}}(R_2)|$ but $\dim(R_1) \neq \dim(R_2)$, then $\boldsymbol{\mathcal{S}}(R_1) \simeq \boldsymbol{\mathcal{S}}(R_2)$ but $\mathrm{Spec}(R_1) \not\simeq \mathrm{Spec}(R_2)$ (an, in particular, $i_1(\mathrm{Spec}(R_1))$ and $i_2(\mathrm{Spec}(R_2))$ does not correspond under any homeomorphism $\boldsymbol{\mathcal{S}}(R_1) \longrightarrow \boldsymbol{\mathcal{S}}(R_2)$). This, for example, happens if we take an infinite UFD R_1 and define $R_2 := R_1[X]$.

Example 2.157. Two rings may have homeomorphic spectra, but their spaces of semigroup primes may still be non-homeomorphic.

Consider a Dedekind domain D such that the class group $\mathrm{Cl}(D)$ of D is not a torsion group (such a domain is guaranteed to exist by [20, Theorem 7]; an explicit example is $D := K[X,Y]/(X^2 - Y^3 + Y + 1)$, where K is an algebraically closed field [51]); then, there is a maximal ideal P of D such that the class $[P]$ has infinite order in $\mathrm{Cl}(D)$, i.e., P^n is never principal or, equivalently, no principal ideal is P-primary.

Let $Y := \mathrm{Spec}(R) \setminus \{P\}$: then, Y is closed in the inverse topology, but we claim that $Y \notin \zeta(\boldsymbol{\mathcal{S}}(D))$. If $Y = \zeta(\mathscr{Q})$, then $\mathscr{Q} \in \boldsymbol{\mathcal{S}}(D)$ must contain every element of Y, but there must be an $x \in P$ such that $x \notin \mathscr{Q}$. However, the ideal xD is not P-primary, and so there also exists a prime ideal Q of R, $Q \neq P$, such that $x \in Q$. This contradicts $Y = \zeta(\mathscr{Q})$, and so ζ is not surjective.

On the other hand, consider a principal ideal domain D' such that the cardinality of $\mathrm{Max}(D')$ is equal to the cardinality of D (it suffices to take $D' := F[X]$, where F is a field with the same cardinality of $\mathrm{Max}(D)$). Then, $\mathrm{Spec}(D')$ and $\mathrm{Spec}(D)$ are homeomorphic (it is enough to take any bijection between $\mathrm{Max}(D')$ and $\mathrm{Max}(D)$ then extend it to a bijection $\rho : \mathrm{Spec}(D') \to \mathrm{Spec}(D)$ by defining $\rho((0)) := (0)$); we claim that $\boldsymbol{\mathcal{S}}(D')$ and $\boldsymbol{\mathcal{S}}(D)$ are not homeomorphic.

Indeed, if they were, take any homeomorphism $\varphi : \boldsymbol{\mathcal{S}}(D) \longrightarrow \boldsymbol{\mathcal{S}}(D')$. Then, $\varphi(\{0_D\}) = \{0_{D'}\}$ (since $\{0\}$ is the minimal semigroup prime in every ring), and thus the minimal elements of $\boldsymbol{\mathcal{S}}(D) \setminus \{\{0_D\}\}$ correspond to the minimal elements of $\boldsymbol{\mathcal{S}}(D') \setminus \{\{0_{D'}\}\}$. However, the minimal elements of $\boldsymbol{\mathcal{S}}(R) \setminus \{\{0\}\}$ are, for any ring R, the minimal nonzero prime ideals; hence, φ sends every maximal ideal of D into a maximal ideal of D'. Consider now $Y := \mathrm{Spec}(R) \setminus \{P\}$, defined above, as a subset of $\boldsymbol{\mathcal{S}}(D)$. Then, since φ is, in particular, an isomorphism of partially ordered sets, $\varphi(\sup Y) = \sup \varphi(Y)$. Clearly, in $\boldsymbol{\mathcal{S}}(R)$, the supremum of a subset Z is the union of elements of Z; therefore,

$$\sup(Y) = \bigcup_{\mathfrak{p} \in Y} \mathfrak{p} = D \setminus U(D) = \bigcup_{\mathfrak{q} \in \mathrm{Spec}(D)} \mathfrak{q},$$

where $U(D)$ denotes the units of D. Hence, $\varphi(\sup(Y)) = \varphi(D \setminus U(D)) = D' \setminus U(D')$, since $R \setminus U(R)$ is always the maximal semigroup prime of R. However, if $\varphi(P) = Q$, then $Q = qD'$ for some q (being D' a principal ideal domain), and so $Q \nsubseteq \bigcup\{\mathfrak{q} \mid \mathfrak{q} \in \varphi(Y)\}$; it follows that $\sup \varphi(Y) \neq D' \setminus U(D')$. This is a contradiction, and the homeomorphism φ cannot exist.

A closer inspection of Example 2.157 shows that the main problem for the existence of φ is that the image of $\mathcal{D}_{\mathrm{Spec}}(a)$ under a map $f^a : \mathrm{Spec}(R_2) \longrightarrow \mathrm{Spec}(R_1)$ (where a is an element of R_2) may not be exprimible as $\mathcal{D}_{\mathrm{Spec}}(b)$, with b an element of R_1; that is, we cannot characterize topologically the open subsets of $\mathrm{Spec}(R)$ that are generated by the principal ideals. This is in striking contrast with the fact that we *can* express topologically the fact that an open set is generated by a finitely-generated ideal: indeed, an open set Ω is compact if and only if $\Omega = \mathcal{D}_{\mathrm{Spec}}(I)$ for some finitely generated I. (Note that open sets generated by principal ideals had a rôle also in Proposition 2.152(f).)

We end this section by characterizing when $\zeta : \boldsymbol{\mathcal{S}}(R) \longrightarrow \boldsymbol{\mathcal{X}}(R)$ is surjective.

Proposition 2.158. *Let R be a ring, and let $\zeta : \boldsymbol{\mathcal{S}}(R) \longrightarrow \boldsymbol{\mathcal{X}}(R)$ be the map defined in Proposition 2.152. The following statements are equivalent.*

(i) ζ is surjective.
(ii) ζ is a homeomorphism.
(iii) The radical of every finitely generated ideal of R is the radical of a principal ideal.

(iv) If I is a finitely generated ideal of R and $I \subseteq \bigcup\{Q_\lambda \mid \lambda \in \Lambda, Q_\lambda \in \mathrm{Spec}(R)\}$, then $I \subseteq Q_{\overline{\lambda}}$ for some $\overline{\lambda} \in \Lambda$.

(v) The collection $\{\boldsymbol{\mathcal{U}}(\mathcal{D}_{\mathrm{Spec}}(x)) \mid x \in R\}$ is a basis for the Zariski topology of $\boldsymbol{\mathcal{X}}(R)$.

Proof. (i) $\iff$ (ii) follows from the fact that ζ is an embedding (Proposition 2.152(d)).

(i) $\Longrightarrow$ (iii) Suppose there is a finitely generated ideal I such that $\mathrm{rad}(I) \neq \mathrm{rad}(aR)$ for every $a \in R$. Consider the open and compact subset $\mathcal{D}_{\mathrm{Spec}}(I) \in \boldsymbol{\mathcal{X}}(R)$: since ζ is surjective, $\mathcal{D}_{\mathrm{Spec}}(I) = \zeta(\mathscr{Q})$ for some $\mathscr{Q} \in \boldsymbol{\mathcal{S}}(R)$. Thus, no prime ideal $P \subseteq \mathscr{Q}$ contains I; as in the proof of Proposition 2.152(a), this implies that, $I \nsubseteq \mathscr{Q}$, i.e., that there is an $a \in I \setminus \mathscr{Q}$. Since $\mathrm{rad}(aR) \neq \mathrm{rad}(I)$, we have $\mathrm{rad}(aR) \subsetneq \mathrm{rad}(I)$, and thus there is a prime ideal Q containing aR but not I; hence, $Q \in \mathcal{D}_{\mathrm{Spec}}(I)$ and so $Q \subseteq \mathscr{Q}$. But this would imply $a \in \mathscr{Q}$, a contradiction. Hence, $\mathcal{D}_{\mathrm{Spec}}(I)$ is not in the range of ζ.

(iii) $\Longrightarrow$ (iv) If $\mathrm{rad}(I) = \mathrm{rad}(aR)$, and P is a prime ideal, then $I \subseteq P$ if and only if $a \in P$; therefore, $I \subseteq \bigcup Q_\lambda$ implies $a \in \bigcup Q_\lambda$, which implies $a \in Q_{\overline{\lambda}}$ for some $\overline{\lambda} \in \Lambda$ and so $I \subseteq Q_{\overline{\lambda}}$.

(iv) $\Longrightarrow$ (i) Let $Y \in \boldsymbol{\mathcal{X}}(R)$; we claim that $Y = \zeta \circ \mathscr{P}(Y)$. By Proposition 2.152(f), $Y \subseteq \zeta \circ \mathscr{P}(Y)$. Conversely, suppose $P \in \zeta \circ \mathscr{P}(Y) \setminus Y$. Then, since Y is closed in the inverse topology, there is an open and compact subset Ω such that $Y \subseteq \Omega$ but $P \notin \Omega$. We can write $\Omega = \mathcal{D}_{\mathrm{Spec}}(I)$, with I finitely generated; $P \in \zeta \circ \mathscr{P}(Y)$ implies that $I \subseteq \bigcup\{Q \mid Q \in Y\}$, and the hypothesis guarantees that $I \subseteq \overline{Q}$ for some $\overline{Q} \in Y$. This is equivalent to $\overline{Q} \notin \mathcal{D}_{\mathrm{Spec}}(I)$, and thus $Y \nsubseteq \Omega$, a contradiction; therefore, P must be in Y, and $Y = \zeta \circ \mathscr{P}(Y)$.

(iii) $\Longrightarrow$ (v) because, for every Ω, $\boldsymbol{\mathcal{U}}(\Omega) = \boldsymbol{\mathcal{U}}(\mathcal{D}_{\mathrm{Spec}}(aR))$ for some $a \in R$, and the $\boldsymbol{\mathcal{U}}(\Omega)$ form a basis; to show (v) $\Longrightarrow$ (iii), let J be a finitely generated ideal of R. Then, $\boldsymbol{\mathcal{U}}(\mathcal{D}_{\mathrm{Spec}}(J)) = \bigcup_{a \in A} \boldsymbol{\mathcal{U}}(\mathcal{D}_{\mathrm{Spec}}(aR))$ for some $A \subseteq R$. Since $\mathcal{D}_{\mathrm{Spec}}(J) \in \boldsymbol{\mathcal{U}}(\mathcal{D}_{\mathrm{Spec}}(J))$, it follows that $\mathcal{D}_{\mathrm{Spec}}(J) \in \boldsymbol{\mathcal{U}}(aR) \subseteq \boldsymbol{\mathcal{U}}(\mathcal{D}_{\mathrm{Spec}}(J))$ for some $a \in A$. But $\mathcal{D}_{\mathrm{Spec}}(J)$ is the maximum of $\boldsymbol{\mathcal{U}}(\mathcal{D}_{\mathrm{Spec}}(J))$, and thus it must be $\boldsymbol{\mathcal{U}}(aR) = \boldsymbol{\mathcal{U}}(\mathcal{D}_{\mathrm{Spec}}(J))$; that is, $\mathrm{rad}(aR) = \mathrm{rad}(J)$. □

2.5.1.2. Semigroup primes of the Nagata ring

Given a ring R, a polynomial $f(T) \in R[T]$ is *primitive* if its content $\boldsymbol{c}(R)$ is equal to R. By the Dedekind-Mertens formula (see e.g. [50, Corollary 28.3], or [72, Section 1.7]), the set of primitive polynomials is a saturated multiplicatively closed set of $R[T]$; more precisely, it is the complement in $R[T]$ of $\bigcup\{M[T] \mid M \in \mathrm{Max}(R)\}$. The localization of R with respect to the set of primitive polynomials is called the *Nagata ring* of R, and it is denoted by $\mathrm{Na}(R)$ or by $R(T)$ [50, Section 33]. The construction of the Nagata ring shares many similarities with the construction of the Kronecker function ring (see Section 2.3.2.1), and, as the latter, can be generalized to construct a ring $\mathrm{Na}(R, *)$, where $*$ is a spectral semistar operation of finite type [45].

Proposition 2.159. *Let R be a ring. Then, the map*

$$\begin{aligned} \eta\colon \boldsymbol{\mathcal{X}}(R) &\longrightarrow \boldsymbol{\mathcal{S}}(R(T)) \\ Y &\longmapsto \bigcup\{PR(T) \mid P \in Y\} \end{aligned}$$

is a spectral embedding.

Proof. To show that η is continuous, let $\mathcal{D}_{\boldsymbol{\mathcal{S}}}(f/p)$ be a basic open set of $\boldsymbol{\mathcal{S}}(R(T))$, where $f, p \in R[T]$ and p is primitive. Then,

$$\begin{aligned}\eta^{-1}\left(\mathcal{D}_{\boldsymbol{\mathcal{S}}}\left(\tfrac{f}{p}\right)\right) &= \{Y \in \boldsymbol{\mathcal{X}}(R) \mid \tfrac{f}{p} \notin \bigcup\{PR(T) \mid P \in Y\}\} = \\ &= \{Y \in \boldsymbol{\mathcal{X}}(R) \mid f \notin PR[T] \text{ for all } P \in Y\} = \boldsymbol{\mathcal{U}}(\mathcal{D}_{\mathrm{Spec}}(\boldsymbol{c}(f))),\end{aligned}$$

and thus η is continuous and spectral. If now $f_0, \dots, f_n \in R$, define $f := f_0 + f_1T + \cdots + f_nT$; then,

$$\eta(\mathcal{D}_{\mathrm{Spec}}(f_0, \dots, f_n)) = \boldsymbol{\mathcal{U}}\left(\mathcal{D}_{\mathrm{Spec}}\left(\frac{f}{1}\right)\right) \cap \eta(\boldsymbol{\mathcal{X}}(R)),$$

and so η is open when seen as a map $\boldsymbol{\mathcal{X}}(R) \longrightarrow \eta(\boldsymbol{\mathcal{X}}(R))$.

To complete the proof, we have to show that η is injective. Suppose $\eta(Y) = \eta(Z)$ for some $Y, Z \in \boldsymbol{\mathcal{X}}(R)$, $Y \neq Z$: then, there is a finitely generated ideal $J = (j_0, \dots, j_n)R$ such that $Y \subseteq \mathcal{D}_{\mathrm{Spec}}(J)$ but $Z \nsubseteq \mathcal{D}_{\mathrm{Spec}}(J)$. The latter condition implies that $J \subseteq P$ for some $P \in Z$; therefore, $J \subseteq \eta(Z)$ and so $J \subseteq \eta(Y)$. Consider $f := j_0 + j_1T \cdots + j_nT^n$; then, $f \in \eta(Y)$ as well, and thus $f \in QR(T)$ for some $Q \in Y$. However, this implies that $\boldsymbol{c}(f) = J \subseteq Q$, i.e., $Q \notin \mathcal{D}_{\mathrm{Spec}}(J)$. This contradicts $Y \subseteq \mathcal{D}_{\mathrm{Spec}}(J)$, and thus η is injective. □

The embedding η ties well with the other maps we considered.

Proposition 2.160. *Let R be a ring, and let $g : R \longrightarrow R(T)$ be the localization map; let ζ, χ, $\mathscr{P}$, i denote the usual maps, and define*

$$\begin{aligned}\omega\colon \boldsymbol{\mathcal{S}}(R(T)) &\longrightarrow \boldsymbol{\mathcal{X}}(R)\\ \mathscr{Q} &\longmapsto \{P \in \mathrm{Spec}(R) \mid g(P) \subseteq \mathscr{Q}\}.\end{aligned}$$

Then:

(a) $\omega = \zeta \circ \boldsymbol{\mathcal{S}}(g)$;
(b) $\mathscr{P} = \boldsymbol{\mathcal{S}}(g) \circ \eta$;
(c) $\mathcal{S}(g) \circ \eta \circ \zeta$ *is the identity on* $\boldsymbol{\mathcal{S}}(R)$;
(d) $\omega \circ \eta = \zeta \circ \mathscr{P}$.

Proof. (a) By definition, for every $\mathscr{Q} \in \boldsymbol{\mathcal{S}}(R(T))$,

$$\begin{aligned}\omega(\mathscr{Q}) &= \{P \in \mathrm{Spec}(R) \mid g(P) \subseteq \mathscr{Q}\} = \\ &= \{P \in \mathrm{Spec}(R) \mid P \subseteq g^{-1}(\mathscr{Q})\} = \\ &= \{P \in \mathrm{Spec}(R) \mid P \subseteq \boldsymbol{\mathcal{S}}(g)(\mathscr{Q})\} = \zeta \circ \boldsymbol{\mathcal{S}}(g)(\mathscr{Q}),\end{aligned}$$

as claimed.

$$\begin{tikzcd}
\mathrm{Spec}(R) \arrow[r, hook, "i"] \arrow[rr, bend left, hook, "\chi"] & \boldsymbol{\mathcal{S}}(R) \arrow[r, hook, shift left, "\zeta_R"] & \boldsymbol{\mathcal{X}}(R) \arrow[l, two heads, shift left, "\mathscr{P}_R"] \arrow[r, hook, shift left, "\eta"] & \boldsymbol{\mathcal{S}}(R(T)) \arrow[l, shift left, "\omega"] \arrow[r, hook, shift left, "\zeta_{R(T)}"] \arrow[ll, bend left, two heads, "\boldsymbol{\mathcal{S}}(g)"] & \boldsymbol{\mathcal{X}}(R(T)) \arrow[l, two heads, shift left, "\mathscr{P}_{R(T)}"] \arrow[ll, bend right, two heads, "\boldsymbol{\mathcal{X}}(g^a)"']
\end{tikzcd}$$

Figure 2.4: Maps between $\boldsymbol{\mathcal{S}}$- and $\boldsymbol{\mathcal{X}}$-type spaces.

(b) For any $Y \in \boldsymbol{\mathcal{X}}(R)$, we have

$$\boldsymbol{\mathcal{S}}(g) \circ \eta(Y) = g^{-1}(\eta(Y)) = g^{-1}\left(\bigcup_{P \in Y} PR(T)\right) = \bigcup_{P \in Y} g^{-1}(PR(T)).$$

However, $g^{-1}(PR(T)) = P$ for every $P \in \mathrm{Spec}(R)$ [50, Proposition 33.1(4)], and thus $\boldsymbol{\mathcal{S}}(g) \circ \eta(Y) = \bigcup\{P \mid P \in Y\}$, which is exactly the definition of $\mathscr{P}(Y)$.

(c) By the point above, $\boldsymbol{\mathcal{S}}(g) \circ \eta \circ \zeta = \mathscr{P} \circ \zeta$, which is the identity on $\boldsymbol{\mathcal{S}}(R)$ by Proposition 2.152(b).

(d) Applying again point (b), we have

$$\omega \circ \eta = \zeta \circ \boldsymbol{\mathcal{S}}(g) \circ \eta = \zeta \circ \mathscr{P},$$

as requested. $\square$

2.5.2. Flat overrings and sublocalizations

The cases of flat overrings and of sublocalizations are quite different from the case of quotient rings, since in both cases there is no obvious way to find "canonical" subsets of the base domain D through which represent these overrings; even in the case of sublocalizations, which we can represent by sets of prime ideals, the correspondence is in general not as nice as we would hope. Yet, the results we can obtain are similar to the ones in the previous case. We start by recalling a characterization of flat modules.

Lemma 2.161 [86, Theorem 7.6]. *Let R be a ring and M be a R-module. Then, M is flat over R if and only if, for every $a_i \in R$, $x_i \in M$ (for $1 \leq i \leq n$) such that*

$$a_1x_1 + \cdots + a_nx_n = 0$$

there are an integer N and elements $b_{ik} \in R$, $y_k \in M$ (for $1 \leq k \leq N$) such that

$$\begin{cases} a_1b_{1k} + \cdots + a_nb_{nk} = 0 & \text{for all } k \\ x_i = b_{i1}y_1 + \cdots + b_{iN}y_N & \text{for all } i. \end{cases}$$

Proposition 2.162. *Let D be an integral domain. Then,* $\mathrm{Over}_{\mathrm{flat}}(D)$ *is a proconstructible subspace of* $\mathrm{Over}(D)$ *if and only if* $\mathrm{Over}_{\mathrm{flat}}(D) \cap B(x_1, \dots, x_n)$ *is compact for every $x_1, \dots, x_n \in K$.*

Proof. If $\mathrm{Over}_{\mathrm{flat}}(D)$ is proconstructible, the compactness of $\mathrm{Over}_{\mathrm{flat}}(D) \cap B(x_1, \dots, x_n)$ follows like in the proof of Proposition 2.142.

Suppose that the compactness property holds, and let $x_1, \dots, x_n \in K$. Consider the canonical subbase $\mathcal{S} := \{B(x) \cap X \mid x \in K\}$ of $X := \mathrm{Over}_{\mathrm{flat}}(D)$, and let $\mathscr{U}$ be an ultrafilter on X. By Proposition 2.111, we need to show that $A_{\mathscr{U}} := \{x \in K \mid B(x) \cap X \in \mathscr{U}\}$ is flat.

Let $a_1, \dots, a_n \in D$, $x_1, \dots, x_n \in A_{\mathscr{U}}$ such that $a_1 x_1 + \cdots + a_n x_n = 0$. For all $C \in \mathrm{Over}_{\mathrm{flat}}(D) \cap B(x_1, \dots, x_n)$, by Lemma 2.161 there are $b_{jk}^{(C)} \in D$, $y_k^{(C)} \in C$ such that

$$\begin{cases} a_1 b_{1k}^{(C)} + \cdots + a_n b_{nk}^{(C)} = 0 & \text{for all } k \\ x_i = b_{i1}^{(C)} y_1^{(C)} + \cdots + b_{iN}^{(C)} y_N^{(C)} & \text{for all } i. \end{cases} \tag{2.8}$$

Let $\Omega(C) := B(y_1^{(C)}, \dots, y_{n_C}^{(C)})$. Then, the family of the $\Omega(C)$ is an open cover of $\mathrm{Over}_{\mathrm{flat}}(D) \cap B(x_1, \dots, x_n)$. Hence, there is a finite subcover $\{\Omega(C_1), \dots, \Omega(C_n)\}$; by the properties of ultrafilters, it follows that $\Omega(C_j) \in \mathscr{U}$ for some j. Thus, $y_i^{(C_j)} \in A_{\mathscr{U}}$ for all i; then, (2.8) holds in $A_{\mathscr{U}}$. Hence, $A_{\mathscr{U}}$ is flat, applying again Lemma 2.161. □

Corollary 2.163. *Let D be an integral domain such that* $\mathrm{Over}_{\mathrm{flat}}(D) = \mathrm{Over}_{\mathrm{sloc}}(D)$. *Then,* $\mathrm{Over}_{\mathrm{flat}}(D)$ *is a proconstructible subset of* $\mathrm{Over}(D)$. *In particular, D is rad-colon coherent.*

Proof. For any $x_1, \dots, x_n$, the set $\mathrm{Over}_{\mathrm{sloc}}(D) \cap B(x_1, \dots, x_n)$ has a minimum (the intersection of all localizations of D containing $x_1, \dots, x_n$), and so it is compact. The first claim follows from the previous proposition and the hypothesis $\mathrm{Over}_{\mathrm{sloc}}(D) = \mathrm{Over}_{\mathrm{flat}}(D)$; the second follows from Proposition 2.143. □

Corollary 2.163 can also be seen in the other way: if $\mathrm{Over}_{\mathrm{sloc}}(D) = \mathrm{Over}_{\mathrm{flat}}(D)$, then $\mathrm{Over}_{\mathrm{sloc}}(D)$ is proconstructible, and in particular a spectral space. However, the ideas of the proof of Corollary 2.163 can be used to say more, showing a striking difference between sublocalizations and the other classes of rings.

Proposition 2.164. *Let D be an integral domain. Then,* $\mathrm{Over}_{\mathrm{sloc}}(D)$ *is a spectral space if and only if it is proconstructible in* $\mathrm{Over}(D)$.

Proof. If $\mathrm{Over}_{\mathrm{sloc}}(D)$ is proconstructible, then it is spectral. Conversely, suppose it is spectral. If $\mathcal{B} := \{B_x \mid x \in K\}$ is the canonical subbasis of $\mathrm{Over}(D)$, then as in the proof of Corollary 2.163 we see that $B_x \cap \mathrm{Over}_{\mathrm{sloc}}(D)$ is compact, since it has a minimum, namely the intersection of the localizations of D that contain $D[x]$. By Lemma 2.119, $\mathrm{Over}_{\mathrm{sloc}}(D)$ is proconstructible in $\mathrm{Over}(D)$. □

We are now tasked to study the spectrality of $\mathrm{Over}_{\mathrm{sloc}}(D)$; the idea is to represent $\mathrm{Over}_{\mathrm{sloc}}(D)$ through $\boldsymbol{\mathcal{X}}(D)$. Indeed, one way to define $\mathrm{Over}_{\mathrm{sloc}}(D)$ is as the image of $\mathrm{SStar}_{sp}(D)$ under the map $\pi : \mathrm{SStar}(D) \longrightarrow \mathrm{Over}(D)$ that sends $*$ to D^*. However, $\mathrm{SStar}_{sp}(D)$ is not a spectral space (or at least, we do not know if it is), while $\mathrm{SStar}_{f,sp}(D) \simeq \boldsymbol{\mathcal{X}}(D)$ is; therefore, to link more closely the latter space with $\mathrm{Over}_{\mathrm{sloc}}(D)$, we introduce the map

$$\begin{aligned} \pi_s\colon \mathrm{SStar}_{f,sp}(D) &\longrightarrow \mathrm{Over}_{\mathrm{sloc}}(D) \\ * &\longmapsto D^*. \end{aligned} \tag{2.9}$$

as the restriction of π to $\mathrm{SStar}_{f,sp}(D)$.

Lemma 2.165. *Let D be an integral domain, and let $*$ be a spectral semistar operation on D.*

(a) If $\mathcal{D}(F \cap D)$ is a compact subset of $\mathrm{Spec}(D)$ for every $F \in \mathcal{F}_f(D)$, then $*_f = \widetilde{*}$.
(b) If D is rad-colon coherent, then $D^{*_f} = D^{\widetilde{*}}$.

Proof. (a) We always have $\widetilde{*} \leq *_f$; since $\widetilde{*}$ is always of finite type, it is enough to show that $F^* = F^{\widetilde{*}}$ if F is finitely generated. Let thus $x \in F^*$, and consider $I := x^{-1}F \cap D$. Then, $xI = F \cap xD \subseteq F$. Moreover,

$$I^* = (x^{-1}F \cap D)^* = x^{-1}F^* \cap D^*$$

since $*$ is stable, and thus $1 \in I^*$. Since $x^{-1}F$ is finitely generated, by hypothesis $\mathcal{D}(I)$ is compact, that is, there is a finitely generated ideal J of D such that $\mathrm{rad}(I) = \mathrm{rad}(J)$; passing, if needed, to a power of J, we can suppose $J \subseteq I$, so that $xJ \subseteq xI \subseteq F$. For any semifinite operation $\sharp$, $\mathrm{rad}(A) = \mathrm{rad}(B)$ implies that $1 \in A^\sharp$ if and only if $1 \in B^\sharp$; therefore, $1 \in J^*$, and thus $x \in (F : J) \subseteq F^{\widetilde{*}}$, and $x \in F^{\widetilde{*}}$. Therefore, $*_f = \widetilde{*}$, as requested.

(b) It is enough to repeat the proof of the previous point by using $F = D$, and noting that $\mathcal{D}(x^{-1}D \cap D)$ is compact since D is rad-colon coherent. □

To complete our reasoning we need a topological result, which can be seen as a slightly generalized form of [29, Proposition 9] (using a subbase instead of all open and compact subspaces).

Proposition 2.166. *Let $\phi : X \longrightarrow Y$ be a continuous surjective map between two topological spaces. Suppose that:*

(a) X is spectral;
(b) Y is T_0;
(c) there is a subbasis $\mathcal{C}$ of Y such that, for every $C \in \mathcal{C}$, $\phi^{-1}(C)$ is compact.

Then, Y is a spectral space.

Proof. Let $\mathscr{U}$ be an ultrafilter on Y. Then, the set $\phi^{-1}(\mathscr{U}) := \{\phi^{-1}(U) \mid U \in \mathscr{U}\}$ has the finite intersection property, and thus there is an ultrafilter $\mathscr{U}_0$ on X that contains $\phi^{-1}(\mathscr{U})$. We claim that, for every $U \subseteq Y$, $\phi^{-1}(U) \in \mathscr{U}_0$ if and only if $U \in \mathscr{U}$. One implication is clear; suppose $\phi^{-1}(U) \in \mathscr{U}_0$ but $U \notin \mathscr{U}$. Then, $V := Y \setminus U \in \mathscr{U}$, and so $\phi^{-1}(V) \in \phi^{-1}(\mathscr{U}) \subseteq \mathscr{U}_0$. This would imply that $\phi^{-1}(U) \cap \phi^{-1}(V) \in \mathscr{U}_0$; however, $\phi^{-1}(U) \cap \phi^{-1}(V) = \phi^{-1}(U \cap V) = \emptyset$, which is absurd.

Consider now the subbasis $\mathcal{B}$ of X composed of the open and compact subsets of X (it exists because X is spectral). Still by spectrality, there is an $x \in X_{\mathcal{B}}(\mathscr{U}_0)$; we claim that $y := \phi(x) \in Y_{\mathcal{C}}(\mathscr{U})$. Indeed, let $C \in \mathcal{C}$.

If $y \in C$, then $x \in \phi^{-1}(C)$; by hypothesis, $\phi^{-1}(C)$ is in $\mathcal{B}$, and so, by definition of $X_{\mathcal{B}}(\mathscr{U}_0)$, $\phi^{-1}(C) \in \mathscr{U}_0$. By the previous claim, $C \in \mathscr{U}$.

Conversely, if $C \in \mathscr{U}$ then $\phi^{-1}(C) \in \mathscr{U}_0$, and so $x \in \phi^{-1}(C)$, i.e., $y \in \phi(\phi^{-1}(C)) = C$. Therefore, $y \in Y_{\mathcal{C}}(\mathscr{U})$, and Y is a spectral space. □

Proposition 2.167. *Let D be an integral domain. If D is rad-colon coherent, then* $\mathrm{Over}_{\mathrm{sloc}}(D)$ *is a spectral space.*

Proof. Suppose D is rad-colon coherent, and consider the map π_s defined in (2.9). If $T \in \mathrm{Over}_{\mathrm{sloc}}(D)$, then there is a $\sharp \in \mathrm{SStar}_{sp}(D)$ such that $T = D^\sharp$; since D is D-finitely generated, moreover, we have $D^\sharp = D^{\sharp_f}$. By Lemma 2.165(b), $D^{\sharp_f} = D^{\widetilde{\sharp}}$; but $\widetilde{\sharp} \in \mathrm{SStar}_{f,sp}(D)$, and thus π_s is surjective.

We know that $\mathrm{SStar}_{f,sp}(D)$ is a spectral space (Theorem 2.45) and that $\mathrm{Over}_{\mathrm{sloc}}(D)$ is always a T_0 space; consider the subbase $\{B(x) \cap \mathrm{Over}_{\mathrm{sloc}}(D) \mid x \in K\}$ of $\mathrm{Over}_{\mathrm{sloc}}(D)$. Then,

$$\begin{aligned}
\pi_s^{-1}(B(x) \cap \mathrm{Over}_{\mathrm{sloc}}(D)) &= \{* \in \mathrm{SStar}_{f,sp}(D) \mid x \in D^*\} = \\
&= \{* \in \mathrm{SStar}_{f,sp}(D) \mid 1 \in x^{-1}D^*\} = \\
&= \{* \in \mathrm{SStar}_{f,sp}(D) \mid 1 \in (x^{-1}D)^*\} = \\
&= \{* \in \mathrm{SStar}_{f,sp}(D) \mid 1 \in (x^{-1}D \cap D)^*\} = \\
&= V_{x^{-1}D \cap D} \cap \mathrm{SStar}_{f,sp}(D) = V_{(D:_D x)} \cap \mathrm{SStar}_{f,sp}(D)
\end{aligned}$$

since $x^{-1}D \cap D = (D :_D x)$. If $U_I := V_I \cap \mathrm{SStar}_{f,sp}(D)$, then $U_I = U_J$ if and only if $\mathrm{rad}(I) = \mathrm{rad}(J)$, and U_I is compact if I is finitely generated; therefore, if D is rad-colon coherent then $\pi_s^{-1}(B(x) \cap \mathrm{Over}_{\mathrm{sloc}}(D))$ is compact for every $x \in K$. By Proposition 2.166, it follows that $\mathrm{Over}_{\mathrm{sloc}}(D)$ is spectral. □

Corollary 2.168. *If D is a domain with Noetherian spectrum (in particular, if it is Noetherian) then* $\mathrm{Over}_{\mathrm{sloc}}(D)$ *is a spectral space.*

Proposition 2.169. *Let D be a Prüfer domain. Then, π_s establishes a homeomorphism between* $\mathrm{SStar}_{f,sp}(D)$ *and* $\mathrm{Over}(D)$.

Proof. By composing it with the inclusion of $\mathrm{Over}_{\mathrm{sloc}}(D)$ into $\mathrm{Over}(D)$, we can consider π_s as a map $\mathrm{SStar}_{f,sp}(D) \longrightarrow \mathrm{Over}(D)$.

A Prüfer domain is rad-colon coherent, and thus, by the proof of Proposition 2.167, π_s is a continuous map whose image is $\mathrm{Over}_{\mathrm{sloc}}(D)$; however, since D is a Prüfer domain, $\mathrm{Over}_{\mathrm{sloc}}(D) = \mathrm{Over}(D)$ [50, Theorem 26.2], and thus π_s is surjective onto $\mathrm{Over}(D)$. If now $*_1, *_2 \in \mathrm{SStar}_{f,sp}(D)$ and $\pi_s(*_1) = \pi_s(*_2) = T$, then at least one between $*_1$ and $*_2$, when restricted to $\mathbf{F}(T)$, becomes a semistar operation $\sharp$ of finite type on T that is different from the identity, and it is such that $T^\sharp = T$. However, every finitely-generated fractional ideal of T is invertible, and thus $\sharp$-closed; it would follow that $\sharp$ is the identity, a contradiction. Hence, π_s is injective.

Finally, if I is a finitely-generated ideal of D,

$$\pi_s(U_I) = \{\pi_s(*) \mid I \subseteq D^*\} = \{D^* \mid I \subseteq D^*\} = B_I,$$

and thus π_s is open. Hence, π_s is a homeomorphism. □

Note that, even if π_s may not be surjective, its range contains always the flat overrings:

Proposition 2.170. *Let D be an integral domain. Then,* $\mathrm{Over}_{\mathrm{flat}}(D) \subseteq \pi_s(\mathrm{SStar}_{f,sp}(D))$.

Proof. Let $T \in \mathrm{Over}_{\mathrm{flat}}(D)$, and let $\Delta := \mathrm{QSpec}^t(T)$. Then, $T = \bigcap\{T_P \mid P \in \Delta\}$. Since T is flat, T_P is a localization of D [105, Theorem 2]; precisely, $T_P = D_{\phi(Q)}$, where $\phi : \mathrm{Spec}(T) \longrightarrow \mathrm{Spec}(D)$ is the canonical map associated to the inclusion $D \hookrightarrow T$. Thus, $T = \bigcap\{D_Q \mid Q \in \phi(\Delta)\}$; but now Δ is compact and ϕ is continuous, so $s_{\phi(\Delta)}$ is of finite type and $T = \pi_s(s_{\phi(\Delta)})$. □

A further question about $\mathrm{Over}_{\mathrm{sloc}}(D)$ is if we can find a way to represent it, in a way similar to how $\boldsymbol{\mathcal{X}}(D)$ represent the spectral semistar operations of finite type. The map π_s is one possibility; however, even when it is surjective, it is usually very far from being injective; indeed, there are in general many ways to represent the ring itself D as a compact intersection of localizations. For example, if D is a Krull domain (e.g., a Noetherian integrally closed domain), and Δ is a subset of the spectrum containing the height-1 primes, then the intersection of the localizations at elements of Δ is D. One way to get around this problem is to use, instead of the whole spectrum, the t-spectrum, that is, the set of prime ideals that are closed by the t-operation (which we recall being the finite-type star – or semistar – operation associated to the v-operation $I \mapsto (D : (D : I))$). Note that $\mathrm{QSpec}^t(D)$ is a proconstructible subspace of $\mathrm{Spec}(D)$ [19, Proposition 2.5], and thus in particular a spectral space: it follows that the space $\boldsymbol{\mathcal{X}}(\mathrm{QSpec}^t(D))$ is defined and spectral.

More explicitly, we will use the map

$$\begin{aligned}\pi_t\colon \boldsymbol{\mathcal{X}}(\mathrm{QSpec}^t(D)) &\longrightarrow \mathrm{Over}_{\mathrm{sloc}}(D)\\ \Delta &\longmapsto D^{s_\Delta}.\end{aligned}$$

Note that π_t is continuous, since it is the composition of the inclusion $\boldsymbol{\mathcal{X}}(\mathrm{QSpec}^t(D)) \hookrightarrow \boldsymbol{\mathcal{X}}(D)$, the homeomorphism $\boldsymbol{\mathcal{X}}(D) \longrightarrow \mathrm{SStar}_{f,sp}(D)$ and the map $\pi_s : \mathrm{SStar}_{f,sp}(D) \longrightarrow \mathrm{Over}(D)$.

We first show that, using π_t instead of π_s, we do not lose anything.

Proposition 2.171. *Let D be an integral domain. Then,*

$$\pi_s(\mathrm{SStar}_{f,sp}(D)) = \pi_t(\boldsymbol{\mathcal{X}}(\mathrm{QSpec}^t(D))).$$

Proof. One containment is obvious. Let $T = \pi_s(s_\Delta)$ for some $\Delta \in \boldsymbol{\mathcal{X}}(D)$; then, Δ is a proconstructible subset of $\mathrm{Spec}(D)$, so that $\Delta_0 := \Delta \cap \mathrm{QSpec}^t(D)$ is proconstructible. In particular, Δ_0 is compact, and so belongs to $\boldsymbol{\mathcal{X}}(\mathrm{QSpec}^t(D))$. We claim that $T = \pi_t(\Delta_0)$.

Indeed, let $P \in \Delta$. Then, $t_P : ID_P \mapsto I^t D_P$ is a star operation of finite type on D_P (see [67] or Definition 3.1 and Proposition 3.3 in Chapter 3), and QD_P is a maximal t_P-ideal if and only if Q is maximal among the t-prime ideals contained in P. Hence, $D_P = \bigcap\{D_Q \mid Q \subseteq P, Q = Q^t\}$, and

$$T = \bigcap\{D_Q \mid Q = Q^t, Q \subseteq P \text{ for some } P \in \Delta\}.$$

The set of primes on the right hand side is exactly Δ_0, since Δ is closed by generizations. Therefore, $T = \pi_t(\Delta_0) \in \pi_t(\boldsymbol{\mathcal{X}}(\mathrm{QSpec}^t(D)))$. □

Corollary 2.172. *Let D be an integral domain. If D is rad-colon coherent, then π_t is surjective.*

Proof. It follows directly from Propositions 2.167 and 2.171. □

The t-spectrum is much less redundant than $\mathrm{Spec}(D)$: indeed, if $D = \bigcap\{D_P \mid P \in \Delta\}$ for some compact $\Delta \subseteq \mathrm{QSpec}^t(D)$, then Δ must contain the t-maximal ideals, since t is the biggest star (or, equivalently, (semi)star) operation of finite type. In general, we are not able to prove that π_t is always injective, but we can prove some special cases. Recall that a domain is *v-coherent* if, for any ideal I, $(D : I) = (D : J)$ for some finitely generated ideal J.

Proposition 2.173. *Let D be a v-coherent domain. Then, π_t is injective.*

Proof. We claim that, for every $Q \in \mathrm{Spec}(D)$, $D_Q = \bigcap\{D_P \mid P \subsetneq Q\}$ if and only if $Q \neq Q^t$. Indeed, D_Q is equal to the intersection if and only if QD_Q is not t_{D_Q}-closed, which happens if and only if $(QD_Q)^{t_{D_Q}} = D_Q$. But v-coherence imply that $(ID_Q)^{t_{D_Q}} = I^t D_Q$ for every ideal I of D (Lemma 3.40), and thus this condition is equivalent to $Q^t D_Q = D_Q$, i.e., to $Q^t \neq Q$.

Suppose now that $\pi_t(\Lambda_1) = \pi_t(\Lambda_2)$ for some $\Lambda_1, \Lambda_2 \in \boldsymbol{\mathcal{X}}(\mathrm{QSpec}^t(D))$; without loss of generality there is a $Q \in \Lambda_2 \setminus \Lambda_1$. Since Λ_1 is compact, we have, by Theorem 2.76,

$$\left(\bigcap_{P \in \Lambda_1} D_P\right) D_Q = \bigcap_{P \in \Lambda_1} D_P D_Q = \bigcap_{P \in \Delta} D_P,$$

where Δ contains only prime ideals that are properly contained in Q. However, the leftmost side is exactly $TD_Q = D_Q$; it would follow that $D_Q = \bigcap\{D_P \mid P \in \Delta\}$, and by the first part of the proof that $Q \neq Q^t$, against the hypothesis. Therefore, π_t is injective. □

The t-dimension $\dim_t(D)$ of D is defined as the supremum of the length of the chain of t-prime ideals [66].

Proposition 2.174. *Let D be an integral domain; suppose that $\mathrm{QSpec}^t(D)$ is Noetherian and that $\dim_t(D) \leq 2$. Then, π_t is injective.*

Proof. Suppose that $T = \pi_t(\Lambda_1) = \pi_t(\Lambda_2)$ for some $\Lambda_1, \Lambda_2 \in \boldsymbol{\mathcal{X}}(\mathrm{QSpec}^t(D))$, and suppose (without loss of generality) that there is a $Q \in \Lambda_2 \setminus \Lambda_1$. We distinguish two cases.

1. Q does not contain other nonzero t-primes; in particular, Q must have height 1, since a prime minimal over a principal ideal is t-closed. Then,
$$D_Q = TD_Q = \left(\bigcap_{P\in\Lambda_1} D_P\right) D_Q = \bigcap_{P\in\Lambda_1} D_P D_Q.$$
In particular, Λ_1 cannot contain any prime containing Q (for otherwise $Q \in \mathrm{Cl}^{\mathrm{inv}_t}(\Lambda_1) = \Lambda_1$, where inv_t denotes the inverse topology in $\mathrm{QSpec}^t(D)$). Hence, since $h(Q) = 1$, we have $D_P D_Q = K$ for all $P \in \Lambda_1$. But this means $D_Q = K$, which is absurd.
2. Q contains nonzero t-primes: since $\dim_t(D) \leq 2$, this means that Q is t-maximal. Since $\mathrm{QSpec}^t(D)$ is Noetherian, $\Delta := \mathrm{QSpec}^t(D) \setminus \{Q\}$ is compact, and so it is in $\boldsymbol{\mathcal{X}}(\mathrm{QSpec}^t(D))$; moreover, $\Lambda_1 \subseteq \Delta$. Therefore,
$$\bigcap_{P\in\Delta} D_P = \bigcap_{P\in\Lambda_1} D_P \cap \bigcap_{P\in\Delta} D_P = T \cap \bigcap_{P\in\Delta} D_P \subseteq$$
$$\subseteq D_Q \cap \bigcap_{P\in\Delta} D_P = \bigcap_{P\in\mathrm{QSpec}^t(D)} D_P = D.$$
Thus, Δ induces on D a spectral star operation of finite type which is strictly smaller than $w = \widetilde{t}$ (since Q is t-maximal, hence w-maximal). But this is absurd.

Hence, Q cannot exist and π_t is injective. □

Beside surjectivity and injectivity, the last property we need to transform π_t into a homeomorphism is openess; for the following results, we denote by t_T and w_T, respectively, the t and the w-operation on T, and we denote by $\boldsymbol{\mathcal{U}}_t(\mathcal{D}(I))$ the subbasic open set of $\boldsymbol{\mathcal{X}}(\mathrm{QSpec}^t(D))$ associated to $\mathcal{D}(I)$, i.e.,
$$\boldsymbol{\mathcal{U}}_t(\mathcal{D}(I)) := \{\Delta \in \boldsymbol{\mathcal{X}}(\mathrm{QSpec}^t(D)) \mid \Delta \subseteq \mathcal{D}(I)\}.$$

Lemma 2.175. *Let D be a rad-colon coherent integral domain, and I an integral ideal of D; let $T \in \pi_t(\boldsymbol{\mathcal{X}}(\mathrm{QSpec}^t(D)))$. Then, $T \in \pi_t(\boldsymbol{\mathcal{U}}_t(\mathcal{D}(I)))$ if and only if $(IT)^{t_T} = T$.*

Proof. Let $T \in \pi_t(\boldsymbol{\mathcal{X}}(\mathrm{QSpec}^t(D)))$.

Suppose $(IT)^{t_T} = T$. The semistar operation $\sigma(t_T)$ (where $\sigma : \mathrm{SStar}(T) \longrightarrow \mathrm{SStar}(D)$ is the canonical map) is a semistar operation of finite type on D; let $* := \Psi_w(\sigma(t_T))$. Then, $D^* = T = T^*$ by Lemma 2.165; moreover,
$$T \supseteq (IT)^* \supseteq (IT : IT) \supseteq T$$

and thus $(IT)^* = T$. Hence, if $* = s_\Delta$ for some $\Delta \in \mathrm{QSpec}^t(D)$, then I cannot be contained in any $P \in \Delta$, and thus $\Delta \subseteq \mathcal{D}(I)$. Hence, $T = \pi_t(\Delta) \subseteq \pi_t(\boldsymbol{\mathcal{U}}_t(\mathcal{D}(I)))$.

Conversely, suppose $T = \pi_t(\Delta)$ for some $\Delta \in \boldsymbol{\mathcal{U}}_t(\mathcal{D}(I))$, i.e., for some $\Delta \subseteq \mathcal{D}(I)$. Then, $T = D^{s_\Delta}$, and thus $\sharp := s_\Delta|_{\mathbf{F}(T)}$ is a (semi)star operation of finite type on T. Therefore, $\sharp \leq t_T$; but $\Delta \subseteq \mathcal{D}(I)$ implies $1 \in I^\sharp$, and so $1 \in (IT)^{t_T}$. □

Proposition 2.176. *Let D be an integral domain such that $\mathrm{QSpec}^t(D)$ is Noetherian and $\dim_t(D) = 1$. Then, π_t is a homeomorphism between $\boldsymbol{\mathcal{X}}(\mathrm{QSpec}^t(D))$ and $\mathrm{Over}_{\mathrm{sloc}}(D)$.*

Proof. A reasoning analogue to the proof of Proposition 2.171 shows that, if $\mathrm{QSpec}^t(D)$ is Noetherian, then π_t is surjective; by Proposition 2.174, π_t is injective.

If now T is a sublocalization of D, then by Proposition 2.171 T is the intersection of a family of localizations at t-primes of D, that are also localizations of T; it follows that $\dim_t(T) \leq \dim_t(D) = 1$, and if $T = \pi_t(\Delta)$ then the t-ideals of T are those in the form $\phi_T^{-1}(P)$, where $\phi_T : \mathrm{Spec}(T) \longrightarrow \mathrm{Spec}(D)$ is the canonical map and P ranges in Δ. It follows that $1 \in (IT)^{t_T}$ if and only if $1 \in I^tT$. We claim that, for every integral ideal J of D, $\{T \in \mathrm{Over}(D) \mid JT = T\}$ is an open set. Indeed, if $JT = T$, then $1 = j_1t_1 + \cdots + j_nt_n$ for some $j_i \in J$, $t_i \in T$; however, $B_{t_1,\ldots,t_n}$ is an open neighborhood of T contained in $\{T \mid JT = T\}$, and thus the latter set is open.

Therefore, by Lemma 2.175, π_t is open. □

Proposition 2.177. *Let D be a Noetherian integral domain. If $\mathrm{Over}_{\mathrm{sloc}}(D) = \mathrm{Over}_{\mathrm{flat}}(D)$, then π_t is a homeomorphism.*

Proof. If D is Noetherian, then it is rad-colon coherent and v-coherent; therefore, π_t is surjective by the proof of Propositions 2.164 and 2.171 and injective by Proposition 2.173. If every sublocalization is flat, then $(IT)^t = I^tT$ by v-coherence (Lemma 3.40), and so the claim follows as in the previous proposition. □

3. Local and global properties of star operations

3.1. Extendable star operations

The starting point of this chapter is the notion of localization of a star operation, originally defined in [67]; another inspiration is [68], which uses the concept somewhat implicitly. We will adopt a more general and more abstract approach.

Definition 3.1. *Let R be an integral domain and T a flat overring of R. We say that a star operation $* \in \mathrm{Star}(R)$ is* extendable to T *if the map*

$$\begin{aligned} *_T\colon \mathcal{F}(T) &\longrightarrow \mathcal{F}(T) \\ IT &\longmapsto I^*T \end{aligned} \tag{3.1}$$

is well-defined. We denote the set of star operation on R that are extendable to T as $\mathrm{ExtStar}(R;T)$*.*

Remark 3.2.

(1) If T is flat over R, then every fractional ideal of T is an extension of a fractional ideal of R (since, if J is an integral ideal of T, $J = (J \cap R)T$); therefore, $*_T$ is (potentially) defined on all $\mathcal{F}(T)$.
(2) When $T = S^{-1}R$ is a localization of R and $*$ is of finite type, Definition 3.1 coincide with the definition of $*_S$ given in [67, Proposition 2.4].
(3) When $T = S^{-1}R$, we denote $*_T$ with $*_S$; when $T = R_P$ for some $P \in \mathrm{Spec}(R)$, we denote $*_T$ with $*_P$.

The following proposition shows the basic properties of extendability.

Proposition 3.3. *Let R be an integral domain, let $* \in \mathrm{Star}(R)$ and let T be a flat overring of R.*

(a) If $*$ is extendable to T, then $*_T$ is a star operation.
(b) $*$ is extendable to T if and only if $I^*T = J^*T$ whenever $IT = JT$.
(c) The identity star operation d is always extendable, and d_T is the identity on T.
(d) If $*$ is of finite type, then it is extendable to T, and $*_T$ is of finite type. (Compare [67, Proposition 2.4].)

Proof. (a) and (c) are obvious, while (b) is just a reformulation of Definition 3.1.

For (d), by symmetry it is enough to show that $J^*T \subseteq I^*T$, or equivalently that $1 \in (I^*T : J^*T)$. Indeed,

$$(I^*T : J^*T) = \left(I^*T : \left(\sum_{\substack{L\subseteq J \\ L \text{ finitely generated}}} L^* \right) T \right) = \left(I^*T : \sum_{\substack{L\subseteq J \\ L \text{ finitely generated}}} L^*T \right) =$$
$$= \bigcap_{\substack{L\subseteq J \\ L \text{ finitely generated}}} (I^*T : L^*T) \supseteq \bigcap_{\substack{L\subseteq J \\ L \text{ finitely generated}}} (I^* : L^*)T.$$

By properties of star operations, $(I^* : L^*) = (I^* : L)$; since L is finitely generated and T is flat, it follows that, for every L,

$$(I^* : L^*)T = (I^* : L)T = (I^*T : LT)$$

which contains 1 since $LT \subseteq JT = IT \subseteq I^*T$. Hence, $1 \in (I^*T : J^*T)$, as requested. □

Example 3.4. Not every star operation is extendable: let R be an almost Dedekind domain (i.e., a one-dimensional non-Noetherian domain such that R_M is a discrete valuation ring for every $M \in \mathrm{Max}(R)$), and suppose that R is not Dedekind; let P be a non-finitely generated prime ideal of R. Then P is not divisorial by [43, Lemma 4.1.8], and thus the v-operation is not extendable to R_P, since otherwise $(PR_P)^{v_P} = P^v R_P = R_P$, while the unique star operation on R_P is the identity.

Beside being of finite type, extension preserves the main properties of a star operation. For the definition of stable and spectral operations see Chapter 2; a star operation $*$ on R is *Noetherian* (a more common terminology is to say that R is $*$-*Noetherian*; see e.g. [4, 115, 31]) if the set of proper $*$-closed ideals satisfies the ascending chain condition.

Proposition 3.5. *Let R be a domain and T be a flat overring of R; suppose $* \in \mathrm{Star}(R)$ is extendable to T. If $*$ is stable (respectively, spectral, Noetherian) then so is $*_T$.*

Proof. Suppose $*$ is stable, and let $I_1 := J_1T$, $I_2 := J_2T$ be ideals of T. Then,

$$\begin{aligned}(I_1 \cap I_2)^{*_T} &= (J_1T \cap J_2T)^{*_T} = [(J_1 \cap J_2)T]^{*_T} = \\ &= (J_1 \cap J_2)^*T = (J_1^* \cap J_2^*)T = J_1^*T \cap J_2^*T = I_1^{*_T} \cap I_2^{*_T}\end{aligned}$$

and thus $*_T$ is stable.

If $*$ is spectral, it is stable, and thus so is $*_T$. Let now I be a proper $*_T$-closed ideal of T, and let $J := I \cap R$; then, $JT = (I \cap R)T = I$, and thus $J^* \subseteq I^{*_T} \cap R = I \cap R = J$, so that J is a $*$-ideal. By definition, there is a $\Delta \subseteq \mathrm{Spec}(R)$ such that $* = *_\Delta$; hence,

$$J = J^* = \bigcap_{P\in\Delta} JR_P = \bigcap_{P\in\Delta} (I \cap R)R_P = \bigcap_{P\in\Delta} IR_P \cap R_P.$$

In particular, there is a $P \in \Delta$ such that $1 \notin IR_P = ITR_P$; hence, there is a $Q \in \mathrm{Spec}(TR_P)$ such that $ITR_P \subseteq Q$. We claim that $Q_0 := Q \cap T$ is a prime $*_T$-ideal containing I. Indeed, it is clear that $I \subseteq Q_0$; moreover, $Q \cap R \subseteq P$ and $QT = QT \cap T = Q_0$, and thus $Q_0^* = QT \cap T = QT \cap T = Q_0$. Therefore, $*_T$ is also semifinite, and by Proposition 2.31(b) (that holds also for star operations; see [3, Theorem 4]) it is spectral.

Suppose $*$ is Noetherian, and let $\{I_\alpha T \mid \alpha \in A\}$ be an ascending chain of $*_T$-ideals. Then, $\{I_\alpha^* \mid \alpha \in A\}$ is an ascending chain of $*$-ideals, which has to stabilize at $I_{\overline{\alpha}}^*$. Hence, the original chain stabilizes at $I_{\overline{\alpha}}^* T = (I_{\overline{\alpha}} T)^{*_T} = I_{\overline{\alpha}} T$, and $*_T$ is Noetherian. □

Extendability works well with the order structure of $\mathrm{Star}(R)$.

Proposition 3.6. *Let R be an integral domain and T be a flat overring of R. Let $*_1, *_2, \{*_\lambda \mid \lambda \in \Lambda\}$ be star operations that are extendable to T.*

(a) If $*_1 \leq *_2 \in \mathrm{Star}(R)$, then $(*_1)_T \leq (*_2)_T$.
(b) $*_1 \wedge *_2$ is extendable to T and $(*_1 \wedge *_2)_T = (*_1)_T \wedge (*_2)_T$.
(c) If each $*_\lambda$ is of finite type, then $\sup_\lambda *_\lambda$ is extendable to T and $(\sup_\lambda *_\lambda)_T = \sup_\lambda (*_\lambda)_T$.

Proof. (a) is obvious: if $*_1 \leq *_2$, then $I^{*_1} \subseteq I^{*_2}$ for every fractional ideal I and thus $(I^{*_1}T) \subseteq (I^{*_2}T)$. Using the definition of $*_T$, we get $(*_1)_T \leq (*_2)_T$.

(b) Let I be an ideal of R. By definition, $I^{*_1 \wedge *_2} = I^{*_1} \cap I^{*_2}$, so that

$$(IT)^{(*_1 \wedge *_2)_S} = (I^{*_1 \wedge *_2})T = (I^{*_1} \cap I^{*_2})T =$$

$$= I^{*_1}T \cap I^{*_2}T = (IT)^{(*_1)_T} \cap (IR_T)^{(*_2)_T} = (IT)^{(*_1)_T \wedge (*_2)_T}$$

and thus $(*_1 \wedge *_2)_T = (*_1)_T \wedge (*_2)_T$.

(c) Let $* := \sup_\lambda *_\lambda$. Since each $*_\lambda$ is of finite type, so is $*$, and thus $*$ is extendable to T by Proposition 3.3(d). Moreover, by Lemma 2.20 (that holds also for star operations; see [5, p.1628]), $I^* = \sum I^{*_1 \circ \cdots \circ *_n}$, as $(*_1, \ldots, *_n)$ ranges among the finite strings of elements of $\{*_\lambda \mid \lambda \in \Lambda\}$; therefore,

$$I^*T = \left(\sum I^{*_1 \circ \cdots \circ *_n}\right) T = \sum I^{*_1 \circ \cdots \circ *_n} T.$$

We claim that $I^{*_1 \circ \cdots \circ *_n} T = (IT)^{(*_1)_T \circ \cdots \circ (*_n)_T}$; we proceed by induction. The case $n = 1$ is just the definition of the extension; suppose the claim holds for $n' < n$. Then,

$$I^{*_1 \circ \cdots \circ *_n} T = (I^{*_1})^{*_2 \circ \cdots \circ *_n} T = (I^{*_1}T)^{(*_2)_T \circ \cdots \circ (*_n)_T} = (IT)^{(*_1)_T \circ \cdots \circ (*_n)_T}$$

as claimed. Thus,

$$I^*T = \sum (IT)^{(*_1)_T \circ \cdots \circ (*_n)_T}$$

which is equal to $(IT)^{\sup_\lambda (*_\lambda)_T}$, again by Lemma 2.20. Hence, $* = \sup_\lambda (*_\lambda)_T$. □

Extendability is also transitive:

Proposition 3.7. *Let R be a domain and $T_1 \subseteq T_2$ be two flat overrings of R. If $* \in \mathrm{Star}(R)$ is extendable to T_1 and $*_{T_1}$ is extendable to T_2, then $*$ is extendable to T_2, and $*_{T_2} = (*_{T_1})_{T_2}$.*

Proof. For every ideal I of R, we have

$$I^*T_2 = (I^*T_1)T_2 = (IT_1)^{*_{T_1}}T_2 = (IT_1T_2)^{(*_{T_1})_{T_2}} = (IT_2)^{(*_{T_1})_{T_2}}$$

and thus if $IT_2 = JT_2$ then $I^*T_2 = J^*T_2$, so that $*$ is extendable to T_2. The previous calculation also shows that $*_{T_2} = (*_{T_1})_{T_2}$. □

Proposition 3.8. *Let R be an integral domain and T be a flat overring of R. Let $\Delta := \{M \cap R \mid M \in \mathrm{Max}(T)\}$. If $* \in \mathrm{Star}(R)$ is extendable to every $P \in \Delta$, then it is extendable to T.*

Proof. Let I, J be ideals of R such that $IT = JT$. Let $P \in \Delta$ and let M be the (necessarily unique, since $T_P = R_{P\cap R}$ for all prime ideal P of T [105, Theorem 2]) maximal ideal of T such that $M \cap R = P$. Then, $T_M = R_P$, and since $*$ is extendable to R_P we have $I^*R_P = J^*R_P$. It follows that

$$I^*T = \bigcap_{P\in\Delta} I^*R_P = \bigcap_{P\in\Delta} J^*R_P = J^*T,$$

and thus $*$ is extendable to T. □

Corollary 3.9. *Let R be a domain, and let $* \in \mathrm{Star}(R)$. The following are equivalent:*

(i) $*$ is extendable to R_P, for every $P \in \mathrm{Spec}(R)$;
(ii) $*$ is extendable to every flat overring of R.

Note that condition (i) of the above corollary cannot be replaced by the version that considers only maximal ideals of T: indeed, if (R, M) is local, then clearly every star operation is extendable to R_M, but it would be implausible that every star operation is extendable to every localization. Indeed, we can build an explicit counterexample tweaking slightly [67, Remark 2.5(3)]. Let $R := \mathbb{Z}_{p\mathbb{Z}} + X\mathbb{Q}(\sqrt{2})[[X]]$ (where p is a prime number). Then, R is a two-dimensional local domain, with maximal ideal $M := p\mathbb{Z}_{p\mathbb{Z}} + X\mathbb{Q}(\sqrt{2})[[X]]$; let $P := X\mathbb{Q}(\sqrt{2})[[X]]$. We claim that the v-operation is not extendable to $R_P = \mathbb{Q}+P$. Let $A := X(\mathbb{Q}+P)$ and $B := XR$: then, $AR_P = BR_P = A$, but $A^vR_P = P$ and $B^vR_P = BR_P \neq P$.

Definition 3.10. *Let R be an integral domain and $* \in \mathrm{Star}(R)$. We say that $*$ is* totally extendable *if it is extendable to every flat overring of R. We denote the set of totally extendable star operation by* $\mathrm{ExtStar}(R)$.

We observe that, by Proposition 3.3(d), $\mathrm{Star}_f(R) \subseteq \mathrm{ExtStar}(R)$.

3.1.1. Map between sets of star operations

The notion of extendability, while interesting on the "local" level (i.e., on individual star operations), can be used to study star operations on a "global" level, i.e., to study $\mathrm{Star}(R)$ (or some of its subsets) as a whole. The following proposition revisits the results obtained above from this point of view.

Proposition 3.11. *Let R be an integral domain and T be a flat overring of R. The map*

$$\begin{aligned}\lambda_{R,T}\colon \mathrm{ExtStar}(R;T) &\longrightarrow \mathrm{Star}(T)\\ * &\longmapsto *_T\end{aligned}$$

is a continuous map (in the Zariski topologies) such that:

(a) $\lambda_{R,T}(\mathrm{ExtStar}(R)) \subseteq \mathrm{ExtStar}(T)$*;*
(b) $\lambda_{R,T}(\mathrm{Star}_f(R)) \subseteq \mathrm{Star}_f(T)$*;*
(c) if U is a flat overring of R containing T, then $\lambda_{T,U} \circ \lambda_{R,T} = \lambda_{R,U}$.

Proof. Let $\mathcal{U}^{(T)} := \{U_I^{(T)} \mid I \in \mathcal{F}(T)\}$ be the canonical subbasis of $\mathrm{Star}(T)$, where $U_I^{(T)} := \{* \in \mathrm{Star}(T) \mid 1 \in I^*\}$. Define $U_I^{(R)}$ analogously and $V_I^{(R)} := U_I^{(R)} \cap \mathrm{ExtStar}(R)$. Then,

$$\lambda_{R,T}^{-1}(U_I^{(T)}) = \{* \in \mathrm{ExtStar}(R) \mid 1 \in J^*T\}.$$

Fix a $* \in \lambda_{R,T}^{-1}(U_I^{(T)})$. Then, $1 = \alpha_1 t_1 + \cdots \alpha_n t_n$, where $\alpha_i \in J^*$ for every i; then, $* \in U_{\alpha_i^{-1}J}$ for every i. We claim that $\Omega := U_{\alpha_1^{-1}J} \cap \cdots U_{\alpha_n^{-1}J} \subseteq \lambda_{R,T}^{-1}(U_I^{(T)})$. Indeed, if $\sharp$ is in the intersection, then $1 \in (\alpha_i^{-1}J)^\sharp$ for every i, that is, $\alpha_i \in J^\sharp$. Hence, $1 \in J^\sharp T = (JT)^{\sharp_T}$ and $\sharp_T \in \lambda_{R,T}^{-1}(U_I^{(T)})$, i.e., $\Omega \subseteq \lambda_{R,T}^{-1}(U_I^{(T)})$. Therefore, $\lambda_{R,T}$ is continuous.

The other claims are a translation of Propositions 3.3(d) and 3.7. □

It is a natural question to ask if (or when) $\lambda_{R,T}$ is injective or surjective. The latter does not always hold: for example, let V be a valuation domain whose maximal ideal M is principal, and let P be a prime ideal such that PV_P is not finitely generated. Then, $\mathrm{Star}(V) = \{d\}$, while $\mathrm{Star}(V_P) = \{d, v\}$ (this is essentially proved in [50, Exercise 12, p.431]); in particular, λ_{V,V_P} cannot be surjective.

In [67, Proposition 4.6] it is proved that, if R is *v-coherent* (that is, if, for every finitely generated I, $(R : I) = J^v$ for some finitely generated ideal J), then every finite-type operation on a localization $S^{-1}R$ is an extension from a finite-type star operation on R. Modifying the proof therein, we can generalize this fact to non-finite operations and to flat overrings:

Proposition 3.12. *Let R be a domain and T be a flat overring of R. Then:*

(a) the map $\lambda_{R,T} : \mathrm{ExtStar}(R,T) \longrightarrow \mathrm{Star}(T)$ is surjective if and only if the v-operation on R is in the image of $\lambda_{R,T}$;
(b) the restriction $\lambda'_{R,T} : \mathrm{Star}_f(R,T) \longrightarrow \mathrm{Star}_f(T)$ of $\lambda_{R,T}$ is surjective if and only if the t-operation on R is in the image of $\lambda'_{R,T}$.

Proof. Let $v^{(T)}$ and $t^{(T)}$ denote, respectively, the v- and the t-operation on T.

(a) If $\lambda_{R,T}$ is surjective, then $v^{(T)}$ is in its range. Conversely, suppose $v^{(T)} = \Phi_T(\sharp)$ for some $\sharp \in \Delta$. Let $*^{(T)}$ be a star operation on T, and define $*$ as the closure operation on R which sends I to $(IT)^{*^{(T)}} \cap I^\sharp$. Clearly, $*$ is a star operation.

Let I be an ideal of R. Then, since $v^{(T)}$ is the maximum of $\mathrm{Star}(T)$,

$$I^*T = ((IT)^{*^{(T)}} \cap I^\sharp)T = (IT)^{*^{(T)}} \cap I^\sharp T = (IT)^{*_T} \cap (IT)^{v^{(T)}} = (IT)^{*^{(T)}} \tag{3.2}$$

so that, if $IT = JT$, then $I^*T = J^*T$. Hence, $*$ is extendable to T and $\lambda_{R,T}(*) = *_T$, and $\lambda_{R,T}$ is surjective.

(b) follows in the same way, by substituting $v^{(T)}$ with $t^{(T)}$ and noting that the map $I \mapsto (IT)^{*^{(T)}} \cap I^\sharp$ is of finite type if both $*^{(T)}$ and $\sharp$ are. □

On the other hand, the injectivity of $\lambda_{R,T}$ is much more rare. This is somewhat natural, since $\lambda_{R,T}$ cuts off the behaviour that lies outside the maximal ideals of T or, in another way, it loses the information about what happens to the ideals I such that $IT = T$. We deal with this problem by considering, instead of a single flat overring, a whole family of them. To shorten the notation, we introduce the following definition.

Definition 3.13. *A set Θ of overrings of R is* complete *if $I = \bigcap_{T\in\Theta} IT$ for every ideal I of R.*

The simplest example of complete set of overrings is $\{R_M : M \in \mathrm{Max}(R)\}$; when Θ is a complete family composed of local flat overrings, then $R_M \in \Theta$ for every $M \in \mathrm{Max}(R)$.

Proposition 3.14. *Let R be an integral domain and let Θ be a complete set of flat overrings. The map*

$$\begin{aligned}\lambda_\Theta\colon \mathrm{ExtStar}(R) &\longrightarrow \prod_{T\in\Theta}\mathrm{Star}(T)\\ * &\longmapsto (*_T)_{T\in\Theta}\end{aligned}$$

is injective and continuous (where $\mathrm{ExtStar}(R)$ *is endowed with the Zariski topology and* $\prod_{T\in\Theta}\mathrm{Star}(T)$ *with the product of the Zariski topologies). If Θ is locally finite, then λ_Θ is a topological embedding.*

Proof. The continuity of λ_Θ follows from the continuity of its components $\lambda_{R,T} : \mathrm{ExtStar}(R) \longrightarrow \mathrm{Star}(T)$. For the injectivity, let I be a fractional ideal of R. Using the completeness, we have

$$I^* = \bigcap_{T\in\Theta} I^*T = \bigcap_{T\in\Theta}(IT)^{*_T}$$

and thus $\lambda_\Theta(*) = \lambda_\Theta(*')$ implies that $I^* = I^{*'}$ for every I, i.e., $* = *'$.

Let I be a fractional ideal of R; we claim that

$$\lambda_\Theta(U_I^{(R)}) = \left(\prod_{T\in\Theta} U_{IT}^{(T)}\right) \cap \lambda_\Theta(\mathrm{Star}(R)).$$

Indeed, if $* \in U_I^{(R)}$ then clearly $\lambda_\Theta(*) \in \lambda_\Theta(\mathrm{Star}(R))$ and $1 \in I^*T = (IT)^{*_T}$ (that is, $*_T \in U_{IT}^{(T)}$) for every $T \in \Theta$. On the other hand, if $(*^{(T)})_{T\in\Theta} = \lambda_\Theta(*)$ is in the right hand side, then $1 \in (IT)^{*^{(T)}} = I^*T$ for every T, and by completeness $1 \in \bigcap\{I^*T \mid T \in \Theta\} = I^*$, i.e., $* \in U_I^{(R)}$.

If now Θ is locally finite, $1 \in IT$ for all but finitely many overrings $T \in \Theta$; hence, $U_{IT}^{(T)} = \mathrm{Star}(T)$ for all but finitely many T. Therefore, $\prod_{T\in\Theta} U_{IT}^{(T)}$ is an open set of $\prod_{T\in\Theta} \mathrm{Star}(T)$, and thus λ_Θ is a topological embedding. □

The surjectivity of λ_Θ – for special families Θ – will be explored in Section 3.3.

The injectivity of λ_Θ allows to bound the cardinality of $\mathrm{ExtStar}(R)$:

Corollary 3.15. *Let R be an integral domain and let Θ be a complete set of localizations of R. Then,*

$$|\mathrm{ExtStar}(R)| \leq \prod_{T\in\Theta} |\mathrm{ExtStar}(T)| \quad \textit{and} \quad |\mathrm{Star}_f(R)| \leq \prod_{T\in\Theta} |\mathrm{Star}_f(T)|.$$

In particular,

$$|\mathrm{ExtStar}(R)| \leq \prod_{M\in\mathrm{Max}(R)} |\mathrm{ExtStar}(R_M)| \quad \textit{and} \quad |\mathrm{Star}_f(R)| \leq \prod_{M\in\mathrm{Max}(R)} |\mathrm{Star}_f(R_M)|.$$

Proof. It is enough to apply the previous proposition, noting that the extension of a totally extendable star operation is again extendable (and similarly for finite-type operations). □

Corollary 3.15 does not generalize to the set of all star operations: in fact, an almost Dedekind domain that is not Dedekind has an infinite number of star operations [67, Corollary 2.2], but each of its localizations at maximal ideals has only one star operation. Indeed, if $\dim(R) = 1$, when considering arbitrary star operations the inequality is reversed (see the following Corollary 3.30).

3.1.2. Restrictions of star operations

Let T be an overring (not necessarily flat) of R. We can always define a map

$$\begin{aligned} \rho_{T,R}\colon \mathrm{Star}(T) &\longrightarrow \mathrm{Star}(R) \\ * &\longmapsto * \wedge v, \end{aligned}$$

where we denote formally with $* \wedge v$ the map $I \mapsto (IT)^* \cap I^v$, as I ranges among the fractional ideals of R; $* \wedge v$ is still a star operation, so $\rho_{T,R}$ is well-defined. The purpose of $\rho_{T,R}$ is to act as an inverse of $\lambda_{R,T}$, as the next proposition shows.

Proposition 3.16. *Let R be an integral domain and T be a flat overring of R. Then:*

(a) $\rho_{T,R}$ is a continuous map;
(b) $\lambda_{R,T} \circ \rho_{T,R}$ is the identity on $\lambda_{R,T}(\mathrm{ExtStar}(R;T))$.

Proof. (a) Let I be an ideal of R. Then,

$$\rho_{T,R}^{-1}(U_I^{(R)}) = \{* \in \mathrm{Star}(T) \mid 1 \in I^{*\wedge v}\} = \{* \in \mathrm{Star}(T) \mid 1 \in (IT)^* \cap I^v\}.$$

Hence, $\rho_{T,R}^{-1}(U_I^{(R)}) = U_{IT}^{(T)}$ if $1 \in I^v$, while $\rho_{T,R}^{-1}(U_I^{(R)}) = \emptyset$ if $1 \notin I^v$. In both cases, $\rho_{T,R}^{-1}(U_I^{(R)})$ is open, and thus $\rho_{T,R}$ is continuous.

(b) Suppose $* = \lambda_{R,T}(\sharp)$ for some $\sharp \in \mathrm{ExtStar}(R;T)$; we have to show that $\rho_{T,R}(*)$ is extendable to T and that $\lambda_{R,T} \circ \rho_{T,R}(*) = *$.

Indeed, let I be an ideal of R. Then,

$$I^{\rho_{T,R}(*)}T = I^{*\wedge v}T = ((IT)^* \cap I^v)T = (IT)^*T \cap I^vT.$$

However, $(IT)^*T = (IT)^*$, while $I^vT \supseteq I^\sharp T = (IT)^*$. Thus, $I^{\rho_{T,R}(*)}T = (IT)^* = I^\sharp T$. Since $\sharp$ is extendable, it follows that if $IT = JT$ then $I^{\rho_{T,R}(*)}T = J^{\rho_{T,R}(*)}T$, and that $(\lambda_{R,T} \circ \rho_{T,R})(*) = *$. □

As in the previous section, we can consider, instead of T alone, a whole family Θ of overrings of R, getting a map

$$\begin{aligned} \rho_\Theta \colon \prod_{T\in\Theta} \mathrm{Star}(T) &\longrightarrow \mathrm{Star}(R) \\ (*^{(T)})_{T\in\Theta} &\longmapsto \bigwedge_{T\in\Theta} *^{(T)} \wedge v \end{aligned}$$

where $\bigwedge_{T\in\Theta} *^{(T)} \wedge v$ denotes, as before, the map $I \mapsto \bigcap_{T\in\Theta}(IT)^{*^{(T)}} \cap I^v$. We are mainly interested in the case when $\bigcap_{T\in\Theta} T = R$; when this happens, the intersection with I^v is superfluous, and ρ_Θ preserves the principal properties of the $*^{(T)}$.

Proposition 3.17. *Let Θ be a locally finite set of overrings of R such that $\bigcap_{T\in\Theta} T = R$, and let $*^{(T)} \in \mathrm{Star}(T)$ for every $T \in \Theta$. Then, if each $*^{(T)}$ is of finite type (respectively, semifinite, Noetherian) then $\rho_\Theta(*^{(T)})$ is of finite type (resp., semifinite, Noetherian). If, moreover, each $T \in \Theta$ is flat over R and each $*^{(T)} \in \mathrm{Star}(T)$ is stable (respectively, spectral) then so is $\rho_\Theta(*^{(T)})$.*

Proof. Let $* := \rho_\Theta((*^{(T)})_{T\in\Theta})$.

If each $*^{(T)}$ is of finite type, then $*$ is of finite type by [5].

Suppose each $*^{(T)}$ is semifinite and $I = I^*$ is a proper ideal of R. Then, $1 \notin I$, so there is a $T \in \Theta$ such that $(IT)^{*^{(T)}} \neq T$, and thus there is a prime ideal P of T containing IT such that $P = P^{*^{(T)}}$. If $Q := P \cap R$, then

$$Q^* \subseteq (QT)^{*^{(T)}} \cap R \subseteq P^{*^{(T)}} \cap R = Q,$$

so that Q is a $*$-prime ideal of R containing I.

Suppose now $*^{(T)}$ is Noetherian for every $T \in \Theta$ and let $\{I_\alpha : \alpha \in A\}$ be an ascending chain of $*$-ideals. If $I_\alpha = (0)$ for every α we are done. Otherwise, there is a $\overline{\alpha}$ such that $I_{\overline{\alpha}} \neq (0)$, and thus $I_{\overline{\alpha}}$ (and, consequently, every I_α for $\alpha > \overline{\alpha}$) extends to a proper ideal in only a finite number of elements of Θ, say $T_1, \ldots, T_n$. For each $i \in \{1, \ldots, n\}$, the set $\{I_\alpha T_i\}$ is an ascending chain of $*^{(T_i)}$-ideals, and thus there is a α_i such that $I_\alpha T_i = I_{\alpha_i} T_i$ for every $\alpha \geq \alpha_i$.

Let thus $\widetilde{\alpha} := \max\{\overline{\alpha}, \alpha_i : 1 \leq i \leq n\}$. For every $\beta \geq \widetilde{\alpha}$, we have $I_\beta T_i = I_{\alpha_i} T_i = I_{\widetilde{\alpha}} T_i$, while, if $T \neq T_i$ for every i, then $I_\beta T = T = I_{\widetilde{\alpha}} T$ since $\beta \geq \overline{\alpha}$. Therefore, $I_\beta = \bigcap_{T\in\Theta} I_\beta T = \bigcap_{T\in\Theta} I_{\widetilde{\alpha}} T = I_{\widetilde{\alpha}}$ and the chain $\{I_\alpha\}$ stabilizes.

Suppose now that each T is flat over R. If each $*^{(T)}$ is stable, then given ideal I, J of R we have

$$(I \cap J)^* = \bigcap_{T\in\Theta} ((I \cap J)T)^{*^{(T)}} = \bigcap_{T\in\Theta} (IT)^{*^{(T)}} \cap \bigcap_{T\in\Theta} (JT)^{*(T)} = I^* \cap J^*.$$

Hence, $*$ is stable. The case of spectral star operation follows since $*$ is spectral if and only if it is stable and semifinite. □

Proposition 3.18. *Let R be an integral domain and let Θ be a locally finite set of overrings of R. Then, ρ_Θ is a continuous map.*

Proof. Let I be a fractional ideal of R. Then,

$$\rho_\Theta^{-1}(U_I^{(R)}) = \{(*^{(T)}) \mid 1 \in \bigcap_{T\in\Theta} (IT)^{*^{(T)}}\} = \prod_{T\in\Theta} U_{IT}^{(T)}.$$

If Θ is locally finite, $U_{IT}^{(T)} = \mathrm{Star}(T)$ for all but finitely many $T \in \Theta$, and thus $\rho_\Theta^{-1}(U_I^{(R)})$ is open. Hence, ρ_Θ is continuous. □

As with λ_Θ, it is natural to ask when ρ_Θ is injective or surjective. We will start with the studying the latter property.

Definition 3.19. *Let Θ be a set of flat overrings of R. We say that a family $(*^{(T)})_{T\in\Theta} \subseteq \prod_{T\in\Theta} \mathrm{ExtStar}(T)$ is* compatible on Θ *if, whenever P is a prime ideal such that $PT_1 \neq T_1$ and $PT_2 \neq T_2$ for some $T_1, T_2 \in \Theta$, we have $(*^{(T_1)})_P = (*^{(T_2)})_P$. We denote the set of compatible families on Θ as $\mathcal{C}(\Theta)$, and by $\mathcal{C}_f(\Theta)$ the set of compatible family such that each $*^{(T)}$ is of finite type. If $\Theta = \{R_P : P \in \Delta\}$ for some $\Delta \subseteq \mathrm{Spec}(R)$, we set $\mathcal{C}(\Delta) := \mathcal{C}(\Theta)$.*

By Proposition 3.7, we have maps

$$\mathrm{ExtStar}(R) \xrightarrow{\lambda_\Theta} \prod_{M\in\mathrm{Max}(R)} \mathrm{ExtStar}(R_M) \xrightarrow{\rho_\Theta} \mathrm{Star}(R)$$

and, using the proof of Proposition 3.14, we see that $\rho_\Theta \circ \lambda_\Theta$ is the identity. Moreover, again by Proposition 3.7, the range of λ_Θ is contained in $\mathcal{C}(\mathrm{Max}(R))$.

Lemma 3.20. *Let R be a domain, $M \in \mathrm{Spec}(R)$, T be a flat overring of R, and define $A := TR_M$. Then, A is a flat overring of R and, for every prime ideal P of R, $PA \neq A$ if and only if $PT \neq T$ and $P \subseteq M$.*

Proof. As a R-module, $A = (R \setminus M)^{-1}T$ is a localization of a flat module, so it is still R-flat. Clearly if $PA \neq A$ then $PT \neq T$ and $P \subseteq M$; conversely, suppose these two condition hold. Since $PT \cap R = P$, we have $PT \cap (R \setminus M) = \emptyset$; thus, $PA \neq A$, as claimed. □

Theorem 3.21. *Let R be an integral domain, let Θ be a locally finite complete set of flat overrings of R, and let $(*^{(T)})_{T\in\Theta}\in\prod_{T\in\Theta}\mathrm{ExtStar}(T)$. Then:*

(a) $*:=\rho_\Theta((*^{(T)})_{T\in\Theta})\in\mathrm{ExtStar}(R)$*;*
(b) if $(*^{(T)})_{T\in\Theta}$ *is compatible on* Θ*, then* $*^{(U)}=(\rho_\Theta(*))_U$ *for every* $U\in\Theta$*;*
(c) ρ_Θ *and* λ_Θ *are homeomorphisms between* $\mathrm{ExtStar}(R)$ *and* $\mathcal{C}(\Theta)$*.*

Proof. (a) Fix a nonzero ideal I; without loss of generality, suppose $I\subseteq R$. Since Θ is locally finite, there are only a finite number of members T of Θ such that $IT\neq T$, say $T_1,\ldots,T_n$. Let Q be a fixed prime ideal of R. Then, since localization commutes with finite intersections,

$$I^*R_Q=\left[\bigcap_{T\in\Theta}(IT)^{*^{(T)}}\right]R_Q=\left[\bigcap_{i=1}^{n}(IT_i)^{*^{(T_i)}}\cap R\right]R_Q=$$

$$=\bigcap_{i=1}^{n}(IT)^{*^{(T_i)}}R_Q\cap R_Q=\bigcap_{T\in\Theta}(IT)^{*^{(T)}}R_Q.$$

By Lemma 3.20,

$$(IT)^{*^{(T)}}R_Q=\bigcap_{\substack{P\subseteq Q\\ PT\neq T}}(IT)^{(*^{(T)})_P}$$

and thus

$$I^*R_Q=\bigcap_{T\in\Theta}\bigcap_{\substack{P\subseteq Q\\ PT\neq T}}(IT)^{(*^{(T)})_P}.$$

If $IR_Q=JR_Q$, then $IR_P=JR_P$ for every prime ideal $P\subseteq Q$, and thus $(IT)^{(*^{(T)})_P}=(JT)^{(*^{(T)})_P}$ for every $T\in\Theta$. Hence, $I^*R_Q=J^*R_Q$, and $*$ is extendable to R_Q. Since Q was arbitrary, $*$ is totally extendable, i.e., $*\in\mathrm{ExtStar}(R)$.

(b) Suppose $(*^{(T)})_{T\in\Theta}\in\mathcal{C}(\Theta)$, and fix an overring $U\in\Theta$. We claim that, for every $T\in\Theta$, $(IT)^{*^{(T)}}U=(IU)^{*^{(U)}}T$: indeed, this is clear if $T=U$. If $T\neq U$ then, using Lemma 3.20, we have

$$(IT)^{*^{(T)}}U=\bigcap_{\substack{P\in\mathrm{Spec}(R)\\ PU\neq U}}(IT)^{*^{(T)}}R_P=\bigcap_{PU\neq U}\bigcap_{\substack{Q\subseteq P\\ QT\neq T}}(IT)^{*^{(T)}}R_Q=\bigcap_{\substack{QU\neq U\\ QT\neq T}}(IR_Q)^{(*^{(T)})_Q}.$$

By compatibility, $(IR_Q)^{(*^{(T)})_Q}=(IR_Q)^{(*^{(U)})_Q}$, and by symmetry we have our claim. Therefore,

$$(IT)^{*^{(T)}}=\bigcap_{U\in\Theta}(IT)^{*^{(T)}}U=\bigcap_{T\in\Theta}(IU)^{*^{(U)}}T.$$

Since Θ is locally finite, we have

$$I^*T=\left[\bigcap_{U\in\Theta}(IU)^{*^{(U)}}\right]T=\bigcap_{U\in\Theta}(IU)^{*^{(U)}}T=\bigcap_{U\in\Theta}(IT)^{*^{(T)}}U=(IT)^{*^{(T)}}$$

and thus (since $*$ is extendable by the previous point) $*_T = *^{(T)}$.

(c) We first claim that

$$\rho_\Theta\left(\prod_{T\in\Theta}\mathrm{ExtStar}(T)\right)=\rho_\Theta(\mathcal{C}(\Theta)).$$

Since every compatible family is in the product $\prod_{T\in\Theta}\mathrm{ExtStar}(T)$ (by definition), the $(\supseteq)$ containment is obvious. On the other hand, if $\underline{*} := (*^{(T)})_{T\in\Theta}$ is in the product, then $\rho_\Theta(\underline{*}) \in \mathrm{ExtStar}(R)$ by point (a), and so $(\lambda_\Theta \circ \rho_\Theta)(\underline{*})$ is compatible. As in Proposition 3.16, $\rho_\Theta \circ \lambda_\Theta \circ \rho_\Theta = \rho_\Theta$, so $\rho_\Theta(\underline{*})$ is the image of the compatible family $(\lambda_\Theta \circ \rho_\Theta)(\underline{*})$.

Let now $(*^{(T)})_{T\in\Theta} \in \mathcal{C}(\Theta)$. By point (b), $\rho_\Theta(*^{(T)})$ is in $\mathrm{ExtStar}(R)$, and $(\rho_\Theta((*^{(T)})_{T\in\Theta}))_U = *^{(U)}$ for every $U \in \Theta$; therefore, $\lambda_\Theta \circ \rho_\Theta$ is the identity on $\mathcal{C}(\Theta)$. Since we had shown that $\rho_\Theta \circ \lambda_\Theta$ is the identity on $\mathrm{ExtStar}(R)$, it follows that ρ_Θ and λ_Θ are bijections, inverse of each other. Moreover, they are both continuous (λ_Θ by Proposition 3.14, ρ_Θ by Proposition 3.18), and thus they are homeomorphism. □

We now consider some consequences of this theorem when $\Theta = \{R_M \mid M \in \mathrm{Max}(R)\}$. In this case, we will denote ρ_Θ and λ_Θ with $\rho_{\mathrm{Max}(R)}$ and $\lambda_{\mathrm{Max}(R)}$, respectively.

Corollary 3.22. *Let R be a locally finite one-dimensional domain. Then, $\lambda_{\mathrm{Max}(R)}$ and $\rho_{\mathrm{Max}(R)}$ are homeomorphisms between $\mathrm{ExtStar}(R)$ and $\prod_{M\in\mathrm{Max}(R)}\mathrm{Star}(R_M)$.*

Proof. If $\dim R = 1$, then every family is compatible on $\mathrm{Max}(R)$; moreover, each R_M is local and one-dimensional, so $\mathrm{Star}(R_M) = \mathrm{ExtStar}(R_M)$. The claim follows from Theorem 3.21. □

The following corollary can be seen as a topological version of [68, Theorem 2.3].

Corollary 3.23. *If R is Noetherian and one-dimensional, then $\lambda_{\mathrm{Max}(R)}$ and $\rho_{\mathrm{Max}(R)}$ are homeomorphisms between $\mathrm{Star}(R)$ and $\prod_{M\in\mathrm{Max}(R)}|\mathrm{Star}(R_M)|$. In particular, $|\mathrm{Star}(R)| = \prod_{M\in\mathrm{Max}(R)}|\mathrm{Star}(R_M)|$.*

Proof. Since R is Noetherian, $\mathrm{Star}_f(R) = \mathrm{Star}(R)$ and thus $\mathrm{ExtStar}(R) = \mathrm{Star}(R)$. Moreover, since it is also one-dimensional, it is locally finite. We can apply Corollary 3.22. □

Corollary 3.24. *Let R be a locally finite domain. Then, $\lambda_{\mathrm{Max}(R)}$ and $\rho_{\mathrm{Max}(R)}$ induce homeomorphisms between $\mathrm{Star}_f(R)$ and $\mathcal{C}_f(\mathrm{Max}(R))$.*

Proof. Since R is locally finite, $\rho_{\mathrm{Max}(R)}(\mathcal{C}_f(\mathrm{Max}(R))) \subseteq \mathrm{Star}_f(R)$; however, we also have $\lambda_{\mathrm{Max}(R)}(\mathrm{Star}_f(R)) \subseteq \mathcal{C}_f(\mathrm{Max}(R))$, and thus we can apply Theorem 3.21. □

Proposition 3.25. *Let R be a domain, and let $\mathcal{U} := \{M \in \mathrm{Max}(R) :$ for all $P \subsetneq M, |\mathrm{ExtStar}(R_P)| = 1\}$. Then,*

$$\mathcal{C}(\mathrm{Max}(R)) = \prod_{M\in\mathcal{U}}\mathrm{ExtStar}(R_M)\times\mathcal{C}(\mathrm{Max}(R)\setminus\mathcal{U}).$$

Proof. Let $(*^{(M)})_{M\in\mathrm{Max}(R)\setminus\mathcal{U}} \in \mathcal{C}(\mathrm{Max}(R)\setminus\mathcal{U})$, and let $(*^{(N)})_{N\in\mathcal{U}}$: we need to show that $(*^{(P)})_{P\in\mathrm{Max}(R)}$ is compatible.

Let N_1, N_2 be maximal ideals of R, $N_1 \neq N_2$, and $Q \subseteq N_1 \cap N_2$. If N_1 or N_2 are in $\mathcal{U}$, then $(*^{(N_1)})_Q$ and $(*^{(N_2)})_Q$ are elements of $\mathrm{ExtStar}(R_Q)$, and thus they coincide since this set is a singleton. Conversely, if neither N_1 nor N_2 are in $\mathcal{U}$, then $(*^{(N_1)})_Q = (*^{(N_2)})_Q$ since $(*^{(M)})_{M\in\mathrm{Max}(R)\setminus\mathcal{U}}$ was compatible. □

Corollary 3.26. *Let R be a locally finite domain of dimension 2. If R_P is a divisorial ring for every $P \in X^1(R)$, then $\rho_{\mathrm{Max}(R)}$ and $\lambda_{\mathrm{Max}(R)}$ induce homeomorphisms between $\mathrm{Star}_f(R)$ and $\prod_{M\in\mathrm{Max}(R)} \mathrm{Star}_f(R_M)$.*

Proof. If $M \neq N$ are maximal ideals of R and $P \subseteq M \cap N$, then P has height 1; therefore, $(*^{(M)})_P = (*^{(N)})_P$ since R_P is divisorial. We can therefore apply Corollary 3.24. □

Corollary 3.27. *Let R be a locally finite Noetherian domain of dimension 2, and suppose that R_M is integrally closed or Gorenstein for all but at most one maximal ideal M. Then, $\rho_{\mathrm{Max}(R)}$ and $\lambda_{\mathrm{Max}(R)}$ are homeomorphisms between $\mathrm{Star}(R)$ and $\prod_{M\in\mathrm{Max}(R)} \mathrm{Star}(R_M)$.*

Proof. If R is integrally closed or Gorenstein, then, for every $P \in X^1(R)$, the localization R_P is a one-dimensional Gorenstein domain, and therefore divisorial. Let M be the exceptional maximal ideal. Then, by Proposition 3.25 and Theorem 3.21,

$$\mathrm{Star}(R) \simeq \prod_{N\in\mathrm{Max}(R)\setminus\{M\}} \mathrm{Star}(R_M) \times \mathcal{C}(\{M\}) = \prod_{M\in\mathrm{Max}(R)} \mathrm{Star}(R_M)$$

since, for a set composed of only one localization, compatibility is an empty condition. □

Corollary 3.28. *Let R be a locally finite Krull domain of dimension 2. Then, $\rho_{\mathrm{Max}(R)}$ and $\lambda_{\mathrm{Max}(R)}$ induce homeomorphisms between $\mathrm{Star}_f(R)$ and $\prod_{M\in\mathrm{Max}(R)} \mathrm{Star}_f(R_M)$.*

Proof. For every $P \in X^1(R)$, R_P is a discrete valuation ring and thus every ideal is divisorial; therefore, we can apply by Corollary 3.26. □

Proposition 3.29. *Let R be a one-dimensional domain. Then,*

$$\mathcal{C}(\mathrm{Max}(R)) = \prod_{M\in\mathrm{Max}(R)} \mathrm{Star}(R_M).$$

Proof. Since R is one-dimensional, a prime ideal contained in two different maximal ideals must be (0). Since $R_{(0)}$ is the quotient field of R, every family $(*^{(T)})$ is compatible. □

Corollary 3.30. *For any domain R of dimension 1, $|\mathrm{Star}(R)| \geq \prod_{M\in\mathrm{Max}(R)} |\mathrm{Star}(R_M)|$.*

Proof. By Proposition 3.29, $|\mathcal{C}(\mathrm{Max}(R))| = \prod_{M\in\mathrm{Max}(R)} |\mathrm{Star}(R_M)|$. By Theorem 3.21, $|\mathcal{C}(\mathrm{Max}(R))| = |\mathrm{ExtStar}(R)| \leq |\mathrm{Star}(R)|$. The claim follows. □

Remark 3.4 shows that, in general, Corollary 3.30 can't be improved to give equality. We will show in Remark 3.73 that, under the hypothesis that $\mathrm{Max}(R)$ is locally finite, equality does indeed hold.

On the other hand, suppose that R is one-dimensional, but not necessarily locally finite. Then, we have a chain of injective maps

$$\mathrm{Star}_f(R) \xrightarrow{\lambda_{\mathrm{Max}(R)}|_{\mathrm{Star}_f(R)}} \prod_{P\in\mathrm{Max}(R)} \mathrm{Star}_f(R_P) \longrightarrow \prod_{P\in\mathrm{Max}(R)} \mathrm{Star}(R_P) \xrightarrow{\rho_{\mathrm{Max}(R)}} \mathrm{Star}(R) \tag{3.3}$$

where the composition $\mathrm{Star}_f(R) \longrightarrow \mathrm{Star}(R)$ is the canonical inclusion.

Proposition 3.31. *Suppose R is one-dimensional. If $\mathrm{Star}(R) = \mathrm{Star}_f(R)$, then $\mathrm{Star}(R_P) = \mathrm{Star}_f(R_P)$ for every $P \in \mathrm{Max}(R)$, and $\lambda_{\mathrm{Max}(R)}$ and $\rho_{\mathrm{Max}(R)}$ are bijections.*

Proof. The condition implies that $\mathrm{Star}_f(R) \longrightarrow \mathrm{Star}(R)$ is the identity, and since all the maps in (3.3) are injective, they must be bijective. □

3.1.3. Star operations with fixed $*$-maximal ideals

In this section, we reverse the point of view of the previous one; that is, instead of passing from a single overring to a complete set to obtain all star operations, we try to find subsets of star operations that can be described using families that are not complete. This approach is similar in some way to the one inspiring [7], although our point of view is different and more focused on sets of star operations.

For the sake of simplicity, instead of using the full generality of flat overrings we restrict to localizations at prime ideals: in particular, if Δ is a subset of $\mathrm{Spec}(R)$, we will denote by ρ_Δ the map ρ_Θ, where $\Theta := \{R_P \mid P \in \Delta\}$.

We start by considering what happens if we use only height-1 primes.

Proposition 3.32. *Let R be an integral domain, and let $\Delta = X^1(R)$ be the set of height-1 prime ideals of R; suppose that $R = \bigcap\{R_P \mid P \in \Delta\}$. Then, ρ_Θ is injective.*

Proof. Let $\underline{*} := (*^{(Q)})_{Q\in\Delta}, \underline{*'} := (*'^{(Q)})_{Q\in\Delta}$ be elements of $\prod_{P\in\Delta}\mathrm{Star}(R_P)$, and suppose $\underline{*} \neq \underline{*'}$. Then, there is a $Q \in \Delta$ such that $*_Q \neq *'_Q$; without loss of generality, there is an ideal I of R_Q such that $I = I^{*_Q} \neq I^{*'_Q}$. Since $\dim R_Q = 1$, I is QR_Q-primary. The ideal $J := I \cap R$ is Q-primary, and thus it is not contained in any $P \in \Delta$ different from Q. Moreover, $JR_Q = I$ by the correspondence between primary ideals of R and R_Q.

Therefore,

$$J^{\rho_\Delta(\underline{*})} = (JR_Q)^{*^{(Q)}} \cap \bigcap_{\substack{P\in\Delta\\ P\neq Q}} (JR_P)^{*^{(P)}} = I^{*_Q} \cap \bigcap_{\substack{P\in\Delta\\ P\neq Q}} R_P = I \cap \bigcap_{P\in\Delta} R_P = I \cap R = J$$

while, analogously, $J^{\rho_\Delta(\underline{*'})} = I^{*'^{(Q)}} \cap R$. However, $I^{*'^{(Q)}}$ is QR_Q-primary and different from I, so that it restricts to a Q-primary ideal different from J. Thus, $\rho_\Delta(\underline{*}) \neq \rho_\Delta(\underline{*'})$, that is, ρ_Δ is injective. □

Note that Proposition 3.32 gives a proof of Corollary 3.30 that does not depend upon Theorem 3.21.

We now restrict to finite-type operations. In this case, we not only need not to care about extendability (since it is automatic for finite-type closures) but we can also use the $*$-spectrum.

Proposition 3.33. *Let $*_1, *_2$ be star operations of finite type on the integral domain R. The following are equivalent:*

(i) $*_1 = *_2$;
(ii) $\mathrm{QMax}^{*_1}(R) = \mathrm{QMax}^{*_2}(R)$ *and* $(*_1)_M = (*_2)_M$ *for every* $M \in \mathrm{QMax}^{*_i}(R)$;
(iii) $(*_1)_M = (*_2)_M$ *for every* $M \in \mathrm{Max}(R)$.

Proof. Both (i $\iff$ ii) and (i $\iff$ iii) follow from the fact that $I^* = \bigcap_{P\in\Delta} I^*R_P$ holds for $\Delta = \mathrm{Max}(R)$ (see Proposition 3.14) and $\Delta = \mathrm{QMax}^*(R)$. $\square$

Definition 3.34. *Let R be an integral domain and $\Delta \subseteq \mathrm{Spec}(R)$. We define*

$$\mathrm{Star}^\Delta(R) := \{* \in \mathrm{Star}_f(R) \mid \mathrm{QMax}^*(R) = \Delta\}$$

Remark 3.35.

(1) Since $\mathrm{QMax}^*(R) \neq \emptyset$ for all finite-type star operation $*$, we have

$$\mathrm{Star}_f(R) = \bigcup_{\Delta\subseteq\mathrm{Spec}(R)} \mathrm{Star}^\Delta(R).$$

(2) If R has dimension 1, then every nonzero prime ideal is minimal over a principal ideal, and thus it is $*$-closed for every finite-type operation $*$. It follows that $\mathrm{Star}_f(R) = \mathrm{Star}^{\mathrm{Max}(R)}(R)$.

An obvious condition for having $\mathrm{Star}^\Delta(R) \neq \emptyset$ is that Δ is an antichain in $\mathrm{Spec}(R)$, i.e., it does not contain any pair of comparable prime ideals. This alone is not a sufficient condition, but we can reach a characterization. Before giving it, we prove an equivalent definition for $\mathrm{Star}^\Delta(R)$.

Proposition 3.36. *Let R be a domain and $\Delta \subseteq \mathrm{Spec}(R)$ be a set of incomparable ideals. Then,* $\mathrm{Star}^\Delta(R) = \{* \in \mathrm{Star}_f(R) \mid \widetilde{*} = s_\Delta\}$.

Proof. If $* \in \mathrm{Star}^\Delta(R)$, then $\widetilde{*}$ is of finite type and $\mathrm{QMax}^{\widetilde{*}}(R) = \mathrm{QMax}^*(R)$, so that $\widetilde{*} = s_\Delta$.

Conversely, if $\widetilde{*} = s_\Delta$, then $\mathrm{QMax}^{\widetilde{*}}(R) = \Delta$ (since elements of Δ are incomparable) and thus $\mathrm{QMax}^*(R) = \Delta$, and $* \in \mathrm{Star}^\Delta(R)$. $\square$

In particular, another way to see $\mathrm{Star}^\Delta(R)$ is as the set $\Psi_w^{-1}(s_\Delta) \cap \mathrm{Star}_f(R)$, where Ψ_w is the map introduced in Section 2.2.3 (strictly speaking, is the analogue of that Ψ_w in the star setting), or, in term of the inverse topology, as the set of star operations of finite type $*$ such that $\mathrm{Cl}^{\mathrm{inv}}(\mathrm{QSpec}^*(R)) = \mathrm{Cl}^{\mathrm{inv}}(\Delta)$.

Corollary 3.37. *Let R be a domain. Then,* $\mathrm{Star}^{\mathrm{QMax}^w(R)}(R) = \{* \in \mathrm{Star}_f(R) \mid * \geq w\}$.

Proof. It is enough to apply the above Proposition, noting that w is the biggest spectral star operation of finite type, and thus, for an arbitrary $* \in \mathrm{Star}_f(R)$, we have $\widetilde{*} = w$ if and only if $* \geq w$. □

Corollary 3.38. *Let R be a domain and $\Delta \subseteq \mathrm{Spec}(R)$. Then,* $\mathrm{Star}^{\Delta}(R) \neq \emptyset$ *if and only if Δ is compact in* $\mathrm{Spec}(R)$*, it is an antichain and* $\bigcap\{R_P \mid P \in \Delta\} = R$.

Proof. Clearly Δ must be an antichain. Moreover, if $* \in \mathrm{Star}^{\Delta}(R)$, then $\widetilde{*} = s_\Delta \in \mathrm{Star}^{\Delta}(R)$. Thus, $R = \bigcap\{R_P \mid P \in \Delta\}$, and Δ is compact since s_Δ is of finite type (see Corollary 2.37).

Conversely, if Δ satisfies the conditions, then $s_\Delta \in \mathrm{Star}^{\Delta}(R)$, which thus is not empty. □

Suppose now Star^{Δ} is nonempty. Then, just like we defined λ_Θ, we can define a map

$$\begin{aligned} \lambda_\Delta \colon \mathrm{Star}^{\Delta}(R) &\longrightarrow \prod_{P\in\Delta} \mathrm{Star}^{PR_P}(R_P) \\ * &\longmapsto (*^{(P)})_{P\in\Delta} \end{aligned}$$

which is well-defined since $(PR_P)^{*^{(P)}} = P^*R_P = PR_P$. As in Proposition 3.14, λ_Δ is a continuous map, and it is injective by Proposition 3.33. Moreover, we have a map

$$\begin{aligned} \rho_\Delta^f \colon \mathrm{Star}^{PR_P}(R_P) &\longrightarrow \mathrm{Star}(R) \\ (*^{(P)})_{P\in\Delta} &\longmapsto \bigwedge_{P\in\Delta} *^{(P)} \end{aligned}$$

obtained restricting the map ρ_Δ. Moreover, if Δ is locally finite, then $\bigwedge_{P\in\Delta} *^{(P)}$ is of finite type and thus the range of ρ_Δ is contained in $\mathrm{Star}^{\Delta}(R)$.

By Proposition 3.33, $\rho_\Delta \circ \lambda_\Delta$ is the identity on $\mathrm{Star}^{\Delta}(R)$. In particular, this allows to repeat the theory of compatibility in the context of $\mathrm{Star}^{\Delta}(R)$; restricting to the case of height-1 primes, we get the following result. Remember that a *Mori domain* is a domain where the integral divisorial ideals satisfy the ascending chain condition, while a *strong Mori domain* is a domain where the w-ideals satisfy the ascending chain condition; in other words, R is Mori if v is a Noetherian star operation, while it is strong Mori if w is Noetherian. In particular, each Noetherian domain is a strong Mori domain, and each strong Mori domain is a Mori domain.

Proposition 3.39. *Suppose R is a strong Mori domain and that* $\mathrm{QMax}^w(R) = X^1(R) =: X^1$. *Then, λ_Θ and ρ_Θ are bijections between* $\mathrm{Star}^{X^1}(R)$ *and* $\prod_{P\in X^1} \mathrm{Star}(R_P)$.

Proof. By Corollary 3.37, we have $\{* \in \mathrm{Star}_f(R) \mid * \geq w\} = \mathrm{Star}^{X^1}(R)$.

Therefore, we have a chain of maps

$$\mathrm{Star}^{X^1}(R) \xrightarrow{\lambda_{X^1}} \prod_{P\in X^1} \mathrm{Star}^{PR_P}(R_P) \xrightarrow{\rho_{X^1}} \mathrm{Star}^{X^1}(R).$$

By Proposition 3.32, ρ_{X^1} is injective; therefore, since $\rho_{X^1} \circ \lambda_{X^1}$ is the identity on $\mathrm{Star}^{X^1}(R)$, the two maps are bijections. Moreover, R_P is Noetherian [34, Proposition 4.6] and one-dimensional, and thus $\mathrm{Star}^{PR_P}(R_P) = \mathrm{Star}_f(R_P) = \mathrm{Star}(R_P)$. Hence, λ_Θ and ρ_Θ are bijections. $\square$

Another consequence is that we can obtain a cleaner proof of the results in the results of [89, Section 2], and we can add another equivalent condition. A *TW-domain* is a domain where the t-operation and the w-operation coincide. We shall denote by $t^{(A)}$ and $w^{(A)}$, respectively, the t- and the w-operation on the domain A.

Lemma 3.40. *Let R be a v-coherent domain and T be a flat overring of R. Then, $\lambda_{R,T}(t^{(R)}) = t^{(T)}$ and $\lambda_{R,T}(w^{(R)}) = w^{(T)}$.*

Proof. Note that $\lambda_{R,T}(t^{(R)})$ and $\lambda_{R,T}(w^{(R)})$ are defined since the t- and w-operations are of finite type. The fact that $\lambda_{R,T}(t^{(R)}) = t^{(T)}$ follows from Proposition 3.12; for w, note that the map $* : I \mapsto (IT)^{w^{(T)}} \cap T$ is a stable star operation of finite type on R, and clearly $\lambda_{R,T}(*) = w^{(T)}$. However, since $w^{(T)}$ is the maximal spectral operation of finite type, $\lambda_{R,T}(w^{(R)}) \geq \lambda_T(*) = w^{(T)}$ by Proposition 3.6; by Propositions 3.5 and 3.3(d), $\lambda_{R,T}(w^{(R)})$ is again stable and of finite type, and thus $\lambda_{R,T}(w^{(R)}) = w^{(T)}$, as requested. $\square$

Proposition 3.41 [89, Theorem 2.2]. *Let R be a v-coherent domain. The following are equivalent:*

(i) R is a TW-domain.
(ii) R_S is a TW-domain for each multiplicatively closed subset S of R.
(iii) R_P is a TW-domain for each prime ideal P of R.
(iv) R_P is a TW-domain for each t-prime ideal P of R.
(v) R_M is a TW-domain for each t-maximal ideal M of R.
(vi) $|\mathrm{Star}_f(R_M)| = 1$ for each t-maximal ideal M of R.

Proof. (i $\Longrightarrow$ ii). Since $t^{(R)} = w^{(R)}$, the localizations $\lambda_{R,R_S}(t^{(R)})$ and $\lambda_{R,R_S}(w^{(R)})$ are equal. Since R is a v-coherent domain, $\lambda_{R,R_S}(t^{(R)}) = t^{(R_S)}$ and $\lambda_{R,R_S}(w^{(R)}) = w^{(R_S)}$ (Lemma 3.40), and thus R_S is a TW-domain.

(ii $\Longrightarrow$ iii $\Longrightarrow$ iv $\Longrightarrow$ v) are trivial.

(v $\Longrightarrow$ vi). Since R_M is a TW-domain, $t^{(R_M)} = w^{(R_M)}$; however, since the maximal ideal MR_M is a $t^{(R_M)}$-ideal (and thus a $w^{(R_M)}$-ideal) the w-operation on R_M is the identity. Hence $t^{(R_M)} = d^{(R_M)}$ and $|\mathrm{Star}_f(R_M)| = 1$.

(vi $\Longrightarrow$ v) is trivial.

(v $\Longrightarrow$ i). We have $\lambda_{R,R_M}(t^{(R)}) = t^{(R_M)} = w^{(R_M)} = \lambda_{R,R_M}(w^{(R)})$ for every $M \in \mathrm{Max}(R)$. By Proposition 3.33, it follows that $t^{(R)} = w^{(R)}$, i.e., R is a TW-domain. $\square$

Proposition 3.42 [89, Theorem 2.4]. *Let R be a Mori domain. The following are equivalent:*

(i) R is a TW-domain;

(ii) $|\mathrm{Star}(R_M)| = 1$ *for each t-maximal ideal M of R.*

Proof. (ii $\Longrightarrow$ i) follows readily from Proposition 3.41. To show (i $\Longrightarrow$ ii) it is enough to note that each R_M is a Mori domain, and thus $t^{(R_M)} = v^{(R_M)}$; by Proposition 3.41, $t^{(R_M)} = d^{(R_M)}$, and hence R_M admits only one star operation. □

Proposition 3.43. *Let R be a Noetherian domain. The following are equivalent:*

(i) *R is a TW-domain;*
(ii) $\mathrm{QMax}^t(R) = X^1(R)$ *and R_P is Gorenstein for every $P \in X^1(R)$.*

Proof. (ii $\Longrightarrow$ i) follows from Proposition 3.42, since a Gorenstein 1-dimensional domain is divisorial (i.e., it has only one star operation).

Suppose (i) holds. If there is a $P \in \mathrm{QSpec}^t(R)$ such that $h(P) \geq 2$, then (since the map $\mathrm{Star}(R) \longrightarrow \mathrm{Star}(R_P)$ is surjective by Proposition 3.12) PR_P would be divisorial, and R_P would be a TW-domain by Proposition 3.41. Hence, PR_P would be w-closed: but this would imply that R_P is divisorial, which is impossible since $\dim(R_P) = h(P) \geq 2$ [58, Corollary 4.3]. Moreover, if $P \in X^1(R)$, then R_P is a TW-domain; however, in a one-dimensional domain $w = d$, and thus R_P is divisorial, and hence Gorenstein. □

3.2. Jaffard families

Theorem 3.21 applies to a very wide range of families of flat overrings, requiring only completeness and local finiteness. However, there are two big problems that restrict its usefulness: firstly, it only applies to the set $\mathrm{ExtStar}(R)$ of totally extendable star operations on R (and without giving any hint on how much this set if far from the whole $\mathrm{Star}(R)$); secondly, it is usually hard to determine the set $\mathcal{C}(\Theta)$ of families of star operations that are compatible on Θ.

The results after the theorem show that, sometimes, one or both problems can sometimes be bypassed: for example, we do not need to check extendability if we restrict to finitely-generated star operations (and in particular, to Noetherian domains), or compatibility is free if we work with domains of low dimension. However, in both cases, we are greatly reducing the generality of Theorem 3.21, especially if we want to avoid both problems at once.

The purpose of this section is to operate in the opposite way: instead of giving conditions on the domains to which we apply our "best results", we restrict our attention to a narrow class of families of flat overrings. While this approach does not always give useful information, nor we are able to cover all the cases considered in Corollaries 3.22-3.28, nevertheless the results we obtain will be applicable to a wider range of domains.

In this section, we will mostly follow [41, Section 6.3]. However, instead of starting with a set of overrings, we will relativize our approach to a base ring R.

Definition 3.44. *Let S and T two domains with common quotient field K. We say that S and T are* independent *if $ST = K$ and no nonzero prime ideal of $S \cap T$ survive in both S and T.*

Note that the latter condition is superfluous if S and T are localizations (or intersection of localizations) of $S \cap T$.

Definition 3.45. *Let R be a domain and Θ be a set of overrings of R such that the quotient field of R is not in Θ. We say that Θ is a* Jaffard family on R *if:*

- $R = \bigcap_{T \in \Theta} T$;
- *Θ is locally finite;*
- *for every ideal I of R, $I = \prod_{T \in \Theta}(IT \cap R)$;*
- *if $T \neq S$ are in Θ, and I is an ideal of R, then $(IT \cap R) + (IS \cap R) = R$.*

We say that an overring T of R is a Jaffard overring *of R if T belongs to a Jaffard family of R.*

Remark 3.46.

(1) By the second axiom, if $I \neq (0)$ then $IT = T$ for all but finitely many $T \in \Theta$, so that the product $I = \prod_{T \in \Theta}(IT \cap R)$ is finite.
(2) Jaffard overrings and Jaffard families have nothing to do with Jaffard domains, that is, domains whose valutative dimension is equal to their Krull dimension (such domains were considered in Section 2.3.3).

Lemma 3.47. *Let R be an integral domain with quotient field K and let A, B two independent overrings of R such that $R = A \cap B$. Then, A and B are flat over R.*

Proof. By symmetry, it is enough to prove the flatness of A. By [105, Theorem 1], A is R-flat if and only if, for every $P \in \mathrm{Spec}(R)$, either $PA = A$ or $A \subseteq R_P$. Suppose $PA \neq A$. Then,

$$R_P = RR_P = (A \cap B)R_P = AR_P \cap BR_P$$

by the properties of flatness. However, if Q is a prime ideal of R that survives in BR_P, then $Q \subseteq P$; therefore, $QB \neq B$ and $QA \neq A$. By independence, it must be $Q = (0)$; hence, $BR_P = K$ and $A \subseteq R_P$. Therefore, A is R-flat. ☐

Proposition 3.48 [41, Theorem 6.3.1]. *Let R be an integral domain with quotient field K, and let Θ be a Jaffard family on R. For each $T \in \Theta$, let $\Theta^\perp(T) := \bigcap\{U \in \Theta \mid U \neq T\}$.*

(a) Θ is complete.
(b) For each $P \in \mathrm{Spec}(R)$, $P \neq (0)$, there is a unique $T \in \Theta$ such that $PT \neq T$.
(c) For each $T \in \Theta$, both T and $\Theta^\perp(T)$ are flat over R.
(d) For each $T \in \Theta$, we have $T \cdot \Theta^\perp(T) = K$.

Proof. (a) Let $I \neq (0)$. Then,

$$I = \prod_{T \in \Theta}(IT \cap R) = (IT_1 \cap R) \cdots (IT_n \cap R) = I_1 \cdots I_n,$$

where $I_i := IT_i \cap R$ and $T_1, \ldots, T_n$ are the overrings in Θ such that $IT \neq T$. However, the I_i are coprime (by definition of Jaffard familily), and thus $I_1 \cdots I_n = I_1 \cap \cdots \cap I_n$. It follows that Θ is complete.

(b) Let $P \in \mathrm{Spec}(R)$, $P \neq (0)$. By the previous point, $P = P_1 \cap \cdots \cap P_n$, where $P_i := PT_i \cap R$ and $PT = T$ if $T \in \Theta \setminus \{T_1, \ldots, T_n\}$. However, each P_i is prime, so the unique possibility for P to be prime is that $P = P_a$ for some a. In particular, the other P_i must be equal to R, i.e., $n = 1$.

(d) Fix a $T \in \Theta$; since $K \notin \Theta$, there must be a prime $P \neq (0)$ of R such that $PT \neq T$. Take an $r \in P$, $r \neq 0$; then, $I := rT \cap R \subseteq P$, and in particular, $IT \neq T$. By definition of Jaffard family, we can write $rR = \prod_{j=1}^{n}(IT_j \cap R)$ for some subset $\{T_j \mid j \in J\} \subseteq \Theta$. Such family must contain T; in particular, since rR is an invertible ideal of R, so must be I. Moreover, $IS = S$ for every $S \in \Theta \setminus \{T\}$.

If now $xI \subseteq R$, then $xIS \subseteq S$ and thus $xS \subseteq S$; i.e., $x \in S$. It follows that $(R : I) \subseteq S$ for every $S \in \Theta \setminus T$, and thus $(R : I) \subseteq \Theta^{\perp}(T)$. However, since I is invertible,

$$R = I(R : I) \subseteq I\Theta^{\perp}(T) \subseteq \Theta^{\perp}(T),$$

and in particular $1 \in I\Theta^{\perp}(T)$, that is, $I\Theta^{\perp}(T) = \Theta^{\perp}(T)$, and thus $P\Theta^{\perp}(T) = \Theta^{\perp}(T)$. It follows that no prime ideal of R survives in both T and $\Theta^{\perp}(T)$; hence, no prime ideal of R survives in $A := T\Theta^{\perp}(T)$. However, this is possible only if $A = K$.

(c) It is enough to apply Lemma 3.47. □

Corollary 3.49. *Let Θ be a family of flat overrings of the domain R, and let K be the quotient field of R. Then, Θ is a Jaffard family if and only if it is complete, locally finite and $TS = K$ for all $T, S \in \Theta$, $T \neq S$.*

Proof. If Θ is a Jaffard family the properties follow by the definition and Proposition 3.48. Conversely, suppose Θ verifies the three properties, let $I \neq (0)$ be an ideal of R and let $T \neq S$ be members of Θ. If $IT \cap R$ and $IS \cap R$ are not coprime, then there would be a prime P of R containing both; since Θ is complete, it would follow that $IT \cap R$ and $IS \cap R$ survive in some $A \in \Theta$. By flatness,

$$(IT \cap R)A = ITA \cap A = IK \cap A = A$$

and analogously for S, against the choice of R. Therefore, $(IT \cap R) + (IS \cap R) = R$. Moreover, $I = \bigcap\{IT \cap R \mid T \in \Theta\} = (IT_1 \cap R) \cap \cdots \cap (IT_n \cap R)$ by local finiteness; since the $IT_i \cap R$ are coprime, their intersection is equal to their product, and thus $I = (IT_1 \cap R) \cdots (IT_n \cap R)$. □

One consequence of points (b) and (d) of the previous proposition is that every Jaffard family Θ defines a partition on $\mathrm{Spec}(R) \setminus \{(0)\}$, whose elements are the families Σ_T of prime ideals P such that $PT \neq T$, as T ranges in Θ. In particular, $\{\Sigma_T\}$ induces also a partition $\{\Lambda_T\}$ on $\mathrm{Max}(R)$.

More precisely, giving a Jaffard family on R is equivalent to giving a *Matlis partition* on R, that is, a partition $\{\Lambda_\alpha \mid \alpha \in A\}$ of $\mathrm{Max}(R)$ such that [41, Section 6.3, p.131]:

- for any $r \in R$, $r \neq 0$, there are only a finite number of Λ_α such that $r \subseteq M$ for some $M \in \Lambda_\alpha$;
- for every $P \in \mathrm{Spec}(R) \setminus \{(0)\}$ there is a unique α such that $P \subseteq M$ for some $M \in \Lambda_\alpha$.

A Matlis partition can be then extended to the whole $\mathrm{Spec}(R) \setminus \{(0)\}$ by taking $\Sigma_\alpha := (\Lambda_\alpha)^\downarrow \setminus \{0\}$.

Example 3.50.

(1) For any domain R, the family composed only by R itself is a Jaffard family.
(2) If (R, M) is local, then the unique Matlis partition of $\mathrm{Max}(R)$ is the trivial one. It follows that R has only one Jaffard family, namely $\{R\}$.
(3) Similarly, if R is semilocal, then $\mathrm{Max}(R)$ admits only a finite number of partition, and thus there are only a finite number of Jaffard families on R.
(4) A h-local domain is an integral domain R such that $\mathrm{Max}(R)$ is locally finite and such that every prime ideal P is contained in only one maximal ideal. In this case, $\{R_M \mid M \in \mathrm{Max}(R)\}$ is a Jaffard family of R. Conversely, if $\{R_M \mid M \in \mathrm{Max}(R)\}$ is a Jaffard family, then $\mathrm{Max}(R)$ is locally finite (by definition) and each prime is contained in only one maximal ideal (by Proposition 3.48(b)), and thus R is h-local.
(5) If we do not suppose that Θ is locally finite, points (a), (b) and (c) of Proposition 3.48 can be fulfilled, but we may not have $T \cdot \Theta^\perp(T) = K$. For example, let R be the ring of entire functions (i.e., functions that are holomorphic on the whole $\mathbb{C}$), and let $\Theta := \{R_M \mid M \in \mathrm{Max}(R)\}$. By [61, Theorem 2], every prime ideal of R is contained in a unique maximal ideal, and thus $R_M R_N = K$ if $M \neq N$ are maximal ideal; Θ is clearly complete; but if M is a *free* maximal ideal (i.e., the elements of M have no common zeros) then $\Theta^\perp(R_M) = \bigcap_{N \neq M} R_N = R$, and so $R_M \cdot \Theta^\perp(R_M) = R_M \neq K$.

We end this section by generalizing to Jaffard families two properties of h-local domains (see [92, Proposition 3.1]).

Proposition 3.51. *Let R be a domain and T be a Jaffard overring of R. Then:*

(a) for every family $\{X_\alpha : \alpha \in A\}$ of R-submodules of K with nonzero intersection, we have $(\bigcap_{\alpha\in A} X_\alpha) T = \bigcap_{\alpha \in A} X_\alpha T$;
(b) if $\{I_\alpha : \alpha \in A\}$ is a family of integral ideals of R with nonzero intersection such that $(\bigcap_{\alpha\in A} I_\alpha) T \neq T$, then $I_{\overline{\alpha}} T \neq T$ for some $\overline{\alpha} \in A$.

Proof. (a) Let Θ be a Jaffard family of R such that $T \in \Theta$. Then, by flatness of T,

$$\left(\bigcap_{\alpha\in A} X_\alpha\right) T = \left(\bigcap_{\alpha\in A}\bigcap_{U\in\Theta} X_\alpha U\right) T = \left(\bigcap_{U\in\Theta}\bigcap_{\alpha\in A} X_\alpha U\right) T =$$

$$= \left(\bigcap_{U\in\Theta'}\bigcap_{\alpha\in A} X_\alpha U\right) T \cap \bigcap_{\alpha\in A} X_\alpha T = K \cap \bigcap_{\alpha\in A} X_\alpha T$$

since $\bigcap_{U\in\Theta'}\bigcap_{\alpha\in A} X_\alpha U$ is a $\Theta^\perp(T)$-module, and thus its product with T is equal to K by Proposition 3.48(d).

(a $\Longrightarrow$ b). Suppose $(\bigcap_{\alpha\in A} I_\alpha)\,T \neq T$. Since $(\bigcap_{\alpha\in A} I_\alpha)\,T \subseteq T$, then 1 is not contained in the left hand side. Hence, 1 is not contained in $\bigcap_{\alpha\in A} I_\alpha T$, i.e., there is a $\overline{\alpha}$ such that $1 \notin I_{\overline{\alpha}}T$, and thus $I_{\overline{\alpha}}T \neq T$. □

3.2.1. Minimal Jaffard families

Given a R, we define $\mathcal{J}(R)$ as the set of Jaffard families of R. We will investigate two problems: the possibility of finding a *minimal* Jaffard family (in this section) and under what hypothesis $\mathcal{J}(R)$ is a singleton (in the next one).

Definition 3.52. *Let Θ_1 and Θ_2 be two Jaffard families, and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the corresponding Matlis partition of* $\mathrm{Max}(R)$. *We say that Θ_1 is* finer *than Θ_2, and we write $\Theta_1 \preceq \Theta_2$, if $\mathcal{P}_1$ is finer than $\mathcal{P}_2$, that is, if for every $\Lambda \in \mathcal{P}_1$ there is a $\Lambda' \in \mathcal{P}_2$ such that $\Lambda \subseteq \Lambda'$.*

Equivalently, Θ_1 is finer than Θ_2 if and only if, for every $T_1 \in \Theta_1$, there is a $T_2 \in \Theta_2$ such that $T_2 \subseteq T_1$.

Proposition 3.53. *Let Θ be a Jaffard family of a domain R, and let $R \subseteq A$ be an extension of domains such that the field extension $Q(R) \subseteq Q(A)$ is algebraic. Then, $\Theta \cdot A := \{TA : T \in \Theta\} \setminus \{Q(A)\}$ is a Jaffard family of A.*

Proof. Note first that, if I is a nonzero ideal of A, then $I \cap R \neq (0)$: indeed, if $x \in I$, then x is algebraic over A, so that $a_nx^n + \cdots + a_0 = 0$ for some $a_i \in A$, $a_0, a_n \neq 0$ (taking n as the minimum degree), and thus $a_0 = -x(a_nx^{n-1} + \cdots + a_1) \in xB \cap A \subseteq I \cap A$. Moreover, if $T \in \Theta$, then T is flat over R and so $TA \simeq T \otimes_R A$ is flat over A.

$\Theta \cdot A$ is complete: if $P \in \mathrm{Spec}(A)$, let $Q := P \cap R$. There is a $T \in \Theta$ such that $QT \neq T$; thus, $QR_M \neq R_M$ for some $M \in \mathrm{Max}(R)$ such that $MT \neq T$. In particular, $Q \cap (R \setminus M) = \emptyset$; but $AR_M = (R \setminus M)^{-1}A$, and thus $QAR_M \neq AR_M$. It follows that $QA \neq A$.

$\Theta \cdot A$ is locally finite: let I be an ideal of A such that $IT \neq T$ for an infinite family $\Lambda \subseteq \Theta \cdot A$; let $\Theta' \subseteq \Theta$ be a subset such that $\Lambda = \Theta' \cdot A$. Then, for every $S \in \Theta'$, $1 \notin ISA$, and thus $1 \notin (I \cap R)S$; it follows that $I \cap R$ survive in every $S \in \Theta'$. Since $I \cap R \neq (0)$, this would imply that Θ' is not locally finite, and thus that neither Θ is locally finite, against the hypothesis that Θ is a Jaffard family.

$\Theta \cdot A$ is independent: if $T_1, T_2 \in \Theta \cdot A$, $T_1 \neq T_2$, then $T_i = S_iA$ for some $S_i \in \Theta$, $S_1 \neq S_2$, and $T_1T_2 = S_1S_2A = Q(R)A = Q(A)$ since $Q(R) \subseteq Q(A)$ is algebraic.

By Corollary 3.49, $\Theta \cdot A$ is a Jaffard family. □

Proposition 3.54. *Let R be a domain and Θ a Jaffard family of R. For every $T \in \Theta$, let $\Theta(T) \in \mathcal{J}(T)$; then, $\bigcup_{T\in\Theta} \Theta(T)$ is a Jaffard family of R.*

Proof. Let $\Theta' := \bigcup_{T\in\Theta} \Theta(T)$. Note that $\Theta(T_1) \cap \Theta(T_2) = \emptyset$ if $T_1 \neq T_2$, since if $U \in \Theta(T_1) \cap \Theta(T_2)$ then T_1 and T_2 are contained in U and so would be T_1T_2, which however is equal to K.

For every ideal I of R,

$$I = \bigcap_{T\in\Theta} IT = \bigcap_{T\in\Theta}\bigcap_{U\in\Theta(T)} ITU = \bigcap_{U\in\Theta'} IU$$

and thus Θ' is complete. If $A \neq B$ are in Θ' then either $A, B \in \Theta(T)$ for some T or $A \in \Theta(T)$ and $B \in \Theta(S)$ for some $T \neq S$: in the latter case, $K = TS \subseteq AB \subseteq K$. In both cases, $AB = K$. Finally, if $x \in R$, then since Θ is locally finite there are only a finite number of elements of Θ, say $T_1, \dots, T_n$, where $xT \neq T$. On the other hand, for every i there are only a finite number of members of $\Theta(T_i)$ where x is not a unit; therefore, Θ' is locally finite. By Corollary 3.49, Θ' is a Jaffard family. □

For every pair $\Theta_1, \Theta_2 \in \mathcal{J}(R)$ of R, we define $\Theta_1 \cdot \Theta_2$ as the set $\{T_1T_2 : T_1 \in \Theta_1, T_2 \in \Theta_2\} \setminus \{K\}$.

Proposition 3.55. *If Θ_1 and Θ_2 are Jaffard families of R, so is $\Theta_1 \cdot \Theta_2$.*

Proof. We have $\Theta_1 \cdot \Theta_2 = \bigcup_{T\in\Theta_1} T \cdot \Theta_2$. By Proposition 3.53, each $T \cdot \Theta_2$ is a Jaffard family of T, and by Proposition 3.54, their union is Jaffard family of R. □

Lemma 3.56. *Let Θ_1, Θ_2 be Jaffard families of the domain R. Then,*

(a) $\Theta_1 \cdot \Theta_2 \preceq \Theta_1$ *and* $\Theta_1 \cdot \Theta_2 \preceq \Theta_2$*;*
(b) $\Theta_1 \preceq \Theta_2$ *if and only if* $\Theta_1 \cdot \Theta_2 = \Theta_1$.

Proof. Let $\iota_S : \mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ be the map of spectra associated to the inclusion $R \hookrightarrow S$; note that if Θ is a Jaffard family then $\{\iota_T(\mathrm{Max}(T)) \mid T \in \Theta\}$ is the Matlis partition associated to Θ.

(a) For every pair of flat overrings T_1, T_2 of R, $\iota_{T_1T_2}(\mathrm{Max}(T_1T_2)) = \iota_{T_1}(\mathrm{Max}(T_1)) \cap \iota_{T_2}(\mathrm{Max}(T_2)) \subseteq \iota_{T_1}(\mathrm{Max}(T_1))$, so that the partition determined by $\Theta_1 \cdot \Theta_2$ is finer than the partition determined by Θ_1, i.e., $\Theta_1 \cdot \Theta_2 \preceq \Theta_1$. Symmetrically, $\Theta_1 \cdot \Theta_2 \preceq \Theta_2$.

(b) If $\Theta_1 \preceq \Theta_2$, then for every $T_1 \in \Theta_1$ there is a $T_2 \in \Theta_2$ such that $\iota_{T_1}(\mathrm{Max}(T_1)) \subseteq \iota_{T_2}(\mathrm{Max}(T_2))$. Hence, $T_1 \supseteq T_2$, $T_2T_1 = T_1$, and $\Theta_1 \cdot \Theta_2 = \Theta_1$.

Conversely, if $\Theta_1 \cdot \Theta_2 = \Theta_1$, for every $T_2 \in \Theta_2$ and $T_1 \in \Theta_1$ we have $T_1T_2 = K$ or $T_1T_2 \in \Theta_1$. In the latter case, however, $T_1T_2 = T_1$, since T_1 is the unique overring of T_1 belonging to Θ_1. Hence, $T_2 \subseteq T_1$, and since T_2 was arbitrary $\Theta_1 \preceq \Theta_2$. □

Proposition 3.57. *Let R be a domain. If Θ is a minimal element, with respect to $\preceq$, of $\mathcal{J}(R)$, then it is the minimum.*

Proof. Suppose Θ is not the minimum; then there is a Jaffard family Λ such that $\Theta \npreceq \Lambda$. By the previous lemma, $\Theta \cdot \Lambda \precneqq \Theta$, against the hypothesis that Θ is a minimal element. Hence, Θ is a minimum. □

3.2.2. Trivially branched domains

Definition 3.58. *A domain R is* trivially branched *if its unique Jaffard family is* $\{R\}$.

As remarked in Example 3.50, every local domain is trivially branched. However, this is not the only example: for example, if the Jacobson radical of R contains a nonzero prime ideal P, then R is trivially branched, because, given any Jaffard family Θ, there is only one $T \in \Theta$ such that $PT \neq T$, while $PR_M \neq R_M$ for every maximal ideal M of R.

Following this idea, given a Jaffard family Θ, we say that two nonzero prime ideals are Θ-*equivalent* (and we write $P \sim_\Theta Q$) if there is a $T \in \Theta$ such that $PT \neq T$ and $QT \neq T$, that is, if P and Q are contained in maximal ideals belonging to the same set relative to the Matlis partition associated to Θ. We say that they are *Jaffard-equivalent* (and we write $P \sim_{\mathcal{J}} Q$) if $P \sim_\Theta Q$ for every Jaffard family Θ of R.

Clearly, if P and Q are comparable (i.e., $P \subseteq Q$ or $Q \subseteq P$) then $P \sim_\Theta Q$ for every $\Theta \in \mathcal{J}(R)$, and thus $P \sim_{\mathcal{J}} Q$. More generally, say that P and Q are *comparably connected* if there is a sequence $P_0 = P, P_1, \ldots, P_n = Q$ of nonzero prime ideals such that P_i and P_{i+1} are comparable for every $0 \leq i \leq n-1$. Clearly, both being Jaffard-equivalent and being comparably connected are equivalence relations on $\mathrm{Spec}(R) \setminus \{(0)\}$; moreover, if P and Q are comparably connected then they are Jaffard-equivalent.

Proposition 3.59. *Let R be a domain, and let let $\Delta \subseteq \mathrm{Spec}(R) \setminus \{(0)\}$ be an equivalence class with respect to the relation of being comparably connected. If Δ is compact (in the Zariski topology), then the ring $T := \bigcap\{R_P \mid P \in \Delta\}$ is flat over R.*

Proof. By [105, Theorem 1], we need to show that, for every prime ideal P of R, either $PT = T$ or $T \subseteq R_P$. Suppose $PT \neq T$; if $P \in \Delta$ then $T \subseteq R_P$ by definition. Hence, suppose $P \notin \Delta$ and $P \neq (0)$.

Consider the semistar operation $*_\Delta$ on R, and let $*$ be the restriction of $*_\Delta$ to $\mathcal{F}(T)$; then, $*$ is a spectral star operation on T. Moreover, $*_\Delta$ is of finite type by Proposition 2.32, and thus so is $*$ (the proof is completely analogous to the one of Proposition 2.53). Since $P \notin \Delta$ and $\Delta \cup \{(0)\}$ is closed by generization (being an equivalence class of the "comparably connected" relation), we have $(PT)^* = T$. However, if $p \in PT$, then (since $*$ is of finite type) every minimal prime ideal of pT is $*$-closed (see [73, Theorem 9, p.30] or [57, Proposition 1.1(5)]); in particular, one of them (say A) is contained in PT. However, $A^* \neq T$ implies that $A \cap R \subseteq Q$ for some $Q \in \Delta$; hence, $A \cap R$ is a nonzero prime contained in both P and Q, so that P and Q are comparably connected. This contradicts the definition of Δ and the assumption $P \notin \Delta$; thus, if $PT \neq T$ then $P \in \Delta$, and T is R-flat. □

Note that, if R admits a minimal Jaffard family Θ, then each $T \in \Theta$ must be trivially branched: if not, then $(\Theta \setminus \{T\}) \cup \Theta(T)$ (where $\Theta(T)$ is a Jaffard family of T) would be a Jaffard family strictly finer than Θ.

However, if T is trivially branched and a flat overring of R, it does not follow that it belongs to a minimal Jaffard family and, indeed, the existence of a complete and independent family of trivially branched domains does not guarantee the existence of the minimal Jaffard family; see Example 3.89.

We next highlight some other cases of trivially branched domains; our results will have the form of transfer properties, i.e., we will show that if some ring is trivially branched so is another one.

Proposition 3.60. *Let $A \subseteq B$ be an integral extension of domains, and suppose that B is the integral closure of A in a field F. If B is trivially branched, so is A.*

Proof. Suppose Θ is a Jaffard family on A. By [41, Theorem 6.3.9], the family $\overline{\Theta} := \{\overline{T} \mid T \in \Theta\}$ is a Jaffard family on B. However, the map $\Theta \longrightarrow \overline{\Theta}$, $T \mapsto \overline{T}$, is injective, and thus Θ is composed of a single element. Since Θ is complete, it must be $\Theta = \{A\}$, and A is trivially branched. □

The converse is not true: for example, if A is a local 1-dimensional domain, then it is trivially branched, but there could exist an integral extension B of A which is semilocal but not local, so that $\{B_M : M \in \mathrm{Max}(B)\}$ is a non-trivial Jaffard family on B.

Proposition 3.61. *Suppose $A \subseteq B$ is an extension of domains. Then, $A + XB[[X]]$ is trivially branched, and in particular so is $A[[X]]$ for every domain A.*

Proof. Let M be a maximal ideal of $R := A + XB[[X]]$: we claim that $XB[[X]] \subseteq M$. In fact, if $f \in XB[[X]] \setminus M$, then $(M, f) = R$, i.e., there are $m \in M$ and $\alpha \in R$ such that $m + \alpha f = 1$. However, the order of αf is at least 1, and thus m must be equal to $1 + Xb$, for some $b \in B[[X]]$. But this would imply that m is invertible in R, a contradiction. In particular, every pair of maximal ideals of R is comparable connected, and thus R is trivially branched. □

Proposition 3.62. *Let R be a one-dimensional domain that is not local. If R is trivially branched, then every nonunit is contained in infinitely many maximal ideals of R.*

Proof. If R is semilocal (i.e., it has a finite number of maximal ideals) then it is enough to take $\Theta = \{R_M : M \in \mathrm{Max}(R)\}$.

Suppose $\mathrm{Max}(R)$ is infinite and the claim does not hold. Then, there is a $x \in R$ such that $x \in M_1, \dots, M_n$ but in no other maximal ideal. Consider the set $\Theta := \{R[\frac{1}{x}], R_{M_1}, \dots, R_{M_n}\}$: clearly it is a complete set of localizations of R. Moreover, since $\dim R = 1$, Θ is independent and, being finite, it is also locally finite. Therefore Θ is a nontrivial Jaffard family on R. □

Example 3.63. There exist examples of one-dimensional domains where each nonunit belongs to infinitely many maximal ideals: an example is the ring of all algebraic integers $\mathbb{A}$. Indeed, if $\alpha \in \mathbb{A}$ is a nonunit, then $\alpha \in \mathcal{O}_K$ for some finite extension K of $\mathbb{Q}$. If $P \in \mathrm{Spec}(\mathcal{O}_K)$ contains α, then there is a finite extension F of K such that P is contained in more than one maximal ideal of $\mathcal{O}_F$, and each one of these is dominated by a different prime ideal of $\mathbb{A}$ (by properties of integral extensions); repeating the process we obtain an infinite number of prime ideals of $\mathbb{A}$ containing α. (See Example 2.44 for more details.) However, $\mathbb{A}$ is not trivially branched, because the set $\Theta := \{\overline{\mathbb{Z}_{(p)}} : p \text{ is a prime of } \mathbb{Z}\}$

is a nontrivial Jaffard family by Proposition 3.53. In the same way, if F is any finite extension of $\mathbb{Q}$, we can consider the Jaffard family $\Theta_F := \{(\mathcal{O}_F)_P : P \in \mathrm{Max}(\mathcal{O}_F)\}$, which induces a Jaffard family $\overline{\Theta_F}$ of $\mathbb{A}$. Finite extensions of F of increasing degree induce finer Jaffard families on $\mathbb{A}$, and these families get progressively closer to the finest possible family, $\{\mathbb{A}_P : P \in \mathrm{Max}(\mathbb{A})\}$. However, the latter is not a Jaffard family, since it is not locally finite. In particular, $\mathbb{A}$ does not admit a minimal Jaffard family.

3.3. Jaffard families and star operations

The reason why we introduced Jaffard families is that they provide a way to decompose $\mathrm{Star}(R)$ as a product of spaces of star operations of overrings of T. Before reaching this objective (Theorem 3.67) we show that weaker properties can lead to a decomposition of at least a subset of $\mathrm{Star}(R)$.

Proposition 3.64. *Let R be an integral domain with quotient field K. Let Θ be a family of flat overrings of R such that $R = \bigcap\{T \mid T \in \Theta\}$ and such that $AB = K$ whenever $A, B \in \Theta$ and $A \neq B$. Then, the map*

$$\begin{aligned} \rho_\Theta\colon \prod_{T\in\Theta} \mathrm{Star}(T) &\longrightarrow \mathrm{Star}(R) \\ (*^{(T)})_{T\in\Theta} &\longmapsto \bigwedge_{T\in\Theta} *^{(T)} \end{aligned}$$

is injective and order-preserving.

Proof. Since $\bigcap_{T\in\Theta} T = R$, $*$ is a star operation; moreover, it is clear that if $*_1^{(T)} \leq *_2^{(T)}$ for all T then $\rho_\Theta((*_1^{(T)})_{T\in\Theta}) \leq \rho_\Theta((*_2^{(T)})_{T\in\Theta})$. Hence, ρ_Θ is an order-preserving map; we need to show that it is injective.

Suppose it is not; then, $* := \rho_\Theta((*_1^{(T)})_{T\in\Theta}) = \rho_\Theta((*_2^{(T)})_{T\in\Theta})$ for some families of star operations such that $*_1^{(U)} \neq *_2^{(U)}$ for some $U \in \Theta$. There is an integral ideal J of U such that $J^{*_1^{(U)}} \neq J^{*_2^{(U)}}$; let $I := J \cap R$. Since U is flat, for both $i = 1$ and $i = 2$ we have

$$I^*U = \left[\bigcap_{T\in\Theta} (IT)^{*_i^{(T)}}\right] U = (IU)^{*_i^{(U)}} U \cap \left[\bigcap_{T\in\Theta\setminus\{U\}} (IT)^{*_i^{(T)}}\right] U.$$

If $T \neq U$, then, since T is flat,

$$(IT)^{*_i^{(T)}} = ((J\cap R)T)^{*_i^{(T)}} = (JT \cap T)^{*_i^{(T)}}.$$

However, $JT = JUT = K$ since $UT = K$ (by hypothesis); therefore, $(IT)^{*_i^{(T)}} = T$, and (since $I \subseteq R$)

$$\begin{aligned} I^*U = (IU)^{*_i^{(U)}} U \cap \left[\bigcap_{T\in\Theta\setminus\{U\}} T\right] U = (IU)^{*_i^{(U)}} U \cap \left[\bigcap_{T\in\Theta} T\right] U = \\ = (IU)^{*_i^{(U)}} \cap RU = (IU)^{*_i^{(U)}} = J^{*_i^{(U)}}. \end{aligned}$$

However, this contradicts the choice of J; hence, ρ_Θ is injective. $\square$

In particular, if R is a one-dimensional domain, we can consider the family $\{R_M : M \in \mathrm{Max}(R)\}$; thus, the previous proposition can be seen as a generalization of Proposition 3.32. In general, we cannot hope that it is a bijection: the usual Example 3.4 of an almost Dedekind non-Dedekind domain applies.

However, if we suppose that Θ is a Jaffard family, everything works. We premise two lemmas.

Lemma 3.65. *Let R be a domain with quotient field K, and let Θ be a Jaffard family on R. For every $U \in \Theta$, let J_U be a U-submodule of K, and define $J := \bigcap_{U\in\Theta} J_U$. If $J \neq (0)$, then for every $T \in \Theta$ we have $JT = J_T$.*

Proof. By Proposition 3.51(a), we have

$$JT = \left(\bigcap_{U\in\Theta} J_U\right) T = \bigcap_{U\in\Theta} J_U T.$$

If $U \neq T$, then $J_U T = J_U U T = J_U K = K$; therefore, $JT = J_T T = J_T$. □

The next lemma can be seen as a generalization of [41, Theorem 6.2.2(2)] and [15, Lemma 2.3].

Lemma 3.66. *Let R be an integral domain, T be a Jaffard overring of R, and let $I, J \in \mathbf{F}(R)$ such that $(I : J) \neq (0)$. Then, $(I : J)T = (IT : JT)$.*

Proof. It is enough to note that $(I : J) = \bigcap_{j\in J} j^{-1} I \neq (0)$, and apply Proposition 3.51. □

Theorem 3.67. *Let R be an integral domain and let Θ be a Jaffard family on R. Then, every $* \in \mathrm{Star}(R)$ is extendable to every $T \in \Theta$, and the maps*

$$\begin{aligned}\lambda_\Theta\colon \mathrm{Star}(R) &\longrightarrow \prod_{T\in\Theta} \mathrm{Star}(T)\\ * &\longmapsto (*_T)_{T\in\Theta}\end{aligned} \quad \textit{and} \quad \begin{aligned}\rho_\Theta\colon \prod_{T\in\Theta} \mathrm{Star}(T) &\longrightarrow \mathrm{Star}(R)\\ (*^{(T)})_{T\in\Theta} &\longmapsto \bigwedge_{T\in\Theta} *^{(T)}\end{aligned}$$

establish homeomorphisms between $\mathrm{Star}(R)$ *and* $\prod\{\mathrm{Star}(T) \mid T \in \Theta\}$.

Proof. We first show extendability. Let $T \in \Theta$ and let I, J be ideals of R such that $IT = JT$. Then, using Lemma 3.66, we have

$$(I^*T : J^*T) = (I^* : J^*)T = (I^* : J)T = (I^*T : JT)$$

and, since $JT = IT \subseteq I^*T$, we have $1 \in (I^*T : J^*T)$, so that $J^*T \subseteq I^*T$. Symmetrically, $I^*T \subseteq J^*T$, and hence $J^*T = I^*T$. It follows that $*_T$ is well-defined, and $*$ is extendable to T; in particular, also λ_Θ is well-defined.

Since each $\lambda_{R,T} : \mathrm{Star}(R) \longrightarrow \mathrm{Star}(T)$ is continuous (Proposition 3.14), so is λ_Θ, while ρ_Θ is continuous by Proposition 3.18 (since Θ is locally finite). With the same proof of Proposition 3.14, we see that, if $* \in \mathrm{Star}(R)$, then

$$I^* = \bigcap_{T\in\Theta} I^*T = \bigcap_{T\in\Theta} (IT)^{*_T}$$

and thus $* = \rho_\Theta \circ \lambda_\Theta(*)$, i.e., $\rho_\Theta \circ \lambda_\Theta$ is the identity. It follows that λ_Θ is injective and ρ_Θ is surjective. But ρ_Θ is injective by Proposition 3.64, so λ_Θ and ρ_Θ must be bijections and thus homeomorphisms. □

These maps respect also the properties of the star operations involved:

Proposition 3.68. *Let R be a domain and Θ be a Jaffard family on R, and let $* \in$ Star(R). Then, $*$ is of finite type (respectively, semifinite, stable, spectral, Noetherian) if and only if $*_T$ is of finite type (resp., semifinite, stable, spectral, Noetherian) for every $T \in \Theta$.*

Proof. By Propositions 3.3(d) and 3.5, if $*$ is of finite type, stable, spectral ot Noetherian so is $*_T$. If $*$ is semifinite, let I be a $*_T$-closed ideal of T, and let $J := I \cap R$. Then $JT = I$, and $J^* \subseteq I^{*_T} \cap R = J$, so that there is a prime ideal $Q \supseteq J$ such that $Q^* = Q$. For every $U \in \Theta$, $U \neq T$, we have $JU = U$; hence $QU = U$, and thus $QT \neq T$; since R is flat, QT is prime (this follows from [105, Theorem 2]). Therefore, $(QT)^{*_T} = Q^*T = QT$ is a proper prime $*_T$-ideal containing I, and $*_T$ is semifinite.

The inverse implications follow from Proposition 3.17. □

Recall from Section 2.3.2.2 that a star operation $*$ is said to be *endlich arithmetisch brauchbar* (*eab* for short) if for every nonzero finitely generated ideals F, G, H such that $(FG)^* \subseteq (FH)^*$, we have $G^* \subseteq H^*$. If this property hold for arbitrary nonzero fractional ideal G, H (but F still finitely generated) then $*$ is said to be *arithmetisch brauchbar* (*ab* for short).

Lemma 3.69. *Let R be an integral domains and let T be a Jaffard overring of R. For all nonzero integral ideals I, J of T,*

$$(I \cap R)(J \cap R) = IJ \cap R.$$

Proof. Let Θ be a Jaffard family containing T. Since Θ is complete, it is enough to show that they are equal when localized on every $U \in \Theta$. We have

$$(I \cap R)(J \cap R)U = (IU \cap U)(JU \cap U) = \begin{cases} IJ & \text{if } U = T \\ U & \text{if } U \neq T \end{cases}$$

while

$$(IJ \cap R)U = IJU \cap U = \begin{cases} IJ & \text{if } U = T \\ U & \text{if } U \neq T \end{cases}$$

and thus $(I \cap R)(J \cap R) = IJ \cap R$. □

Lemma 3.70. *Let R be an integral domain, T a Jaffard overring of R, and let I be a finitely generated integral ideal of T. Then, $I \cap R$ is finitely generated (over R).*

Proof. Let $S := \Theta^\perp(T)$, where Θ is the Jaffard family whom T belongs. Then, by Proposition 3.48, $(I\cap R)S = IS\cap S = ITS\cap S = S$, and thus there are $i_1,\ldots,i_n \in I\cap R$, $s_1,\ldots,s_n \in S$ such that $1 = i_1s_1+\cdots+i_ns_n$; let $I_0 := (i_1,\ldots,i_n)$.

Let $x_1,\ldots,x_m$ be the generators of I in S. Since $(I\cap R)T = IT = I$, for every x_i there are $j_{1i},\ldots,j_{n_ii} \in I\cap R$, $t_{1i},\ldots,t_{n_ii} \in T$ such that $x_i = j_{1i}t_{1i}+\cdots+j_{n_ii}t_{n_ii}$; let $I_i := (j_{1i},\ldots,j_{n_ii})$. Then, $J := I_0+I_1+\cdots+I_n$ is a finitely generated ideal contained in $I\cap R$ (since it is generated by elements of $I\cap R$) such that $(I\cap R)T \subseteq JT$ and $(I\cap R)S \subseteq JS$; thus, $I\cap R\subseteq J$. Therefore, $I\cap R = J$ is finitely generated, as claimed. □

Proposition 3.71. *Let R be an integral domain, let Θ be a Jaffard family on R and let $* \in \mathrm{Star}(R)$. Then, $*$ is eab (resp., ab) if and only if $*_T$ is eab (resp., ab) for every $T\in\Theta$.*

Proof. ($\Longrightarrow$). Suppose $(IJ)^{*_T} \subseteq (IL)^{*_T}$ for some finitely generated ideal I,J,L of T (which we can suppose contained in T). Since

$$(IJ\cap R)^*T = ((IJ\cap R)T)^{*_T} = (IJ)^{*_T}$$

(and the same happens for IL), we have $(IJ\cap R)^*T \subseteq (IL\cap R)^*T$, and so

$$(IJ\cap R)^*T\cap R \subseteq (IL\cap R)^*T\cap R.$$

However, both $IJ\cap R$ and $IL\cap R$ survive (among the ideals of Θ) only in T, so that

$$(IJ\cap R)^*T\cap R = (IJ\cap R)^* = ((I\cap R)(J\cap R))^*$$

by Lemma 3.69, and thus

$$((I\cap R)(J\cap R))^* \subseteq ((I\cap R)(L\cap R))^*.$$

Since I is finitely generated, by Lemma 3.70 so is $I\cap R$; the same happens for $J\cap R$ and $L\cap R$. Hence, since $*$ is eab, $(J\cap R)^*\subseteq (L\cap R)^*$, and thus

$$J^{*_T} = (J\cap R)^*T \subseteq (L\cap R)^*T = L^{*_T}.$$

Hence, $*_T$ is eab.

($\Longleftarrow$). Suppose $(IJ)^*\subseteq (IL)^*$. Then, $(IJ)^*T\subseteq (IL)^*T$, i.e., $(IJT)^{*_T}\subseteq (ILT)^{*_T}$ for every $T\in\Theta$. Since $*_T$ is eab, this implies that $(JT)^{*_T}\subseteq (LT)^{*_T}$ for every $T\in\Theta$; since $H^* = \bigcap_{T\in\Theta}(HT)^{*_T}$, it follows that $J^*\subseteq L^*$, and $*$ is eab.

The same reasoning applies for the ab case. □

Corollary 3.72. *Let R be a domain and Θ be a Jaffard family on R. If every $T\in\Theta$ is Noetherian, so is R.*

Proof. A domain A is Noetherian if and only if the identity star operation $d^{(A)}$ is Noetherian. If every $T\in\Theta$ is Noetherian, each d_T is a Noetherian star operation, and thus (by Proposition 3.68) $\rho_\Theta(d^{(T)})$ is Noetherian. However, by Theorem 3.67, $\rho_\Theta(d_T) = d_R$, and thus R is a Noetherian domain. □

Remark 3.73. Theorem 3.67 can be specialized in two different directions.

On the one hand, we can specialize the ring R: for example, if R is h-local (see Example 3.50), then $\{R_M \mid M \in \mathrm{Max}(R)\}$ is a Jaffard family of R, and thus we can apply the theorem. Specializing again, we see that the theorem holds if R is a one-dimensional domain such that $\mathrm{Max}(R)$ is locally finite; for example, if R is Noetherian and one-dimensional (in the latter case, we find a result we have already obtained in Corollary 3.23).

On the other hand, the homeomorphism can also be thought of as an isomorphism of partially ordered set; moreover, it induces an equality between the number $|\mathrm{Star}(R)|$ of the star operations on R and the product $\prod\{|\mathrm{Star}(T)| : T \in \Theta\}$ of the cardinality of the $\mathrm{Star}(T)$. The same applies for the sets of finite-type, spectral, stable, Noetherian and eab star operations. In particular, we shall be interested in the cardinality in Theorem 3.99, and in the order structure in Section 3.5.5.

The two specializations can, obviously, be used together: for example, if R is one-dimensional and $\mathrm{Max}(R)$ is locally finite, then $|\mathrm{Star}(R)| = \prod\{|\mathrm{Star}(R_M)| : M \in \mathrm{Max}(R)\}$. (Compare with Corollary 3.30.)

3.3.1. Invertible ideals and class group

Let $*$ be a star operation on a domain R. A fractional ideal I of R is $*$-*invertible* if there is an ideal J such that $(IJ)^* = R$ or, equivalently, if $(I(R : I))^* = R$. The set of $*$-invertible $*$-ideals, indicated with $\mathrm{Inv}^*(R)$, is a group under the natural "$*$-product" $I \times_* J \mapsto (IJ)^*$ [73, 53, 116, 54].

If $*_1 \leq *_2$, then by definition $I^{*_1} \subseteq I^{*_2}$, so that every $*_1$-invertible ideal is also $*_2$-invertible. Moreover, for every star operation $*$, any $*$-invertible $*$-ideal is divisorial [116, Theorem 1.1 and Observation C], and thus, if $*_1 \leq *_2$, there is a natural inclusion $\mathrm{Inv}^{*_1}(R) \subseteq \mathrm{Inv}^{*_2}(R)$.

Proposition 3.74. *Let R be an integral domain and Θ be a Jaffard family on R. The map*

$$\begin{aligned}\Gamma\colon \mathrm{Inv}^*(R) &\longrightarrow \bigoplus_{T\in\Theta} \mathrm{Inv}^{*_T}(T)\\ I &\longmapsto (IT)_{T\in\Theta}\end{aligned}$$

is well-defined and a group isomorphism.

Proof. Let $\mathcal{F}(R)$ be the set of nonzero fractional ideals of R. Define a map

$$\begin{aligned}\widehat{\Gamma}\colon \mathcal{F}(R) &\longrightarrow \prod_{T\in\Theta} \mathcal{F}(T)\\ I &\longmapsto (IT)_{T\in\Theta}\end{aligned}$$

For every $*$-ideal I, $\widehat{\Gamma}(I) = (IT)_{T\in\Theta}$ is a sequence such that IT is $*_T$-closed. Moreover, if I is $*$-invertible, then $(I(R : I))^* = R$ and thus $(I(R : I)T)^{*_T} = T$, so that IT is $*_T$-invertible. Thus $\widehat{\Gamma}(\mathrm{Inv}^*(R)) \subseteq \prod_{T\in\Theta} \mathrm{Inv}^{*_T}(T)$, and indeed $\widehat{\Gamma}(\mathrm{Inv}^*(R)) \subseteq \bigoplus_{T\in\Theta} \mathrm{Inv}^{*_T}(T)$

since Θ is locally finite, by Theorem 3.67. Hence, Γ is well-defined since it is the restriction of $\widehat{\Gamma}$ to $\mathrm{Inv}^*(R)$.

It is straightforward to verify that Γ is a group homomorphism, and since $I = \bigcap_{T\in\Theta} IT$, we have that Γ (or even $\widehat{\Gamma}$) is injective.

We need only to show that Γ is surjective. Let $(I_T)_{T\in\Theta} \in \bigoplus_{T\in\Theta} \mathrm{Inv}^{*_T}(T)$, and define $I := \bigcap I_T$. Since $I_T = T$ for all but a finite number of elements of Θ, say $T_1, \ldots, T_n$, there are $d_1, \ldots, d_n \in R$ such that $d_i I_{T_i} \subseteq T_i$. Defining $d := d_1 \cdots d_n$, we have $dI_T \subseteq T$ for every T, and thus $dI \subseteq \bigcap_{T\in\Theta} T = R$, so that I is indeed a fractional ideal of R. Moreover, since I_T is $*_T$-closed, $I_T \cap R$ is $*$-closed, and thus I, being the intersection of a family of $*$-closed ideals, is $*$-closed. It is also $*$-invertible, since

$$(I(R:I))^* = \bigcap_{T\in\Theta} (I(R:I)T)^{*_T} = \bigcap_{T\in\Theta} (IT(T:IT))^{*_T} = \bigcap_{T\in\Theta} T = R.$$

Therefore, $(I_T)_{T\in\Theta} = \Gamma(I) \in \Gamma(\mathrm{Inv}^*(R))$, and thus Γ is an isomorphism. □

For every star operation $*$ on R, every principal ideal is $*$-closed and $*$-invertible; thus the set of nonzero principal fractional ideals forms a subgroup of $\mathrm{Inv}^*(R)$, denoted by $\mathrm{Prin}(I)$. The quotient between $\mathrm{Inv}^*(R)$ and $\mathrm{Prin}(R)$ is called the *$*$-class group* of R [8], and it is denoted by $\mathrm{Cl}^*(R)$. Since $\mathrm{Prin}(R) \subseteq \mathrm{Inv}^{*_1}(R) \subseteq \mathrm{Inv}^{*_2}(R)$ if $*_1 \leq *_2$, in this case there is an injective homomorphism $\mathrm{Cl}^{*_1}(R) \subseteq \mathrm{Cl}^{*_2}(R)$.

Of particular interest are the class group of the identity star operation (usually called the *Picard group* of R, denoted by $\mathrm{Pic}(R)$) and the t-class group, which is linked to the factorization properties of the group (see for example [107, 18, 116]).

Theorem 3.75. *Let R be an integral domain and let Θ be a Jaffard family on R. Then, the map*

$$\begin{aligned} \Lambda\colon \frac{\mathrm{Cl}^*(R)}{\mathrm{Pic}(R)} &\longrightarrow \bigoplus_{T\in\Theta} \frac{\mathrm{Cl}^{*_T}(T)}{\mathrm{Pic}(T)} \\ [I] &\longmapsto ([IT])_{T\in\Theta} \end{aligned}$$

is well-defined and a group isomorphism.

Proof. By the proof of Proposition 3.74, there are two isomorphisms $\Gamma^* : \mathrm{Inv}^*(R) \longrightarrow \bigoplus_{T\in\Theta} \mathrm{Inv}^{*_T}(T)$ and $\Gamma^d : \mathrm{Inv}^d(R) \longrightarrow \bigoplus_{T\in\Theta} \mathrm{Inv}^{d_T}(T)$.

Consider the chain of maps

$$\mathrm{Inv}^*(R) \xrightarrow{\Gamma^*} \bigoplus_{T\in\mathrm{Max}(T)} \mathrm{Inv}^{*_T}(T) \xrightarrow{\pi} \bigoplus_{T\in\mathrm{Max}(T)} \frac{\mathrm{Inv}^{*_T}(T)}{\mathrm{Inv}^{d_T}(T)}$$

where π is the componentwise quotient; then, the kernel of π is exactly $\bigoplus_{T\in\Theta} \mathrm{Inv}^{d_T}(T)$. However, Γ^* and Γ^d coincide on $\mathrm{Inv}^d(R) \subseteq \mathrm{Inv}^*(R)$; hence,

$$\ker(\pi \circ \Gamma^*) = (\Gamma^d)^{-1}(\ker \pi) = \mathrm{Inv}^d(R).$$

Therefore, there is an isomorphism $\dfrac{\mathrm{Inv}^*(R)}{\mathrm{Inv}^d(R)} \simeq \displaystyle\bigoplus_{T\in\mathrm{Max}(T)} \dfrac{\mathrm{Inv}^{*_T}(T)}{\mathrm{Inv}^{d_T}(T)}$. However, for an arbitrary domain A and an arbitrary $\sharp \in \mathrm{Star}(A)$, we have $\mathrm{Prin}(A) \subseteq \mathrm{Inv}^d(A) \subseteq \mathrm{Inv}^\sharp(A)$, and thus

$$\frac{\mathrm{Inv}^\sharp(A)}{\mathrm{Inv}^d(A)} \simeq \frac{\mathrm{Inv}^\sharp(A)/\mathrm{Prin}(A)}{\mathrm{Inv}^d(A)/\mathrm{Prin}(A)} \simeq \frac{\mathrm{Cl}^\sharp(A)}{\mathrm{Pic}(A)}$$

so that Λ becomes an isomorphism between $\dfrac{\mathrm{Cl}^*(R)}{\mathrm{Pic}(R)}$ and $\displaystyle\bigoplus_{T\in\Theta} \dfrac{\mathrm{Cl}^{*_T}(T)}{\mathrm{Pic}(T)}$, as claimed. □

3.3.2. Principal star operation, irreducibility and primality

Recall from Chapter 1 (more specifically, Section 1.8.1) that, for any (fractional) ideal I, the *principal star operation* generated by I is

$$*_I : J \mapsto J^v \cap (I : (I : J))$$

or, equivalently, the biggest star operation $*$ such that I is $*$-closed. A star operation $*$ is said to be *principal* if $* = *_I$ for some ideal I; we denote the set of principal star operations as $\mathrm{PStar}(R)$.

Lemma 3.76. *Let R be an integral domain and T be a Jaffard overring of R. For any ideal I of R, the localization $(*_I)_T$ is equal to $*_{IT}$.*

Proof. Let $L = JT$ be an ideal of T. Then,

$$L^{(*_I)_T} = J^{*_I}T = (J^v \cap (I : (I : J)))T = (R : (R : J))T \cap (I : (I : J))T.$$

Applying Lemma 3.66, we have

$$L^{(*_I)_T} = (T : (T : JT)) \cap (IT : (IT : JT)) = (JT)^{*_{IT}} = L^{*_{IT}}$$

and the claim is proved. □

For the following proposition, we define a "direct sum"-like construction of sets of principal ideals as

$$\bigoplus_{T\in\Theta} \mathrm{PStar}(T) := \{(*_T)_{T\in\Theta} : *_T \neq v^{(T)} \text{ for all but a finite number of } T\}.$$

Proposition 3.77. *Let R be an integral domain and Θ be a Jaffard family on R. Then, the map*

$$\begin{aligned} \Upsilon\colon \mathrm{PStar}(R) &\longrightarrow \bigoplus_{T\in\Theta} \mathrm{PStar}(T) \\ *_I &\longmapsto (*_{IT})_{T\in\Theta} \end{aligned}$$

is well-defined and a homeomorphism.

Proof. The map Υ is just the restriction of λ_Θ to $\mathrm{PStar}(R)$, so we have only to show that it is well-defined and surjective.

By Lemma 3.76, $(*_I)_T = *_{IT}$ for every $T \in \Theta$; moreover, $IT = T$ for all but a finite number of T, so that $*_{IT} = *_T = v^{(T)}$ for all but a finite number of T. In particular, the image of Υ lies inside the direct sum $\bigoplus_{T\in\Theta} \mathrm{PStar}(T)$.

Suppose, conversely, that $(*_{J_T})_{T\in\Theta} \in \bigoplus_{T\in\Theta} \mathrm{PStar}(T)$. We can suppose that $J_T \subseteq T$ for every T, and that $J_T = T$ if $*_{J_T} = v^{(T)}$. Define thus $I := \bigcap_{T\in\Theta} J_T$: then, I is nonzero (since $J_T \neq T$ for only a finite number of T) and $IT = J_T$ for every T. Therefore, $(*_I)_T = *_{IT} = *_{J_T}$, and the image of Υ is exactly $\bigoplus_{T\in\Theta} \mathrm{PStar}(T)$. □

Remember from Chapter 1 that $*$ is *irreducible* if, whenever $* = *_1 \wedge *_2$ for some $*_1, *_2 \in \mathrm{Star}(R)$, $*$ is equal to $*_1$ or $*_2$, while it is *prime* if $* \geq *_1 \wedge *_2$ implies that $* \geq *_1$ or $* \geq *_2$. We denote by $\mathrm{IrrStar}(R)$ and $\mathrm{PrimeStar}(R)$, respectively, the sets irreducible and prime star operations on R strictly smaller than the v-operation of R.

It is easy to see that every prime star operation is irreducible, i.e., that $\mathrm{PrimeStar}(R) \subseteq \mathrm{IrrStar}(R)$.

Proposition 3.78. *Let R be a domain and Θ be a Jaffard family on R. Then:*

- *(a) if $* \in \mathrm{Star}(R)$ is irreducible (respectively, prime), then $*_T$ is irreducible (resp., prime) for every $T \in \Theta$, and $*_T \neq v^{(T)}$ for at most one $T \in \Theta$;*
- *(b) if $*^{(T)} \in \mathrm{Star}(T)$ is irreducible (resp., prime), then $\rho_{T,R}(*^{(T)})$ is irreducible (resp., prime);*
- *(c) λ_Θ and ρ_Θ induce bijections*

$$\mathrm{IrrStar}(R) \leftrightarrow \bigsqcup_{T\in\Theta} \mathrm{IrrStar}(T) \quad \textit{and} \quad \mathrm{PrimeStar}(R) \leftrightarrow \bigsqcup_{T\in\Theta} \mathrm{PrimeStar}(T).$$

Proof. (a) Suppose $*_T$ is not irreducible. Then there are $*_1, *_2 \in \mathrm{Star}(T)$ such that $*_T = *_1 \wedge *_2$ but $*_T \neq *_1, *_2$. However, $\rho_{T,R}(*_T) = \rho_{T,R}(*_1) \wedge \rho_{T,R}(*_2)$, while $\rho_{T,R}(*_T) \neq \rho_{T,R}(*_i)$ since $\lambda_{R,T} \circ \rho_{T,R}$ is the identity (by Proposition 3.16) and so

$$\lambda_{R,T}(\rho_{T,R}(*_T)) = *_T \neq *_i = \lambda_{R,T}(\rho_{T,R}(*_i)).$$

Suppose there are two rings $T_1, T_2 \in \Theta$ such that $*_{T_i} \neq v^{(T_i)}$. Define two star operations $*_i$ by

$$J^{*_i} = (JT_i)^{v^{(T_i)}} \cap \bigcap_{U\in\Theta\setminus\{T_i\}} (JU)^{*_U}.$$

Then, $*_U = (*_1)_U \wedge (*_2)_U$ for every $U \in \Theta$, so that $* = *_1 \wedge *_2$, but $* \neq *_1$ and $* \neq *_2$ (respectively because $*_{T_1} \neq v^{(T_1)} = (*_1)_{T_1}$ and $*_{T_2} \neq v^{(T_2)} = (*_1)_{T_2}$). Hence, $*$ would not be irreducible, a contradiction.

(b) Suppose $\rho_{T,R}(*_T)$ is not irreducible. Then, there are $*_1, *_2 \in \mathrm{Star}(T)$, $*_i \neq \rho_{T,R}(*_T)$, such that $\rho_{T,R}(*_T) = *_1 \wedge *_2$; in particular, $(\rho_{T,R}(*_T))_U = (*_1)_U \wedge (*_2)_U$ for every $U \in \Theta$. However, $(\rho_{T,R}(*_T))_U = v^{(U)}$ if $U \neq T$, and thus we must have $(*_i)_U = v^{(U)}$. Hence,

$$(*_1)_T \wedge (*_2)_T = (\rho_{T,R}(*_T))_T = *_T,$$

which implies (since $*_T$ is irreducible) that $(\rho_{T,R}(*_T))_T = (*_i)_T$ for $i = 1$ or $i = 2$. However, if $U \neq T$, then $(\rho_{T,R}(*_T))_U = (*_i)_U$ for both $i = 1$ and $i = 2$, and thus $\rho_{T,R}(*_T) = *_i$, against the choice of $*_1$ and $*_2$. Therefore, $\rho_{T,R}(*_T)$ is irreducible.

(c) Let $*$ be an irreducible star operation. By point (a), the components of $\lambda_\Theta(*)$ in the product $\prod \mathrm{Star}(T)$ are all equal to the v-operation, with the exception of exactly one $*^{(S)}$. Hence, we can define a map

$$\begin{aligned} \lambda' \colon \mathrm{IrrStar}(R) &\longrightarrow \bigsqcup_{T\in\Theta} \mathrm{IrrStar}(T) \\ * &\longmapsto *^{(S)}, \end{aligned}$$

where S depends on $*$ in the above way. By point (b), we can construct an inverse ρ' by sending a $*_S \in \mathrm{IrrStar}(S)$ into the map $\rho_{S,T}(*_S)$. It follows that the two sets $\mathrm{IrrStar}(R)$ and $\bigsqcup_{T\in\Theta} \mathrm{IrrStar}(T)$ are in (canonical) bijective correspondence.

The case of prime star operations is handled in a completely analogous way. □

Corollary 3.79. *Let R be a domain and Θ be a Jaffard family on R. Then,* $\mathrm{IrrStar}(R)$ *(resp.,* $\mathrm{PrimeStar}(R)$*) is nonempty if and only if* $\mathrm{IrrStar}(T)$ *(resp.,* $\mathrm{PrimeStar}(T)$*) in nonempty for some $T \in \Theta$.*

Remark 3.80. We mentioned in Remark 1.32 that we do not have an example of a star operation on a numerical semigroup that is principal but not irreducible. However, in the case of rings, we do have such examples: indeed, if R is h-local with maximal ideals M_1 and M_2, and I_i is an ideal of R_{M_i} for $i \in \{1, 2\}$, then $J := I_1 \cap I_2$ generates a principal star operation $*_J$ that is not irreducible, since $*_J = *_{I_1} \wedge *_{I_2}$ by the proof of Proposition 3.77. Indeed, comparing Proposition 3.77 with 3.78, we see that the number of principal star operations on R is the product of the cardinaliies of $\mathrm{PStar}(T)$ (as T ranges in a Jaffard family of R), while the number of irreducible star operation on R is the sum of the cardinality of the $\mathrm{IrrStar}(T)$.

However, we note that this example relies exclusively on the fact that $\mathrm{Star}(R)$ can be decomposed (through a nontrivial Jaffard family), while (given a numerical semigroup S) $\mathrm{Star}(S)$ does not have any such decomposition, since S is "local" (i.e., it has a unique maximal ideal).

Corollary 3.81. *Let R be an integral domain and let Θ be a Jaffard family on R. If I is an integral ideal of R such that $*_I$ is irreducible, then $IT \cap R$ is divisorial for all but one $T \in \Theta$.*

Proof. By Proposition 3.77, $(*_I)_T = *_{IT}$ for every $T \in \Theta$. By Proposition 3.78(a), $*_{IT} \neq v^{(T)}$ for at most one $T \in \Theta$; hence, IT is divisorial (as a T-ideal) for all but at most one T. However, if IT is divisorial in T, then $IT \cap R$ is divisorial in R. Hence, there is at most one T such that of $IT \cap R$ is not divisorial. □

Recall that, following [60], we say that an ideal A is m-canonical if $(A : A) = R$ and $*_A$ is the identity. The following proposition can be seen as a generalization of [60, Theorem 6.7] to domains that are not necessarily integrally closed.

Proposition 3.82. *Let R be a domain. Then R admits an m-canonical ideal if an only if R is h-local, R_M admits an m-canonical ideal for every $M \in \mathrm{Max}(R)$ and $|\mathrm{Star}(R_M)| \neq 1$ for only a finite number of maximal ideals of M.*

Proof. Suppose A is a m-canonical ideal of R. Then, R is h-local by [60, Proposition 2.4], and thus $*_{AR_M} = (*_A)_M = d_{R_M}$ for every $M \in \mathrm{Max}(R)$. Moreover, $R_M = (A : A)R_M = (AR_M : AR_M)$ and thus AR_M is m-canonical for every M. If M does not contain A, then $AR_M = R_M$, so that R_M is m-canonical for R_M. But then $v_{R_M} = *(R_M) = d_{R_M}$, and thus $|\mathrm{Star}(R_M)| = 1$ for every M not containing A. Since A is contained in only finitely many maximal ideals, the result follows.

Conversely, suppose that the three hypotheses hold. For every $M \in \mathrm{Max}(R)$, let J_M be an m-canonical ideal of R_M, and define

$$I_M := \begin{cases} R_M & \text{if } |\mathrm{Star}(R_M)| = 1 \\ J_M & \text{if } |\mathrm{Star}(R_M)| > 1 \end{cases}$$

Note that, if $|\mathrm{Star}(R_M)| = 1$, then R_M is m-canonical for R_M, and thus I_M is m-canonical for every M.

Then $J := \bigcap_{P \in \mathrm{Max}(R)} I_P$ is a nonzero ideal of R and $*_J = \rho_\Theta((*_J)_M) = \rho_\Theta(*_{JR_M})$, where $\Theta := \{R_M : M \in \mathrm{Max}(R)\}$, and thus it is enough to prove that $JR_M = I_M$ for every M. However, this follows readily from Lemma 3.65, since Θ is a Jaffard family. □

3.4. Extension of other closure operations

The idea of extending a closure operation need not to be limited to star operation, but can also be applied to other classes of closure operations. In this section, we highlight what happens if we consider semiprime or semistar operations; we proceed by considering both at the same time.

The definition of extendability is completely analogous when the obvious modifications are made: a semiprime operation c on R is *extendable* to the flat overring T of R if the map

$$\begin{aligned} c_T \colon \mathcal{I}(T) &\longrightarrow \mathcal{I}(T) \\ IT &\longmapsto I^cT \end{aligned}$$

is well-defined, while a semistar operation $*$ is *extendable* to T is the map

$$\begin{aligned} *_T \colon \mathbf{F}(T) &\longrightarrow \mathbf{F}(T) \\ IT &\longmapsto I^*T \end{aligned}$$

is well-defined. The proofs of Propositions 3.3-3.9 carry over without modifications, noting that the equalities $(I^c : J^c) = (I^c : J)$ and $(I^* : J^*) = (I^* : J)$ holds when c and $*$ are, respectively, a semiprime or a semistar operation.

We can define maps

$$\begin{aligned} \lambda^{\mathrm{sp}}_{R,T} \colon \mathrm{ExtSp}(R;T) &\longrightarrow \mathrm{Sp}(T) \\ c &\longmapsto c_T \end{aligned} \quad\text{and}\quad \begin{aligned} \lambda^{\mathrm{sm}}_{R,T} \colon \mathrm{ExtSStar}(R;T) &\longrightarrow \mathrm{SStar}(T) \\ * &\longmapsto *_T \end{aligned}$$

that are continuous in the Zariski topology. We can also define restriction maps

$$\begin{aligned} \rho^{\mathrm{sp}}_{T,R}\colon \mathrm{Sp}(T) &\longrightarrow \mathrm{Sp}(R) \\ c &\longmapsto \rho^{\mathrm{sp}}_{T,R}(c) \end{aligned} \quad \text{and} \quad \begin{aligned} \rho^{\mathrm{sm}}_{R,T}\colon \mathrm{SStar}(T) &\longrightarrow \mathrm{SStar}(R) \\ * &\longmapsto \rho^{\mathrm{sm}}_{R,T}(*) \end{aligned}$$

by defining

$$I^{\rho^{\mathrm{sp}}_{T,R}(c)} := (IT)^c \cap R \quad \text{and} \quad I^{\rho^{\mathrm{sm}}_{T,R}(*)} := (IT)^*.$$

Both the extension maps and the restriction maps can be also extended to cover a whole family of flat overrings instead of only one.

Proposition 3.12 can be made stronger: indeed, every restriction to R of a semiprime (respectively, semistar) operation on T is extendable to R, and $\lambda^{\mathrm{sp}}_{R,T}\circ\rho^{\mathrm{sp}}_{T,R}$ and $\lambda^{\mathrm{sm}}_{R,T}\circ\rho^{\mathrm{sm}}_{T,R}$ are the identity, as can be seen noting that

$$[(IT)^c \cap R]T = (IT)^c \cap T = (IT)^c$$

if c is a semiprime operation (since I must be an integral ideal of R) and

$$(IT)^*T = (IT)^*$$

if $*$ is a semistar operation.

Proposition 3.17 remains true in both the semiprime and semistar setting (with the exception of the Noetherian semistar operations, that are not defined). Analogously, the definition of compatibility (Definition 3.19) needs only to take care of the obvious differences, and Theorem 3.21 still holds. Corollaries 3.22-3.24 carry over, while Corollaries 3.26-3.28 break down because, even if V is a discrete valuation ring (so, for example, if $V = R_P$ for some Krull domain R and some height-1 prime ideal P) the set $\mathrm{Sp}(R)$ is infinite [112, Proposition 3.2].

The results from Proposition 3.32 to Corollary 3.38 apply to semiprime operations as well, while they can be improved for semistar operations by removing the conditions on the intersection of the prime ideals; that is, we can remove the hypothesis that $R = \bigcap\{R_P \mid P \in X^1(R)\}$ (in Proposition 3.32), while $\mathrm{SStar}^{\Delta}(R)$ (the analogue of $\mathrm{Star}^{\Delta}(R)$ in the semistar setting) is nonempty if and only if Δ is a compact antichain. Propositions 3.41, 3.42 and 3.43, as the results at the end of Section 3.1.2, do not apply to semiprime or semistar operations.

However, when we come to Jaffard families, the behaviour of semiprime and semistar operations become different. Semiprime operations, as long as we stick to the whole $\mathrm{Sp}(R)$, are essentially analogous to star operations: using the same proof, we get an analogous of Theorem 3.67, that is, if Θ is a Jaffard family for R then the maps

$$\begin{aligned} \lambda^{\mathrm{sp}}_{\Theta}\colon \mathrm{Sp}(R) &\longrightarrow \prod_{T\in\Theta} \mathrm{Sp}(T) \\ c &\longmapsto (c_T)_{T\in\Theta} \end{aligned} \quad \text{and} \quad \begin{aligned} \rho^{\mathrm{sp}}_{\Theta}\colon \prod_{T\in\Theta} \mathrm{Sp}(T) &\longrightarrow \mathrm{Sp}(T) \\ (c^{(T)})_{T\in\Theta} &\longmapsto \bigwedge_{T\in\Theta} c^{(T)} \end{aligned}$$

are homeomorphisms between $\mathrm{Sp}(R)$ and $\prod\{\mathrm{Sp}(T) \mid T \in \Theta\}$. The correspondence then extends to the other classes of semiprime operations as in Propositions 3.68 and 3.71, and we can as well specialize the hypothesis like in Remark 3.73. In particular, we can analyze the structure of the semiprime operation on a Dedekind domain D almost directly from the structure of $\mathrm{Sp}(V)$, for V a discrete valuation ring, shortening the analysis done in [112, Section 3]. On the other hand, the definition of the class group cannot be readily adapted to semiprime operations (since principal ideals may not be closed and since I can be closed while xI is not), and it is not clear how to write principal semiprime operation explicitly enough to use the correspondence.

The case of semistar operation, on the other hand, is much more delicate. The first and foremost problem is that the result corresponding to Theorem 3.67 is *not* true: that is, even if Θ is a Jaffard family of R and $T \in \Theta$ is different from R, there are semistar operations that are not extendable to T. For example, let $*$ be the trivial extension of the identity of R, that is,

$$I^* = \begin{cases} I & \text{if } I \in \mathcal{F}(R) \\ K & \text{otherwise.} \end{cases}$$

If T is a Jaffard (proper) overring of R, then it is not a fractional ideal of R (for otherwise $T \cdot \Theta^{\perp}(T) = K$ would imply $\Theta^{\perp}(T) = K$); however, we have $RT = TT$, while

$$R^*T = T \neq K = T^*T.$$

Hence, $*$ is not extendable to T. The exact point in which the proof of Theorem 3.67 fails is the possibility of using Lemma 3.66, because the equality $IT = JT$ does not imply that $(I : J) \neq (0)$. In the same way, the analogue of Lemma 3.76 does not hold for principal semistar operations, since we can't apply Lemma 3.66. On the other hand, the analogue of Theorem 3.67 does hold for finite-type operations: that is, the restrictions of $\lambda_\Theta^{\mathrm{sm}}$ and $\rho_\Theta^{\mathrm{sm}}$ are homeomorphisms between $\mathrm{SStar}_f(R)$ and $\prod\{\mathrm{SStar}_f(T) \mid T \in \Theta\}$: this can be seen using the analogue of Theorem 3.21 and noting that $\mathcal{C}_f(\Theta) = \prod\{\mathrm{SStar}_f(T) \mid T \in \Theta\}$ since no nonzero prime ideal P of R can survive in two overrings $T_1 \neq T_2$ belonging to Θ.

3.5. Applications to Prüfer domains

The result in Section 3.3 allows to subdivide the study of the set $\mathrm{Star}(R)$ of star operations on R into the study of the sets $\mathrm{Star}(T)$, as T ranges among the members of a Jaffard family Θ. Obviously, this result isn't quite useful if we don't know how to find Jaffard families, or if studying $\mathrm{Star}(T)$ is as complex as studying $\mathrm{Star}(R)$. The purpose of this section is to show that, in the case of (some classes of) Prüfer domains, we can resolve the first question, and we can at least do some progress on the second in order to prove more explicit results on $\mathrm{Star}(R)$.

3.5.1. Treed domains

Definition 3.83. *A domain R is* treed *if its prime spectrum* $\mathrm{Spec}(R)$ *is a tree, i.e., if* $\mathrm{Spec}(R_P)$ *is linearly ordered for every prime ideal P of R.*

Prüfer domains constitute probably the most important class of treed domains. More generally, treed domains include *going-down domains* [26, Theorem 2.2] (and thus the smaller classes of *divided domains* [27, Proposition 2.1] and *i-domains* [101, Corollary 2.13]) and *stable domains* [93, Theorem 4.11]. (See the references for the respective definitions.)

Let now R be a treed domain. We say that two maximal ideals M, N of R are *dependent*, and we write $M \sim N$, if there is a nonzero prime ideal $P \subseteq M \cap N$. Clearly dependence is reflexive and symmetric; we show that it is also transitive. Suppose $M \sim N$ and $N \sim Q$, and let P_1, P_2 be nonzero prime ideals such that $P_1 \subseteq M \cap N$ and $P_2 \subseteq N \cap Q$. Then P_1 and P_2 are both contained in N, so that P_1R_N and P_2R_N are proper prime ideals of R_N. However, since R is treed, this imply that $P_1R_N \subseteq P_2R_N$ (or conversely) and thus that $P_1 \subseteq P_2$ (or conversely). In particular, $P_1 \subseteq M \cap Q$ (respectively, $P_2 \subseteq M \cap Q$) and thus M and Q are dependent.

Note that, in a general domain, dependence is a weak form of being comparably connected; indeed, if R is treed, then $P \sim Q$ if and only if P and Q are comparably connected (but note that the two relations are defined on different sets, namely $\mathrm{Max}(R)$ and $\mathrm{Spec}(R) \setminus \{(0)\}$).

If $M \in \mathrm{Max}(R)$ is dependent only to itself, we say that M is *isolated*.

Hence, we can divide $\mathrm{Max}(R)$ into the equivalence classes $\{\Delta_\lambda \mid \lambda \in \Lambda\}$ relative to $\sim$. For any class Δ_λ, define the overring T_λ as the intersection of R_M, as M ranges among Δ_λ. If T_λ is R-flat and its maximal ideals restrict exactly to the elements of Δ_λ, we say that T_λ is a *branch* of R.

Lemma 3.84. *Let R be a treed domain, and let Δ_λ be an equivalence class of $\mathrm{Max}(R)$ with respect to dependence. If Δ_λ is compact, then T_λ is a branch of R.*

Proof. The claim follows directly from Proposition 3.59, noting that $\Delta_\lambda^\downarrow \setminus \{(0)\}$ is an equivalence class with respect to the "comparably connected" relation. □

Lemma 3.85. *Let R be a treed domain of finite dimension, and suppose T is a branch of R. Then, T has a unique prime ideal P of height 1, and there is a bijection between the nonzero prime ideals of T and the prime ideals of R containing $P \cap R$.*

Proof. Let P be a prime ideal of height 1 of T (it exists because $\dim R < \infty$): to prove the theorem it suffices to show that $P \subseteq Q$ for every nonzero prime ideal Q of T. Suppose $P \nsubseteq Q$, let M_1 be a maximal ideal of T containing P and $N_1 := M_1 \cap R$: then (since T is a flat overring of R) $M_1 = N_1T$, and $T_{M_1} = R_{N_1}$ is a treed domain, and thus its prime ideal are linearly ordered. It follows that Q is not contained in M_1 and, if M_2 is a maximal ideal containing Q, in the same way we can show that M_2 does not contain P. Let $N_2 := M_2 \cap R$.

The definition of branch implies that N_1 and N_2 are maximal ideals, and also that $N_1 \sim N_2$. Hence there is a nonzero prime ideal $O' \subseteq N_1 \cap N_2$. Let $O \in \mathrm{Spec}(T)$ such that $O \cap R = O'$. Then, $O \neq (0)$ and it is contained in both M_1 and M_2. Since P is of height 1 and is contained in M_1, it follows that $P \subseteq O$, and thus $P \subseteq M_2$, which is absurd. Hence $P \subseteq Q$, and the lemma is proved. □

Corollary 3.86. *Let R be a treed domain of finite character. If $\dim R < \infty$, then each Δ_λ is finite.*

Proof. Each Δ_λ corresponds to an height 1 prime ideal of R. However, since R is of finite character, the number of maximal ideals containing a nonzero ideal is finite, and thus Δ_λ is finite. □

Proposition 3.87. *Preserve the notation above. Let R be a treed domain such that*

(a) $\mathrm{Max}(R)$ *is a Noetherian space; or*
(b) R *is semilocal.*

Then, the set $\Theta := \{T_\lambda \mid \lambda \in \Lambda\}$ is the minimal Jaffard family of R.

Proof. Clearly, Θ is complete. Moreover, if P is a prime ideal of R, then there is a unique T_λ such that $PT_\lambda \neq T_\lambda$: therefore, if $T_\lambda \neq T_\mu$ are in Θ, then $T_\lambda T_\mu = K$.

Suppose $\mathrm{Max}(R)$ is Noetherian. Then, each Δ_λ is compact, and thus each T_λ is a branch of R by Lemma 3.84. Moreover, if $x \in R$, then the ideal xR has only a finite number of minimal primes (this follows, for example, from the proof of [17, Chapter 4, Corollary 3, p.102] or [12, Chapter 6, Exercises 5 and 7]); in particular, since each prime survives in only one $T \in \Theta$, the family Θ is of finite character.

On the other hand, if R is semilocal, then each Δ_λ is finite, and thus every T_λ is a branch of R; moreover, Λ is fintie, and thus Θ is also locally finite.

Hence, in both cases, Θ is a Jaffard family by Corollary 3.49.

Suppose now that Θ is not minimal. Then, there is a Matlis partition $\mathcal{P}$ of $\mathrm{Max}(R)$ that is strictly finer than the partition determined by Θ; in particular, there are maximal ideals M, N of R such that $M \sim N$ but M, N belong to different classes with respect to $\mathcal{P}$. However, $M \sim N$ means that there is a nonzero prime ideal $P \subseteq M \cap N$, against the definition of Matlis partition. Therefore, Θ is minimal, and thus the minimum (by Proposition 3.57). □

Corollary 3.88. *Let R be a finite-dimensional treed domain of finite character. Then, the minimal Jaffard family Θ of R is in bijective correspondence with $X^1(R)$.*

Proof. If R is finite-dimensional and of finite character, then $\mathrm{Spec}(R)$ (and thus $\mathrm{Max}(R)$) is Noetherian: indeed, if I is a nonzero radical ideal of R, then $V(I)$ is finite, and thus every ascending chain of radical ideals must stop; by [12, Chapter 6, Exercise 5], this implies Noetherianity.

Hence, by Proposition 3.87, the minimal Jaffard family on R is the set of branches of R. However, each branch has a unique height-1 ideal (by Lemma 3.85), and each height-1 ideal survives in exactly one branch. Therefore, the map $\Theta \longrightarrow X^1(R)$, $T \mapsto P_T \cap R$ (where P_T is the height-1 prime ideal of T) is bijective. □

As in the case of h-local domains, the existence of a minimal Jaffard family implies that all the results of Section 3.3 can be applied to the class of locally finite treed domains; for example, $\mathrm{Star}(R)$ is homeomorphic to $\prod \mathrm{Star}(T_\lambda)$, as T_λ varies among the branches of R.

Example 3.89. If R is treed, but not of finite character, it could be that every T_λ is a branch, but that $\Theta := \{T_\lambda \mid \lambda \in \Lambda\}$ is not a Jaffard family on R.

For example, let R be an almost Dedekind domain which is not Dedekind. Since R is one-dimensional, the relation $\sim$ is trivial, i.e., every maximal ideal is isolated; therefore, each T_λ has the form R_M for some maximal ideal M, and thus it is a branch. However, if Θ were a Jaffard family, then every star operation would be extendable, and this is not true (see Example 3.4).

Further along this road, we can construct a domain without a minimal Jaffard family: let R be an almost Dedekind such that $\mathrm{Max}(R) = \{M_i : i \in \mathbb{N}\} \cup \{M_\infty\}$, where each $M_i = (m_i)$ is principal and M_∞ is not finitely generated. (Such domains do indeed exist: see [85, Example 6.13].) Let $R_n := R\left[\frac{1}{m_0}, \dots, \frac{1}{m_n}\right]$: then, R_n is a localization of R whose maximal ideals are the extension of M_∞ and of M_i, for $i > n$. It is easy to see that the family $\Theta_n := \{R_{M_0}, R_{M_1}, \dots, R_{M_n}, R_n\}$ is a Jaffard family of R, and that $\Theta_n \preceq \Theta_k$ if $n \geq k$; moreover, each R_{M_t} is a member of Θ_k, for $t \leq k$. Hence, if Θ is the minimal Jaffard family of R, then it should contain every R_{M_t}, and thus (since every maximal ideal survives in exactly one element of Θ) it should contain also R_{M_∞}, and thus it should be $\{R_M : M \in \mathrm{Max}(R)\}$. However, we have seen that this is not a Jaffard family, so that $\mathcal{J}(R)$ does not admit a minimum.

3.5.2. Cutting the branch

Let R be a treed domain, and let T be a branch of R; suppose $\mathrm{Max}(T) = \{M_1, \dots, M_n\}$ is finite. Since T is a branch, for all $i = 2, \dots, n$ there is a prime ideal $P_i \subseteq M_1 \cap M_i$; in particular, the ideal $P_2, \dots, P_n$ are all contained in M_1, and thus they must be linearly ordered. In particular, one of them (call it P) must contain the others; thus, P is contained in all the maximal ideals of T. The ring T/P is still a treed domain, and it is semilocal; in particular, we can apply the results in the previous section, finding a Jaffard family for T and repeating the construction until we arrive to local domains. The purpose of this section is to see how we can use this process to better investigate the set of star operations on R; we are mainly interested in Prüfer domains, although few of the results need only slightly more general hypotheses.

Recall that, if V is a valuation domain with maximal ideal M, then M is said to be *branched* if there is a prime ideal $P \subsetneq M$ of V such that there are no primes strictly contained between P and M. In particular, if V is finite-dimensional then M is branched.

Lemma 3.90. *Let R be a domain and let $P \in \mathrm{Spec}(R)$ be a nonzero prime ideal such that $P = PR_P$ and such that R_P is a valuation domain. Suppose also that $P \notin \mathrm{Max}(R)$. Then:*

(a) R_P is a fractional ideal of R;
(b) $(R : P) = R_P$ and $(R : R_P) = P$: in particular, P and R_P are divisorial over R;
(c) every fractional ideal of R is comparable with P and with R_P.

Proof. For (a), note that $P = PR_P \subseteq R$, so that $P \subseteq (R : R_P)$, which in particular is different from (0). Hence, R_P is a fractional ideal.

Since P is not maximal, $R \neq R_P$. Therefore, $(R : R_P) \subsetneq R \subsetneq R_P$. However, $(R : R_P)$ is a R_P-module containing P, and since P is the maximal ideal of the valuation ring R_P, the containments $P \subseteq (R : R_P) \subsetneq R_P$ imply that $(R : R_P) = P$. In the same way, $(R : P)$ is a R_P-module containing R_P. If $x \notin R_P$, then $\mathbf{v}(x) < 0$ (where $\mathbf{v}$ is the valuation associated to R_P), and thus xP is a R_P-module properly containing P. Hence, $R_P \subseteq xP$, and $x \notin (R : P)$. Therefore, $(R : P) = R_P$. From this it follows that $(R : (R : P)) = (R : R_P) = P$ and $(R : (R : R_P)) = (R : P) = R_P$, and P and R_P are divisorial over R.

Next, suppose $I \nsubseteq R_P$. Then there is a $x \in I \setminus R_P$, and thus (since R_P is a valuation domain) $x^{-1} \in PR_P = P$. Thus, $x^{-1}R_P \subseteq P$ and $R_P \subseteq xP \subseteq xR \subseteq I$; hence, R_P is comparable with every ideal of R. In the same way, if $J \nsubseteq P$, then there is a $x \in J \setminus P$, which verifies $\mathbf{v}(x) \leq 0$. If $\mathbf{v}(x) = 0$, then x is a unit in R_P, and thus $xPR_P = PR_P$, i.e., $xP = P$, and thus $P \subseteq xR \subseteq J$. On the other hand, if $\mathbf{v}(x) < 0$, then $x^{-1} \in P \subseteq R$, so that $P \subseteq R \subseteq xR \subseteq J$. Therefore, P is comparable with every ideal of R. This shows (c). $\square$

We say that a collection $\mathcal{A} \subseteq \mathbf{F}(R)$ is a $*$-*family* if $(0), K \in \mathcal{A}$, $\mathcal{A}$ is closed by arbitrary intersections and $xI \in \mathcal{A}$ whenever $I \in \mathcal{A}$ and $x \in K \setminus \{0\}$. It is straightforward to see that $\mathcal{A}$ is a $*$-family if and only if there is a semistar operation $\sharp$ such that $\mathcal{A}$ is the set of $\sharp$-closed submodules. Moreover, if $\mathcal{A} \subseteq \mathcal{F}(R) \cup \{K\}$ and $R \in \mathcal{A}$, then $\mathcal{A}$ is a $*$-family if and only if $\mathcal{A} \setminus \{K\}$ is the set of $\sharp$-closed ideals for some star operation $\sharp$ on R.

Lemma 3.91. *Let R be a domain with quotient field K and let $P \in \operatorname{Spec}(R)$ be a prime ideal such that $P = PR_P$ and R_P is a valuation domain. Let $\mathbf{v}$ be the valuation associated to R_P and $G := \mathbf{v}(R_P)$ be the valuation group of R_P. Let I be a fractional ideal of R. Then:*

- *(a) if $\mathbf{v}(I)$ has an infimum α in G, then there is an $a \in K$ such that $\mathbf{v}(a) = \alpha$ and $P \subseteq a^{-1}I \subseteq R_P$;*
- *(b) if $\mathbf{v}(I)$ has not an infimum in G, then I is a R_P-module, and there is a set $A \subseteq K$ such that $I = \bigcap_{a \in A} aR_P$ and $R_P = \bigcap_{a \in A} a^{-1}I$.*

Moreover, if I is divisorial over R_P and not principal, then $\mathbf{v}(I)$ has not an infimum in G.

Proof. Suppose $\mathbf{v}(I)$ has an infimum α. If $\alpha \in \mathbf{v}(I)$, choose $a \in I$ such that $\mathbf{v}(a) = \alpha$; otherwise, take any $a \in K$ such that $\mathbf{v}(a) = \alpha$. In both cases, $IR_P \subseteq aR_P$, and so $I \subseteq aR_P$, i.e., $a^{-1}I \subseteq R_P$. Moreover, if $\mathbf{v}(b) > \alpha$, then there is an $i \in I$ such that $\mathbf{v}(b) > \mathbf{v}(i) \geq \alpha$, so that $bi^{-1} \in P \subseteq R$ and $b \in I$. Hence, if $c \in P$, then $\mathbf{v}(ac) > \mathbf{v}(a)$ and thus $ac \in I$, i.e., $c \in a^{-1}I$; therefore, $P \subseteq a^{-1}I$.

Suppose $\mathbf{v}(I)$ has not an infimum in G. Let $A := \{\alpha \in R : \mathbf{v}(\alpha) < \mathbf{v}(i)$ for every $i \in I\}$. We claim that $I = \bigcap_{\alpha \in A} \alpha R_P$. Clearly $I \subseteq \alpha R_P$. Suppose $x \in \bigcap_{\alpha \in A} \alpha R_P$, and suppose there is an $i \in I$ such that $\mathbf{v}(i) \leq \mathbf{v}(x)$. Then, since $\mathbf{v}(I)$ has no infimum, there is a $j \in I$ such that $\mathbf{v}(j) < \mathbf{v}(x)$; hence, $\mathbf{v}(xj^{-1}) > 0$, i.e., $xj^{-1} \in P \subseteq R$, and $x \in jR \subseteq I$. On the other hand, if $\mathbf{v}(i) > \mathbf{v}(x)$ for every $i \in I$, $I \subseteq xR_P$; since $\mathbf{v}(I)$ has no infimum, $I \subsetneq xR_P$, and if $I \subseteq yR_P \subsetneq xR_P$ then $y \in A$ while $x \notin yR_P$, against the choice of x. Hence, $I = \bigcap_{\alpha\in A} \alpha R_P$ and, in the same way, we can prove that $R_P = \bigcap_{\beta\in A} \beta^{-1} I$.

For the last statement, note that if I is not principal but $\mathbf{v}(I)$ has an infimum α, then α is not a minimum, and $I^v = \{x \mid \mathbf{v}(x) \geq \alpha\} \neq I$. □

Proposition 3.92. *Let R be a domain with quotient field K, and let $P \in \mathrm{Spec}(R)$ be a nonzero prime ideal such that $P = PR_P$ and R_P is a valuation domain. Suppose also that $P \notin \mathrm{Max}(R)$. Then, there is a bijection between $\{* \in \mathrm{SStar}(R) : P^* = P\}$ and $\mathrm{SStar}(R/P)$, which restricts to a bijection between $\mathrm{Star}(R)$ and $(\mathrm{S})\mathrm{Star}(R/P)$.*

Proof. The quotient field F of $D := R/P$ is equal to $D_{P/P} \simeq R_P/PR_P = R_P/P$. Therefore, the quotient $\pi : R_P \longrightarrow F$ establishes a bijection between $\mathcal{F}_0(R) := \{I \in \mathcal{F}(R) : P \subseteq I \subseteq R_P\}$ and $\mathbf{F}(D)$.

Let $\mathfrak{A}$ be the set of $*$-families $\mathcal{A} \subseteq \mathcal{F}(R) \cup \{K\}$ such that $P \in \mathcal{A}$ (in particular, $\mathcal{A} \neq \{K\}$), and let $\mathfrak{B}$ be the set of $*$-families of D-submodules of F. We will prove the theorem by showing that there is a (natural) bijection between $\mathfrak{A}$ and $\mathfrak{B}$.

Let $\mathcal{A} \subseteq \mathcal{F}(R)$ be a $*$-family and suppose $P \in \mathcal{A}$. Let $\mathcal{A}_0 := \mathcal{A} \cap \mathcal{F}_0(R)$ and define $\pi^\diamondsuit(\mathcal{A})$ as the family $\pi(\mathcal{A}_0) = \{\pi(I) : I \in \mathcal{A}_0\}$. We claim that $\mathcal{B} := \pi^\diamondsuit(\mathcal{A})$ is a $*$-family of D. Note that, since $P \in \mathcal{A}$, then $R_P \in \mathcal{A}$ (since $R_P = (P : P)$) and thus $P/P = (0_D)$ and $R_P/P = F$ are in $\pi^\diamondsuit(\mathcal{A})$.

Let $\{J_\gamma : \gamma \in C\} \subseteq \mathcal{B}$. For every $\gamma \in C$ there is an $I_\gamma \in \mathcal{A}'$ such that $\pi(I_\gamma) = J_\gamma$. Hence, since π is surjective,

$$\bigcap_{\gamma\in\Gamma} J_\gamma = \bigcap_{\gamma\in\Gamma} \pi(I_\gamma) = \pi\left(\bigcap_{\gamma\in\Gamma} I_\gamma\right) \in \mathcal{B} \tag{3.4}$$

because $\mathcal{A}$ is a $*$-family. Suppose now $J \in \mathcal{B}$ and $y \in F \setminus \{0\}$. Then $J = \pi(I)$ for some $I \in \mathcal{A}'$ and $y = \pi(x)$ for some $x \in R_P \setminus P$. Therefore, x is a unit in R_P, and thus $P \subseteq xI \subseteq R_P$, and it follows that $yJ = \pi(x)\pi(I) = \pi(xI) \in \mathcal{B}$ since $xI \in \mathcal{A}$. Thus, $\mathcal{B}$ is a $*$-family, and $\pi^\diamondsuit$ is a map $\mathfrak{A} \longrightarrow \mathfrak{B}$.

Suppose $\mathcal{B} \subseteq \mathbf{F}(D)$ is a $*$-family. Let $\mathcal{N}(R)$ be the set of ideals of R such that $\mathbf{v}(I)$ has not an infimum in $\mathbf{v}(R_P)$, and define

$$\pi^\square(\mathcal{B}) := \{x\pi^{-1}(J) : x \in K, J \in \mathcal{B}\} \cup \mathcal{N}(R) \cup \{K\}.$$

Clearly, $\mathcal{A} := \pi^\square(\mathcal{B})$ is closed by product with elements of K, and it contains $P = \pi^{-1}(0_D)$. Note also that $\pi^{-1}(J)$ is a fractional ideal of R, since it is contained in the fractional ideal R_P, and thus $\mathcal{A} \subseteq \mathcal{F}(R) \cup \{K\}$.

We first show that if $P \subsetneq x\pi^{-1}(J) \subseteq R_P$ and $J \in \mathcal{B}$, $J \neq (0)$, then $x\pi^{-1}(J) = \pi^{-1}(\pi(x)J)$. Let $w := \pi(x)$. If $z \in x\pi^{-1}(J)$ then $z = xj$ and $\pi(z) = w\pi(j) \in wJ$, i.e., $z \in \pi^{-1}(wJ)$. On the other hand, if $z \in \pi^{-1}(wJ)$, then $\pi(z) = wj$ and $\pi(zx^{-1}) \in J$, so that $zx^{-1} \in \pi^{-1}(J)$ and $z \in x\pi^{-1}(J)$.

Let now $\{I_\gamma : \gamma \in C\} \subseteq \mathcal{A}$, and let $I := \bigcap_{\gamma\in C} I_\gamma$. If $I = K$ or $\mathbf{v}(I)$ has not an infimum in $\mathbf{v}(R_P)$ then $I \in \mathcal{A}$, and we are done. Suppose $\mathbf{v}(I)$ admits an infimum. Then there is an $a \in K$ such that $P \subseteq a^{-1}I \subseteq R_P$, and thus $P \subseteq \bigcap_{\gamma\in C} a^{-1}I_\gamma \subseteq R_P$.

If $a^{-1}I=R_P$ or $a^{-1}I=P$ then $a^{-1}I\in\mathcal{A}$ since P and R_P are divisorial over R, by Lemma 3.90. Otherwise, let $C':=\{\gamma\in C : a^{-1}I_\gamma\subseteq R_P\}$. For every $\gamma\notin C'$, $a^{-1}I_\gamma\supseteq R_P$ (since $a^{-1}I_\gamma$ is always comparable with R_P) and thus $a^{-1}I=\bigcap_{\gamma\in C'}a^{-1}I_\gamma$. If some I_γ is divisorial over R_P, then $I_\gamma=\bigcap_{x^{-1}\in(R_P:I)}xR_P$, and with the same argument we can suppose that each $a^{-1}I_\gamma$ is either in the form $x\pi^{-1}(J)$ for some $J\in\mathcal{B}$ and $x\in K$ or yR_P for some $y\in R_P$. On the other hand, since $F\in\mathcal{B}$, $yR_P=y\pi^{-1}(F)$, and the latter case is contained in the former. Hence, defining $y_\gamma:=\pi(x_\gamma)$, we have

$$a^{-1}I=\bigcap_{\gamma\in C'}a^{-1}I_\gamma=\bigcap_{\gamma\in C'}x_\gamma\pi^{-1}(J_\gamma)=\bigcap_{\gamma\in C'}\pi^{-1}(y_\gamma J_\gamma)\in\mathcal{A}.$$

Therefore, $a^{-1}I=\pi^{-1}(J)$ for some $J\in\mathcal{B}$, i.e., $\pi^\square(\mathcal{B})$ is a $*$-family and $(\pi^\square(\mathcal{B}))_0=\pi^{-1}(\mathcal{B})$.

Hence we have maps

$$\begin{aligned}\Diamond\colon \mathfrak{A}&\longrightarrow\mathfrak{B}\\ \mathcal{A}&\longmapsto\pi(\mathcal{A}_0)\end{aligned}\qquad\text{and}\qquad\begin{aligned}\square\colon\mathfrak{B}&\longrightarrow\mathfrak{A}\\ \mathcal{B}&\longmapsto\pi^\square(\mathcal{B})\end{aligned}$$

such that

$$\pi^\Diamond\circ\pi^\square(\mathcal{B})=\pi(\pi^\square(\mathcal{B})\cap\mathcal{F}_0(R))=\pi(\pi^{-1}(\mathcal{B}))=\mathcal{B}\tag{3.5}$$

and thus $\Diamond\circ\square$ is the identity on $\mathfrak{B}$.

On the other hand,

$$\pi^\square\circ\pi^\Diamond(\mathcal{A})=\pi^\square(\pi(\mathcal{A}_0))$$

contains all the ideals in $\mathcal{N}$ and all the ideals in the form $x\pi^{-1}(\pi(J))=xJ$ for any $J\in\mathcal{A}$, and thus $\pi^\square\circ\pi^\Diamond(\mathcal{A})\subseteq\mathcal{A}$. However, if $J\in\mathcal{A}$, then either $J\in\mathcal{N}$ (and thus $J\in\pi^\square\circ\pi^\Diamond(\mathcal{A})$) or there is an $a\in K$ such that $a^{-1}J\in\mathcal{A}\cap\mathcal{F}_0(R)$. In the latter case, $a^{-1}J\in\pi^\square\circ\pi^\Diamond(\mathcal{A})$ and so does J. Hence, also $\square\circ\Diamond$ is the identity, and $\square$ and $\Diamond$ are bijections between $\mathfrak{A}$ and $\mathfrak{B}$, which induces a bijection between $\{*\in\mathrm{SStar}(R):P^*=P\}$ and $\mathrm{SStar}(D)$.

Moreover $R\in\mathcal{A}$ if and only if $D\in\pi^\Diamond(\mathcal{A})$, and if $R\in\mathcal{A}$ then $P\in\mathcal{A}$ (since P is R-divisorial), so the above correspondence restricts to a bijection between $\mathrm{Star}(R)$ and (S)Star(D). □

The correspondence outlined above can also be extended to the topological level. For every $I\in\mathbf{F}(R)$, let $\mathcal{Z}(I):=\{J\in\mathbf{F}(R)\mid J\supseteq I,1\notin J\}$. Then, a semistar operation $\sharp$ is in the subbasic open set $V_I:=\{*\in\mathrm{SStar}(R)\mid 1\in I^*\}$ if and only if $\mathcal{Z}(I)\cap\mathbf{F}^\sharp(R)=\emptyset$: indeed, if it is empty then $I^\sharp$ is an ideal containing I but not in $\mathcal{Z}(I)$, and thus $1\in I^\sharp$; conversely, if the intersection contains J, then $I^\sharp\subseteq J$ and so $\sharp\notin V_I$. Hence, the Zariski topology on $\mathrm{SStar}(R)$ can be transferred to the set of $*$-families of $\mathbf{F}(R)$ by declaring open the sets of the form

$$\mathfrak{V}(I):=\{\mathcal{A}\mid\mathcal{A}\cap\mathcal{Z}(I)=\emptyset\}.$$

Corollary 3.93. *Preserve the notation and the hypotheses of Proposition 3.92. There is a homeomorphism between* $\{*\in\mathrm{SStar}(R):P^*=P\}$ *and* $\mathrm{SStar}(R/P)$*, which restricts to a homeomorphism between* $\mathrm{Star}(R)$ *and* (S)Star(R/P).

Proof. It is enough to prove that the maps $\pi^\diamondsuit$ and $\pi^\square$ (defined in (3.5), in the previous proof) are homeomorphisms.

Let $I \in \mathbf{F}(R)$. If $I \subseteq P$ or $R \subseteq I$, then $\mathfrak{A} \subseteq \mathfrak{V}(I)$, so the image of $\mathfrak{V}(I)$ is the whole $\mathfrak{B}$. Suppose that $P \subsetneq I \subseteq R_P$. Then, $\mathcal{B} \in \pi^\diamondsuit(\mathfrak{V}(I))$ if and only if $\pi^\square(\mathcal{B}) \in \mathfrak{V}(I)$, that is, if and only if $\pi^\square(\mathcal{B}) \cap \mathcal{Z}(I) = \emptyset$.

Since $P \subsetneq I$, $\mathbf{v}(I)$ has 0 as a minimum; thus the above condition is equivalent to $\{x\pi^{-1}(J) \mid x \in K, J \in \mathcal{B}\} \cap \mathcal{Z}(I) = \emptyset$. But if $x\pi^{-1}(J)$ contains I and does not contain I, then it must be between P and R_P, and in particular it must be equal to $\pi^{-1}(L)$ for some $L \in \mathcal{B}$ (see the proof of Proposition 3.92); hence, the intersection is empty if and only if $\mathcal{B} \cap \pi(\mathcal{Z}(I)_0) = \emptyset$. But now $\pi(\mathcal{Z}(I)_0) = \mathcal{Z}(I/P)$, and thus $\pi^\diamondsuit(\mathfrak{V}(I)) = \mathfrak{V}(I/P)$. Hence, $\pi^\diamondsuit$ is open and, since all the R/P-submodule of $F = R_P/P$ are images of some R-submodule of K, it is also continuous, Being bijective, it is a homeomorphism. $\square$

Remark 3.94. The construction presented in the proof of Proposition 3.92 can be seen as a variant of the construction presented in [9] (for the domains of the type $D + M$) and in [48] (for general pullbacks). It is also a generalization of [69, Lemmas 2.3 and 2.4] to the case of general semistar operations (although we use a different approach).

Following [48], suppose $\phi : R \longrightarrow D$ is a surjective ring homomorphism, where R and D are an integral domains; let P be the kernel of ϕ. Then, for every star operation $*$ on D we can define a star operations $*^\phi$ on R by defining, for every fractional ideal I of R, [48, Corollary 2.4]

$$I^{*^\phi} := \bigcap \left\{ x^{-1}\phi^{-1}\left(\phi(xI + P)^*\right) \mid x \in (R : I), x \neq 0 \right\};$$

similarly, for every star operation $\sharp$ on R, we can define a star operation $\sharp_\phi$ on D by defining, for every $F \in \mathcal{F}(D)$, [48, Proposition 2.6]

$$F^{\sharp_\phi} := \bigcap \{ y^{-1}\phi(\phi^{-1}(yF))^* \mid y \in (D : F), y \neq 0 \}.$$

These two construction yield two maps

$$\begin{aligned} (-)^\phi \colon \mathrm{Star}(R) &\longrightarrow \mathrm{Star}(D) \\ * &\longmapsto *^\phi \end{aligned} \qquad \text{and} \qquad \begin{aligned} (-)_\phi \colon \mathrm{Star}(D) &\longrightarrow \mathrm{Star}(R) \\ \sharp &\longmapsto \sharp_\phi, \end{aligned}$$

such that $(-)_\phi$ is surjective and $(-)^\phi$ is injective [48, Corollary 2.11].

We can interpret these maps in terms of our construction. Suppose R verifies the hypotheses of Proposition 3.92 and, with a slight abuse of notation, denote by $\square$ and $\diamondsuit$ the bijections (S)Star$(D) \longrightarrow$ Star(R) and Star$(R) \longrightarrow$ (S)Star(D), respectively.

Suppose that I is a fractional ideal of R contained between R and R_P, such that $J := \phi(I)$ is *not* a fractional ideal of D. Then, $(R : I) = P$, and thus

$$I^{*^\phi} = \bigcap \left\{ x^{-1}\phi^{-1}\left((0)^*\right) \mid x \in P, x \neq 0 \right\} = R_P$$

for every $* \in \mathrm{Star}(D)$. Hence, $*^\phi = \square \circ \iota(*)$, where $\iota : * \mapsto *_e$ is the trivial extension map defined in Proposition 2.52. In a similar way, we can see that, if $\diamondsuit(*) \in \iota \circ (-)_\phi(\mathrm{Star}(R))$, then $\diamondsuit(*) = \iota \circ (-)_\phi(*)$; that is, $\diamondsuit = \iota \circ (-)_\phi$ on the set $(\iota \circ (-)_\phi)^{-1}(\mathrm{Star}(R))$.

We proceed to apply Proposition 3.92 to Prüfer domains.

Proposition 3.95. *Let R be a Prüfer domain and let P be a nonzero prime ideal such that P is contained in every maximal ideal of R. Suppose also that $P \notin \mathrm{Max}(R)$. Then, there is a homeomorphism between* $\mathrm{Star}(R)$ *and* $(\mathrm{S})\mathrm{Star}(R/P)$.

Proof. It is enough to show that R and P verify the hypotheses of Proposition 3.92 (or, equivalently, of Lemma 3.90). Clearly R_P is a valuation domain. Moreover, we have $P = \bigcap_{M\in\mathrm{Max}(R)} PR_M$ and, since $P \subseteq M$, $PR_M = P(R_M)_{PR_M} = PR_P$ and thus $P = \bigcap_{M\in\mathrm{Max}(R)} PR_P = PR_P$. □

Corollary 3.96. *Let R be a Prüfer domain and let P be a nonzero prime ideal that is contained in every maximal ideal of R. Then, there is a homeomorphism between* $\mathrm{SStar}(R)$ *and the disjoint union*

$$\Delta_P := \mathrm{SStar}(R/P) \sqcup (\mathrm{SStar}(R_P) \setminus \{d\}),$$

where the topology on the right hand side is generated by the two families

- $V_1(I/P) := \{* \in \mathrm{SStar}(R/P) \mid 1 \in (I/P)^*\} \sqcup (\mathrm{SStar}(R_P) \setminus \{d^{(R_P)}\})$ *and*
- $V_2(IR_P) := \{\sharp \in \mathrm{SStar}(R_P) \setminus \{d\} \mid 1 \in (IR_P)^\sharp\}$.

Proof. We first claim that there is a homeomorphism between $\mathrm{SStar}(R_P) \setminus \{d^{(R_P)}\}$ and $\{* \in \mathrm{SStar}(R) : P \neq P^*\}$. Indeed, consider the topological embedding $\iota_P : \mathrm{SStar}(R_P) \longrightarrow \mathrm{SStar}(R)$ such that $I^{\iota_P(*)} := (IR_P)^*$ (see Section 2.2.4). If $* \neq d^{(R_P)}$, then $P^* \neq P$: indeed, if $P^* = P$ then $(P : P) = R_P$ is $*$-closed, and thus $*$ must be the v-operation on R_P, and it must not be equal to the identity; but in this case, $*$ would not close the maximal ideal of R_P, i.e., P. Hence $P \neq P^*$.

Conversely, if $*$ is a semistar operation on R and $P^* \neq P$, then R^* cannot be properly contained in R_P (since P is S-divisorial for every $S \subsetneq R_P$ – apply Lemma 3.90 to S), and thus it must contain R_P (since every overring of R, different from K, is a fractional ideal of R_P and thus of R). In particular, $*|_{\mathbf{F}(R_P)}$ is a semistar operation on R_P such that $* = \iota_P(*|_{\mathbf{F}(R_P)})$. Note also that the Zariski topology on $\mathrm{SStar}(R_P) \setminus \{d^{(R_P)}\}$ coincides with the subspace topology induced by Δ_P.

Proposition 3.92 gives also a homeomorphism between $\mathrm{Star}(R)$ and $\mathrm{SStar}(R/P)$, and its proof shows that the Zariski topology on $\mathrm{SStar}(R/P)$ is the same that the subspace topology induced by Δ_P. To complete the proof, we have to show that, if $P \subseteq I \subseteq R_P$, then the image of V_I into Δ_P contains the whole $\mathrm{SStar}(R_P) \setminus \{d\}$, or equivalently that if $*_1$ and $*_2$ are semistar operations on R such that $P^{*_1} = P$ and $P^{*_2} \neq P$ then $*_1 \leq *_2$. However, if $P^{*_1} = P$, then $*_1|_{\mathbf{F}(R_P)}$ is a semistar operation on R_P which closes its maximal ideal; since R_P is a valuation ring, it follows that $*_1|_{\mathbf{F}(R_P)} = d^{(R_P)}$, and thus $*_1|_{\mathbf{F}(R_P)} \leq *_2$. Therefore, also $*_1 \leq *_2$. □

We end this section by showing that, using induction, we can prove that, if R is a Prüfer domain with finite spectrum, then it has only a finite number of star and semistar operation. We will give a more precise version of this result in Theorem 3.130, where we will show that in this case $\mathrm{Star}(R)$ depends only on the order structure of R and on the valuation ring R_P. We need only another lemma.

Lemma 3.97 [69, Lemma 4.2]. *Let R be a semilocal Prüfer domain and I a R-submodule of its quotient field K. Then I is a fractional ideal of the ring $(I : I)$.*

Proof. We can suppose $I \neq K$. Let $\mathrm{supp}(I) := \{M \in \mathrm{Spec}(R) : IR_M \neq K\}$. Since $I = \bigcap_{M\in\mathrm{supp}(I)} IR_M$, the set $\mathrm{supp}(I)$ is not empty, and since R is semilocal it is finite. Let $T := \bigcap_{M\in\mathrm{supp}(I)} R_M$. Then T is the localization of R with respect to $S := R \setminus \bigcup_{M\in\mathrm{supp}(I)} M$, the maximal ideals of T are the MT, for $M \in \mathrm{supp}(I)$, and $T_{MT} = R_M$ for every $M \in \mathrm{supp}(I)$.

Therefore, $IT = \bigcap_{N\in\mathrm{Max}(R)} IT_N = \bigcap_{M\in\mathrm{supp}(I)} IR_M = I$, and since $(I : I)$ is the biggest ring such that I is a $(I : I)$-module, we have $T \subseteq (I : I)$. Every R_M is a valuation ring, and thus IR_M is a R_M-fractional ideal for every $M \in \mathrm{supp}(I)$, i.e., for each $M \in \mathrm{supp}(I)$ there is a d_M such that $d_M IR_M \subseteq R_M$. Hence, if $d := \prod_{M\in\mathrm{supp}(I)} d_M$, we have $dI \subseteq R_M$ for every $M \in \mathrm{supp}(I)$, and thus $dI \subseteq T \subseteq (I : I)$, so that I is a $(I : I)$-fractional ideal. $\square$

Example 3.98. Lemma 3.97 does not hold when R is not semilocal: for example, let $R = \mathbb{Z}$, let $\mathbb{P}$ be the set of prime numbers, and define $I := \sum_{p\in\mathbb{P}} \frac{1}{p}\mathbb{Z}$. If $q \in \mathbb{P}$ and $q^2|b$, then $\frac{1}{b} \notin I$, for otherwise there would be integers $a_1, \ldots, a_n$ and primes $p_1, \ldots, p_n$ such that $\frac{1}{b} = \frac{a_1p_1+\cdots+a_np_n}{p_1\cdots p_n}$. However, $a_1p_1 + \cdots + a_np_n$ is an integer and q^2 cannot divide $p_1 \cdots p_n$, and thus the equality cannot hold. On the other hand, we claim that $(I : I) = \mathbb{Z}$: indeed, suppose $\frac{a}{b} \in (I : I)$, with $b > 1$ and a, b coprime integers. Then, there is a prime p dividing b, and thus $\frac{a}{b}\frac{1}{p} = \frac{a}{bp} \in I$. However, we can choose a prime q and find a c such that $\frac{a}{bp} + \frac{c}{q} = \frac{1}{bpq}$: since $p^2|pbq$, $\frac{1}{bpq} \notin I$, and since $\frac{c}{q} \in I$, we have $\frac{a}{bp} \notin I$. Hence $\frac{a}{b}$ cannot belong to $(I : I)$, and thus $(I : I) = \mathbb{Z}$.

The following result was already proved in [69, Theorem 4.4]; we present a different proof.

Theorem 3.99. *Let R be a semilocal Prüfer domain of finite dimension. Then, the sets* $\mathrm{Star}(R)$, $\mathrm{(S)Star}(R)$ *and* $\mathrm{SStar}(R)$ *are finite.*

Proof. For every $k \in \mathbb{N}$, $k > 0$, let:

(A_k) for every semilocal Prüfer domain R such that $\dim(R) \leq k$, $\mathrm{Star}(R)$ is finite;

(B_k) for every semilocal Prüfer domain R such that $\dim(R) \leq k$, $\mathrm{(S)Star}(R)$ is finite;

(C_k) for every semilocal Prüfer domain R such that $\dim(R) \leq k$, $\mathrm{SStar}(R)$ is finite.

We will show that (A_1) is true and that, for every $n \in \mathbb{N}$, $n > 0$, $(A_n) \Longrightarrow (C_n) \Longrightarrow (B_n) \Longrightarrow (A_{n+1})$: by induction, this will prove the theorem.

To prove (A_1), note that a semilocal domain of dimension 1 is h-local, and thus $|\mathrm{Star}(R)| = \prod\{|\mathrm{Star}(R_M)| : M \in \mathrm{Max}(R)\}$ by Theorem 3.67. However, $|\mathrm{Star}(R_M)| \leq 2$ for each M, and since there are only a finite number of maximal ideals we have $|\mathrm{Star}(R)| < \infty$.

Since $\mathrm{(S)Star}(R) \subseteq \mathrm{SStar}(R)$, it is clear that $(C_n) \Longrightarrow (B_n)$.

$(A_n) \implies (C_n)$. Let $\mathrm{PSStar}(R)$ be the set of principal semistar operations $\wedge_I$; then, $X^{\wedge_I} := (I : (I : X))$ for every $X \in \mathbf{F}(R)$, and by definition $\wedge_I$ is the biggest semistar operation which fixes I. For every semistar operation $*$, we have $* = \inf\{\wedge_I \mid I = I^*\}$ (the proof is analogue to the one of Proposition 1.5), and it follows that if $\mathrm{PSStar}(R)$ is finite then so is $\mathrm{SStar}(R)$.

Let now $I \in \mathbf{F}(R)$ and define $T := (I : I)$ and $* := \wedge_I|_{\mathcal{F}(T)}$. Note that $*$ is a star operation on T. Then, for each $X \in \mathbf{F}(R)$, $(I : X)$ is a T-module, so that $(I : X) = (I : XT)$ and thus $X^{\wedge_I} = (I : (I : XT))$. In particular, if XT is a fractional ideal of T, then $X^{\wedge_I} = (XT)^*$. Conversely, if XT is not a fractional ideal of T, then $(I : X) = (0)$, for otherwise $aX \subseteq I$ for some $a \neq 0$ and $aXT \subseteq I$, which (since I is a T-fractional ideal) would imply $dXT \subseteq T$ for some $d \in K \setminus \{0\}$. Hence, if $XT \notin \mathcal{F}(T)$ we have $X^{\wedge_I} = (I : 0) = K$, and thus $*$ is determined by $\wedge_I$.

Therefore, if $\wedge_I \neq \wedge_J$, then the associated star operations $\wedge_I|_{(I:I)}$ and $\wedge_J|_{(J:J)}$ are different. Hence, there is an injective map $\mathrm{PSStar}(R) \hookrightarrow \bigsqcup_{T \in \mathrm{Over}(R)} \mathrm{Star}(R)$. However, since R is a semilocal Prüfer domain, each overring is a localization of R, and since R has only a finite number of prime ideals, $\mathrm{Over}(R)$ is finite. But each $T \in \mathrm{Over}(R)$ is a Prüfer domain such that $\dim(T) \leq \dim(R)$, and thus, by inductive hypothesis, each $\mathrm{Star}(T)$ is finite. Hence $\mathrm{PSStar}(R)$ is finite, and thus $\mathrm{SStar}(R)$ is finite.

$(B_n) \implies (A_{n+1})$. Let R be a Prüfer domain of dimension $n+1$. By Theorem 3.67, and the results in Section 3.5.1 about treed domains, $|\mathrm{Star}(R)| = \prod_{\lambda\in\Lambda} |\mathrm{Star}(T_\lambda)|$, where $\{T_\lambda \mid \lambda \in \Lambda\}$ is the minimal Jaffard family of R. By construction, since R is finite-dimensional, every T_λ has a unique prime ideal of height 1, say P_λ, which is contained in every maximal ideal of T_λ. By Proposition 3.95, $|\mathrm{Star}(T_\lambda)| = |\mathrm{(S)Star}(T_\lambda/P_\lambda)|$. Now T_λ/P_λ is a Prüfer domain of dimension less or equal than n (since T_λ is a localization of R), and thus $\mathrm{(S)Star}(T_\lambda/P_\lambda)$ is finite by inductive hypothesis. Therefore, each $\mathrm{Star}(T_\lambda)$ is finite, and since Λ is finite also $\mathrm{Star}(R)$ is finite. □

3.5.3. h-local Prüfer domains

If R is both a Prüfer domain and a h-local domain, then its minimal Jaffard family $\Theta := \{R_M \mid M \in \mathrm{Max}(R)\}$ is composed of valuation domains. In this case, star operations behave particularly well. We start by re-proving [67, Theorem 3.1] using our general theory.

Proposition 3.100. *Let R be an h-local Prüfer domain, and let $\mathcal{M}$ be the set of non-divisorial maximal ideals of R. Then $|\mathrm{Star}(R)| = 2^{|\mathcal{M}|}$.*

Proof. Since R is h-local, a maximal ideal M is divisorial if and only if MR_M is divisorial and, in this case, $\mathrm{Star}(R_M) = \{d\}$. Therefore, we have a chain of maps

$$\mathrm{Star}(R) \xrightarrow{\lambda_\Theta} \prod_{M\in\mathcal{M}} \mathrm{Star}(R_M) \xrightarrow{\rho_\Theta} \mathrm{Star}(R). \tag{3.6}$$

and $\mathrm{Star}(R_M) = \{d, v\}$, with $d \neq v$, for every $M \in \mathcal{M}$. Hence, by Theorem 3.67, $|\mathrm{Star}(R)| = 2^{|\mathcal{M}|}$. □

Our next goal is to show that star operations distribute over arbitrary intersections. We single out the case of valuation rings.

Lemma 3.101. *Let V be a valuation domain, and let $\mathcal{A} := \{I_\alpha\}_{\alpha\in A}$ be a family of ideals of V with nonzero intersection. Then, $\bigcap_{\alpha\in A} I_\alpha^v = \left(\bigcap_{\alpha\in A} I_\alpha\right)^v$.*

Proof. If $\mathcal{A}$ has a minimum $I_{\overline{\alpha}}$, then $I_{\overline{\alpha}}^v \subseteq I_\beta^v$ for every $\beta \in A$, and thus $\left(\bigcap_{\alpha\in A} I_\alpha\right)^v = I_{\overline{\alpha}}^v = \bigcap_{\alpha\in A} I_\alpha^v$.

Suppose $\mathcal{A}$ has not a minimum: since $\left(\bigcap_{\alpha\in A} I_\alpha\right)^v \subseteq I_\alpha^v$ for every $\alpha \in A$, we have $\left(\bigcap_{\alpha\in A} I_\alpha\right)^v \subseteq \bigcap_{\alpha\in A} I_\alpha^v$.

Let $x \in \bigcap_{\alpha\in A} I_\alpha^v$: if $x \in \bigcap_{\alpha\in A} I_\alpha$ then $x \in \left(\bigcap_{\alpha\in A} I_\alpha\right)^v$. On the other hand, if $x \notin \bigcap_{\alpha\in A} I_\alpha$, then there is an $\overline{\alpha}$ such that $x \in I_{\overline{\alpha}}^v \setminus I_{\overline{\alpha}}$, i.e., $w(x) = \inf w(I_{\overline{\alpha}})$ (where w is the valuation associated to V). However, since $\mathcal{A}$ has no minimum, there are $\beta, \gamma \in A$ such that $I_\alpha \supsetneq I_\beta \supsetneq I_\gamma$; in particular, $w(x) > \inf w(I_\gamma)$, and thus $x \notin I_\gamma^v$, a contradiction. Therefore, $x \in \bigcap_{\alpha\in A} I_\alpha$. □

Proposition 3.102. *Let R be an h-local Prüfer domain. Then, every star operation on R distributes over intersections; i.e., if $\{I_\alpha\}_{\alpha\in A}$ is a family of ideals with nonzero intersection, and $* \in \mathrm{Star}(R)$, then $\left(\bigcap_{\alpha\in A} I_\alpha\right)^* = \bigcap_{\alpha\in A} I_\alpha^*$.*

Proof. Let $I := \bigcap_{\alpha\in A} I_\alpha$. By Theorem 3.67 and Proposition 3.51, we have

$$I^*R_M = (IR_M)^{*_M} = \left(\left(\bigcap_{\alpha\in A} I_\alpha\right) R_M\right)^{*_M} = \left(\bigcap_{\alpha\in A} I_\alpha R_M\right)^{*_M}.$$

However, R_M is a valuation domain, and $*_M$ is either the identity or the v-operation; applying Lemma 3.101,

$$\left(\bigcap_{\alpha\in A} I_\alpha R_M\right)^{*_M} = \bigcap_{\alpha\in A} (I_\alpha R_M)^{*_M} = \bigcap_{\alpha\in A} I_\alpha^* R_M.$$

Therefore,

$$I^* = \bigcap_{M\in\mathrm{Max}(R)} I^*R_M = \bigcap_{M\in\mathrm{Max}(R)} \bigcap_{\alpha\in A} I_\alpha^* R_M = \bigcap_{\alpha\in A} \bigcap_{M\in\mathrm{Max}(R)} I_\alpha^* R_M = \bigcap_{\alpha\in A} I_\alpha^*$$

which concludes the proof. □

In fact, this property (almost) characterize h-local domains among the Prüfer domains; we first state a lemma.

Lemma 3.103. *Let R be a Prüfer domain such that its Jacobson radical $\mathrm{Jac}(R)$ contains a nonzero prime ideal. Then, there is a prime ideal $Q \subseteq \mathrm{Jac}(R)$ such that $\mathrm{Jac}(R/Q)$ does not contain nonzero prime ideals.*

Proof. Let $\Delta := \{P \in \mathrm{Spec}(R), P \subseteq \mathrm{Jac}(R), P \neq (0)\}$. By hypothesis, Δ is nonempty. Let $Q := \bigcup_{P \in \Delta} P$.

Δ is a chain of prime ideals: indeed, since R is treed, if P_1 and P_2 were noncomparable elements of Δ, then P_1 would not be contained in the maximal ideals containing P_2, against $P_1 \subseteq \mathrm{Jac}(R)$. Being the union of a chain of prime ideals, Q is itself a prime ideal, and it is contained in every maximal ideal of R. Suppose $\mathrm{Jac}(R/Q)$ contains a nonzero prime ideal $\overline{Q}$. Then, $\overline{Q} = Q'/P$ for some prime ideal Q' of R, and Q' is contained in every maximal ideal of R. It follows that $Q \subsetneq Q' \subseteq \mathrm{Jac}(R)$, against the construction of Q. □

Proposition 3.104. *Let R be a Prüfer domain and suppose that R is either:*

(a) semilocal; or
(b) locally finite and finite-dimensional.

Then, the following are equivalent:

(i) R is h-local;
(ii) every star operation on R distributes over arbitrary intersections;
(iii) every star operation on R distributes over finite intersections;
(iv) the v-operation on R distributes over arbitrary intersections;
(v) the v-operation on R distributes over finite intersections.

Proof. The (i $\Longrightarrow$ ii) implication is Proposition 3.102; (ii $\Longrightarrow$ iii $\Longrightarrow$ v) and (ii $\Longrightarrow$ iv $\Longrightarrow$ v) are clear, and so we only have to show that (v $\Longrightarrow$ i).

Suppose (v) holds and let Θ be the minimal Jaffard family of R. If R is not h-local, then a branch $T \in \Theta$ is not local; the hypotheses on R guarantee that T has only one height-1 prime, which is contained in every maximal ideal of T. Therefore, we can apply Lemma 3.103 and find a prime ideal Q such that $\mathrm{Jac}(T/Q)$ contains no prime ideals. By Proposition 3.95, there is an order-preserving bijection between $\mathrm{Star}(T)$ and $\mathrm{(S)Star}(T/Q)$, where the v-operation on T corresponds to the semistar operation $*$ which is the trivial extension of the v-operation on T/Q.

Since $\mathrm{Jac}(T/Q)$ does not contain nonzero primes, T/Q admits a nontrivial Jaffard family Λ; let $Z \in \Lambda$, and define $Z' := \bigcap_{W \in \Lambda \setminus \{Z\}} W$. Then, Z and Z' are not fractional ideals of T/Q, and thus $Z^* = Z'^* = F$, where F is the quotient field of T/Q; on the other hand, $Z \cap Z' = T/Q$ and thus $(Z \cap Z')^* = T/Q$.

If $\pi : R_Q \longrightarrow T/Q$ is the canonical quotient, it follows that $\pi^{-1}(Z)^v = \pi^{-1}(Z')^v = T_Q$, while $\pi^{-1}(Z \cap Z')^v = \pi^{-1}(T/Q)^v = T^v = T$. Since T is not local, $T \neq T_Q$, and thus v does not distribute over finite intersections, against the hypothesis. □

It is noted in the proof of [92, Theorem 3.10] that, if R is an h-local Prüfer domain and I, J are divisorial ideals of R, then $I + J$ is also divisorial. We can extend this result to arbitrary star operations:

Proposition 3.105. *Let R be an h-local Prüfer domain, let $* \in \mathrm{Star}(R)$ and let I, J be $*$-closed ideals. Then, $I + J$ is $*$-closed.*

Proof. Since R is h-local, $I + J$ is $*$-closed if and only if $(I + J)R_M$ is $*_M$-closed for every $M \in \mathrm{Max}(R)$. However, since R_M is a valuation domain, either $IR_M \subseteq JR_M$ or $JR_M \subseteq IR_M$; hence, $(I+J)R_M = IR_M + JR_M$ is equal either to IR_M or to JR_M, both of which are $*_M$-closed. □

The last result does not hold if we drop the hypothesis that R is h-local: in fact, let $R = \mathbb{Z} + X\mathbb{Q}[[X]]$ and let $R_p := \mathbb{Z}[1/p] + X\mathbb{Q}[[X]]$ for each prime number p. Then, the star operation generated by v, R_2 and R_3 (which are fractional ideals of R) closes both R_2 and R_3, but $(R_2 + R_3)^* = \mathbb{Q}[[X]] \neq R_2 + R_3$: in fact, $1/5 \notin R_2 + R_3$, and if $a(R_2 + R_3) \subseteq R_2$ for some $a \in \mathbb{Q}((X))$, then $a/3^n \in R_2$ for every $n \in \mathbb{N}$, and thus $a \in X\mathbb{Q}[[X]]$. Hence, if $R_2 + R_3 \subseteq a^{-1}R_2$, then $(R_2 + R_3)^v = \mathbb{Q}[[X]] \subseteq a^{-1}R_2$, so that closing R_2 (and, symmetrically, R_3) has no effect on the closure of $R_2 + R_3$.

Proposition 3.106. *Let R be an h-local Prüfer domain, let $* \in \mathrm{Star}(R)$ and let I, J be ideals of R. If I is not $*$-closed, then at least one of $I + J$ and $I \cap J$ is not $*$-closed.*

Proof. For every $M \in \mathrm{Max}(R)$, $(I + J)R_M = IR_M + JR_M = \max\{IR_M, JR_M\}$, while $(I\cap J)R_M = IR_M\cap JR_M = \min\{IR_M, JR_M\}$. Since I is not $*$-closed, there is a maximal ideal N such that IR_N is not $*_N$-closed; however, at least one of $(I+J)R_N$ and $(I\cap J)R_N$ is equal to IR_N, and thus at least one is not $*_N$-closed. Therefore, at least one between $I + J$ and $I \cap J$ is not $*$-closed. □

In Proposition 3.117 we will show that the sum of two $*$-invertible $*$-ideals is $*$-invertible, provided that the Prüfer domain R is locally finite and finite-dimensional.

As in the case of Propositions 3.102 and 3.104, also Proposition 3.106 is almost an equivalence:

Proposition 3.107. *Let R be a Prüfer domain and suppose that R is either:*

(a) semilocal; or
(b) locally finite and finite-dimensional.

Then, the following are equivalent:

(i) R is h-local;
(ii) for every $* \in \mathrm{Star}(R)$, $I \in \mathcal{F}(R) \setminus \mathcal{F}^*(R)$ and $J \in \mathcal{F}(R)$, at least one of $I \cap J$ and $I + J$ is not $*$-closed;
(iii) for every $I \in \mathcal{F}(R) \setminus \mathcal{F}^v(R)$ and $J \in \mathcal{F}(R)$, at least one of $I \cap J$ and $I + J$ is not divisorial.

Proof. (i $\Longrightarrow$ ii) is Proposition 3.106, while (ii $\Longrightarrow$ iii) is obvious.

To show (iii $\Longrightarrow$ i), like in the proof of the Proposition 3.104 we reduce to the case that R is trivially branched, and subsequently to R/P, where P is the biggest prime ideal contained in $\mathrm{Jac}(R)$. Let Λ be a nontrivial Jaffard family of $T := R/P$, $Z \in \Lambda$ and $Z' := \bigcap_{W\in\Lambda\setminus\{Z\}} W$. As before, $Z \cap Z' = T$; moreover, for every maximal ideal M of T, either $ZT_M = K$ or $Z'T_M = K$. Therefore, $Z + Z' = \bigcap_{M\in\mathrm{Max}(T)}(Z + Z')T_M = K$. Hence, both $\pi^{-1}(Z \cap Z')$ and $\pi^{-1}(Z + Z')$ are divisorial, while $\pi^{-1}(Z)$ and $\pi^{-1}(Z')$ are not. □

3.5.4. The class group of a Prüfer domain

If $*$ is a (semi)star operation, we can define the $*$-class group by mirroring the definition of the case of star operations: we say that I is $*$-invertible if $(I(R:I))^* = R$, and we define $\mathrm{Cl}^*(R)$ as the quotient between the group of the $*$-invertible $*$-ideals (endowed with the $*$-product $I \times_* J := (IJ)^*$) and the subgroup of principal ideals. Since $(R : I) = (0)$ if I is not a fractional ideal of R, every $*$-invertible ideal is a fractional ideal, and thus $\mathrm{Cl}^*(R)$ coincides with $\mathrm{Cl}^{*'}(R)$, where $*' := *|_{\mathcal{F}(R)}$ is the restriction of $*$.

The first result of this section is that the method used in Section 3.5.2 can be extended to the class group.

Proposition 3.108. *Let R be a domain and let $P \in \mathrm{Spec}(R)$ be a prime ideal such that $P = PR_P$ and R_P is a valuation domain. Suppose also that $P \notin \mathrm{Max}(R)$. Let $* \in \mathrm{Star}(R)$ and let $\sharp$ be the corresponding (semi)star operation on $D := R/P$. Then, $\mathrm{Cl}^*(R)$ is naturally isomorphic to $\mathrm{Cl}^\sharp(D)$.*

Proof. Let $\pi : R_P \longrightarrow F = Q(D)$ be the quotient map, and let I be a fractional ideal of R contained between P and R_P. We claim that $\pi((R:I)) = (D : \pi(I))$. In fact, if $y \in \pi((R:I))$ then $y = \pi(x)$ for some $x \in (R:I)$, and thus $y\pi(I) = \pi(x)\pi(I) = \pi(xI) \subseteq \pi(R) = D$, and thus $x \in (D : \pi(I))$. Conversely, if $y \in (D : \pi(I))$ and $y = \pi(x)$ then $y\pi(I) \subseteq D$, i.e., $\pi(xI) \subseteq D$. By the correspondence between R-submodules of R_P and D-submodules of F we have $xI \subseteq R$ and $y \in \pi((R:I))$.

Let J be a $\sharp$-invertible ideal of D. Hence, there are no $\sharp$-closed ideals E such that $J(D:J) \subseteq E \subsetneq D$ and thus (using what we have proved in the previous paragraph), if $I := \pi^{-1}(J)$, there are no $*$-closed ideals L such that $I(R:I) \subseteq L \subsetneq R$, and hence I is $*$-invertible. Therefore, we have an injective map $\theta : \mathrm{Inv}^\sharp(D) \longrightarrow \mathrm{Inv}^*(R)$. It is also straightforward to see that θ is a group homomorphism.

By the proof of Proposition 3.92, if J, J' are D-submodules of F, and $I := \pi^{-1}(J)$, $I' := \pi^{-1}(J')$, then $J = zJ'$ for some $z \in F$ if and only if $I = wI'$ for some $w \in K$. Therefore, θ induces an injective map $\overline{\theta} : \mathrm{Cl}^\sharp(D) \longrightarrow \mathrm{Cl}^*(R)$ which clearly is a group homomorphism.

Let now I be a $*$-invertible ideal of R, and let $\mathbf{v}$ be the valuation associated to R_P. If $\mathbf{v}(I)$ has not an infimum, then by Lemma 3.91 I is a R_P-module, and thus so is $(I:I)$: hence $(I:I) \neq R$, and I is not v-invertible [50, Proposition 34.2(2)], and in particular it is not $*$-invertible, for any $* \in \mathrm{Star}(R)$. Therefore, applying again Lemma 3.91 we can find an $a \in K$ such that $P \subseteq a^{-1}I \subseteq R_P$. Moreover, P is not $*$-invertible (since $(P:P) = R_P$) and thus $P \neq a^{-1}I$. Therefore, $[I] = \overline{\theta}([\pi(a^{-1}I)])$, and in particular $[I]$ is in the image of $\overline{\theta}$. Since I was arbitrary, $\overline{\theta}$ is surjective and $\mathrm{Cl}^\sharp(D) \simeq \mathrm{Cl}^*(R)$. □

Corollary 3.109. *Let V be a valuation domain and let P be a prime ideal of V such that P is nonzero and nonmaximal and such that PV_P is branched. Then, $\mathrm{Cl}^v(V) \simeq \mathrm{Cl}^v(V/P)$.*

Proof. The hypotheses of Proposition 3.108 are satisfied, so $\mathrm{Cl}^v(V) \simeq \mathrm{Cl}^\sharp(V/P)$, where $\sharp$ is the (semi)star operation corresponding to v. The maximal ideal M of V is finitely generated if and only if M/P is finitely generated in V/P, and it is v-closed if and only if V/P is $\sharp$-closed; since, for the maximal ideal of a valuation domain, being divisorial is equivalent to being finitely generated, $\sharp$ must be the v-operation on V/P. $\square$

We can now treat the case where R is a Prüfer domain with only finitely many maximal ideals.

Theorem 3.110. *Let R be a Prüfer domain, and suppose that R is semilocal and that every maximal ideal of R is branched. Consider a star operation $*$ on R. Then,*

$$\mathrm{Cl}^*(R) \simeq \bigoplus_{\substack{M\in\mathrm{Max}(R)\\ M\neq M^*}} \mathrm{Cl}^v(R_M).$$

Proof. We proceed by induction on the number n of maximal ideals of R. If $n = 1$, the conclusion is trivial, since if $M = M^*$ then $* = d$ and $\mathrm{Cl}^*(R) = (0)$, while if $M \neq M^*$ then $* = v$.

$* \neq v$ if and only if $M \neq M^*$.

Suppose $n > 1$ and let Θ be the minimal Jaffard family of R (whose existence is guaranteed by Proposition 3.87). By Theorem 3.75, we have

$$\frac{\mathrm{Cl}^*(R)}{\mathrm{Pic}(R)} \simeq \bigoplus_{T\in\Theta} \frac{\mathrm{Cl}^{*_T}(T)}{\mathrm{Pic}(T)};$$

however, $\mathrm{Pic}(R) = (0) = \mathrm{Pic}(T)$ for every $T \in \Theta$, since R and each T are semilocal. Thus, $\mathrm{Cl}^*(R) \simeq \bigoplus_{T\in\Theta} \mathrm{Cl}^{*_T}(T)$. Since a maximal ideal M of R is $*$-closed if and only if MT is $*_T$-closed, it follows that it suffices to prove the theorem when R has only one branch.

Let $\mathrm{Jac}(R)$ be the Jacobson radical of R. Since R is semilocal, $\mathrm{Jac}(R) \neq (0)$, and, with the same reasoning of the beginning of Section 3.5.2, $\mathrm{Jac}(R)$ must contain a nonzero prime ideal. By Lemma 3.103, there is a prime ideal $Q \subseteq \mathrm{Jac}(R)$ such that $\mathrm{Jac}(R/Q)$ does not contain nonzero prime ideals. Let $A := R/Q$.

The minimal Jaffard family Θ' of A is nontrivial, and thus every $B \in \Theta'$ is a semilocal Prüfer domain with less than n maximal ideals. Moreover, by the proof of Proposition 3.95 and by Proposition 3.108, $\mathrm{Cl}^*(R) \simeq \mathrm{Cl}^\sharp(A)$, where $\sharp$ is the restriction to $\mathcal{F}(A)$ of the (semi)star operation corresponding to $*$. Therefore, by inductive hypothesis,

$$\mathrm{Cl}^\sharp(A) \simeq \bigoplus_{B\in\Theta'} \mathrm{Cl}^v(B) \simeq \bigoplus_{B\in\Theta'} \bigoplus_{\substack{N\in\mathrm{Max}(B)\\ N\neq N^{\sharp_B}}} \mathrm{Cl}^v(B_N) \simeq \bigoplus_{\substack{N\in\mathrm{Max}(A)\\ N\neq N^\sharp}} \mathrm{Cl}^v(A_N).$$

Thus,

$$\mathrm{Cl}^*(R) \simeq \mathrm{Cl}^\sharp(A) \simeq \bigoplus_{\substack{N\in\mathrm{Max}(A)\\ N\neq N^\sharp}} \mathrm{Cl}^v(A_N).$$

However, if M is the maximal ideal of R which correspond to the maximal ideal N of A, then $R_M/QR_M \simeq A_N$ by Corollary 3.109, and thus $\mathrm{Cl}^v(R_M) \simeq \mathrm{Cl}^v(A_N)$; the claim follows. $\square$

Corollary 3.111. *Let R be a Prüfer domain, and suppose that R is finite-dimensional and of finite character. Consider a star operation $*$ on R. Then,*

$$\frac{\mathrm{Cl}^*(R)}{\mathrm{Pic}(R)} \simeq \bigoplus_{\substack{M\in\mathrm{Max}(R)\\ M\neq M^*}} \mathrm{Cl}^v(R_M).$$

Proof. Let Θ be the minimal branch decomposition of R. By Corollary 3.88, there is a bijective correspondence between Θ and the height 1 prime ideals of R, and every $T \in \Theta$ is semilocal. Hence, by Proposition 3.87 and Theorem 3.110,

$$\frac{\mathrm{Cl}^*(R)}{\mathrm{Pic}(R)} \simeq \bigoplus_{T\in\Theta} \frac{\mathrm{Cl}^{*_T}(T)}{\mathrm{Pic}(T)} \simeq \bigoplus_{T\in\Theta} \mathrm{Cl}^{*_T}(T) \simeq \bigoplus_{T\in\Theta} \bigoplus_{\substack{M\in\mathrm{Max}(T)\\ M\neq M^{*_T}}} \mathrm{Cl}^v(T_M).$$

The conclusion now follows since $T_M = R_N$ (where $N := M \cap R$) and $N = N^*$ if and only if $M = M^{*_T}$. □

Corollary 3.112. *Let R be a Bézout domain, and suppose that R is either:*

(a) semilocal and with every maximal ideal branched; or
(b) finite-dimensional and of finite character.

Let $*$ be a star operation on R. Then,

$$\mathrm{Cl}^*(R) \simeq \bigoplus_{\substack{M\in\mathrm{Max}(R)\\ M\neq M^*}} \mathrm{Cl}^v(R_M).$$

Proof. It is enough to note that $\mathrm{Pic}(R) = 0$ if R is a Bézout domain, and then apply the previous results. □

Corollary 3.113. *Let R be a Bézout domain, and suppose that R is either*

(a) semilocal and with every maximal ideal branched;
(b) finite-dimensional and of finite character.

Let S be a multiplicatively closed subset of R. Then, there is a natural surjective group homomorphism $\mathrm{Cl}^v(R) \longrightarrow \mathrm{Cl}^v(S^{-1}R)$, $[I] \mapsto [S^{-1}I]$.

Proof. Let $\Delta := \{M \in \mathrm{Max}(R) : M \cap S = \emptyset\}$. Then, for every $M \in \Delta$, $R_M = (S^{-1}R)_{S^{-1}M}$, and thus the isomorphism of Theorem 3.110 and Corollary 3.111 reduces to a surjective map $\mathrm{Cl}^v(R) \longrightarrow \bigoplus_{M\in\Delta} \mathrm{Cl}^v(R_M) \simeq \mathrm{Cl}^v(S^{-1}R)$, where the last equality comes from the fact that the maximal ideals of $S^{-1}R$ are the extensions of the ideals belonging to Δ. □

Therefore, under the hypotheses of Theorem 3.110 or Corollary 3.111, the determination of $\mathrm{Cl}^*(R)/\mathrm{Pic}(R)$ is reduced to the calculation of $\mathrm{Cl}^v(V)$, where V is a valuation domain. This can be calculated:

Proposition 3.114 [11, Corollaries 3.6 and 3.7]. *Let V be a valuation domain with maximal ideal M branched, and let P be the prime ideal directly below M. Let G be the value group of V/P, represented as a subgroup of $\mathbb{R}$. Then,*

$$\mathrm{Cl}^v(V) \simeq \begin{cases} 0 & \text{if } G \simeq \mathbb{Z} \\ \mathbb{R}/G & \text{otherwise.} \end{cases}$$

Proof. By Corollary 3.109, $\mathrm{Cl}^v(V) \simeq \mathrm{Cl}^v(V/P)$, and thus we need only to prove the claim in the case $\dim(V) = 1$. In this case, the value group of V is $\mathbb{Z}$ if and only if V is a discrete valuation ring, in which case the v-operation is the identity and $\mathrm{Cl}^v(V) = (0)$.

Suppose V is not discrete and let $G \subseteq \mathbb{R}$ be the value group of V and let $\mathbf{v}$ be the valuation relative to V. For every $\alpha \in \mathbb{R}$, define I_α as the ideal

$$I_\alpha := \{x \in V \mid \mathbf{v}(x) \geq \alpha\} = \bigcap_{\substack{y \in V \\ \mathbf{v}(y) \leq \alpha}} xV.$$

Every I_α is divisorial. Moreover, all divisorial ideals are of this form: indeed, the only other ideals have the form $J_\beta := \{x \in V \mid \mathbf{v}(x) > \beta\}$, and $J_\beta \neq I_\beta$ if and only if $\beta \in G$; in this case, $J_\beta^v = I_\beta$ and J_β is not divisorial.

We claim that $(I_\alpha I_\beta)^v = I_{\alpha+\beta}$ for all $\alpha, \beta \in V$. Indeed, the $(\subseteq)$ containment is obvious. On the other hand, suppose $\mathbf{v}(x) = \alpha + \beta + \epsilon$ for some $\epsilon > 0$. Since V is not discrete, there are elements y, z such that $\alpha < \mathbf{v}(y) < \alpha + \frac{\epsilon}{2}$ and $\beta < \mathbf{v}(z) < \beta + \frac{\epsilon}{2}$; then, $yz \in I_\alpha I_\beta$ and $x \in yzV$ since $\mathbf{v}(x) > \mathbf{v}(y) + \mathbf{v}(z)$. In particular, $J_{\alpha+\beta} \subseteq I_\alpha I_\beta$ and thus $I_{\alpha+\beta} = (J_{\alpha+\beta})^v \subseteq (I_\alpha I_\beta)^v$.

In particular, $(I_\alpha I_{-\alpha})^v = I_0 = V$, and thus every I_α is also v-invertible; moreover, it follows that the v-product in $\mathrm{Inv}^v(V)$ is just the sum in $\mathbb{R}$, so that $\mathrm{Inv}^v(V) \simeq \mathbb{R}$. The kernel of the map $\mathrm{Inv}^v(V) \twoheadrightarrow \mathrm{Cl}^v(V)$ corresponds to the α such that I_α is principal; however, it is clear that $I_\alpha = xV$ if and only if $\mathbf{v}(x) = \alpha$, and thus I_α is principal if and only if $\alpha \in G$. Therefore, $\mathrm{Cl}^v(V) \simeq \mathbb{R}/G$. □

Corollary 3.115. *Let R be a Bézout domain, and suppose that R is either:*

(a) semilocal; or
(b) finite-dimensional and of finite character.

For every $* \in \mathrm{Star}(R)$, $\mathrm{Cl}^*(R)$ is an injective group (equivalently, an injective $\mathbb{Z}$-module).

Proof. By Corollary 3.112 and Proposition 3.114, $\mathrm{Cl}^*(R) \simeq \bigoplus \mathbb{R}/G_\alpha$, for a family $\{G_\alpha : \alpha \in A\}$ of additive subgroups of $\mathbb{R}$. Each $\mathbb{R}/G_\alpha$ is a divisible group, and thus so is their direct sum; however, a divisible group is injective, and thus so is $\mathrm{Cl}^*(R)$. □

Example 3.116. We can use Theorem 3.110 and Proposition 3.114 to build Prüfer domains with v-class group isomorphic to $H := \bigoplus_{i=1}^n \mathbb{R}/H_i$, where each H_i is an additive subgroup of $\mathbb{R}$ not isomorphic to $\mathbb{Z}$. In fact, let $\mathcal{H}$ be the disjoint union of the H_i, and let K be a field. On the field $K(\mathcal{H}) = K(X_h : h \in \mathcal{H})$ define the valuations

$$v_i(X_h) := \begin{cases} h & \text{if } h \in H_i \\ 0 & \text{if } h \notin H_i \end{cases} \tag{3.7}$$

Let V_i be the valuation ring of v_i and $R := \bigcap_{i=1}^n V_i$. Then V_i has value group H_i and, by [50, Theorem 22.8], R is a Bézout domain with n maximal ideals, say $M_1, \ldots, M_n$, such that $R_{M_i} = V_i$, and no M_i is divisorial. By the results above, $\mathrm{Cl}^v(R) = H$.

We end this section with a result similar in spirit to those proved in Section 3.5.3.

Proposition 3.117. *Let R be a Prüfer domain and suppose that R is either:*

(a) semilocal; or
(b) finite-dimensional and of finite character.

Let $* \in \mathrm{Star}(R)$. If $I, J \in \mathrm{Inv}^*(R)$, then $I + J \in \mathrm{Inv}^*(R)$.

Proof. Suppose first that R is semilocal, and proceed by induction on $n := |\mathrm{Max}(R)|$. If $n = 1$, then R is a valuation domain and $I + J$ is equal either to I or to J, and the claim is proved.

Suppose the claim is true up to rings with $n-1$ maximal ideals, let $|\mathrm{Max}(R)| = n$ and consider the minimal Jaffard family Θ of R. By Proposition 3.74, $I + J \in \mathrm{Inv}^*(R)$ if and only if $(I + J)T \in \mathrm{Inv}^{*_T}(T)$ for every $T \in \Theta$; therefore, if Θ is not trivial, then we can use the inductive hypothesis. Suppose Θ is trivial: then $\mathrm{Jac}(R)$ contains nonzero prime ideals, and by Lemma 3.103 there is a nonzero prime ideal $Q \subseteq \mathrm{Jac}(R)$ such that $\mathrm{Jac}(R/Q)$ does not contain nonzero prime ideals. By Proposition 3.108, I/Q and J/Q are $\sharp$-invertible $\sharp$-ideals of R/Q (where $\sharp$ is the (semi)star operation induced by $*$), and in particular I/Q and J/Q are fractional ideals of R/Q.

By construction, R/Q admits a nontrivial Jaffard family Λ: for every $U \in \Lambda$, $(I/Q)U$ and $(J/Q)U$ are $\sharp_U$-invertible $\sharp_U$-ideals, and thus by inductive hypothesis so is $(I/Q)U + (J/Q)U = ((I + J)/Q)U$. Hence $(I + J)/Q$ is a $\sharp$-invertible $\sharp$-ideal, and so $I + J$ is a $*$-invertible $*$-ideal, i.e., $I + J \in \mathrm{Inv}^*(R)$.

If now R is locally finite and finite-dimensional, we see that if Θ is the minimal Jaffard family of R then every $T \in \Theta$ is semilocal. The ideal $I + J$ is $*$-invertible if and only if $(I + J)T$ is $*_T$-invertible for every $T \in \Theta$; however, since IT and JT are $*_T$-invertible $*_T$-ideals, the previous part of the proof shows that so is $IT + JT = (I + J)T$. Therefore, $I + J \in \mathrm{Inv}^*(R)$. $\square$

3.5.5. The poset of star operations of a semilocal Prüfer domain

Theorem 3.99 was proved by controlling semistar operations on a Prüfer domain R through the set of R-submodule of its quotient field, and linking the semistar operation on R with the star operations on its overrings. The aim of this section is to strengthen this connection by showing that the set of the star operations (and the set of semistar operations) of a Prüfer domain that is both semilocal and finite-dimensional is completely determined by its spectrum.

We start with a definition.

Definition 3.118. *A* fractional overring *of a domain R is an overring T of R which is also a fractional ideal of R. We denote the set of fractional overrings of R as* $\mathrm{FOver}(R)$.

Proposition 3.119. *Let R be a semilocal Prüfer domain of finite dimension. There is a unique set $\Lambda \subseteq \mathrm{Over}(R)$ such that $\mathrm{Over}(R) = \bigsqcup_{T\in\Lambda} \mathrm{FOver}(T)$. Moreover, every $I \in \mathbf{F}(R)$ is the fractional ideal of exactly one $T \in \Lambda$.*

Proof. We first show uniqueness. Suppose there are two sets $\Lambda, \Lambda' \subseteq \mathrm{Over}(R)$ with those properties. Without loss of generality, there is a $T \in \Lambda \setminus \Lambda'$, and thus there is a $T_1 \in \Lambda'$ such that $T \in \mathrm{FOver}(T_1)$. If $T_1 \in \Lambda$, we would have $\mathrm{FOver}(T) \cap \mathrm{FOver}(T_1) \neq \emptyset$; therefore $T_1 \notin \Lambda$, and thus $T_1 \in \mathrm{FOver}(T_2)$ for some $T_2 \in \Lambda$. If now $dT \subseteq T_1$ and $cT_1 \subseteq T_2$ (with $c, d \neq 0$) then $cdT \subseteq T_2$, and thus $T \in \mathrm{FOver}(T_2)$; this implies that $T = T_2$. However, $T_2 \subsetneq T_1 \subsetneq T$ implies that $T_2 \neq T$, a contradiction.

Let now $X^1(R)$ be the set of height-1 prime ideals of R and, for every $\mathcal{Y} \subseteq X^1(R)$, let $M(\mathcal{Y}) := \{M \in \mathrm{Max}(R) \mid M \supseteq P \text{ for some } P \in \mathcal{Y}\}$ and $R_{\mathcal{Y}} := \bigcap_{M\in M(\mathcal{Y})} R_M$. Let $\Lambda := \{R_{\mathcal{Y}} \mid \mathcal{Y} \subseteq X^1(R)\}$: we claim that Λ has the right properties.

Suppose $T \in \mathrm{Over}(R)$ and let $\mathcal{Z} := \{Q \cap R \mid Q \in X^1(T)\}$. For every maximal ideal $M \in M(\mathcal{Z})$, if $P \in X^1(R)$ is contained in M, there is a $d_P \in R$ such that $d_P R_P \subseteq R_M$. If $d_M := \prod_{M\in M(\mathcal{Z})} d_P$, then (since $T_Q = R_{Q\cap R}$)

$$dT \subseteq \bigcap_{Q\in X^1(T)} dT_Q = \bigcap_{P\in\mathcal{Z}} dR_P \subseteq \bigcap_{M\in M(\mathcal{Z})} R_M = R_{\mathcal{Z}}$$

and thus $T \in \mathrm{FOver}(R_{\mathcal{Z}})$. Hence, $\mathrm{Over}(R)$ is the union of $\mathrm{FOver}(S)$ as S ranges in Λ.

Suppose $T \in \mathrm{FOver}(R_{\mathcal{Y}}) \cap \mathrm{FOver}(R_{\mathcal{Z}})$ for some subsets $\mathcal{Y} \neq \mathcal{Z}$ of $X^1(R)$; by the previous paragraph, we can suppose that $T = \bigcap_{P\in\mathcal{Z}} R_P$. Then,

$$(R_{\mathcal{Y}} : T) = \left(\bigcap_{M\in M(\mathcal{Y})} R_M : T\right) = \bigcap_{M\in M(\mathcal{Y})} (R_M : T) = \bigcap_{M\in M(\mathcal{Y})} (R_M : TR_M)$$

Since $R_{\mathcal{Y}} \subseteq T$, every prime ideal which survives in T survives also in $R_{\mathcal{Y}}$; in particular, each $P \in \mathcal{Z}$ survive in $R_{\mathcal{Y}}$, and thus $\mathcal{Y} \supseteq \mathcal{Z}$.

By Theorem 2.76, we have

$$TR_M = \left(\bigcap_{P\in\mathcal{Z}} R_P\right) R_M = \bigcap_{P\in\mathcal{Z}} R_P R_M.$$

If there exist $Q \in \mathcal{Y} \setminus \mathcal{Z}$, then every maximal ideal N of R containing Q does not contain any $P \in \mathcal{Z}$; it follows that $R_P R_N = K$ for every such P, and thus $TR_N = K$. But this implies $(R_N : TR_N) = (0)$, and it follows that $(R_{\mathcal{Y}} : T) = (0)$. Therefore, $T \notin \mathrm{FOver}(R_{\mathcal{Y}})$, a contradiction. Hence, $\mathrm{FOver}(R_{\mathcal{Y}}) \cap \mathrm{FOver}(R_{\mathcal{Z}}) = \emptyset$ if $\mathcal{Y} \neq \mathcal{Z}$.

For the last statement, note that if I is a R-submodule of its quotient field K, then I is a fractional ideal of $(I : I)$ by Lemma 3.97; hence, if $(I : I) \in \mathrm{FOver}(R_{\mathcal{Y}})$, then I is a fractional ideal of $R_{\mathcal{Y}}$, and of no other $R_{\mathcal{Z}}$. $\square$

Definition 3.120. *If Λ is the (unique) set found in Proposition 3.119 , we call $\Lambda \setminus \{K\}$ the* pseudo-Jaffard family *of R.*

We start building our correspondence by the overrings. Note that, when $\mathrm{Spec}(R)$ and $\mathrm{Spec}(T)$ are finite, giving a homeomorphism between them is equivalent to giving an isomorphism of partially ordered sets.

Proposition 3.121. *Let R, T be semilocal and finite-dimensional Prüfer domains, and suppose there is an homeomorphism $\phi : \mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(T)$. Then, there is an homeomorphism $\overline{\phi} : \mathrm{Over}(R) \longrightarrow \mathrm{Over}(T)$ such that $\overline{\phi}(R_P) = T_{\phi(P)}$ and $\overline{\phi}\left(\bigcap_{P\in\Delta} R_P\right) = \bigcap_{P\in\Delta} T_{\phi(P)}$.*

Proof. By Propositions 2.169 and 2.130, and being $\mathrm{Spec}(D)$ finite (in particular, a Noetherian space) $\mathrm{Over}(D)$ is order-antiisomorphic to the set of subsets of $\mathrm{Spec}(R)$ that is closed by generization.

Since Δ is closed by generizations if and only if so is $\phi(\Delta)$, we can build a map $\overline{\phi} : \mathrm{Over}(R) \longrightarrow \mathrm{Over}(T)$ sending S to the ring defined by $\phi(\Delta_S)$, whose inverse is the map induced this way by $\phi^{-1} : \mathrm{Spec}(T) \longrightarrow \mathrm{Spec}(R)$. Being the composition of two antiisomorphism, $\overline{\phi}$ is an isomorphism of partially ordered sets, and thus (since $\mathrm{Over}(R)$ and $\mathrm{Over}(T)$ are finite) a homeomorphism when the spaces are endowed with their Zariski topologies. □

Proposition 3.122. *Let R and T be two semilocal Prüfer domains, and let $\Lambda(R)$ and $\Theta(R)$ be, respectively, the pseudo-Jaffard and the minimal Jaffard family of R (and define similarly $\Lambda(T)$ and $\Theta(T)$). Suppose that there is a homeomorphism $\phi : \mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(T)$, and let $\overline{\phi}$ be the map defined in Proposition 3.121. Then:*

(a) $\overline{\phi}(\Lambda(R)) = \Lambda(T)$;
(b) $\overline{\phi}(\Theta(R)) = \Theta(T)$;

Proof. The proof of Proposition 3.119 shows that the elements of $\Lambda(R)$ are determined by the order structure of $\mathrm{Spec}(R)$ and by intersection of localizations. By Proposition 3.121, $\overline{\phi}$ respects them. For the second part, it is enough to note that the elements of $\Theta(T)$ are exactly the maximal elements of $\Lambda(R) \setminus \{K\}$. □

We now introduce a different generalization of the concept of star operations.

Definition 3.123. *A* fractional star operation *on a domain R is a map $* : \mathcal{F}(R) \longrightarrow \mathcal{F}(R)$, $I \mapsto I^*$ such that, for all $I, J \in \mathcal{F}(R)$, $x \in K \setminus \{0\}$,*

(a) $I \subseteq I^*$;
(b) $I \subseteq J$ implies $I^* \subseteq J^*$;
(c) $(I^*)^* = I^*$;
(d) $x \cdot I^* = (x \cdot I)^*$.

We denote the set of fractional star operations on R by $\mathrm{FStar}(R)$.

Remark 3.124.

(1) Fractional star operations can be seen as a middle step between star and semistar operations, parallel to the other middle step constitued by (semi)star operations; indeed, a fractional star operation is the restriction to $\mathcal{F}(R)$ of a semistar operation such that the closure of any fractional ideal is still a fractional ideal.
(2) If $*$ is a fractional star operation on R, R^* is a fractional overring of R.
(3) As to semiprime and semistar operations, we can apply the methods of Sections 3.1 and 3.3 also to fractional star operations, without changing much the proofs. In particular, the analogue of Theorem 3.67 holds: if Θ is a Jaffard family on R, λ_Θ and ρ_Θ are homeomorphisms between $\mathrm{FStar}(R)$ and $\prod\{\mathrm{FStar}(T) \mid T \in \Theta\}$.

Proposition 3.125. *Let R_1 and R_2 be two semilocal Prüfer domains of finite dimension, and let $\Lambda(R_i)$ be the pseudo-Jaffard families of R_i. Suppose that there is a homeomorphism $\phi : \mathrm{Over}(R_1) \longrightarrow \mathrm{Over}(R_2)$, and that for every branch T of R_1 there is an order-preserving bijection $\psi_T : \mathrm{FStar}(T) \longrightarrow \mathrm{FStar}(\phi(T))$. Then, for every $U \in \Lambda(R_1)$, there is an order-preserving bijection $\psi_U : \mathrm{FStar}(U) \longrightarrow \mathrm{FStar}(\phi(U))$ such that, for every branch T of R_1,*

$$\lambda_{R_2,\phi(T)} \circ \psi_U = \psi_T \circ \lambda_{R_1,T},$$

where $\lambda_{A,B}$ is the localization map.

Proof. By Remark 3.124(3), there is a homeomorphism $\mathrm{FStar}(R_1) \simeq \prod\{\mathrm{FStar}(T) \mid T \in \Theta\}$. Now, if $U \in \Lambda(R_1)$, the branches of U are exactly the branches of R_1 that contain U; hence, if $\{T_1, \dots, T_k\}$ is the minimal Jaffard family of U, we have bijections

$$\mathrm{FStar}(U) \leftrightarrow \prod_{i=1}^{k} \mathrm{FStar}(T_i) \quad \text{and} \quad \mathrm{FStar}(\phi(U)) \leftrightarrow \prod_{i=1}^{k} \mathrm{FStar}(\phi(T_i))$$

since, by Proposition 3.122, the branches of U are sent into branches of $\phi(U)$ (and considering that ϕ restricts to a homeomorphism $\phi' : \mathrm{Over}(U) \longrightarrow \mathrm{Over}(\phi(U))$). Therefore, we can define ψ_U as the product of the bijections ψ_{T_i}. The last property follows directly from this definition. □

We would like to define a semistar operation by "gluing" a family of fractional star operations, each one defined over a member of the pseudo-Jaffard family of R; the natural candidate family associated to a semistar operation $*$ would be composed of the restriction of $*$ to $\mathcal{F}(U)$, as U ranges among $\Lambda(R)$. Unfortunately, $*|_{\mathcal{F}(U)}$ need not to be a fractional star operation; hence, we have to introduce the notation $\widehat{\mathrm{FStar}}(R)$ to denote the set $\mathrm{FStar}(R) \cup \{\infty_R\}$, where ∞_R is a placeholder. We can extend naturally the order of $\mathrm{FStar}(R)$ to $\widehat{\mathrm{FStar}}(R)$ defining $* \leq \infty_R$ for every $* \in \mathrm{FStar}(R)$.

Therefore, if R is an arbitrary domain, and $T \in \mathrm{Over}(R)$, we can define a map

$$\begin{aligned} \xi_{R,T} \colon \mathrm{SStar}(R) &\longrightarrow \widehat{\mathrm{FStar}}(T) \\ * &\longmapsto \begin{cases} *|_{\mathcal{F}(T)} & \text{if } T^* \in \mathcal{F}(T) \\ \infty_T & \text{otherwise.} \end{cases} \end{aligned}$$

Note that $\xi_{R,T}(*) = \infty_T$ if and only if there are no T-fractional ideals which are $*$-closed. In the same way, if $\Theta \subseteq \mathrm{Over}(R)$, then it is defined a map

$$\begin{aligned} \Xi_\Theta \colon \mathrm{SStar}(R) &\longrightarrow \prod_{T\in\Theta} \widehat{\mathrm{FStar}}(T) \\ * &\longmapsto (\xi_{R,T}(*))_{T\in\Theta}. \end{aligned} \tag{3.8}$$

If R is a semilocal Prüfer domain of finite dimension, and $\Theta = \Lambda(R)$ is the pseudo-Jaffard family of R, we will write in the following Ξ instead of Ξ_Θ.

Our first target is to determine the range of the map Ξ. To do so, we introduce two definitions.

Definition 3.126. *Let $\underline{*} = (*^{(T)})_{T\in\Lambda(R)} \in \prod_{T\in\Lambda(R)} \widehat{\mathrm{FStar}}(R)$. The set $\underline{*}^0$ is the set of overrings U of R such that $U^{*^{(T)}} = U$ for some $T \in \Lambda(R)$ such that $*^{(T)} \neq \infty_T$.*

Definition 3.127. *Let $T, U \in \Lambda(R)$ and let $*_1 \in \mathrm{FStar}(T)$, $*_2 \in \mathrm{FStar}(U)$. We say that $*_1 \preceq *_2$ if, for every branch $S \in \Lambda(R)$ such that $T, U \subseteq S$, we have $(*_1)_S \leq (*_2)_S$.*

The relation $\preceq$ is a preorder on the disjoint union $\bigsqcup\{\mathrm{FStar}(T) \mid T \in \Lambda(R)\}$. When $T = U$, $*_1 \preceq *_2$ if and only if $*_1 \leq *_2$, i.e., on each $\mathrm{FStar}(T)$ the relation $\preceq$ coincides with the usual order relation.

Proposition 3.128. *Let R be a semilocal finite-dimensional Prüfer domain and let Ξ be the map defined in* (3.8)*. Then,*

$$\begin{aligned} \Xi(\mathrm{SStar}(R)) \quad = \quad & \{\underline{*} = (*^{(T)})_{T\in\Lambda(R)} \in \textstyle\prod_{T\in\Lambda(R)} \widehat{\mathrm{FStar}}(T) \mid \underline{*}^0 \text{ is closed by intersections} \\ & \text{and if } U_1, U_2 \in \Lambda(R), *_{U_1} \neq \infty_{U_1}, *_2 \neq \infty_{U_2}, U_1 \subseteq U_2 \text{ then } *_{U_1} \preceq *_{U_2}\}. \end{aligned}$$

Proof. ($\subseteq$) Let $*$ be a semistar operation. We first claim that $\Xi(*)^0$ is exactly the set of $*$-closed overrings. Indeed, if $U^* = U$ then $U^{\xi_{R,T}(*)} = U$, where T is the element of $\Lambda(R)$ such that $U \in \Lambda(R)$. Conversely, if $U^{\xi_{R,T}(*)} = U$ for some T then $U = U^*$. In particular, $\Xi(*)^0$ is closed by intersections.

Suppose now that $U_1, U_2 \in \Lambda(R)$ are such that $*_{U_i} := \lambda_{R,U_i}(*) \neq \infty_{U_i}$ and $U_1 \subseteq U_2$. If $U_1 = U_2$ then $*_{U_1} = *_{U_2}$ and in particular $*_{U_1} \preceq *_{U_2}$. Suppose $U_1 \neq U_2$ and $*_{U_1} \npreceq *_{U_2}$: then, by definition, there is a branch T of R such that $\lambda_{R,T}(*_{U_1}) \nleq \lambda_{R,T}(*_{U_2})$, that is, there is a T-integral ideal I such that $I^{\lambda_{U_1,T}(*_{U_1})} \nsubseteq I^{\lambda_{U_2,T}(*_{U_2})}$. Let $I_i := I \cap U_i$; since $U_1 \subseteq U_2$, we have $I_1 \subseteq I_2$.

By the properties of Jaffard families, $I_iT = I$; therefore,

$$I^{\lambda_{U_i,T}(*_{U_i})} = (I_iT)^{\lambda_{U_i,T}(*_{U_i})} = I_i^{*_{U_i}}T = I_i^*T,$$

the last passage coming from the fact that $*_{U_i} \neq \infty_{U_i}$. Hence, $I_1 \subseteq I_2$ implies $I^{\lambda_{U_1,T}(*_{U_1})} \subseteq I^{\lambda_{U_2,T}(*_{U_2})}$, against our choice of I. Therefore, $*_{U_1} \preceq *_{U_2}$.

($\supseteq$) Let $\underline{*} := (*_T)_{T\in\Lambda(R)}$ be a sequence satisfying the hypotheses. For every $T \in \Lambda(R)$, let $\mathcal{F}_T^* := \{I \in \mathcal{F}(T) : I^{*_T} = I\}$, and let $\mathcal{F}^* := \bigcup_{T\in\Lambda(R)} \mathcal{F}_T$. To show that $\mathcal{F}$ is a $*$-family, i.e., the set of closed ideal of a semistar operation (and thus that $(*_T)$ is in the range of Ξ), it is enough to show that it is closed by intersections (since clearly $xI \in \mathcal{F}^*$ if $I \in \mathcal{F}^*$).

Let I be the intersection of a family of elements of $\mathcal{F}^*$. Since each $\mathcal{F}_T^*$ is closed by intersections, without loss of generality we can suppose $I := I_{T_1} \cap \cdots \cap I_{T_k}$, where $I_{T_i} \in \mathcal{F}_{T_i}^*$ for every i and $T_i \neq T_j$ if $i \neq j$. Then, I is a fractional ideal of $U := T_1 \cap \cdots \cap T_k$, which itself is in $\Lambda(R)$; moreover, $U' := T_1^{*T_1} \cap \cdots \cap T_k^{*T_k}$ is a U-fractional ideal, and since $\underline{*}^0$ is closed by intersections it contains U', that thus is $*_U$-closed. In particular, $*_U \neq \infty_U$.

The pseudo-Jaffard family of U is exactly $\Lambda(U) := \Lambda(R) \cap \mathrm{Over}(U)$, where $\Lambda(R)$ is the pseudo-Jaffard family of R; in particular, each T_i is in $\Lambda(U)$ and, with the notation of Proposition 3.119, $T_i = U_{\mathcal{Y}_i}$ for some $\mathcal{Y}_i \subseteq X^1(U)$. Taking $\mathcal{Z}_i := X^1(U) \setminus \mathcal{Y}_i$, it is not hard to see that $\{T_i, U_{\mathcal{Z}_i}\}$ is a Jaffard family for U; therefore,

$$I^{*U} \subseteq (IT_i)^{\lambda_{U,T_i}(*_U)} = I_{T_i}^{\lambda_{U,T_i}(*_U)} \subseteq I_{T_i}^{*T_i} = I_{T_i}$$

by the hypothesis of compatibility and remembering that $I_{T_i} \in \mathcal{F}_{T_i}^*$. Therefore,

$$I^{*U} \subseteq I_{T_1} \cap \cdots \cap I_{T_n} = I$$

and thus $I \in \mathcal{F}^*$, as requested. □

Corollary 3.129. *Let R be a semilocal principal ideal domain with quotient field K. Then,* $\mathrm{SStar}(R)$ *is in bijective correspondence with the subsets of* $\mathrm{Over}(R) \setminus \{K\}$ *that are closed under intersections.*

Proof. If T is a PID, then $\widehat{\mathrm{FStar}}(T) = \{d_T, \infty_T\}$; therefore, since every overring of R is a PID, the condition on U_1 and U_2 of Proposition 3.128 is empty. Thus, the only condition is for the set of closed overrings to be closed by intersection and to contain the quotient field K, and we have the claim. □

We are ready to prove the main result of this subsection.

Theorem 3.130. *Let R and T be semilocal Prüfer domains of finite dimension, and suppose there is a homeomorphism $\nu : \mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(T)$ such that, for every prime ideal P of R, PR_P is divisorial if and only if so is $\nu(P)R_{\nu(P)}$. Then, there is an order-preserving bijection between* $\mathrm{SStar}(R)$ *(respectively,* $\mathrm{FStar}(R)$*,* $\mathrm{(S)Star}(R)$*,* $\mathrm{Star}(R)$*) and* $\mathrm{SStar}(T)$ *(respectively,* $\mathrm{FStar}(T)$*,* $\mathrm{(S)Star}(T)$*,* $\mathrm{Star}(T)$*).*

Proof. By Proposition 3.121, we can associate to ν a homeomorphism $\phi : \mathrm{Over}(R) \longrightarrow \mathrm{Over}(T)$ that preserves localizations and their intersections.

For every $k \in \mathbb{N}$, $k > 0$, let

(SS_k) if R and T verify the hypotheses of the theorem and $\dim(R), \dim(T) \leq k$, then there is an order-preserving bijection $\Psi : \mathrm{SStar}(R) \longrightarrow \mathrm{SStar}(T)$ such that if $U^* = U$ for some $U \in \mathrm{Over}(R)$ then $\phi(U)^{\Psi(*)} = \phi(U)$;

(FS_k) if R and T verify the hypotheses of the theorem and $\dim(R), \dim(T) \leq k$, then then there is an order-preserving bijection $\Psi : \mathrm{FStar}(R) \longrightarrow \mathrm{FStar}(T)$ such that if $U^* = U$ for some $U \in \mathrm{FOver}(R)$ then $\phi(U)^{\Psi(*)} = \phi(U)$;

We will show that (FS_1) is true and that $(FS_n) \Longrightarrow (SS_n) \Longrightarrow (FS_{n+1})$; by induction, this will prove (FS_n) and (SS_n) for every $n \in \mathbb{N}$.

(FS_1). If $\dim(R) = 1$, then R is completely integrally closed; hence, the unique fractional overring of R is R itself (if $cT \subseteq R$ and $x \in T \setminus R$, then $cx^n \in R$, so x is almost integral over R and R would not be completely integrally closed) and thus $\mathrm{FStar}(R) = \mathrm{Star}(R)$; the same happens for T. By Theorem 3.67 (see also Proposition 3.100), $\mathrm{Star}(R) \simeq \prod\{\mathrm{Star}(R_M) \mid M \in \mathrm{Max}(R)\}$, and the same happens for T. Since R_M is a valuation domain, $\mathrm{Star}(R_M)$ is equal to $\{d\}$ or $\{d, v\}$ (the latter with $d \neq v$) according to whether MR_M is divisorial or not; but this condition is preserved by ν, by hypothesis, and thus we have a bijection between $\mathrm{Star}(R_M)$ and $\mathrm{Star}(T_{\nu(M)})$, which can be used to define an order-preserving bijection between $\mathrm{Star}(R)$ and $\mathrm{Star}(T)$. Moreover, since the only fractional overring of R is R, it is always closed; the same happens for T. Therefore, the last condition of (FS_1) holds.

$(FS_n) \Longrightarrow (SS_n)$. Suppose $\dim(R) = n$, let $\Theta(R)$ be the minimal Jaffard family of R and $\Lambda(R)$ be the pseudo-Jaffard family of R (and define accordingly $\Theta(T)$ and $\Lambda(T)$). The inductive hypothesis guarantees the existence of maps $\psi_U : \mathrm{FStar}(U) \longrightarrow \mathrm{FStar}(\phi(U))$ for every $U \in \Theta(R)$; use Proposition 3.125 to extend it to every $U \in \Lambda(R)$, and subsequently to a bijection

$$\pi : \prod_{U \in \Lambda(R)} \widehat{\mathrm{FStar}}(U) \longrightarrow \prod_{S \in \Lambda(T)} \widehat{\mathrm{FStar}}(S).$$

Let $\lambda_R := \lambda_{\mathrm{Max}(R)}$ and $\lambda_T := \lambda_{\mathrm{Max}(T)}$ be the localization maps; then, we have functions

$$\mathrm{SStar}(R) \xrightarrow{\Xi_R} \prod_{U \in \Lambda(R)} \widehat{\mathrm{FStar}}(U) \xrightarrow{\pi} \prod_{S \in \Lambda(T)} \widehat{\mathrm{FStar}}(S) \xleftarrow{\Xi_T} \mathrm{SStar}(T).$$

We want to show that $\pi(\Xi_R(\mathrm{SStar}(R))) = \Xi_T(\mathrm{SStar}(T))$, that is, if we show that π preserves the range of Ξ.

Let now $\underline{*} \in \prod_{U \in \Lambda(R)} \widehat{\mathrm{FStar}}(U)$. We first show that $\pi(\underline{*})^0 = \phi(\underline{*}^0)$. Indeed, let $A \in \mathrm{FOver}(B) \subseteq \mathrm{Over}(R)$. Then, $\phi(A)$ is $\pi(\underline{*})$-closed if and only if $\phi(A)$ is closed by $\psi_B(\sharp_B)$, where $\sharp_B$ is the component of $\underline{*}$ along B; however, by (FS_n), the bijection $\psi_B : \widehat{\mathrm{FStar}}(B) \longrightarrow \widehat{\mathrm{FStar}}(\phi(B))$ preserves the closed fractional overrings, and thus $\phi(A)$ is closed if and only if it is the image of a closed overring, that is, $\phi(A) \in \pi(\underline{*})^0$ if and only if $\phi(A) \in \phi(\underline{*}^0)$.

Let now $\underline{*} := (*_U)_{U \in \Lambda(R)}$, and define $\underline{\sharp} := \pi(\underline{*}) := (\sharp_{\phi(U)})_{U \in \Lambda(R)}$. Suppose $U_1, U_2 \in \Lambda(R)$ are overrings such that $U_1 \subseteq U_2$ and such that $*_{U_1} \neq \infty_{U_1}$, $*_{U_2} \neq \infty_{U_2}$; we want to show that $*_{U_1} \preceq *_{U_2}$ if and only if $\sharp_{\phi(U_1)} \preceq \sharp_{\phi(U_2)}$. By definition, $*_{U_1} \preceq *_{U_2}$ if and only if, for every $S \in \Lambda(R)$ such that $U_1, U_2 \subseteq \Lambda(R)$, we have $(*_{U_1})_S \leq (*_{U_2})_S$, that is, if and only if, for every such S,

$$\lambda_{U_1,S}(*_{U_1}) \leq \lambda_{U_2,S}(*_{U_2}).$$

Since ψ_S is an order isomorphism, we can apply it to both sides without affecting the equivalence; that is, $*_{U_1} \preceq *_{U_2}$ if and only if, for every appropriate S,

$$\psi_S \circ \lambda_{U_1,S}(*_{U_1}) \leq \psi_S \circ \lambda_{U_2,S}(*_{U_2}).$$

Applying Proposition 3.125, this is equivalent to

$$\lambda_{\phi(U_1),\phi(S)} \circ \psi_{U_1}(*_{U_1}) \leq \lambda_{\phi(U_2),\phi(S)} \circ \psi_{U_2}(*_{U_2})$$

for every appropriate S. However, if $S \in \Lambda(R)$, then $U_i \subseteq S$ if and only if $\phi(U_i) \subseteq \phi(S)$; thus, the last condition is equivalent to $\psi_{U_1}(*_{U_1}) \preceq \psi_{U_2}(*_{U_2})$, that is, $\sharp_{U_1} \preceq \sharp_{U_2}$. In particular, by the characterization in Proposition 3.128, we have that $\underline{*}$ is in the range of Ξ_R if and only if $\underline{\sharp}$ is in the range of Ξ_T, as requested.

Therefore, Ξ_T can be inverted on $\pi(\Xi_T(\mathrm{SStar}(T)))$, and we can define a map

$$\begin{aligned} \Psi\colon \mathrm{SStar}(R) &\longrightarrow \mathrm{SStar}(T) \\ * &\longmapsto \Xi_T^{-1} \circ \pi \circ \Xi_R(*) \end{aligned}$$

which is easily seen to be an order-preserving bijection. Hence, (SS_n) holds.

$(SS_n) \Longrightarrow (FS_{n+1})$. Suppose $\dim(R), \dim(T) \leq n+1$. By Remark 3.124(3), we have a bijection between $\mathrm{FStar}(R)$ and $\prod_{U\in\Theta(R)} \mathrm{FStar}(U)$. Moreover, for every overring U of R, $\mathrm{SStar}(U)$ can be identified with the subset of $\mathrm{SStar}(R)$ such that no overring T such that $R \subseteq T \subsetneq U$ is closed; by the hypothesis in (SS_n), and since ϕ sends the pseudo-Jaffard family of R into the pseudo-Jaffard family of T, it follows that it is enough to prove the theorem when the minimal Jaffard decomposition $\Theta(R)$ is the trivial one, i.e., it is $\{R\}$.

Suppose thus that R is trivially branched, and let P be its (necessarily unique) height-1 prime. By inductive hypothesis, there is a bijection between $\mathrm{SStar}(R/P)$ and $\mathrm{SStar}(T/\nu(P))$. For a prime ideal Q, let

$$A(Q) := \begin{cases} \emptyset & \text{if } QR_Q \text{ is divisorial} \\ \{\infty^-\} & \text{if } QR_Q \text{ is not divisorial} \end{cases}$$

Since PR_P is divisorial if and only if $\nu(P)R_{\nu(P)}$ is divisorial, $A(P) = A(\nu(P))$. Moreover, $A(P)$ is in bijective correspondence with $\mathrm{Star}(R_P)\setminus\{d^{(R_P)}\}$ and, by Corollary 3.96, there is a bijection between $\mathrm{SStar}(R)$ and $\mathrm{SStar}(R/P) \cup A(P)$. Finally, since P is the unique height-1 prime of R, R_P is a R-fractional ideal, and thus the unique semistar operation that is not a fractional star operation is the trivial extension $\wedge_K$. Hence, we have bijections

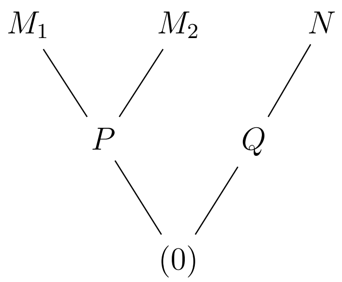
$$\mathrm{FStar}(R) \cup \{\infty\} \leftrightarrow \mathrm{SStar}(R) \leftrightarrow \mathrm{SStar}(R/P) \cup A(P)$$

and analogously $\mathrm{FStar}(T) \leftrightarrow \mathrm{SStar}(T/\nu(P)) \cup A(\nu(P))$. However, $\dim(R/P), \dim(T/\nu(P)) \leq n$, and thus by (SS_n) there is a bijection $\mathrm{SStar}(R/P) \leftrightarrow \mathrm{SStar}(T/\nu(P))$; composing everything, we get a bijection between $\mathrm{FStar}(R)\cup\{\infty\}$ and $\mathrm{FStar}(T)\cup\{\infty\}$, and thus a bijection between $\mathrm{FStar}(R)$ and $\mathrm{FStar}(T)$. Moreover, the quotient maps $\pi_U : U \longrightarrow U/P_U$

and $\pi' : \phi(U) \longrightarrow \phi(U)/Q_{\phi(U)}$ restrict to bijections between the respective set of overrings, so that the set of closed overrings is preserved along the bijection. Hence, (FS_{n+1}) holds.

By induction, (SS_n) and (FS_n) hold for every $n > 0$. Moreover, (S)Star$(R) = \{* \in \mathrm{SStar}(R) \mid R = R^*\}$, and since the set of closed overring is preserved (SS_n) implies the existence of a bijection between (S)Star(R) and (S)Star(T); analogously, Star$(R) = \{* \in \mathrm{FStar}(R) \mid R = R^*\}$, and (FS_n) implies the existence of a bijection between Star(R) and Star(T). $\square$

Example 3.131. Let R be a Prüfer domain whose spectrum is isomorphic to the following partially ordered set:



Suppose moreover that, for every $A \in \mathrm{Spec}(R)$, $A \neq N$, AR_A is principal, while NR_N is not. We will use the methods of Proposition 3.128 and Theorem 3.130 to calculate the cardinality of SStar(R). (Note that the hypothesis on the prime ideals can be changed without affecting the method much.)

Since $|X^1(R)| = 2$, Proposition 3.119 shows that $\Lambda(R) = \{R, R_N, R_{\{P\}}\}$, where $R_{\{P\}} := R_{M_1} \cap R_{M_2}$. We then determine $\widehat{\mathrm{FStar}}(A)$ for $A \in \Lambda(R)$; we will indicate the identity operation on B as d_B, while v_B indicate the v-operation on B.

- $\widehat{\mathrm{FStar}}(R_N) = \{d_{R_N} < v_{R_N} < d_{R_Q} < \infty_{R_N}\}$.
- By the proof of Theorem 3.130, $\widehat{\mathrm{FStar}}(R_{\{P\}}) \leftrightarrow \mathrm{SStar}(R_{\{P\}}/P)$; let $T := R_{\{P\}}/P$. By Corollary 3.129, SStar(T) corresponds to the subsets of $\mathrm{Over}(T) \setminus \{Q(T)\}$ that are closed by intersections; these are 5, namely $\{T\}$, $\{T, T_A, T_B\}$, $\{T_A\}$, $\{T_B\}$ and $\emptyset$ (where A and B be the maximal ideals of T). On R, these correspond to the fractional star operation d_{R_P}, $d_{R_{\{P\}}}$, $d_{R_{M_1}}$, $d_{R_{M_2}}$ and $\infty_{R_{\{P\}}}$, where the last one is counterimage of the trivial extension of the identity star operation on T. As a

partially ordered set, we can represent $\widehat{\mathrm{FStar}}(R_{\{P\}})$ as

$$\begin{tikzcd}
& \infty_{R_{\{P\}}} \arrow[d, dash] & \\
& d_{R_P} \arrow[dl, dash] \arrow[dr, dash] & \\
d_{R_{M_1}} \arrow[dr, dash] & & d_{R_{M_2}} \arrow[dl, dash] \\
& d_{R_{\{P\}}} &
\end{tikzcd}$$

- Since $\{R_{\{P\}}, R_N\}$ is a Jaffard family on R, we have

$$\widehat{\mathrm{FStar}}(R) \leftrightarrow \Big(\mathrm{FStar}(R_{\{P\}}) \times \mathrm{FStar}(T)\Big) \cup \{\infty_R\}.$$

In particular, $\mathrm{SStar}(R)$ is contained (isomorphically) in the product $\widehat{\mathrm{FStar}}(R) \times \widehat{\mathrm{FStar}}(R_{\{P\}}) \times \widehat{\mathrm{FStar}}(R_N)$, which has $21 \cdot 5 \cdot 4 = 420$ elements. Consider now an element $(*_0, *_P, *_N)$ of the product. We consider two cases.

- $*_0 \neq \infty_R$. Then, $*_0 = (\sharp_P, \sharp_N)$ for some $\sharp_P \in \mathrm{FStar}(R_{\{P\}})$, $\sharp_N \in \mathrm{FStar}(R_N)$; by Proposition 3.128, we have $\sharp_P \leq *_P$ and $\sharp_N \leq *_N$. In particular:
  - if $\sharp_P = d_{R_{\{P\}}}$, there are 5 choices for $*_P$;
  - if $\sharp_P \in \{d_{R_{M_1}}, d_{R_{M_2}}\}$ there are 3 choices for $*_P$;
  - if $\sharp_P = d_{R_P}$, there are 2 choices for $*_P$.

  On the other hand, there are 4 choices for $*_N$ if $\sharp_N = d_{R_N}$, 3 if $\sharp_N = v_{R_N}$ and 2 if $\sharp_N = d_{R_Q}$. The choices on P and N are independent; therefore, the total number of possibilities is

$$(5 + 2 \cdot 3 + 2) \cdot (4 + 3 + 2) = 13 \cdot 9 = 117.$$

- $*_0 = \infty_R$. In this case, at least one between $*_P$ and $*_N$ must be equal to ∞, since otherwise there would be a $A \in \mathrm{FOver}(R_{\{P\}})$ and a $B \in \mathrm{FOver}(R_N)$ closed and the intersection $A \cap B$ would be closed and in $\mathrm{FOver}(R)$, against $*_0 = \infty_R$. Thus, if $*_P = \infty_{R_{\{P\}}}$ we have 4 choices, if $*_N = \infty_{R_N}$ we have 5 choices and we must subtract the case $(\infty_{R_{\{P\}}}, \infty_{R_N})$ which is counted in both cases. Hence, we have 8 possibilities.

Adding, we see that $|\mathrm{Star}(R)| = 125$.

Example 3.132. Let R be a Prüfer domain of dimension 1 with three maximal ideals; suppose that MR_M is not principal for exactly one maximal ideal M of R. We calculate the cardinality of $\mathrm{SStar}(R)$.

To fix the notation, denote by V_1, V_2 and V_3 the three valuation overrings of R, and suppose that only V_1 is not discrete; define also $T_1 := V_2 \cap V_3$, $T_2 := V_1 \cap V_3$ and $T_3 := V_1 \cap V_2$ (so that $V_iT_i = K$ for $i = 1, 2, 3$). The partially ordered set $\mathrm{Over}(R)$ can thus be so pictured:

$$\begin{tikzcd}
 & K \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
V_1 \arrow[d, dash] \arrow[dr, dash] & V_2 \arrow[dl, dash] \arrow[dr, dash] & V_3 \arrow[dl, dash] \arrow[d, dash] \\
T_3 \arrow[dr, dash] & T_2 \arrow[d, dash] & T_1 \arrow[dl, dash] \\
 & R &
\end{tikzcd}$$

We have

$$|\mathrm{Star}(V_2)| = |\mathrm{Star}(V_3)| = |\mathrm{Star}(T_1)| = 1$$

while

$$|\mathrm{Star}(V_1)| = |\mathrm{Star}(T_2)| = |\mathrm{Star}(T_3)| = |\mathrm{Star}(R)| = 2;$$

for the latter ones, the star operations are exactly the d- and the v-operation, and the v-operation always extend to the v-operation. Let $\Lambda := \{T_1, V_2, V_3\}$. Note that the action of a semistar operation $*$ on $\mathcal{F}(T_1) \cup \mathcal{F}(V_2) \cup \mathcal{F}(V_3)$ depends only on what elements of Λ are $*$-closed (see Corollary 3.129).

We will divide the semistar operations according to which subset Δ of $\{R, T_1, T_3, V_1\} = \mathrm{Over}(R) \setminus (\Lambda \cup \{K\})$ they close; since $T_2 \cap T_3 = R$, if T_2 and T_3 are present in the subset so does R. Thus, we have the following 12 possibilities.

- $\Delta = \emptyset$. We need only to count the subsets of Λ that are closed by intersections, and there are exactly 6 of them: $\emptyset$, $\{T_1\}$, $\{V_2\}$, $\{V_3\}$, $\{T_1, V_2\}$, $\{T_1, V_3\}$ and $\{T_1, V_2, V_3\}$.
- $\Delta = \{R\}$. This case is analogous to the previous one, with the only difference that each subset correspond to two semistar operation (the one whose restriction to $\mathcal{F}(R)$ is d, and the one that restricts to v). Hence, we get 12 operations.
- $\Delta = \{T_2\}$. Since $R = T_1 \cap T_2 = V_2 \cap T_2$, and R is not closed, nor T_1 nor V_2 can be closed. Hence, the unique possible subsets of Λ are $\emptyset$ and $\{V_3\}$, and each corresponds to two semistar operations (corresponding to the two star operations on T_2; this follows from the fact that $|\mathrm{Star}(V_3)| = 1$), giving 4 semistar operations.
- $\Delta = \{T_3\}$. This case is symmetrical to the previous one, and we get other 4 semistar operations.
- $\Delta = \{V_1\}$. If any $A \in \Lambda$ is closed, then so would be $A \cap V_1$, which is different from V_1 and not in Λ; therefore, the only possible subset of Λ is $\emptyset$, and we get two semistar operations.
- $\Delta = \{R, V_1\}$. Nor V_2 nor V_3 can be closed (since $V_1 \cap V_2 = T_3$ and $V_1 \cap V_3 = T_2$, and neither is closed); therefore, the unique possible subsets of Λ are $\emptyset$ and $\{T_1\}$.

Let $*$ be a semistar operation that is comprised in this case and let $*_R$, $*_{V_1}$ be the restrictions to $\mathcal{F}(R)$ and $\mathcal{F}(V_1)$ respectively. By Proposition 3.128, we must have $\lambda_{R,V_1}(*_R) \leq *_{V_1}$; therefore, we have the three possibilities (d, d), (d, v) and (v, v) (where the first element indicate $*_R$ and the second $*_{V_1}$). Thus, this case generates 6 operations.

- $\Delta = \{T_2, V_1\}$. We have that V_2 cannot be closed (since $V_1 \cap V_2 = T_3$) and nor can T_1 (since $T_1 \cap T_2 = R$, which is not closed). Hence, the only possible subsets of Λ are $\emptyset$ and $\{V_3\}$. As in the previous case, we have three possibilities for $(*_{T_2}, *_{V_1})$, and again we have 6 semistar operations.
- $\Delta = \{T_3, V_1\}$. This case is the symmetrical of the previous one, and thus we get other 6 operations.
- $\Delta = \{R, T_2, T_3\}$. For the subsets of Λ we have the full array of possibilities, namely 6; for the subsets of $\mathrm{Star}(R)\times\mathrm{Star}(T_2)\times\mathrm{Star}(T_3)$, we have the possibilities (d, d, d), (d, d, v), (d, v, d), (d, v, v) and (v, v, v), that are 5. Hence this case generates 30 semistar operations.
- $\Delta = \{R, T_2, V_1\}$. The overring V_2 cannot be closed (otherwise $T_3 = V_2 \cap V_1$ would be closed), and so we get only 4 possibilities. On the other hand, we have to consider both the compatibility between $*_R$ and $*_{T_2}$ and the one between $*_{T_2}$ and $*_{V_1}$, so we get the four cases (d, d, d), (d, d, v), (d, v, v) and (v, v, v), for a total of 16 semistar operations.
- $\Delta = \{R, T_3, V_1\}$ is again symmetrical with the previous one, and we have 16 semistar operations.
- $\Delta = \{R, T_2, T_3, V_1\}$. All the 6 subsets of Λ are acceptable; on the other hand, the possibilities for $(*_R, *_{T_2}, *_{T_3}, *_{V_1})$ are (d, d, d, d), (d, d, d, v), (d, d, v, v), (d, v, d, v), (d, v, v, v) and (v, v, v, v), for a total of $6 \cdot 6 = 36$ semistar operations.

Adding everything, we have

$$|\mathrm{SStar}(R)| = 6 + 12 + 4 + 4 + 2 + 6 + 6 + 6 + 30 + 16 + 16 + 36 = 144.$$

A. The topology of spectral spaces

In this appendix, we collect several definitions and results about topological aspects of the theory of spectral spaces, in view of the results proved in the main body of the thesis. We will not give proofs; we defer the interested reader to [64], [35], [24, 0.9], [25, 1.8-1.9].

A.1. T_0 spaces

Let $(X, \mathcal{T})$ be a topological space. (In the following, we will often use the common abuse of notation of using X to denote the topological space $(X, \mathcal{T})$.) Given a subset $Y \subseteq X$, we denote by $\mathrm{Cl}(Y)$ the *closure* of Y, that is, the intersection of all closed set of X containing Y. If $Y = \{y\}$ is a singleton, we will shorten $\mathrm{Cl}(\{y\})$ as $\mathrm{Cl}(y)$, and talk of it as the closure of y (instead of the closure of $\{y\}$).

We can define on X a relation $\leq$ by saying that, given $x, y \in X$, $x \leq y$ if and only if $y \in \mathrm{Cl}(x)$. This relation is a preorder on X, since it is reflexive ($x \in \mathrm{Cl}(x)$) and transitive (if $x \in \mathrm{Cl}(y)$ and $y \in \mathrm{Cl}(z)$, then $x \in \mathrm{Cl}(y) \subseteq \mathrm{Cl}(\mathrm{Cl}(z)) = \mathrm{Cl}(z)$). If it is antisymmetric, that is, if $\leq$ is a partial order, then X is said to be a T_0 *topological space* (sometimes called a *Kolmogoroff space*; for shortness, we will say simply that X is T_0); in this case, we will refer to $\leq$ as the order *induced* by the topology. Equivalently, X is T_0 if and only if, given any two points, there is an open subset of X containg one of the two, but not the other; that is, X is T_0 if and only if the topology distingushes the points (see [55, d-1] or [114, Definition 13.1]).

When X is a T_1 space (that is, when all the points of X are closed), the relation $\leq$ is trivial; that is, if $x \leq y$ then $x = y$. In particular, this happens when X is a Hausdorff space.

The relation $\sim$ defined by $x \sim y$ if and only if $x \leq y$ and $y \leq x$ is an equivalence relation, which is nontrivial if and only if X is not T_0; the quotient space $X/\sim$ (or, more precisely, the canonical map $\pi : X \longrightarrow X/\sim$) is called the *canonical* T_0 *quotient* of X [114, Example 13.2(c) and Problem 13C]. The space $X/\sim$ is always a T_0 space, and π satisfies the following universal property: π is the only continuous function such that, for every continuous map $\phi : X \longrightarrow Y$, with Y a T_0 space, there is a unique continuous map $\phi' : X/\sim \longrightarrow Y$ such that $\phi = \phi' \circ \pi$ [55, d-1].

Let Y be a subset of X. The *generization* of Y is the set

$$Y^{\mathrm{gen}} := \{z \in X \mid z \leq y \text{ for some } y \in Y\};$$

if $Y = Y^{\mathrm{gen}}$, then Y is said to be *closed by generization*. Dually, the *specialization* of Y

is the set
$$Y^{\mathrm{sp}} := \{z \in X \mid y \leq z \text{ for some } y \in Y\},$$
and Y is said to be *closed by specialization* if $Y = Y^{\mathrm{sp}}$. If $Y = \{y\}$ is a singleton, then $\{y\}^{\mathrm{sp}}$ coincides with the closure of y; more generally, Y^{sp} is equal to the union of the sets $\mathrm{Cl}(y)$, as y ranges in Y.

A.2. Spectral spaces

Let R be a ring, and let $\mathrm{Spec}(R)$ be its spectrum, i.e., the set of its prime ideals. The *Zariski topology* on $\mathrm{Spec}(R)$ is the topology whose closed sets have the form

$$\mathcal{V}(I) := \{P \in \mathrm{Spec}(R) \mid I \subseteq P\},$$

as I ranges among the ideals of R (or, equivalently, among the radical ideals of R). We denote the open set $\mathrm{Spec}(R) \setminus \mathcal{V}(I)$ as $\mathcal{D}(I)$. Under this topology, $\mathrm{Spec}(R)$ is a T_0 space, and the order induced by the Zariski topology coincides with the order given by the set-theoretic containment; in particular, a prime ideal P is a closed point if and only if P is a maximal ideal. Therefore, $\mathrm{Spec}(R)$ is a T_1 space if and only if R has dimension 0. In this case, $\mathrm{Spec}(R)$ is also a Hausdorff space.

A topological space X is *irreducible* if it cannot be written as the union of two (not necessarily disjoint) proper closed subsets. A subset $Y \subseteq X$ is an *irreducible component* of X if it is irreducible (in the induced topology) and no subspace of X properly containing Y is irreducible. Every irreducible component of X is closed in X. In $\mathrm{Spec}(R)$, a subset is irreducible if and only if it is in the form $\mathcal{V}(P)$ for some prime ideal P, i.e., if and only if it is the closure of a point. More generally, if x is a point of X, then the closure $\mathrm{Cl}(x)$ of x is always irreducible.

A topological space X is a *spectral space* if there is a ring R such that X is homeomorphic to $\mathrm{Spec}(R)$, endowed with the Zariski topology. A *spectral map* is a map $\phi : X \longrightarrow Y$ between two spectral spaces such that $\phi^{-1}(\Omega)$ is open and compact for every open and compact subspace Ω of Y. Any spectral map is continuous; moreover, any map $f^a : \mathrm{Spec}(R_2) \longrightarrow \mathrm{Spec}(R_1)$ induced by a ring homomorphism $f : R_1 \longrightarrow R_2$ is a spectral map.

Spectral spaces can be characterized topologically.

Theorem A.1 [64, Proposition 4]. *Let X be a topological space. Then, X is a spectral space if and only if the following properties hold:*

- *X is T_0;*
- *X is compact;*
- *every irreducible closed subset has a generic point, i.e., it is the closure of a single point;*
- *there is a basis of open and compact subset of X that is closed by finite intersections.*

Note that that the last point can be substitued by the property that the family of open and compact subsets of X is a basis and it is closed by finite intersections.

A.3. The inverse topology

Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a family of closed sets for $(X, \mathcal{T})$. We say that $\mathcal{B}$ is a *basis of closed sets* if $\{X \setminus B \mid B \in \mathcal{B}\}$ is a basis of open sets for X, that is, if every closed set of $(X, \mathcal{T})$ is the intersection of a family of members of $\mathcal{B}$. In the same way, we say that $\mathcal{B}$ is a *subbasis of closed sets* if $\{X \setminus B \mid B \in \mathcal{B}\}$ is a subbasis of open sets, that is, if $(X, \mathcal{T})$ is the coarsest topology such that every member of $\mathcal{B}$ is a closed set.

Let $(X, \mathcal{T})$ be a spectral space. The family of open and compact subspaces of X is closed by finite unions, and thus it is a basis of closed sets for a topology, which we call the *inverse topology* on X. We will denote the set X endowed with the inverse topology as X^{inv}; in the following, when dealing with a spectral space X and its inverse topology X^{inv}, we will treat X as our "default" topology, adding the superscript "inv" when something is considered relative to the inverse topology.

The space X^{inv} is again a spectral space [64, Proposition 8].

The order $\leq_{\mathrm{inv}}$ induced by the inverse topology is the opposite of the order induced by the topology $\mathcal{T}$; that is, $x \in \mathrm{Cl}(y)$ if and only if $y \in \mathrm{Cl}^{\mathrm{inv}}(x)$. In particular, $\mathrm{Cl}^{\mathrm{inv}}(x) = \{x\}^{\mathrm{gen}}$. More generally, a subspace Y of X is closed in the inverse topology if and only if Y is compact and closed by generization in the original topology (this follows, for example, from [37, Remark 2.2 and Proposition 2.6]).

If $\phi : X \longrightarrow Y$ is a spectral map of spectral spaces, then the map

$$\begin{aligned} \phi^{\mathrm{inv}} \colon X^{\mathrm{inv}} &\longrightarrow Y^{\mathrm{inv}} \\ x &\longmapsto \phi(x) \end{aligned}$$

is again a spectral map.

Since X^{inv} is a spectral space, we can also construct the inverse topology on it; the topological space $(X^{\mathrm{inv}})^{\mathrm{inv}}$ coincides with the original space X (see [30, Proposition 3.1(c)] or [38, Corollary 4.8(4)]).

Note that the inverse topology could be, in principle, defined on every topological space X. However, outside the realm of spectral spaces, many of its properties does not hold: for example, if X is a compact connected Hausdorff space, then the unique open and compact subset of X are the empty set and X itself (since a compact subspace of a Hausdorff space is closed, and an open and closed subspace of a connected space must be either empty or the whole X); hence, the inverse topology on X is just the trivial topology.

A.4. The constructible topology

Let X be a topological space. The *constructible topology* on X (also called *patch topology*) is the coarsest topology such that the open and compact subsets of X are both open and closed; we denote by X^{cons} the set X, endowed with the constructible topology.

If X has a basis of open and compact subsets that is closed by finite intersections, then X is a spectral space if and only if X^{cons} is compact and Hausdorff [64, Corollary, p.54].

Let now and in the following be X a spectral space. Then, X^{cons} is Hausdorff and totally disconnected. Moreover, it is again a spectral space; indeed, if $X = \mathrm{Spec}(R)$, then X^{cons} can be realized as the spectrum of the ring

$$T(R) := \frac{R[Z_r \mid r \in R]}{(r^2 Z_r - r, r Z_r^2 - Z_r \mid r \in R)}$$

where Z_r are independent indeterminates (see [100, Propositions 3 and 5], [99] or [46, Proposition 5]).

The constructible topology is finer than both the given (spectral) topology of X and the inverse topology X^{inv}; moreover, the topological space $(X^{\mathrm{inv}})^{\mathrm{cons}}$ (i.e., the constructible topology of the inverse topology) coincides with X^{cons} [38, Corollary 4.8(1)].

As with the inverse topology, if $\phi : X \longrightarrow Y$ is a spectral map of spectral spaces, then the map

$$\begin{aligned}\phi^{\mathrm{cons}} \colon X^{\mathrm{cons}} &\longrightarrow Y^{\mathrm{cons}}\\ x &\longmapsto \phi(x)\end{aligned}$$

is again a spectral map [12, Chapter 3, Exercise 29]. In particular, since X is Hausdorff and Y is compact, ϕ^{cons} is also a closed map.

If $Y \subseteq X$, we say that Y is *proconstructible* in X if Y is a closed set of X^{cons}. A subspace Y is proconstructible in $\mathrm{Spec}(R)$ if and only if there is a ring homomorphism $f : R \longrightarrow T$ such that $Y = f^a(\mathrm{Spec}(T))$, where $f^a : \mathrm{Spec}(T) \longrightarrow \mathrm{Spec}(R)$ is the spectral map canonically associated to f (see [12, Chapter 3, Exercises 27 and 28] or [25, 1.9.5(ix)]). Note that, by definition, any intersection of proconstructible subsets of X is again proconstructible.

If Y is proconstructible, then, when endowed with the topology induced by X, it is a spectral space, and the constructible topology on Y is exactly the topology induced by the constructible topology on X (this follows from [25, 1.9.5(vi-vii)]). In particular, any open and compact subset of X is a spectral space.

Note that a subset $Y \subseteq X$ may be spectral without being proconstructible; see Example 2.140.

A.5. Ultrafilters

Let X be a set, and denote by $\mathscr{P}(X)$ its power set. A *filter* on X is a nonempty family $\mathscr{F} \subseteq \mathscr{P}(X)$ such that:

- $\emptyset \notin \mathscr{F}$;
- if $Y, Z \in \mathscr{F}$, then $Y \cap Z \in \mathscr{F}$;
- if $Z \in \mathscr{F}$ and $Z \subseteq Y \subseteq X$, then also $Y \in \mathscr{F}$.

In particular, a filter cannot contain two disjoint subsets of X, and thus it cannot contain both Y and $X \setminus Y$. If, for any $Y \subseteq X$, the filter $\mathscr{F}$ contains either Y or $X \setminus Y$, then $\mathscr{F}$ is called an *ultrafilter*; equivalently, an ultrafilter is a filter that is maximal under inclusion, i.e., a filter $\mathscr{F}$ such that, if $\mathscr{G}$ satisfies $\mathscr{F} \subsetneq \mathscr{G} \subseteq \mathscr{P}(X)$, then $\mathscr{G}$ is not a filter. Assuming the Axiom of Choice, every filter is contained in an ultrafilter; the latter property is, however, strictly weaker than the Axiom of Choice [75, Theorems 2.2 and 7.1].

Let now X be a set, $Y \subseteq X$ and let $\mathcal{F}$ be a nonempty collection of subsets of X. For every ultrafilter $\mathscr{U}$ on Y, the set of *ultrafilter limit points* of Y with respect to $\mathscr{U}$ and $\mathcal{F}$ is

$$Y_{\mathcal{F}}(\mathscr{U}) := \{x \in X \mid [\forall F \in \mathcal{F}, x \in F \iff F \cap Y \in \mathscr{U}]\}.$$

The subset Y is said to be *$\mathcal{F}$-stable under ultrafilters* if $Y_{\mathcal{F}}(\mathscr{U}) \subseteq Y$ for every ultrafilter $\mathscr{U}$ on Y.

For any nonempty $\mathcal{F}$, the family of subsets that are $\mathcal{F}$-stable under ultrafilters is the collection of closed sets of a topology, called the *$\mathcal{F}$-ultrafilter topology* and denoted by $X^{\mathcal{F}-\mathrm{ultra}}$ [36, Proposition 2.6]. In general, $X^{\mathcal{F}-\mathrm{ultra}}$ is compact if and only if $X_{\mathcal{F}}(\mathscr{U}) \neq \emptyset$ for every ultrafilter $\mathscr{U}$ on X.

When X is a spectral space and $\mathcal{F} = \mathscr{B}$ is the family of open and compact subspaces of X, then the $\mathscr{B}$-ultrafilter topology (called also the *ultrafilter topology*, without specifications) coincides with the constructible topology [46, Theorem 8]. More generally, the ultrafilter topology gives a criterion for a topological space to be spectral.

Theorem A.2 [36, Corollary 3.3]. *Let X be a topological space. The following are equivalent:*

(i) *X is a spectral space;*
(ii) *$X^{\mathscr{B}-\mathrm{ultra}}$ is compact and Hausdorff for some basis $\mathscr{B}$ of X;*
(iii) *X is T_0 and there is a basis $\mathscr{B}$ such that $X_{\mathscr{B}}(\mathscr{U}) \neq \emptyset$ for every ultrafilter $\mathscr{U}$ on X;*
(iv) *X is T_0 and there is a subbasis $\mathscr{S}$ such that $X_{\mathscr{S}}(\mathscr{U}) \neq \emptyset$ for every ultrafilter $\mathscr{U}$ on X.*

Suppose now $X = \mathrm{Spec}(R)$ is realized as the spectrum of a ring; let $\mathscr{S}$ be a subbasis of open and compact subsets, and let $Y \subseteq X$. For a given ultrafilter $\mathscr{U}$ on Y, the set $X_{\mathscr{S}}(\mathscr{U})$ is always composed by a single element [36, Corollary 2.11(2)], which can be written as [46, p.2918-2919]

$$P_{\mathscr{U}} := \{x \in R \mid \mathcal{V}(x) \cap Y \in \mathscr{U}\} = \{x \in R \mid \mathcal{D}(x) \cap Y \notin \mathscr{U}\}.$$

If X is a spectral space, then Y is proconstructible if and only if it is ultrafilter closed; it follows that Y is proconstructible in X if and only if $y_{\mathscr{S}}(\mathscr{U}) \in Y$ for every ultrafilter $\mathscr{U}$ on Y, where $\mathscr{S}$ is a subbasis of open and compact subset of X and $Y_{\mathscr{S}}(\mathscr{U}) = \{y_{\mathscr{S}}(\mathscr{U})\}$. In particular, a subset Y of $X = \mathrm{Spec}(R)$ is proconstructible if and only if $P_{\mathscr{U}} \in Y$ for every ultrafilter $\mathscr{U}$ on Y.

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