Lecture 3.
REDUCED ARAKELOV DIVISORS

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ICTP–CIMPA summer school 2016
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Review

We have studied:

- Arakelov divisors (I, u).
- The degree.
- Ideal lattices: $(I, u) \leftrightarrow (I, q_D)$.
- The Arakelov class group $Pic_F^0 = Div_F^0/Princ_F$.
- $\Lambda = L(O_F^\times)$ and $T^0 = H/\Lambda$ ...
- $0 \longrightarrow T^0 \xrightarrow{\phi_1} Pic_F^0 \xrightarrow{\phi_2} Cl_F \longrightarrow 0$ is exact.
- $Pic_F^0 \xrightarrow{1:1}$ $\{\text{Isometry classes of ideal lattices of covol.} \sqrt{|\Delta_F|}\}$.

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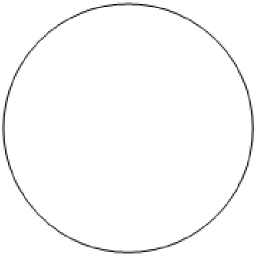

Figure: Pic_F^0 of a real quadratic field, $h_F =?$

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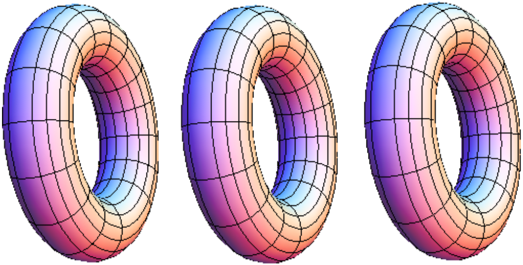

Figure: Pic_F^0 of a totally real cubic field, $h_F = ?$

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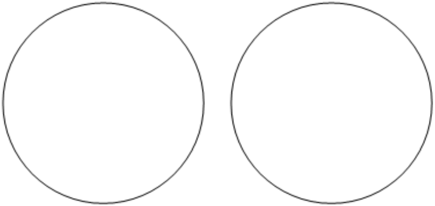

Figure: Pic_F^0 of a real quadratic field $F = \mathbb{Q}(\sqrt{10})$

$$vol(Pic_F^0) = 2\sqrt{2}\log(3+\sqrt{10}), \qquad R_F = ?$$

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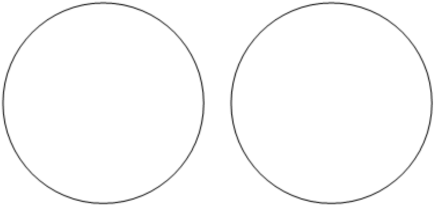

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$vol(T^0) = \sqrt{n}2^{-r_2/2}R_F$ and $vol(Pic_F^0) = h_F vol(T^0)$.

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Today: Metric on Pic_F^0 and Reduced Arakelov divisors.

Content

1. Metric on the Arakelov class group Pic_F^0
2. Reduced Arakelov divisors
3. Properties of reduced Arakelov divisors

Metric on Pic_F^0

Let F be a number field of degree n and $\Lambda = L(O_F^\times)$. Let $u = (u_\sigma) \in \prod_\sigma \mathbb{R}_{>0}$. Denote by

$$L(u) := (\log(u_\sigma))_\sigma \in \prod_\sigma \mathbb{R} \subset F_\mathbb{R}.$$

$$\|u\|_{Pic} = \min_\Lambda \|L(u)\|.$$

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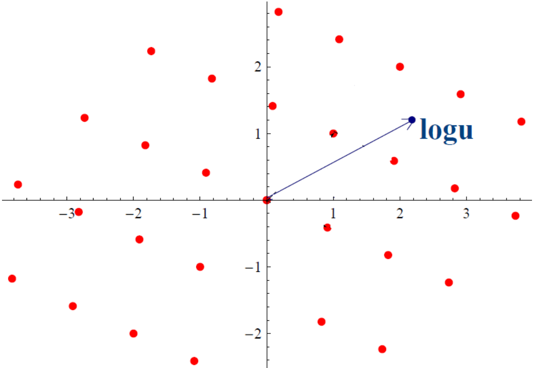

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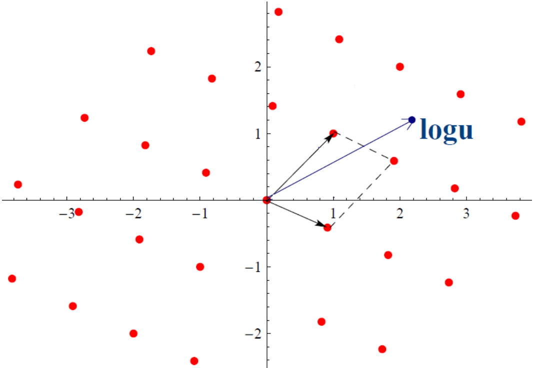

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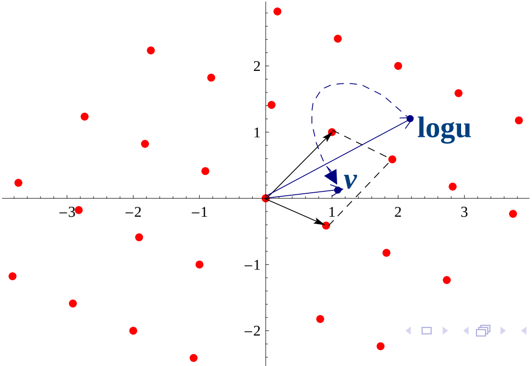

Metric on Pic_F^0

Let $[D]$ and $[D']$ be 2 divisor classes on the same connected component of Pic_F^0. Then there exists unique $u = (u_\sigma) \in \prod_\sigma \mathbb{R}_{>0}$ (up to multiplication by units) such that $D - D' = (O_F, u)$.

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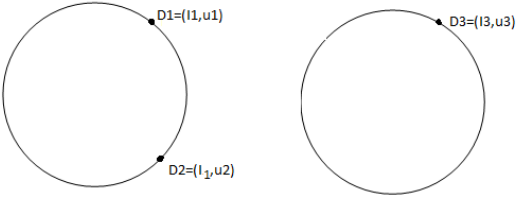

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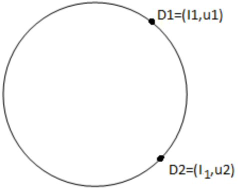

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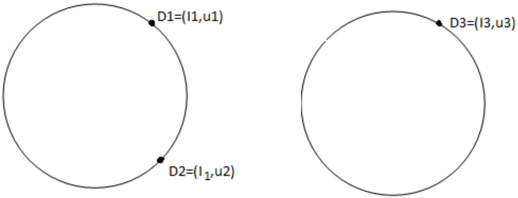

We define

$$\|D - D'\|_{Pic} := \|u\|_{Pic}.$$

The function $\| \quad \|_{Pic}$ induces the natural topology of Pic_F^0.

Metric on Pic_F^0

Ex: $F = \mathbb{Q}(\sqrt{15})$ and $f = \frac{7+\sqrt{\Delta}}{2} \in F^*$
and $I = 1/4(6, \sqrt{15})$ a fractional ideal of F, $u = (10, 1/10)$.
Let $D_1 = (O_F, 1)$, $D_2 = (f)$, $D_3 = (O_F, u)$ and $D_4 = d(I)$.

- $\|D_2 - D_1\|_{Pic} =?$
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- R. Schoof [2008]: The first description of infrastructure in terms of reduced Arakelov divisors and Arakelov class groups.

Reduced Arakelov divisors of real quadratic fields

- Let a real quadratic form $f(X, Y) = aX^2 + bXY + cY^2$ where
  $a, b, c \in \mathbb{Z}$ and $\gcd(a, b, c) = 1$.
  The discriminant of f is $\Delta = b^2 - 4ac > 0$.

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Ex: $f(X, Y) = X^2 + 7XY - 6Y^2$ is reduced ($\Delta = 73$).

Reduced quadratic forms

$f(X, Y) = X^2 + 7XY - 6Y^2$ is reduced where
$a = 1, b = 7$ and $c = -6$, $\Delta = 73$ st:

$(\star)$ $\boxed{\Delta = b^2 - 4ac > 0 \text{ and } \gcd(a, b, c) = 1}$

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???

- How many reduced quadratic forms of discriminant $\Delta = 73$?
- Can find all of them?

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The reduction algorithm can find all reduced quadratic forms of given discriminant.

	a	b	c	distance
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$a > 0$

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- Let $F = \mathbb{Q}(\sqrt{\Delta})$ with $\Delta = 73 > 0$. Then

$$O_F = \mathbb{Z}\left[\frac{1+\sqrt{73}}{2}\right] = 1 \cdot \mathbb{Z} \oplus \frac{b+\sqrt{\Delta}}{2a} \cdot \mathbb{Z}.$$

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- So, O_F corresponds to a reduced quadratic form $f(X, Y) \equiv (1, 7, -6)$ with discriminant $\Delta = 73$ and $a > 0$. The Arakelov divisor $d(O_F) = (O_F, N(O_F)^{-1/n})$ is called reduced.

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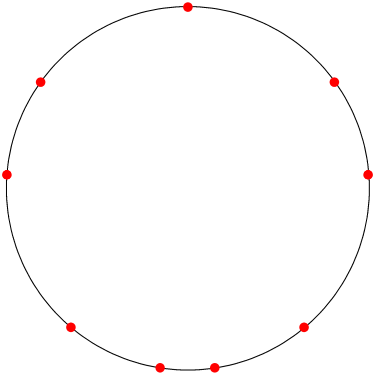

Reduced Arakelov divisors

How to generalize the reducedness?

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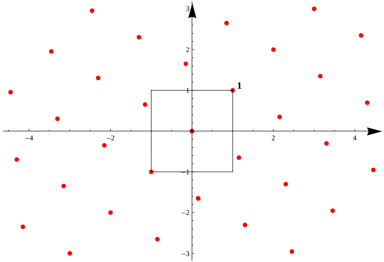

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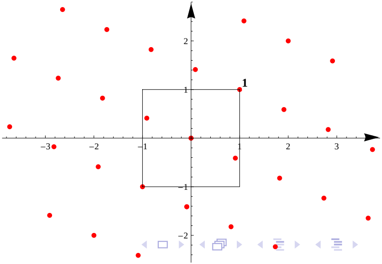

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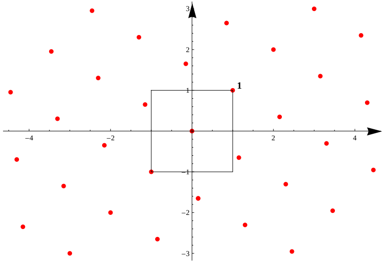

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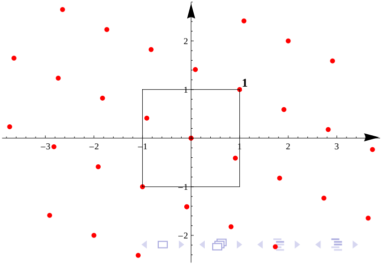

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An Arakelov divisor D is called reduced if
$D = d(I) := (I, N(I)^{-\frac{1}{n}})$ for some reduced ideal I.

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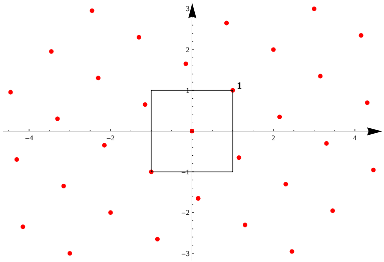

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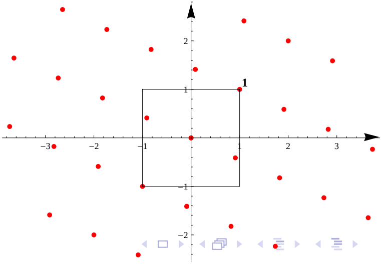

Examples of reduced Arakelov divisors

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2. Let $F = \mathbb{Q}(\sqrt{\Delta})$ with $\Delta > 0$ and $I = \mathbb{Z} + \frac{b+\sqrt{\Delta}}{2a}\mathbb{Z}$ with $a, b, c \in \mathbb{Z}$, $b^2 - 4ac = \Delta$ and $|\sqrt{\Delta} - 2a| < b < \sqrt{\Delta}$. Then $d(I)$ is reduced.

Examples of reduced Arakelov divisors

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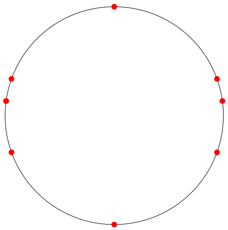
3. Reduced Arakelov divisors on T^0 with $F = \mathbb{Q}(\sqrt{983})$.



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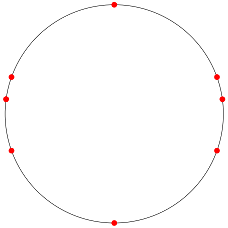

Ex: Find all reduced Arakelov divisors of $\mathbb{Q}(\sqrt{10})$.

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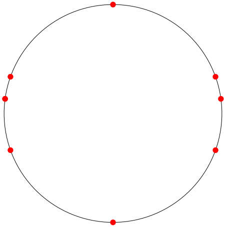

Denote the set of all reduced Arakelov divisors of F is Red_F.
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Red_F (Schoof 2008)

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Proposition 1. (cardinality of Red_F)

Let $D = d(I)$ be a reduced Arakelov divisor. Then
(i) $I^{-1} \subset O_F$ and $N(I^{-1}) \leq \partial_F$ where $\partial_F = (\frac{2}{\pi})^{r_2}\sqrt{|\Delta|}$.
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The number of reduced Arakelov divisors contained in a ball of radius 1 in Pic_F^0 is at most $\left(\frac{2}{\log 2}\right)^n \approx 2.8854^n$.

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$\partial_F = (\frac{2}{\pi})^{r_2}\sqrt{|\Delta|}$.

Lemma

Let $D = (I, u)$ be of deg 0. Then there exists $0 \neq f \in I$ st

$$|u_\sigma \sigma(f)| < \partial_F^{1/n} \text{ for every } \sigma$$

$$(\Rightarrow \|f\|_D \leq \sqrt{n}\partial_F^{1/n}).$$

Proof. Use the Minkowskis Convex Body Theorem with the bounded symmetric convex set

$$V = \{(y_\sigma)_\sigma \in F_\mathbb{R} : |y_\sigma| \leq \partial_F^{1/n} \text{ for all } \sigma\}.$$

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Proof.

i) $I \subset O_F$ since $1 \in I$, we have $I^{-1} \subset O_F$.
By the lemma, there is a nonzero element $f \in I$ such that

$$N(I)^{-1/n}|\sigma(f)| \le \partial_F^{1/n} \text{ for all } \sigma.$$

If $N(I)^{-1} > \partial_F$ then we have $|\sigma(f)| < 1$ for all σ, contradicting the minimality of 1. This proves part (i).

ii) It follows (i) because the number integral ideals of bounded norm is finite.

Red_F (Schoof 2008)

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Proof.

- $deg(D) = 0$, $\exists$ minimal element $f \in I$ (lemma) st

$$u_\sigma |\sigma(f)| < \partial_F^{1/n} \text{ for all } \sigma.$$

- Let $J = f^{-1}I$. Then $D' = d(J)$ is reduced.
- D' is on the same connected component of Pic_F^0 as D bc $D - D' = (f) + (O_F, v)$ with $v = u|f|N(J)^{1/n}$.
- $\|D - D'\|_{Pic_F} = \|v\|_{Pic} \leq log(\partial_F)$ since $v_\sigma = u_\sigma |\sigma(f)| N(J)^{1/n}$ for all σ and $\sum_\sigma \log v_\sigma = 0$.

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$B_{red}^1 =$ the reduced Arakelov divisors contained in a ball of radius 1 in Pic_F^0.

- $n = 1$, $\#B_{red}^1 \leq 2$.
- $n = 2$, $\#B_{red}^1 \leq 8$.
- $n = 3$, $\#B_{red}^1 \leq 24$.
- $n = 4$, $\#B_{red}^1 \leq 69$.
- ...

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  for some $f \in F^*$ such that $\sigma(f) > 0$ for all real σ.

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  There are at most 2^n reduced divisors in the ball of radius $\log 2$ in Pic_F^0.
  Bc if not, then fix one of them: D_0 and consider $D - D_0$ where D runs through the other D. They are all equal to $(f) + (O_F, u)$ for some $u \in \prod_\sigma \mathbb{R}_{>0}$.
  By the box principle, for two distinct divisors, let's say D, D', the signatures of g and g' are equal.
  Then $D - D' = (f) + (O_F, u)$ for some $u \in \prod_\sigma \mathbb{R}_{>0}$, and $f = g/g'$ totally positive.

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- Let $\lambda = N(I/I')^{\frac{1}{n}} = |N(f)|^{\frac{1}{n}}$. Then $\lambda \geq \frac{1}{2}$.

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- $I = fI'$.
- Let $\lambda = N(I/I')^{\frac{1}{n}} = |N(f)|^{\frac{1}{n}}$. Then $\lambda \geq \frac{1}{2}$.
- Assume that $\lambda \leq 1$. $\Rightarrow f - 1 \in I$ satisfies that

  $$|\sigma(f) - 1| \leq |\sigma(f) - \lambda| + |\lambda - 1| < \lambda + 1 - \lambda = 1 \text{ for all } \sigma.$$

  By the minimality of 1, we must have $f - 1 = 0$, so $I = I'$ and then $D = D'$.

How to find a reduced divisor?

The reduction algorithm.

Recap

- Metric on Pic_F^0. Let $D, D' \in Pic_F^0$ st $D - D' = (O_F, u)$. Then
$$\|D' - D\|_{Pic} = \|u\|_{Pic} = \min_{\Lambda} \|L(u)\|.$$
- A fractional ideal I is reduced if $1 \in I$ is minimal.
- Red_F is finite.
- There is at least one reduced Arakelov divisor in the ball of radius $log(\partial_F)$ in Pic_F^0.
- The number of reduced Arakelov divisors contained in a ball of radius 1 in Pic_F^0 is at most 2.8854^n.

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