LATTICES IN NUMBER THEORY

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ICTP–CIMPA summer school 2016
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Lattices in Number Theory

We will study ideal lattices.

- They are lattices with structure of ideals.
- They are used in coding theory and cryptography.
- They have nice algebraic structure (so have nice and cheap representation, allows fast arithmetic,....)

Lattices in Number Theory

We will

1. Review about number fields, ideal lattices and ...
   Study Arakelov divisors.
2. Study the Arakelov class group.
3. Discuss about reduced Arakelov divisors and their properties.
   Discuss the reduction algorithm.

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PS: Questions are very welcome :)

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Lecture 1.
NUMBER FIELDS AND ...

HCM University of Science– Saigon University

Content

1. Review
   - Number fields and the ring of integers
   - Fractional ideals
   - The class group
   - $F_{\mathbb{R}}$
   - The Φ map
   - The L map
2. Ideal lattices

Review

Let F be a number field of degree n.

1. $F = \mathbb{Q}$,
2. $F = \mathbb{Q}(\sqrt{2})$,
3. $F = \mathbb{Q}(i)$ (the Gaussian field),
4. $F = \mathbb{Q}(\sqrt{5})$,
5. F is the splitting field of $x^3 + mx^2 - (m+3)x + 1$ ($m \geq -1, m \not\equiv 3 \mod 9$) (simplest cubic field),
6. $F = \mathbb{Q}(\sqrt[4]{2})$?
7. $F = \mathbb{Q}(\zeta_m)$ (cyclotomic field)?

Review

Let F be a number field of degree n.

$\sigma_1, \ldots, \sigma_{r_1}$ are r_1 real infinite primes

$\sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}$ are r_2 complex infinite primes.

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O_F: The ring of integers of F $(n = r_1 + 2r_2)$.

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Fractional ideals

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1. $I_1 = \{m_1 + \sqrt{2}m_2 : m_1, m_2 \in \mathbb{Z}\}$?

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4. $I_4 = \{2m_1 + (1-\sqrt{5})m_2 : m_1, m_2 \in \mathbb{Z}\}$?

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6. $I_6 = \{\frac{\sqrt{5}}{5}m_1 + \frac{5-\sqrt{5}}{10}m_2 : m_1, m_2 \in \mathbb{Z}\}$? $= \frac{1}{2\sqrt{5}}I_4$.

The class group

Id_F: The group of all fractional ideals of F with multiplication.
Let $Princ_F :=$ the subgroup of principal ideals of F.
The class group of F is

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Ex: $Cl_F = ?, h_F = ?$ if

1. $F = \mathbb{Q}$.
2. $F = \mathbb{Q}(\sqrt{2})$.
3. $F = \mathbb{Q}(i)$.
4. $F = \mathbb{Q}(\sqrt{10})$.

$F_{\mathbb{R}}$

- Let $F_{\mathbb{R}} := F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{\sigma \text{ real}} \mathbb{R} \times \prod_{\sigma \text{ complex}} \mathbb{C} \simeq \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ an $\mathbb{R}$-algebra.

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- A scalar product on $F_{\mathbb{R}}$, for any $u = (u_\sigma), v = (v_\sigma) \in F_{\mathbb{R}}$,
  $\langle u, v \rangle := Tr(u\overline{v})$

$$= \sum_{\sigma \text{ real}} Re(u_\sigma \overline{v_\sigma}) + 2 \sum_{\sigma \text{ complex}} Re(u_\sigma \overline{v_\sigma}).$$

$\Rightarrow \|u\|^2 = \sum_{\sigma \text{ real}} u_\sigma^2 + 2 \sum_{\sigma \text{ complex}} |u_\sigma|^2.$

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The Φ map

Let $\Phi : F \longrightarrow F_{\mathbb{R}}$ with $\Phi(f) = (\sigma_1(f), \ldots, \sigma_{r_1+r_2}(f))$ for $f \in F$.

Ex: $\Phi =?$ if

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Lattices

Definition (Lattices)

A lattice is a pair (L, q) where

- L is a free $\mathbb{Z}$-module of finite rank, and
- $q : L \times L \longrightarrow \mathbb{R}$ is a non-degenerate symmetric bilinear form.

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Ex: $L = \mathbb{Z}^n$ with the standard metric inherited from $\mathbb{R}^n$ is a lattice.
Ex: Let $F = \mathbb{Q}(\sqrt{5})$. Then $\Phi(O_F)$ is a lattice in $F_{\mathbb{R}} \simeq \mathbb{R}^2$.

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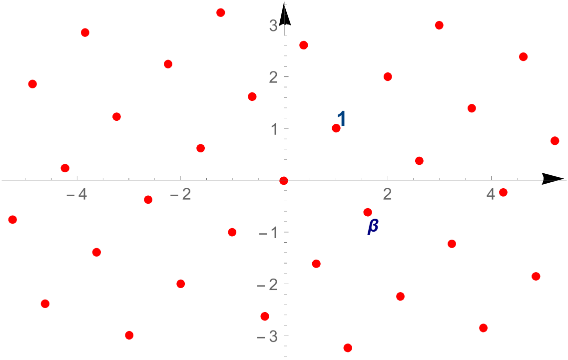

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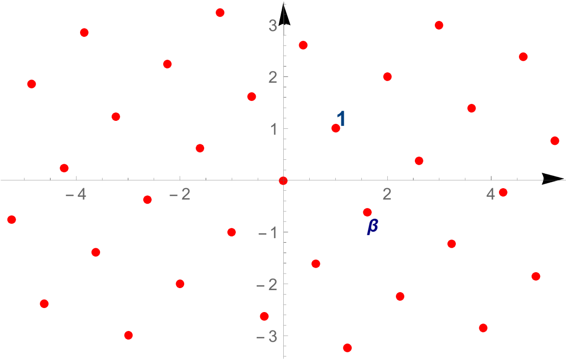

$\Phi(O_F)$ is a lattice in $F_{\mathbb{R}}$.

The Φ map and the images of fractional ideals

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Ex: Let $F = \mathbb{Q}(\sqrt{5})$ and $I_5 = \{\frac{1}{2}m_1 + \frac{1-\sqrt{5}}{4}m_2 : m_1, m_2 \in \mathbb{Z}\}$: a fractional ideal of F.

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Ex: Draw the lattice $\Phi(I_5)$.

The L map

Consider to the map

$$L : F^\times \longrightarrow F_{\mathbb{R}} \text{ where } L(f) = (\log(|\sigma(f)|))_\sigma.$$

Ex: F is the splitting field of $x^3 - x^2 - 2x + 1$ (simplest cubic field). The roots: $\alpha_1 = \alpha, \alpha_2 = \frac{1}{1-\alpha}, \alpha_3 = 1 - \frac{1}{\alpha}$. Then

$$O_F^\times =< \pm 1, \alpha_1, \alpha_2 > .$$

What $L(O_F^\times)$ looks like?

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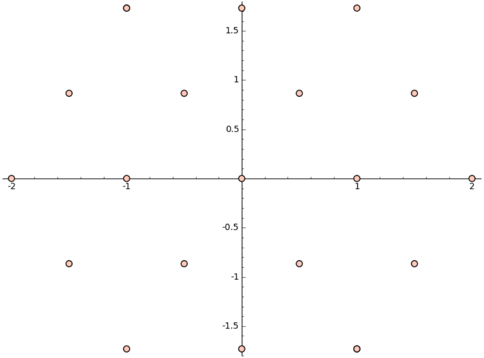

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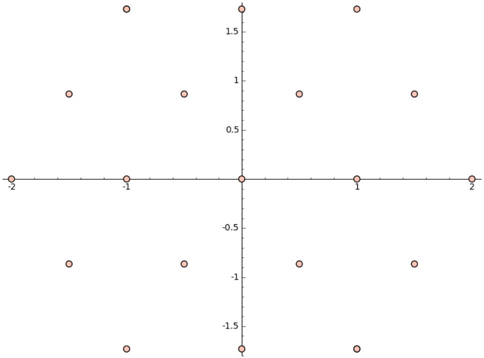

$L(O_F^\times)$ is the hexagonal lattice.

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$$\Lambda := L(O_F^\times) = \{L(x) : x \in O_F^\times\} \subset H.$$

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Λ is a lattice contained in the vector space H.

Let $T^0 = H/\Lambda$. Then T^0 is a compact real torus of dimension $r_1 + r_2 - 1$ (Dirichlet).

T^0 and $\Lambda = L(O_F^\times)$

Ex: Let $F = \mathbb{Q}(\sqrt{-2})$. Then $r_1 = 0, r_2 = 1$ and
$T^0 =?$ $\quad dim(T^0) =?$

T^0 and $\Lambda = L(O_F^\times)$

Ex: Let $F = \mathbb{Q}(\sqrt{2})$. Then $r_1 = 2, r_2 = 0$ and

$$H = \{(x, y) \in \mathbb{R}^2 : x + y = 0\} \simeq \mathbb{R}.$$

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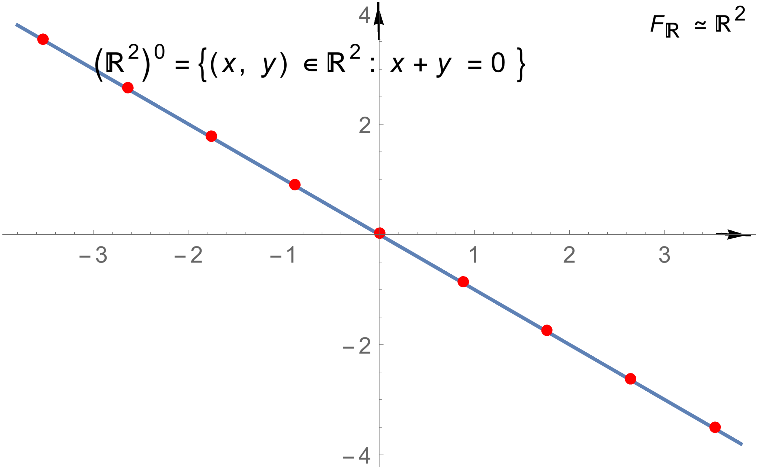

T^0 and $\Lambda = L(O_F^\times)$

Ex: Let F be a totally real cubic field. Then $r_1 = 3, r_2 = 0$ and

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Ex: $F = \mathbb{Q}(\sqrt[4]{2})$?

Ideal lattices

Definition (Ideal lattices)

An ideal lattice is a lattice (I, q), where

- I is a (fractional) O_F-ideal and
- $q : I \times I \longrightarrow \mathbb{R}$ is st
  $q(\lambda x, y) = q(x, \bar{\lambda} y)$ (Hermitian property)
  for all $x, y \in I$ and for all $\lambda \in O_F$.

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Ex 1: Let I be a factional ideal of F.

$$q(x, y) := \langle x, y \rangle \text{ for any } x, y \in I.$$

(the scalar product defined on $F_{\mathbb{R}}$)

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(the scalar product defined on $F_{\mathbb{R}}$)
Then (I, q) is an ideal lattice.

Ideal lattices

Ex 2: Let I be a factional ideal of F and
$u = (u_\sigma) \in (\mathbb{R}_{>0})^{r_1+r_2}$.
$z \in I$, $uz := (u_\sigma \cdot \sigma(z))_\sigma \in F_\mathbb{R}$.
We define

$$q_u(x, y) := \langle ux, uy \rangle \text{ for any } x, y \in I.$$

(the scalar product defined on $F_\mathbb{R}$)

Ideal lattices

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Proposition

(I, q_u) is an ideal lattice.

Isometric ideal lattices

Definition

Two ideal lattices (I, q) and (I', q') are called **isometric** if

- there exists $f \in F^\times$ such that $I' = fI$ and
- $q'(fx, fy) = q(x, y)$ for all $x, y \in I$.

Lattices from fractional ideals

A full rank lattice in $F_{\mathbb{R}}$ is also a full rank lattice in $\mathbb{R}^n$ via

$$F_{\mathbb{R}} \simeq \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \longrightarrow \mathbb{R}^n$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_{r_1} \\ x_{r_1+1} \\ \cdots \\ x_{r_1+r_2} \end{bmatrix} \longmapsto \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_{r_1} \\ Re(x_{r_1+1}) \\ Im(x_{r_1+1}) \\ \cdots \\ Re(x_{r_1+r_2}) \\ Im(x_{r_1+r_2}) \end{bmatrix}.$$

EX: $F = \mathbb{Q}(\sqrt[4]{2})$?

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- Let $I = \frac{1}{(1-\zeta_9)^4} O_F$: principal ideal of $F = \mathbb{Q}(\zeta_9)$. Then (I, q_1) is an ideal lattice $\simeq$?

Lattices from fractional ideals

- Let $I = \frac{1}{(1-\zeta_9)^4} O_F$: principal ideal of $F = \mathbb{Q}(\zeta_9)$. Then (I, q_1) is an ideal lattice $\simeq$?

```
[ 2.00 -1.00        1.00       -1.00       -1.00       -1.00]
[-1.00  2.00       -1.00        1.00        1.00        1.00]
[ 1.00 -1.00        2.00 -4.06 E-38       -1.00 1.47 E-39]
[-1.00  1.00 -4.06 E-38        2.00 4.35 E-38        1.00]
[-1.00  1.00       -1.00  4.35 E-38        2.00 7.35 E-39]
[-1.00  1.00  1.47 E-39        1.00 7.35 E-39        2.00]
```

Figure: The Gram matrix of (I, q_1).

Lattices from fractional ideals

- Let $F = \mathbb{Q}(\zeta_{15})$ and $I = \beta O_F$ for some $\beta \in F$. Then (O_F, q_β) is an ideal lattice $\simeq$?

Lattices from fractional ideals

- Let $F = \mathbb{Q}(\zeta_{15})$ and $I = \beta O_F$ for some $\beta \in F$.
  Then (O_F, q_β) is an ideal lattice $\simeq$?
- Let $F = \mathbb{Q}(\zeta_{35})$ and
  $\alpha = (\zeta_{35}^{-3} + \zeta_{35}^{3})(\zeta_{35}^{-6} + \zeta_{35}^{6})(\zeta_{35}^{-9} + \zeta_{35}^{9})(\zeta_{35}^{-12} + \zeta_{35}^{12})/\psi'(\zeta_{35} + \zeta_{35}^{-1})$.
  Then (O_F, q_α) is an ideal lattice $\simeq$?

Recap

- Number field, the ring of integers.
- Fractional ideals: $1/\alpha J$ for some $\alpha \in O_F$ and $J \subset O_F$ is an ideal.
- The class group $Cl_F = Id_F/Princ_F$ and class number $h_F = \#Cl_F$.
- The Φ map:
  $\Phi = (\sigma_1, \cdots, \sigma_{r_1}, \sigma_{r_1+1}, \cdots, \sigma_{r_1+r_2})$.
- The L map: $L(x) = (\log|\sigma(f)|)_\sigma$, $\forall x \in F^\times$.
- Ideal lattices: (I, q), where
  - I is a (fractional) O_F-ideal and
  - $q : L \times L \longrightarrow \mathbb{R}$ is a non-degenerate symmetric bilinear form with Hermitian property.

  I: factional ideal; $u = \in (\mathbb{R}_{>0})^{r_1+r_2}$.
  Then (I, q_u) is an ideal lattice.

Oh, no : (: (: (

Exercise: Let $F = \mathbb{Q}(\sqrt[4]{2})$. Prove that $O_F = \mathbb{Z}[\sqrt[4]{2}]$.

Oh, no : (: (: (

Exercise: Let $F = \mathbb{Q}(\sqrt[4]{2})$. Prove that $O_F = \mathbb{Z}[\sqrt[4]{2}]$.

There will be a gift for this :).

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