THE CANONICAL HEIGHT OF A FINITE étale K-ALGEBRA

VALERIO TALAMANCA

Università di L'Aquila

ABSTRACT. In this paper we study height functions attached to adelic norms on finite dimensional vector spaces over number fields. We show that for a finite étale algebra over a number field there exists an intrinsically defined adelic norm whose associated height function can be regarded as the canonical height of A.

INTRODUCTION

In this paper we continue the study of heights on linear spaces that we started in [Ta1] and [Ta2], with particular emphasis towards finite étale K-algebras, K being a number field. Let $\mathbf{X}$ be a vector space over number field K. We study height functions which are defined by adelic norms on $\mathbf{X}$, where an adelic norm is a family of v-adic norms on $\mathbf{X}$ satisfying certain compatibility conditions, see section 1 for the precise definition. Adelic norms on line bundles were used by Peyre [Pe] and Zhang [Zh], among others, to define height functions on projective varieties. It is worth noting that the height functions commonly used in the literature, i.e., naive or Northcott-Weil heights, Arakelov heights, and twisted heights, can all be recovered as heights defined by suitable adelic norms on some vector space; see the examples in section 1 for details.

Let A be a (commutative) finite étale K-algebra. In [Ta1] we introduced a height function H_A on A which was invariant under K-algebra isomorphisms. The main results of this paper, Theorem 2.4 and 2.5, show that H_A can be regarded as the canonical height of A. Indeed, Theorem 2.4 exhibits a construction of H_A by an averaging process starting with any adelic norm which is compatible with the algebra structure. It must be noted that this averaging process is the exact analogue of that employed by Tate in constructing the canonical height associated to a symmetric ample line bundle on an abelian variety, see [La]. Theorem 2.5 shows that the only K-linear surjective maps between two finite étale K-algebras which leave the height invariant (i.e., such that $H_A = H_B \circ T$) are, up to multiplication by an invertible element of height 1, the K-algebra isomorphisms of A into B. The paper is organized as follows: section 1 contains the bulk of definitions as well as some general results on heights associated to adelic norms on vector spaces. The results about finite étale K-algebras are proven in section 2.

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1. Heights and adelic norms

Let K be a number field of degree d over $\mathbb{Q}$. We denote by $\mathcal{M}_K$ the set of equivalence classes of absolute values of K; by $\mathcal{M}_K^0$ (respectively $\mathcal{M}_K^\infty$) the subset of $\mathcal{M}_K$ consisting of the equivalence classes of non-archimedean (respectively archimedean) absolute values. If $v \in \mathcal{M}_K^0$, $v|p$, we normalize $|\cdot|_v$ by requiring that $|p|_v = p^{-1}$; while if $v \in \mathcal{M}_K^\infty$ we normalize $|\cdot|_v$ by requiring that restricted to $\mathbb{Q}$ it coincides with the standard archimedean absolute value. Let K_v be the completion of K with respect to $|\cdot|_v$. We denote by n_v the local degree, and set $d_v = n_v/d$.

Before introducing the height functions that we will use, we need to recall a few facts about lattices and norms for vector spaces over number fields and their completions. See [We2] chapters 3 and 5 for more details. Given $v \in \mathcal{M}_K^0$ we denote by $\mathcal{O}_v$ the closure of $\mathcal{O}_K$ (the ring of integers of K) in K_v. Let $\mathbf{X}$ be a finite dimensional K_v-vector space. An $\mathcal{O}_v$-module $\Lambda \subset \mathbf{X}$ is called a $\mathcal{O}_v$-lattice if it is compact and open. The norm $N_\Lambda : \mathbf{X} \to \mathbb{R}$ associated to Λ is defined by

$$N_\Lambda(\mathbf{x}) = \inf_{\gamma \in K_v^\times, \gamma\mathbf{x} \in \Lambda} |\gamma|_v^{-1}.$$

On the other hand to any $|\cdot|_v$-norm on $\mathbf{X}$ one can associate the lattice of elements of norm at most 1. Amongst all the norms associated to the same lattice, N_Λ can be characterized as the only one such that the value group of N coincides with the value group of $|\cdot|_v$.

Let $\mathbf{X}$ be a K-vector space. As customary, in order to be able to define height functions on $\mathbf{X}$ we need to provide $\mathbf{X}$ with some extra structure. In our case this extra structure is that of an adelic norm on $\mathbf{X}$. An $\mathcal{O}_K$-module $\Lambda \subset \mathbf{X}$ is called an $\mathcal{O}_K$-lattice if it is finitely generated and contains a basis of $\mathbf{X}$ over K.

An *adelic norm* on $\mathbf{X}$ is a collection $\mathcal{F} = \{N_v : \mathbf{X}_v = \mathbf{X} \otimes_K K_v \longrightarrow \mathbb{R},\ v \in \mathcal{M}_K\}$ of $|\cdot|_v$-norms, having the following properties:

(a) If $v \in \mathcal{M}_K^0$, then N_v is ultrametric, i.e., $N_v(\mathbf{x}+\mathbf{y}) \le \max\{N_v(\mathbf{x}), N_v(\mathbf{y})\}$.
(b) There exists an $\mathcal{O}_K$-lattice $\Lambda \subset \mathbf{X}$, such that N_v is the norm associated to the $\mathcal{O}_v$-lattice $\Lambda_v = \Lambda \otimes_{\mathcal{O}_K} \mathcal{O}_v$ for all but finitely many $v \in \mathcal{M}_K^0$.

The $\mathcal{O}_K$-lattice $\Lambda_\mathcal{F}$ defined by requiring that $(\Lambda_\mathcal{F})_v = \{\mathbf{x} \in \mathbf{X}_v \mid N_v(\mathbf{x}) \le 1\}$ for all $v \in \mathcal{M}_K^0$ is uniquely determined by $\mathcal{F}$, for $\mathcal{O}_K$-lattices are uniquely determined by their local completions as proven in [We2] Theorem V.2.2. By definition $N_v = N_{(\Lambda_\mathcal{F})_v}$ for almost all $v \in \mathcal{M}_K^0$. Thus, for a given $\mathbf{x} \in \mathbf{X}$, the set $\{v \in \mathcal{M}_K^0 \mid N_v(\mathbf{x}) \ne 1\}$ is finite. Therefore we can define $H_\mathcal{F}$, the *height* associated to $\mathcal{F}$, by setting

$$H_\mathcal{F}(\mathbf{x}) = \prod_{v \in \mathcal{M}_K} N_v(\mathbf{x})^{d_v},$$

for all non zero $\mathbf{x} \in \mathbf{X}$. We set by definition $H_\mathcal{F}(\mathbf{0}) = 1$.

Examples. (a) Let $\mathbf{X}$ be a K-vector space, $\Lambda \subset \mathbf{X}$ an $\mathcal{O}_K$-lattice, $\underline{b}$ a basis of $\mathbf{X}$ over K and $1 \le q \le \infty$. For each $v \in \mathcal{M}_K^\infty$, let $N_{\underline{b},v}^q$ be the $\ell^q - norm$ on $\mathbf{X}_v$ associated to $\underline{b}$ regarded as a K_v-basis. Then $\mathcal{E}_{\Lambda,\underline{b}}^q = \{N_{\Lambda_v}, v \in \mathcal{M}_K^0\} \cup \{N_{\underline{b},v}^q, v \in \mathcal{M}_K^\infty\}$ is an adelic norm. In particular for $\mathbf{X} = K^n$, $\Lambda = \mathcal{O}_K^n$, $\underline{b} = \underline{e}$ the canonical basis of K^n, and $q = 1, 2$ or ∞ we recover the classical naive heights as defined by Northcott ($q = 1$) [No], Weil ($q = \infty$)[We1] and Schmidt ($q = 2$)[Sc].
(b) Let $\mathcal{T} = (T_v)$ be an element of $\mathrm{GL}_n(\mathbb{A}_K)$, the adele group of $\mathrm{GL}_n(K)$. Let $\Lambda_\mathcal{T}$ be the $\mathcal{O}_K$-lattice defined by requiring that $(\Lambda_\mathcal{T})_v = T_v(\mathcal{O}_v^n)$. Set $N_v^\mathcal{T} = N_{(\Lambda_\mathcal{T})_v}$ for all $v \in \mathcal{M}_K^0$, and consider the adelic norm $\mathcal{F}_\mathcal{T} = \{N_v^\mathcal{T}, v \in \mathcal{M}_K^0\} \cup \{N_{\underline{e},v}^\mathcal{T}, v \in \mathcal{M}_K^\infty\}$, where $N_v^\mathcal{T}(\mathbf{x}) = N_{\underline{e},v}^2(T_v(\mathbf{x}))$. The height $H_\mathcal{T} = H_{\mathcal{F}_\mathcal{T}}$ associated to this adelic norm was used by D.Roy and J.Thunder in [RT] and referred to as the twisted height associated to $\mathcal{T}$.
(c) Let $\overline{E} = \big(E, \{<;>_v\}_{v \in \mathcal{M}_K^\infty}\big)$ be an Hermitian vector bundle over $\mathrm{Spec}(\mathcal{O}_K)$. This means that E is a locally free projective $\mathcal{O}_K$-module of finite rank, and $<;>_v$ is a metric on $E_v = E \otimes_{\mathcal{O}_K} K_v$ which is a

scalar product if v is real and Hermitian if v is complex. As a by-product of Arakelov theory[1] one gets a (logarithmic) height theory for the K-vector space $E_K = E \otimes_{\mathcal{O}_K} K$ which can be computed as follows (see [Ga]): given $s \in E_K = E \otimes_{\mathcal{O}_K} K$, let L_s be the Hermitian line bundle $(Ks) \cap E$ (we are regarding E as contained in E_K, via $\mathbf{x} \mapsto \mathbf{x} \otimes 1$) with the induced metric, then $h_{ar}(s) = -\widehat{\deg}(L_s)$. On the other hand to $\overline{E}$ we can associate an adelic norm on E_K, namely $\mathcal{F}_{\overline{E}} = \{N_{E_v}, v \in \mathcal{M}_K^0\} \cup \{\sqrt{<\,;\,>_v}, v \in \mathcal{M}_K^\infty\}$. Then, Lemma 1 of [Vi] yields: $\widehat{\deg}(L_s) = [K:\mathbb{Q}] \cdot \log H_{\mathcal{F}_{\overline{E}}}(s)$.

The height function $H_{\mathcal{F}}$ enjoys the following properties:

(H1) $H_{\mathcal{F}}(\lambda \mathbf{x}) = H_{\mathcal{F}}(\mathbf{x})$ for all $\lambda \in K^\times$ and all $\mathbf{x} \in \mathbf{X}$.
(H2) Let $\mathbb{P}(\mathbf{X})$ be the set of 1-dimensional subspaces of $\mathbf{X}$. Then for all $C > 0$ the set $\{l \in \mathbb{P}(\mathbf{X}) \mid H_{\mathcal{F}}(l) \leq C\}$ is finite.

The first property follows from the product formula. It implies that $H_{\mathcal{F}}(l)$ is well defined for $l \in \mathbb{P}(\mathbf{X})$. To prove (H2), recall that given an adelic norm $\mathcal{F}$ on $\mathbf{X}$, there exists an isomorphism $\iota : \mathbf{X} \to K^n$, $n = \dim_K \mathbf{X}$, such that

$$C_1 H_{\mathcal{E}}\big(\iota(\mathbf{x})\big) \leq H_{\mathcal{F}}(\mathbf{x}) \leq C_2 H_{\mathcal{E}}\big(\iota(\mathbf{x})\big)$$

for some positive constants C_1, C_2. Here $\mathcal{E} = \mathcal{E}^\infty_{\Lambda,\underline{e}}$, as defined in Example (a) above, with $\underline{e}$ the canonical basis of K^n and $\Lambda = \mathcal{O}_K^n$ (see [Ta2] Lemma 2.1). Therefore the classical Northcott's theorem [No] yields the conclusion.

If $\mathcal{S} \subset \mathcal{M}_K$ is a finite set we set $\mathbf{X}_{\mathcal{S}} = \prod_{v \in \mathcal{S}} \mathbf{X}_v$ and we consider $\mathbf{X}$ as embedded diagonally in $\mathbf{X}_{\mathcal{S}}$. Moreover we denote by $d_{\mathcal{F}}$ the metric on $\mathbf{X}_{\mathcal{S}}$ defined by: $d_{\mathcal{F}}\big(\{\alpha_v\}_{v \in \mathcal{S}}, \{\beta_v\}_{v \in \mathcal{S}}\big) = \sup_{v \in \mathcal{S}} N_v(\alpha_v - \beta_v)$.

Lemma 1.1. *Let $\mathcal{F}$ be an adelic norm on $\mathbf{X}$. If $\mathcal{S} \subset \mathcal{M}_K$ is a finite set, then $\mathbf{X}$ is dense in $\mathbf{X}_{\mathcal{S}}$ with respect to the metric $d_{\mathcal{F}}$.*

Proof. This follows directly from the weak approximation theorem for number fields and the fact that all the norms on a finite dimensional vector space over a complete field are equivalent. ∎

For the rest of this section we let $\mathcal{F} = \{N_v\,,\, v \in \mathcal{M}_K\}$ and $\mathcal{G} = \{P_v\,,\, v \in \mathcal{M}_K\}$ be adelic norms on $\mathbf{X}$ and $\mathbf{Y}$ respectively.

Lemma 1.2. *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathcal{F}$ and $\mathcal{G}$ be as above. Assume that $T \in \mathrm{Hom}_K(\mathbf{X}, \mathbf{Y})$ is injective. Then there exists a finite set $\mathcal{S}_T \subset \mathcal{M}_K$ such that: $\tilde{T} : (\mathbf{X}_v, N_v) \to (\mathbf{Y}_v, P_v)$ is norm preserving if and only if $v \notin \mathcal{S}_T$.*

Proof. First note that by an abuse of notation we denote by the same letter the map that a linear transformation induces on the various completions of a vector space. Next, by the definition of an adelic norm, there exists a K-lattice $\Lambda \subset \mathbf{X}$ (respectively $\Omega \subset \mathbf{Y}$) such that N_v (respectively M_v) coincides with the norm associated to Λ_v (respectively Ω_v) for all $v \notin \mathcal{P}$, where $\mathcal{P}$ is a finite subset of $\mathcal{M}_K$ containing $\mathcal{M}_K^\infty$. Thus, for all $v \notin \mathcal{P}$, T is norm preserving if and only if $T^{-1}(\Omega_v) = \Lambda_v$, and this happens for all but finitely many $v \in \mathcal{M}_K \setminus \mathcal{P}$. ∎

Given $T \in \mathrm{Hom}_K(\mathbf{X}, \mathbf{Y})$ we set:

$$H^{op}_{\mathcal{F},\mathcal{G}}(T) = \sup_{\mathbf{x} \in \mathbf{X} \setminus \{\mathbf{0}\}} \frac{H_{\mathcal{G}}\big(T(\mathbf{x})\big)}{H_{\mathcal{F}}(\mathbf{x})}.$$

The function $H^{op}_{\mathcal{F},\mathcal{G}} : \mathrm{Hom}_K(\mathbf{X}, \mathbf{Y}) \to \mathbb{R}$, $T \mapsto H^{op}_{\mathcal{F},\mathcal{G}}(T)$, is called the *operator height* on $\mathrm{Hom}_K(\mathbf{X}, \mathbf{Y})$ associated to $\mathcal{F}$ and $\mathcal{G}$. If $\mathbf{X} = \mathbf{Y}$ and $\mathcal{F} = \mathcal{G}$ we will use $H^{op}_{\mathcal{F}}$ instead of $H^{op}_{\mathcal{F},\mathcal{F}}$. On $\mathrm{Hom}_K(\mathbf{X}, \mathbf{Y})$ it is also

[1]For a comprehensive treatment of Arakelov theory see [GS]; for one of the relations between Arakelov theory and heights (of cycles) on projective varieties see [BGS].

possible to define an adelic norm associated to $\mathcal{F}$ and $\mathcal{G}$, precisely: $\mathcal{L}_{\mathcal{F}}^{\mathcal{G}} = \{M_{N_v, P_v}\,,\ v\in\mathcal{M}_K\}$, where:

$$M_{N_v,P_v}(T) = \sup_{\mathbf{x}\in\mathbf{X}_v\setminus\{\mathbf{0}\}} \frac{P_v\big(T(\mathbf{x})\big)}{N_v(\mathbf{x})}.$$

It is clear that $H_{\mathcal{L}_{\mathcal{F}}^{\mathcal{G}}}(T) \geq H^{op}_{\mathcal{F},\mathcal{G}}(T)$, but in many cases the equality actually holds:

Proposition 1.3. *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathcal{F}$ and $\mathcal{G}$ be as above. Let $T\in \mathrm{Hom}_K(\mathbf{X},\,\mathbf{Y})$ be injective. Then:*

$$H^{op}_{\mathcal{F},\mathcal{G}}(T) = \prod_{v\in\mathcal{M}_K} M_{N_v,P_v}(T)^{d_v}.$$

Proof. The argument is identical to the one used in [Ta2] Proposition 3.1, where the same result is proved for the special case $\mathbf{X} = \mathbf{Y} = K^n$ and $\mathcal{F} = \mathcal{E}^2_{\mathcal{O}^n_K,\underline{e}}$, $\underline{e}$ being the canonical basis of K^n. One only needs to use Lemma 1.1 and Lemma 1.2 instead of Lemma 3.2 of [Ta2].

Proposition 1.4. *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathcal{F}$ and $\mathcal{G}$ be as above. Suppose T is such that there exists $\gamma\in\mathbb{R}$ such that $\gamma H_{\mathcal{F}}(\mathbf{x}) = H_{\mathcal{G}}\big(T(\mathbf{x})\big)$ for all $\mathbf{x}\in\mathbf{X}\setminus\{\mathbf{0}\}$. Then:*

(a) $\prod_{v\in\mathcal{M}_K} M_{N_v,P_v}(T)^{d_v} = \gamma$.

(b) $P_v\big(T(\mathbf{x})\big) = M_{N_v,P_v}(T)N_v(\mathbf{x})$ *for all* $\mathbf{x}\in\mathbf{X}\setminus\{\mathbf{0}\}$ *and all* $v\in\mathcal{M}_K$.

Proof. Since $H_{\mathcal{F}}$ is constant on the subspaces generated by $\mathbf{y}$ and $\ker T$ for each $\mathbf{y}\notin\ker T$, property (H2) implies that T is injective. For ease of notation we set $M_v = M_{N_v,P_v}$. Then by Proposition 1.3 we have

$$\prod_{v\in\mathcal{M}_K} M_{N_v,P_v}(T)^{d_v} = H^{op}_{\mathcal{F},\mathcal{G}}(T) = \gamma$$

proving (a). Next Lemma 1.2 implies that for all $\mathbf{x}\in\mathbf{X}\setminus\{\mathbf{0}\}$:

$$(\bullet)\qquad \gamma = \prod_{v\in\mathcal{S}_T} \left(\frac{P_v\big(T(\mathbf{x})\big)}{N_v(\mathbf{x})}\right)^{d_v}.$$

Suppose that there exists $\mathbf{x}_0\in\mathbf{X}\setminus\{\mathbf{0}\}$ which contradicts (b), i.e., one can find $w\in S_T$ for which $P_w\big(T(\mathbf{x}_0)\big) < M_w(T)N_w(\mathbf{x}_0)$. Then $(\bullet)$ yields:

$$\gamma = \prod_{v\in\mathcal{S}_T} \left(\frac{P_v\big(T(\mathbf{x})\big)}{N_v(\mathbf{x})}\right)^{d_v} \leq \left(\prod_{\substack{v\in\mathcal{S}\\ v\neq w}} M_v(T)^{d_v}\right)\left(\frac{P_w\big(T(\mathbf{x})\big)}{N_w(\mathbf{x})}\right)^{d_w} < \prod_{v\in\mathcal{S}_T} M_v(T)^{d_v} = \gamma,$$

which is a contradiction. ∎

Corollary 1.5. *Let $\mathcal{F} = \{N_v\,,\ v\in\mathcal{M}_K\}$ and $\mathcal{G} = \{P_v\,,\ v\in\mathcal{M}_K\}$ be two adelic norms on $\mathbf{X}$. Then:*

(a) *The ratio $H_{\mathcal{F}}/H_{\mathcal{G}}$ is constant on $\mathbf{X}\setminus\{\mathbf{0}\}$ if and only if the ratio N_v/P_v is constant on $\mathbf{X}_v\setminus\{\mathbf{0}\}$ for all v's.*

(b) *$H_{\mathcal{F}} = H_{\mathcal{G}}$ if and only if $N_v = \gamma_v P_v$, and $\prod_{v\in\mathcal{M}_K}\gamma_v^{d_v} = 1$.*

Proof. Apply the proposition to the identity map.

2. Heights on finite étale K-Algebras

Let A be a K-algebra which is always assumed to be finite dimensional, commutative and with unit usually denoted by 1_A. An adelic norm $\mathcal{F} = \{N_v\,,\ v\in\mathcal{M}_K\}$ on A is said to be *compatible* (with the algebra structure) if

(C1) $N_v(ab) \leq N_v(a)N_v(b)$ for all $a,\, b\in A_v$ and for all $v\in\mathcal{M}_K$.

Therefore, given a compatible adelic norm $\mathcal{F}$ on A, the inequality

$$H_{\mathcal{F}}(ab) \leq H_{\mathcal{F}}(a)H_{\mathcal{F}}(b) \tag{HC1}$$

holds for all $a, b\in A$ such that $ab \neq 0$. We will denote by $\mathcal{C}(A)$ the set of compatible adelic norms on A.

Suppose now that A is a finite étale K-algebra, i.e., that A is a product of finite extensions of K. In [Ta1] we introduced the following family of v-adic norms: let $(X_v,\, \mathcal{O}_{X_v})$ be the affine K_v-scheme associated to the K_v-algebra $A_v = A\otimes_K K_v$. Let $a \mapsto \hat{a}$ denote the canonical isomorphism $A_v \simeq \Gamma(X_v,\, \mathcal{O}_{X_v})$. We then set

$$N_v^A(a) = \sup_{x\in X_v} |\hat{a}(x)|_x$$

where $|\cdot|_x$ is the unique extension of $|\cdot|_v$ to $K_v(x)$.

Lemma 2.1. *Let A be a finite étale K-algebra. The family $\mathcal{F}_A = \{N_v^A,\ v\in\mathcal{M}_K\}$ is a compatible adelic norm on A.*

Proof. In [Ta1] proposition 2.2(b) it was shown that property (C1) holds for $\mathcal{F}_A$. Thus it only remains to check that there exists an $\mathcal{O}_K$-lattice $\Lambda \subset A$ such that $N_v^A = N_{\Lambda_v}$ for almost all $v\in\mathcal{M}_K$. Assume first that A is simple so $A = F$ is a finite extension of K. Our choice of $\mathcal{O}_K$-lattice is $\Lambda = \mathcal{O}_F$. Let us denote by $S_0 \subset \mathcal{M}_K^0$ the subset formed by the absolute values whose associated prime ideal ramifies in the extension $F \supset K$. We want to show that for all $v\in\mathcal{M}_K^0\backslash S$ we have $N_v^F = N_{\Lambda_v}$. A moment of reflection shows that

$$N_v^F(a) = \sup_{w\in\mathcal{M}_F^v} |a|_w,$$

where $\mathcal{M}_F^v = \{w\in\mathcal{M}_F \mid |\cdot|_w|_K = |\cdot|_v\}$. Given $a\in F$ and $\lambda\in K_v^\times$ note that γa belongs to Λ_v if and only if $|\gamma a|_w \leq 1$ for all $w\in\mathcal{M}_F^v$. Thus $\gamma a\in\Lambda_v$ implies $|a|_w \leq |\gamma^{-1}|_v$ which yields

$$|a|_w \leq \inf_{\gamma\in K_v^\times,\gamma a\in\Lambda_v} |\gamma|_v^{-1} = N_{\Lambda_v}(a).$$

The above inequality yields $N_v^F(a) \leq N_{\Lambda_v}(a)$. Note that this holds also for the ramified primes. It is in the proof of the reverse inequality that we use the assumption on v. So let us assume that v is unramified in the extension $K \subset F$. Then the value groups of $|\cdot|_v$ and $|\cdot|_w$ are the same for all $w\in\mathcal{M}_F^v$. Therefore, there exists $\gamma_0\in K_v^\times$ such that $|\gamma_0|_v = (\sup_{w\in\mathcal{M}_F^v} |a|_w)^{-1}$. It follows that $|\gamma_0 a|_w \leq 1$ for all $w\in\mathcal{M}_F^v$. Thus $\gamma_0 a\in\Lambda_v$, yielding $N_{\Lambda_v}(a) \leq N_v^F(a)$. For the general case it suffices to recall that N_v^A behaves well under direct products, i.e., if $A \simeq \prod_{i=1}^f A_i$, then $N_v^A(a) = \sup_{1\leq i\leq f} N_v^{A_i}(\pi_i(a))$, where π_i is the canonical projection (see [Ta1] proposition 1.1(c) for a proof). Thus, if we decompose A as the product of simple K-algebras as above, then the sought $\mathcal{O}_K$-lattice is the product of the $\mathcal{O}_{A_i}$. ∎

For ease of notation we set $H_A = H_{\mathcal{F}_A}$, $H_A^{op} = H_{\mathcal{F}_A}^{op}$. The function H_A enjoys the following properties (see [Ta1] Proposition 2.2 and corollary 2.4 for a proof):

(A1) $H_A(a) \geq 1$;
(A2) $H_A(a^k) = \big(H_A(a)\big)^k$;
(A3) If $\varphi : A \longrightarrow B$ is K-algebra isomorphism, then $H_A = H_B\circ\varphi$.

Remark. The adelic norm $\mathcal{F}_A$ has also the property that $N_v(1_A) = 1$ for all $v \in \mathcal{M}_K$, which means that (A_v, N_v^A) is a Banach algebra (either real, complex or non-archimedean) for all $v \in \mathcal{M}_K$. A compatible adelic norm enjoying this property will be called a *Banach* adelic norm.

Example. Let $K \subset F$ be a finite extension. Let $R \subseteq \mathcal{O}_F$ be an order, i.e., R is a subring of $\mathcal{O}_F$ containing 1 which is also a $\mathcal{O}_K$-lattice. Let $R_v = R \otimes_{\mathcal{O}_K} \mathcal{O}_v$, then $\mathcal{F}_R = \{N_{R_v}; v \in \mathcal{M}_K^0\} \cup \{N_v^F; v \in \mathcal{M}_K^\infty\}$, is an adelic norm on F. Set $H_R = H_{\mathcal{F}_R}$. For each $v \in \mathcal{M}_K^0$, R_v is a subring of $F \otimes_K K_v$. Thus for every pair of elements $a, b \in F \otimes_K K_v$ we have

$$\{\lambda \in K_v^\times \mid \lambda ab \in R_v\} \supseteq \{\gamma \in K_v^\times \mid \gamma = \mu\nu \text{ and } \mu a \in R_v, \nu b \in R_v\}.$$

Therefore $N_{R_v}(ab) \le N_{R_v}(a) N_{R_v}(b)$. Since 1_A belongs to R we have $N_v(1_A) \le 1$ for all $v \in \mathcal{M}_K^0$. Moreover, the inclusion $R \subset \mathcal{O}_F$ yields $H_{\mathcal{O}_F}(a) \le H_R(a)$. From the proof of Lemma 2.1 we know that $H_F(a) \le H_{\mathcal{O}_F}(a)$, so $H_F(a) \le H_R(a)$. In particular $H_F(1_A) \ge 1$ so we must have $N_{R_v}(1_A) = 1$ for all $v \in \mathcal{M}_K^0$, i.e., $\mathcal{F}_R$ is a Banach adelic norm. Conversely, suppose that $\mathcal{F} = \{N_v; v \in \mathcal{M}_K\}$ is a Banach adelic norm on the finite étale K-algebra A. Recall that the $\mathcal{O}_K$-lattice $\Lambda_{\mathcal{F}}$ associated to $\mathcal{F}$ can be computed as (again by Theorem V.2.2 of [We2])

$$\Lambda_{\mathcal{F}} = \bigcap_{v \in \mathcal{M}_K^0} \big(\{a \in A_v \mid N_v(a) \le 1\} \cap A\big).$$

Now, for $v \in \mathcal{M}_K^0$, given $a, b \in \Lambda_{\mathcal{F}}$ we have $N_v(ab) \le N_v(a) N_v(b) \le 1$. Therefore $\Lambda_{\mathcal{F}}$ is a subring of A. Moreover, $\mathcal{F}$ is Banach adelic norm and so $N_v(1_A) = 1$ for all $v \in \mathcal{M}_K^0$, therefore $1_A \in \Lambda_{\mathcal{F}}$ and so $\Lambda_{\mathcal{F}}$ is an order in A.

To better exploit the canonical nature of H_A it is convenient to bring into the picture the spectral height. In order to do so we need some definitions: first the *spectral height* $H_s : \mathrm{End}_K(A) \to \mathbb{R}$ is defined as follows: $H_s(T) = 1$ if T is nilpotent, otherwise we set

$$H_s(T) = \prod_{v \in \mathcal{M}_K} \rho_v(T)^{d_v}.$$

Here $\rho_v(T)$ denotes the v-adic spectral radius[2] of T. Precisely

$$\rho_v(T) = \sup_{\lambda \in \mathrm{sp}(T)} |\lambda|_{K_v(\lambda)},$$

where $\mathrm{sp}(T)$ is the set of characteristic roots of T in an algebraic closure of K_v, and $|\cdot|_{K_v(\lambda)}$ is the unique extension of $|\cdot|_v$ to $K_v(\lambda)$. Second, to any adelic norm $\mathcal{F}$ we can also associate the height function $H_{\mathcal{F}}^{op}$ on $\mathrm{End}_K(A)$, defined in section 1. Moreover, the adelic norm $\mathcal{L}_{\mathcal{F}}^{\mathcal{F}}$, also defined in section 1, gives rise to another height function, $H_{\mathcal{L}_{\mathcal{F}}^{\mathcal{F}}}$, on $\mathrm{End}_K(A)$. In the following, given $a \in A$ we let $m_a : A \to A$ denote the *multiplication* $-$ *by* $-$ *a* map.

Lemma 2.2. *Let A be a finite dimensional commutative K-algebra. Suppose $\mathcal{F}$ is a compatible adelic norm on A. Then for all $a \in A$,*

$$H_{\mathcal{F}}^{op}(m_a) \le H_{\mathcal{L}_{\mathcal{F}}^{\mathcal{F}}}(m_a) \le H_{\mathcal{F}}(a) \le H_{\mathcal{F}}^{op}(m_a) H_{\mathcal{F}}(1_A).$$

In particular if $H_{\mathcal{F}}(1_A) = 1$ we have $H_{\mathcal{F}}^{op}(m_a) = H_{\mathcal{L}_{\mathcal{F}}^{\mathcal{F}}}(m_a) = H_{\mathcal{F}}(a)$.

Proof. By definition of $H_{\mathcal{F}}^{op}$ we have:

$$H_{\mathcal{F}}(1_A) H_{\mathcal{F}}^{op}(m_a) \ge H_{\mathcal{F}}\big(m_a(1_A)\big) = H_{\mathcal{F}}(a).$$

On the other hand, the inequalities $H_{\mathcal{L}_{\mathcal{F}}^{\mathcal{F}}}(m_a) \ge H_{\mathcal{F}}^{op}(m_a)$ holds in an even greater generality as remarked before Proposition 1.3. Since $\mathcal{L}_{\mathcal{F}}^{\mathcal{F}} = \{M_{N_v, N_v}, v \in \mathcal{M}_K\}$ and $M_{N_v, N_v}(m_a) \le N_v(a)$, it follows that $H_{\mathcal{L}_{\mathcal{F}}^{\mathcal{F}}}(m_a) \le H_{\mathcal{F}}(a)$, completing the proof of the lemma.

[2] For a comprehensive treatment of spectral theories for a non-archimedean Banach algebra the reader is referred to [Be].

Lemma 2.3. *Let A be an étale K-algebra. Then $H_A(a) = H_s(m_a)$ for all $a \in A$.*

Proof. Let $\mathcal{F}$ be any adelic norm on A. The limit formula

$$\text{(GBF)} \qquad H_s(m_a) = \lim_{k\to\infty} H^{op}_{\mathcal{F}}(m_a^k)^{\frac{1}{k}}$$

is a special[3] case of Theorem A of [Ta2]. Combining (GBF) with lemma 2.2 and (A2) we obtain:

$$H_s(m_a) = \lim_{k\to\infty} H^{op}_{A}(m_a^k)^{\frac{1}{k}} = \lim_{k\to\infty} H_A(a^k)^{\frac{1}{k}} = H_A(a). \quad \blacksquare$$

The next theorem shows that, when A is a finite étale K-algebra, H_A can be obtained by an averaging process starting from any compatible adelic norm on A. We also prove that H_A is minimal among the heights arising from compatible adelic norms.

Theorem 2.4. *Let A be a finite étale K-algebra. Then*

(a) *$\lim_{k\to\infty} H_{\mathcal{F}}(a^k)^{\frac{1}{k}} = H_A(a)$, for all $\mathcal{F} \in \mathcal{C}(A)$ and all $a \in A$.*
(b) *$H_A(a) = \min_{\mathcal{F}\in\mathcal{C}(A)} H_{\mathcal{F}}(a)$, for all $a \in A$.*

Proof. (a) We have:

$$\begin{aligned}
\lim_{k\to\infty} H_{\mathcal{F}}(a^k)^{\frac{1}{k}} &= \lim_{k\to\infty} H^{op}_{\mathcal{F}}(m_a^k)^{\frac{1}{k}} && \text{by Lemma 2.2}\\
&= H_s(m_a) && \text{by } (GBF)\\
&= H_A(a) && \text{by Lemma 2.3}
\end{aligned}$$

(b) Since $\mathcal{F}_A \in \mathcal{C}(A)$ we only have to prove that $H_A(a) \le H_{\mathcal{F}}(a)$ for all $\mathcal{F} \in \mathcal{C}(A)$. But this follows from (a) and (HC1). $\blacksquare$

In the following $[A]$ denotes the K-vector space underlying A and $A^\times$ denotes the set of invertible elements of A.

Theorem 2.5. *Let A and B be two finite étale K-algebras. Suppose that $T \in \mathrm{Hom}_K([A], [B])$ is surjective and such that $H_B\big(T(a)\big) = H_A(a)$ for all $a \in A$. Then A and B are isomorphic as K-algebras. Moreover, if we let $b = T(1_A)$ then $m_{b^{-1}} \circ T$ is a K-algebra isomorphism.*

Proof. We can apply Proposition 1.4 with $\gamma = 1$ finding that T is a bijection and that for each $v \in \mathcal{M}_K$ there exists $\gamma_v \in \mathbb{R}$, such that $\gamma_v N_v^A(a) = N_v^B\big(T(a)\big)$. In particular for $v \in \mathcal{M}_K^\infty$ we have that the K_v-linear map $\gamma_v^{-1} T$ is an isometry between (A_v, N_v^A) and (A_v, N_v^B). Now A_v (respectively B_v) is either a complex Banach algebra or a real Banach algebra or a real function algebra (see [Ta1]) and they have to be of the same type simultaneously, by the existence of the linear isometry. For each of these cases there is an appropriate version of the Banach-Stone theorem[4] which we can apply, as in [Ta1] Proposition 1.3 and Corollary 1.5, to this setting. We thus find that $\gamma_v^{-1} T(1_A) = c$ is invertible and that $\gamma_v^{-1} m_{c^{-1}} \circ T$ is a K_v-algebra isomorphism of A_v into B_v. Now it is straightforward to verify that $b = T(1_A) = \gamma_v c$ is invertible and that $m_{b^{-1}} \circ T$ is an isomorphism of K-algebras. $\blacksquare$

To conclude the paper we state, as a corollary, a special case of the above proposition which generalizes Theorem 3.8 of [Ta1] by removing the condition of A being isotypical (i.e., of the form B^n for some simple K-algebra B).

Corollary 2.6. *Let A be a finite étale K-algebras. Suppose $T \in \mathrm{End}_K([A])$ is such that $H_A\big(T(a)\big) = H_A(a)$ for all $a \in A$, then $b = T(1_A)$ is invertible and $m_{b^{-1}} \circ T$ is a K-algebra automorphism.*

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[3]In [Ta2] the operator height associated to an adelic norm $\mathcal{F}$ was defined including $\mathbf{0}$ in the set over which we take the superior, see section 3 of [Ta2]. Now it is easy to verify that for every $T \in \mathrm{End}_K(\mathbf{X})$ that is not nilpotent we have $H^{op}_{\mathcal{F}}(T^k) \ge 1$ for all $k \ge \dim_K \mathbf{X}$. Therefore the discrepancy between the two definitions does not have any bearings on the limit formula.

[4]For the precise statement and the proof see [BD] I.5.8 and I.13.7 for the case of real or complex Banach Algebras, and [KL] Theorem 5.1.4 for the real function algebras case.

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Dipartimento di Matematica, Università di L'Aquila, Via Vetoio snc, Coppito, 67010, Italy
E-mail address: valerio@dm.univaq.it, valerio@matrm3.mat.uniroma3.it