THE CANONICAL HEIGHT OF A FINITE étale $K$-ALGEBRA

VALERIO TALAMANCA

Università di L’Aquila

ABSTRACT. In this paper we study height functions attached to adelic norms on finite dimensional vector spaces over number fields. We show that for a finite étale algebra over a number field there exists an intrinsically defined adelic norm whose associated height function can be regarded as the canonical height of $A$.

INTRODUCTION

In this paper we continue the study of heights on linear spaces that we started in [Ta1] and [Ta2], with particular emphasis towards finite étale $K$-algebras, $K$ being a number field. Let $X$ be a vector space over number field $K$. We study height functions which are defined by adelic norms on $X$, where an adelic norm is a family of $v$-adic norms on $X$ satisfying certain compatibility conditions, see section 1 for the precise definition. Adelic norms on line bundles were used by Peyre [Pe] and Zhang [Zh], among others, to define height functions on projective varieties. It is worth noting that the height functions commonly used in the literature, i.e., naïve or Northcott-Weil heights, Arakelov heights, and twisted heights, can all be recovered as heights defined by suitable adelic norms on some vector space; see the examples in section 1 for details.

Let $A$ be a (commutative) finite étale $K$-algebra. In [Ta1] we introduced a height function $H_A$ on $A$ which was invariant under $K$-algebra isomorphisms. The main results of this paper, Theorem 2.4 and 2.5, show that $H_A$ can be regarded as the canonical height of $A$. Indeed, Theorem 2.4 exhibits a construction of $H_A$ by an averaging process starting with any adelic norm which is compatible with the algebra structure. It must be noted that this averaging process is the exact analogue of that employed by Tate in constructing the canonical height associated to a symmetric ample line bundle on an abelian variety, see [La]. Theorem 2.5 shows that the only $K$-linear surjective maps between two finite étale $K$-algebras which leave the height invariant (i.e., such that $H_A = H_B$) are, up to multiplication by an invertible element of height 1, the $K$-algebra isomorphisms of $A$ into $B$. The paper is organized as follows: section 1 contains the bulk of definitions as well as some general results on heights associated to adelic norms on vector spaces. The results about finite étale $K$-algebras are proven in section 2.

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1. Heights and adelic norms

Let $K$ be a number field of degree $d$ over $\mathbb{Q}$. We denote by $\mathcal{M}_K$ the set of equivalence classes of absolute values of $K$; by $\mathcal{M}_K^0$ (respectively $\mathcal{M}_K^\infty$) the subset of $\mathcal{M}_K$ consisting of the equivalence classes of non-archimedean (respectively archimedean) absolute values. If $v \in \mathcal{M}_K^0$, $v|p$, we normalize $|-|_v$ by requiring that $|p|_v = p^{-1}$; while if $v \in \mathcal{M}_K^\infty$ we normalize $|-|_v$ by requiring that restricted to $\mathbb{Q}$ it coincides with the standard archimedean absolute value. Let $K_v$ be the completion of $K$ with respect to $|-|_v$. We denote by $n_v$ the local degree, and set $d_v = n_v/d$.

Before introducing the height functions that we will use, we need to recall a few facts about lattices and norms for vector spaces over number fields and their completions. See [We2] chapters 3 and 5 for more details. Given $v \in \mathcal{M}_K$ we denote by $O_v$ the closure of $O_K$ (the ring of integers of $K$) in $K_v$. Let $X$ be a finite dimensional $K_v$-vector space. An $O_v$-module $\Lambda \subset X$ is called a $O_v$-lattice if it is compact and open. The norm $N_\Lambda : X \to \mathbb{R}$ associated to $\Lambda$ is defined by

$$N_\Lambda(x) = \inf_{\gamma \in K_v^*, \gamma x \in \Lambda} |\gamma|_v^{-1}.$$ 

On the other hand to any $|-|_v$-norm on $X$ one can associate the lattice of elements of norm at most 1. Amongst all the norms associated to the same lattice, $N_\Lambda$ can be characterized as the only one such that the value group of $N$ coincides with the value group of $|-|_v$.

Let $X$ be a $K$-vector space. As customary, in order to be able to define height functions on $X$ we need to provide $X$ with some extra structure. In our case this extra structure is that of an adelic norm on $X$. An $O_K$-module $\Lambda \subset X$ is called an $O_K$-lattice if it is finitely generated and contains a basis of $X$ over $K$.

An **adelic norm** on $X$ is a collection $\mathcal{F} = \{N_v : X_v = X \otimes_k K_v \rightarrow \mathbb{R}, v \in \mathcal{M}_K\}$ of $|-|_v$-norms, having the following properties:

(a) If $v \in \mathcal{M}_K^0$, then $N_v$ is ultrametric, i.e., $N_v(x + y) \leq \max\{N_v(x), N_v(y)\}$.

(b) There exists an $O_K$-lattice $\Lambda \subset X$, such that $N_v$ is the norm associated to the $O_v$-lattice $\Lambda_v = \Lambda \otimes_{O_v} O_v$ for all but finitely many $v \in \mathcal{M}_K^0$.

The $O_K$-lattice $\Lambda_\mathcal{F}$ defined by requiring that $(\Lambda_\mathcal{F})_v = \{x \in X_v \mid N_v(x) \leq 1\}$ for all $v \in \mathcal{M}_K^0$ is uniquely determined by $\mathcal{F}$, for $O_K$-lattices are uniquely determined by their local completions as proven in [We2] Theorem V.2.2. By definition $N_v = N|_{\Lambda_\mathcal{F}}$ for almost all $v \in \mathcal{M}_K^0$. Thus, for a given $x \in X$, the set $\{v \in \mathcal{M}_K^0 \mid N_v(x) \neq 1\}$ is finite. Therefore we can define $H_\mathcal{F}$, the **height** associated to $\mathcal{F}$, by setting

$$H_\mathcal{F}(x) = \prod_{v \in \mathcal{M}_K} N_v(x)^{d_v},$$

for all non zero $x \in X$. We set by definition $H_\mathcal{F}(0) = 1$.

**Examples.** (a) Let $X$ be a $K$-vector space, $\Lambda \subset X$ an $O_K$-lattice, $b$ a basis of $X$ over $K$ and $1 \leq q \leq \infty$. For each $v \in \mathcal{M}_K^0$, let $N^q_{b,v}$ be the $\ell^q$ - norm on $X_v$ associated to $b$ regarded as a $K_v$-basis. Then $E^q_{\Lambda,b} = \{N_{\Lambda,v}, v \in \mathcal{M}_K^0\} \cup \{N^q_{b,v}, v \in \mathcal{M}_K^\infty\}$ is an adelic norm. In particular for $X = K^n$, $\Lambda = O_K^n$, $b = e$ the canonical basis of $K^n$, and $q = 1,2$ or $\infty$ we recover the classical naive heights as defined by Northcott ($q = 1$) [No], Weil ($q = \infty$)[We1] and Schmidt ($q = 2$)[Sc].

(b) Let $T = (T_v)$ be an element of $\text{GL}_n(\mathbb{A}_K)$, the adele group of $\text{GL}_n(K)$. Let $\Lambda_T$ be the $O_K$-lattice defined by requiring that $(\Lambda_T)_v = T_v(O_K^\infty)$. Set $N^q_{\Lambda_T} = N|_{\Lambda_T}$, for all $v \in \mathcal{M}_K^0$, and consider the adelic norm $F_\mathcal{F} = \{N^q_{\Lambda_T}, v \in \mathcal{M}_K^0\} \cup \{N^q_{\Lambda_T}, v \in \mathcal{M}_K^\infty\}$, where $N^q_{\Lambda_T}(x) = N^q_{\Lambda_T}(T_v(x))$. The height $H_\mathcal{F} = H_\mathcal{F}_\mathcal{F}$ associated to this adelic norm was used by D.Roy and J.Thunder in [RT] and referred to as the twisted height associated to $T$.

(c) Let $E = (E_v, \{\langle <, > \rangle_v\}_{v \in \mathcal{M}_K^\infty})$ be an Hermitian vector bundle over $\text{Spec}(O_K)$. This means that $E$ is a locally free projective $O_K$-module of finite rank, and $\langle <, > \rangle_v$ is a metric on $E_v = E \otimes_{O_K} K_v$ which is a
scalar product if \( v \) is real and Hermitian if \( v \) is complex. As a by-product of Arakelov theory\(^1\) one gets a (logarithmic) height theory for the \( K \)-vector space \( E_K = E \otimes_{\mathbb{Q}} K \) which can be computed as follows (see [Ga]): given \( s \in E_K = E \otimes_{\mathbb{Q}} K \), let \( L_s \) be the Hermitian line bundle \((K,s) \cap E\) (we are regarding \( E \) as contained in \( E_K \), via \( x \mapsto x \otimes 1 \)) with the induced metric, then \( h_{ar}(s) = -\overline{\deg}(L_s) \). On the other hand to \( \mathcal{F} \) we can associate an adelic norm on \( E_K \), namely \( \mathcal{F}_E = \{ N_{E_v}, v \in M_k^\infty \} \cup \{ \sqrt{\langle \xi, t \rangle} ; t \in \mathcal{M}_k^\infty \}. \) Then, Lemma 1 of [Vi] yields: \( \overline{\deg}(L_s) = [K: \mathbb{Q}] \cdot \log H_{\mathcal{F}_E}(s). \)

The height function \( H_{\mathcal{F}} \) enjoys the following properties:

\[
\begin{align*}
(\text{H1}) & \quad H_{\mathcal{F}}(\lambda x) = H_{\mathcal{F}}(x) \text{ for all } \lambda \in K^\times \text{ and all } x \in X. \\
(\text{H2}) & \quad \text{Let } \mathcal{P}(X) \text{ be the set of 1-dimensional subspaces of } X. \text{ Then for all } C > 0 \text{ the set } \{ l \in \mathcal{P}(X) \mid H_{\mathcal{F}}(l) \leq C \} \text{ is finite.}
\end{align*}
\]

The first property follows from the product formula. It implies that \( H_{\mathcal{F}}(l) \) is well defined for \( l \in \mathcal{P}(X) \). To prove (H2), recall that given an adelic norm \( \mathcal{F} \) on \( X \), there exists an isomorphism \( \iota : X \to K^n \), \( n = \dim_k X \), such that

\[
C_1 H_{\mathcal{E}}(\iota(x)) \leq H_{\mathcal{F}}(x) \leq C_2 H_{\mathcal{E}}(\iota(x))
\]

for some positive constants \( C_1, C_2 \). Here \( \mathcal{E} = E_{K,\mathcal{E}}^\infty \) as defined in Example (a) above, with \( \mathcal{E} \) the canonical basis of \( K^n \) and \( \Lambda = \mathcal{O}_K^n \) (see [Ta2] Lemma 2.1). Therefore the classical Northcott’s theorem [No] yields the conclusion.

If \( S \subset M_k^\infty \) is a finite set we set \( X_S = \prod_{v \in S} X_v \) and we consider \( X \) as embedded diagonally in \( X_S \). Moreover we denote by \( d_{\mathcal{F}} \) the metric on \( X_S \) defined by:

\[
d_{\mathcal{F}}(\{ \alpha_v \}_{v \in S}, \{ \beta_v \}_{v \in S}) = \sup_{v \in S} N_v(\alpha_v - \beta_v).
\]

**Lemma 1.1.** Let \( \mathcal{F} \) be an adelic norm on \( X \). If \( S \subset M_k^\infty \) is a finite set, then \( X \) is dense in \( X_S \) with respect to the metric \( d_{\mathcal{F}} \).

**Proof.** This follows directly from the weak approximation theorem for number fields and the fact that all the norms on a finite dimensional vector space over a complete field are equivalent. \( \blacksquare \)

For the rest of this section we let \( \mathcal{F} = \{ N_v, v \in M_k^\infty \} \) and \( \mathcal{G} = \{ P_v, v \in M_k^\infty \} \) be adelic norms on \( X \) and \( Y \) respectively.

**Lemma 1.2.** Let \( X, Y, \mathcal{F} \) and \( \mathcal{G} \) be as above. Assume that \( T \in \text{Hom}_K(X, Y) \) is injective. Then there exists a finite set \( S_T \subset M_k^\infty \) such that: \( T : (X_v, N_v) \to (Y_v, P_v) \) is norm preserving if and only if \( v \not\in S_T \).

**Proof.** First note that by an abuse of notation we denote by the same letter the map that a linear transformation induces on the various completions of a vector space. Next, by the definition of an adelic norm, there exists a \( K \)-lattice \( \Lambda \subset X \) (respectively \( \Omega \subset Y \)) such that \( N_v \) (respectively \( M_v \)) coincides with the norm associated to \( \Lambda_v \) (respectively \( \Omega_v \)) for all \( v \not\in \mathcal{P} \), where \( \mathcal{P} \) is a finite subset of \( M_k^\infty \) containing \( M_k^\infty \). Thus, for all \( v \not\in \mathcal{P} \), \( T \) is norm preserving if and only if \( T^{-1}(\Omega_v) = \Lambda_v \), and this happens for all but finitely many \( v \in M_k^\infty \backslash \mathcal{P} \). \( \blacksquare \)

Given \( T \in \text{Hom}_K(X, Y) \) we set:

\[
H_{\mathcal{F}_T, \mathcal{G}}^p(T) = \sup_{x \in X(a)} \frac{H_{\mathcal{G}}(T(x))}{H_{\mathcal{F}}(x)}.
\]

The function \( H_{\mathcal{F}_T, \mathcal{G}}^p : \text{Hom}_K(X, Y) \to \mathbb{R}, T \mapsto H_{\mathcal{F}_T, \mathcal{G}}^p(T) \), is called the **operator height** on \( \text{Hom}_K(X, Y) \) associated to \( \mathcal{F} \) and \( \mathcal{G} \). If \( X = Y \) and \( \mathcal{F} = \mathcal{G} \) we will use \( H_{\mathcal{F}_T}^p \) instead of \( H_{\mathcal{F}_T, \mathcal{F}}^p \). On \( \text{Hom}_K(X, Y) \) it is also

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\(^1\)For a comprehensive treatment of Arakelov theory see [GS]; for one of the relations between Arakelov theory and heights (of cycles) on projective varieties see [BGS].
possible to define an adelic norm associated to $\mathcal{F}$ and $\mathcal{G}$, precisely: $\mathcal{L}_{\mathcal{F}}^{\mathcal{G}} = \{M_{N,v}, v \in \mathcal{M}_K\}$, where:

$$M_{N,v}(T) = \sup_{x \in X_v \setminus \{0\}} \frac{P_v(T(x))}{N_v(x)}.$$

It is clear that $H_{\mathcal{L}_{\mathcal{F}}^{\mathcal{G}}}(T) \geq H_{\mathcal{F}_{\mathcal{G}}}(T)$, but in many cases the equality actually holds:

**Proposition 1.3.** Let $X$, $Y$, $\mathcal{F}$ and $\mathcal{G}$ be as above. Let $T \in \text{Hom}_K(X, Y)$ be injective. Then:

$$H_{\mathcal{F}_{\mathcal{G}}}(T) = \prod_{v \in \mathcal{M}_K} M_{N_v, p_v}(T)^{d_v}.$$

**Proof.** The argument is identical to the one used in [Ta2] Proposition 3.1, where the same result is proved for the special case $X = Y = K^n$ and $\mathcal{F} = \mathcal{G} = \mathcal{O}_{K^n}$, $\mathcal{E}$ being the canonical basis of $K^n$. One only needs to use Lemma 1.1 and Lemma 1.2 instead of Lemma 3.2 of [Ta2].

**Proposition 1.4.** Let $X$, $Y$, $\mathcal{F}$ and $\mathcal{G}$ be as above. Suppose $T$ is such that there exists $\gamma \in \mathbb{R}$ such that $\gamma H_{\mathcal{F}}(x) = H_{\mathcal{G}}(T(x))$ for all $x \in X \setminus \{0\}$. Then:

(a) $\prod_{v \in \mathcal{M}_K} M_{N_v, p_v}(T)^{d_v} = \gamma$.

(b) $P_v(T(x)) = M_{N_v, p_v}(T)N_v(x)$ for all $x \in X \setminus \{0\}$ and all $v \in \mathcal{M}_K$.

**Proof.** Since $H_{\mathcal{F}}$ is constant on the subspaces generated by $y$ and $\ker T$ for each $y \notin \ker T$, property (H2) implies that $T$ is injective. For ease of notation we set $M_v = M_{N_v, p_v}$. Then by Proposition 1.3 we have

$$\prod_{v \in \mathcal{M}_K} M_{N_v, p_v}(T)^{d_v} = H_{\mathcal{F}_{\mathcal{G}}}(T) = \gamma$$

proving (a). Next Lemma 1.2 implies that for all $x \in X \setminus \{0\}$:

$$\gamma = \prod_{v \in \mathcal{M}_K} \left( \frac{P_v(T(x))}{N_v(x)} \right)^{d_v}.$$

Suppose that there exists $x_0 \in X \setminus \{0\}$ which contradicts (b), i.e., one can find $w \in \mathcal{S}_T$ for which $P_w(T(x_0)) < M_w(T)N_w(x_0)$. Then (a) yields:

$$\gamma = \prod_{v \in \mathcal{S}_T} \left( \frac{P_v(T(x))}{N_v(x)} \right)^{d_v} \leq \prod_{v \in \mathcal{S}_T} M_v(T)^{d_v} \left( \frac{P_w(T(x_0))}{N_w(x)} \right)^{d_w} < \prod_{v \in \mathcal{S}_T} M_v(T)^{d_v} = \gamma,$$

which is a contradiction. 

**Corollary 1.5.** Let $\mathcal{F} = \{N_v, v \in \mathcal{M}_K\}$ and $\mathcal{G} = \{P_v, v \in \mathcal{M}_K\}$ be two adelic norms on $X$. Then:

(a) The ratio $H_\mathcal{F} / H_\mathcal{G}$ is constant on $X \setminus \{0\}$ if and only if the ratio $N_v / P_v$ is constant on $X_v \setminus \{0\}$ for all $v$’s.

(b) $H_\mathcal{F} = H_\mathcal{G}$ if and only if $N_v = \gamma_v P_v$, and $\prod_{v \in \mathcal{M}_K} \gamma_v^{d_v} = 1$.

**Proof.** Apply the proposition to the identity map.
2. Heights on finite étale \( K \)-Algebras

Let \( A \) be a \( K \)-algebra which is always assumed to be finite dimensional, commutative and with unit usually denoted by \( 1_A \). An adelic norm \( \mathcal{F} = \{ N_v, v \in \mathcal{M}_K \} \) on \( A \) is said to be compatible (with the algebra structure) if

(C1) \( N_v(ab) \leq N_v(a)N_v(b) \) for all \( a, b \in A_v \) and for all \( v \in \mathcal{M}_K \).

Therefore, given a compatible adelic norm \( \mathcal{F} \) on \( A \), the inequality

\[
(HC1) \quad H_F(ab) \leq H_F(a)H_F(b)
\]

holds for all \( a, b \in A \) such that \( ab \neq 0 \). We will denote by \( C(A) \) the set of compatible adelic norms on \( A \).

Suppose now that \( A \) is a finite étale \( K \)-algebra, i.e., that \( A \) is a product of finite extensions of \( K \). In [Ta1] we introduced the following family of \( v \)-adic norms: let \( (X_v, \mathcal{O}_{X_v}) \) be the affine \( K_v \)-scheme associated to the \( K_v \)-algebra \( A_v = A \otimes_K K_v \). Let \( a \mapsto \hat{a} \) denote the canonical isomorphism \( A_v \simeq \Gamma(X_v, \mathcal{O}_{X_v}) \). We then set

\[
N_v^A(a) = \sup_{x \in X_v} |\hat{a}(x)|_v
\]

where \( | \cdot |_v \) is the unique extension of \( | \cdot |_v \) to \( K_v(x) \).

**Lemma 2.1.** Let \( A \) be a finite étale \( K \)-algebra. The family \( \mathcal{F}_A = \{ N_v^A, v \in \mathcal{M}_K \} \) is a compatible adelic norm on \( A \).

**Proof.** In [Ta1] proposition 2.2(b) it was shown that property (C1) holds for \( \mathcal{F}_A \). Thus it only remains to check that there exists an \( \mathcal{O}_K \)-lattice \( \Lambda \subset A \) such that \( N_v^A = N_{\Lambda_v} \) for almost all \( v \in \mathcal{M}_K \). Assume first that \( A \) is simple so \( A = F \) is a finite extension of \( K \). Our choice of \( \mathcal{O}_K \)-lattice is \( \Lambda = \mathcal{O}_F \). Let us denote by \( S_0 \subset \mathcal{M}_K^\times \) the subset formed by the absolute values whose associated prime ideal ramifies in the extension \( F \supset K \). We want to show that for all \( v \in \mathcal{M}_K^\times \setminus S \) we have \( N_v^F = N_{\Lambda_v} \). A moment of reflection shows that

\[
N_v^F(a) = \sup_{w \in \mathcal{M}_K^\times} |a|_w
\]

where \( \mathcal{M}_K^\times = \{ w \in \mathcal{M}_K \mid | \cdot |_w = | \cdot |_v \} \). Given \( a \in F \) and \( \lambda \in K_v^\times \) note that \( \gamma \lambda \) belongs to \( \Lambda_v \) if and only if \( |\gamma \lambda|_v \leq 1 \) for all \( w \in \mathcal{M}_K^\times \). Thus \( \gamma \lambda \in \Lambda_v \) implies \( |a|_w \leq |\gamma^{-1}|_v \) which yields

\[
|a|_w \leq \inf_{\gamma \in \mathcal{K}_v^\times, \gamma \in \Lambda_v} |\gamma|_v^{-1} = N_{\Lambda_v}(a).
\]

The above inequality yields \( N_v^F(a) \leq N_{\Lambda_v}(a) \). Note that this holds also for the ramified primes. It is in the proof of the reverse inequality that we use the assumption on \( v \). So let us assume that \( v \) is unramified in the extension \( K \subset F \). Then the value groups of \( | \cdot |_v \) and \( | \cdot |_w \) are the same for all \( w \in \mathcal{M}_K^\times \). Therefore, there exists \( \gamma \in \mathcal{K}_v^\times \) such that \( |\gamma|_v = (\sup_{w \in \mathcal{M}_K^\times} |a|_w)^{-1} \). It follows that \( |\gamma a|_w \leq 1 \) for all \( w \in \mathcal{M}_K^\times \). Thus \( \gamma a \in \Lambda_v \), yielding \( N_{\Lambda_v}(a) \leq N_v^F(a) \). For the general case it suffices to recall that \( N_v^A \) behaves well under direct products, i.e., if \( A \simeq \prod_{i=1}^f A_i \) then \( N_v^A(a) = \sup_{1 \leq i \leq f} N_v^{A_i}(\pi_i(a)) \), where \( \pi_i \) is the canonical projection (see [Ta1] proposition 1.1(c) for a proof). Thus, if we decompose \( A \) as the product of simple \( K \)-algebras as above, then the sought \( \mathcal{O}_K \)-lattice is the product of the \( \mathcal{O}_{A_i} \).

For ease of notation we set \( H_A = H_{\mathcal{F}_A}, H_A^{op} = H_{\mathcal{F}_A}^{op} \). The function \( H_A \) enjoys the following properties (see [Ta1] Proposition 2.2 and corollary 2.4 for a proof):

(A1) \( H_A(a) \geq 1; \)
(A2) \( H_A(a^k) = (H_A(a))^k; \)
(A3) If \( \varphi : A \rightarrow B \) is \( K \)-algebra isomorphism, then \( H_A = H_B \circ \varphi. \)
Remark. The adelic norm $F_A$ has also the property that $N_v(1_A) = 1$ for all $v \in \mathcal{M}_K$, which means that $(\mathcal{A}_v, N_v^A)$ is a Banach algebra (either real, complex or non-archimedean) for all $v \in \mathcal{M}_K$. A compatible adelic norm enjoying this property will be called a Banach adelic norm.

Example. Let $K \subset F$ be a finite extension. Let $R \subseteq O_F$ be an order, i.e., $R$ is a subring of $O_F$ containing $1$ which is also a $O_K$-lattice. Let $R_v = R \otimes_{O_F} O_v$, then $F_R = \{N_{R_v} : v \in \mathcal{M}_K \} \cup \{N_v^F : v \in \mathcal{M}_K^\infty \}$, is an adelic norm on $F$. For each $v \in \mathcal{M}_K$, $R_v$ is a subring of $F \otimes_K K_v$. Thus for every pair of elements $a, b \in F \otimes_K K_v$ we have

$$\{\lambda \in K_v^\times \mid \lambda ab \in R_v \} \supseteq \{\gamma \in K_v^\times \mid \gamma = \mu \nu \text{ and } \mu \in R_v, \nu b \in R_v \}.$$ 

Therefore $N_{R_v}(ab) \leq N_{R_v}(a) N_{R_v}(b)$. Since $1_A$ belongs to $R$ we have $N_v(1_A) \leq 1$ for all $v \in \mathcal{M}_K$. Moreover, the inclusion $R \subseteq O_F$ yields $H_{O_F}(a) \leq H_F(a)$. From the proof of Lemma 2.1 we know that $H_F(a) \leq H_{O_F}(a)$, so $H_F(a) \leq H_R(a)$. In particular $H_F(1_A) \geq 1$ so we must have $N_{R_v}(1_A) \leq 1$ for all $v \in \mathcal{M}_K$, i.e., $F_R$ is a Banach adelic norm. Conversely, suppose that $F = \{N_v : v \in \mathcal{M}_K \}$ is a Banach adelic norm on the finite étale $K$-algebra $A$. Recall that the $O_K$-lattice $\Lambda_F$ associated to $F$ can be computed as (again by Theorem V.2.2 of [We2])

$$\Lambda_F = \bigcap_{v \in \mathcal{M}_K^\infty} \left\{a \in A_v \mid N_v(a) \leq 1 \right\} \cap A.$$

Now, for $v \in \mathcal{M}_K^\infty$, given $a, b \in \Lambda_F$ we have $N_v(ab) \leq N_v(a) N_v(b) \leq 1$. Therefore $\Lambda_F$ is a subring of $A$. Moreover, $F$ is Banach adelic norm and so $N_v(1_A) = 1$ for all $v \in \mathcal{M}_K$, therefore $1_A \in \Lambda_F$ and so $\Lambda_F$ is an order in $A$.

To better exploit the canonical nature of $H_\Lambda$ it is convenient to bring into the picture the spectral height. In order to do so we need some definitions: first the spectral height $H_s : \operatorname{End}_K(A) \rightarrow \mathbb{R}$ is defined as follows: $H_s(T) = 1$ if $T$ is nilpotent, otherwise we set

$$H_s(T) = \prod_{v \in \mathcal{M}_K} \rho_v(T)^{d_v}.$$ 

Here $\rho_v(T)$ denotes the $v$-adic spectral radius of $T$. Precisely

$$\rho_v(T) = \sup_{\lambda \in \text{sp}(T)} |\lambda|_{K_v}(\lambda),$$

where $\text{sp}(T)$ is the set of characteristic roots of $T$ in an algebraic closure of $K_v$, and $| \cdot |_{K_v}(\lambda)$ is the unique extension of $| \cdot |_v$ to $K_v(\lambda)$. Second, to any adelic norm $F$ we can also associate the height function $H_F^{op}$ on $\operatorname{End}_K(A)$, defined in section 1. Moreover, the adelic norm $L_F^\infty$, also defined in section 1, gives rise to another height function, $H_{L_F^\infty}$, on $\operatorname{End}_K(A)$. In the following, given $a \in A$ we let $m_a : A \rightarrow A$ denote the multiplication $-b y -a$ map.

Lemma 2.2. Let $A$ be a finite dimensional commutative $K$-algebra. Suppose $F$ is a compatible adelic norm on $A$. Then for all $a \in A$,

$$H_F^{op}(m_a) \leq H_{L_F^\infty}(m_a) \leq H_F(a) \leq H_F^{op}(m_a) H_F(1_A).$$

In particular if $H_F(1_A) = 1$ we have $H_F^{op}(m_a) = H_{L_F^\infty}(m_a) = H_F(a)$.

Proof. By definition of $H_F^{op}$ we have:

$$H_F(1_A) H_F^{op}(m_a) \geq H_F(m_a(1_A)) = H_F(a).$$

On the other hand, the inequalities $H_{L_F^\infty}(m_a) \geq H_F^{op}(m_a)$ holds in an even greater generality as remarked before Proposition 1.3. Since $L_F^\infty = \{M_{N_v} : v \in \mathcal{M}_K \}$ and $M_{N_v}(m_a) \leq N_v(a)$, it follows that $H_{L_F^\infty}(m_a) \leq H_F(a)$, completing the proof of the lemma.

\textsuperscript{2}For a comprehensive treatment of spectral theories for a non-archimedean Banach algebra the reader is referred to [Be].
Theorem 2.4. Let $A$ be a finite étale $K$-algebra. Then
(a) $\lim_{k \to \infty} H_F'(a^k)_A = H_A(a)$, for all $F \in \mathcal{C}(A)$ and all $a \in A$.
(b) $H_A(a) = \min_{F \in \mathcal{C}(A)} H_F(a)$, for all $a \in A$.

Proof. (a) We have:
\[ \lim_{k \to \infty} H_F'(a^k)_A = \lim_{k \to \infty} H_F'(a^k)_A \]
\[ = H_A(m_a) \quad \text{by (GBF)} \]
\[ = H_A(a) \quad \text{by Lemma 2.3} \]

(b) Since $F_A \in \mathcal{C}(A)$ we only have to prove that $H_A(a) \leq H_F(a)$ for all $F \in \mathcal{C}(A)$. But this follows from (a) and (HCl). \[\blacksquare\]

In the following $[A]$ denotes the $K$-vector space underlying $A$ and $A^\times$ denotes the set of invertible elements of $A$.

Theorem 2.5. Let $A$ and $B$ be two finite étale $K$-algebras. Suppose that $T \in \text{Hom}_K([A],[B])$ is surjective and such that $H_B(T(a)) = H_A(a)$ for all $a \in A$. Then $A$ and $B$ are isomorphic as $K$-algebras. Moreover, if we let $b = T(1_A)$ then $m_{b^{-1}} \circ T$ is a $K$-algebra isomorphism.

Proof. We can apply Proposition 1.4 with $\gamma = 1$ finding that $T$ is a bijection and that for each $v \in M_k$ there exists $\gamma_v \in \mathbb{R}$ such that $\gamma_v N_v^A(a) = N_v^B(T(a))$. In particular for $v \in M_k^\times$ we have that the $K_v$-linear map $\gamma_v^{-1}T$ is an isometry between $(A_v,N_v^A)$ and $(A_v,N_v^B)$. Now $A_v$ (respectively $B_v$) is either a complex Banach algebra or a real Banach algebra or a real function algebra (see [Ta1]) and they have to be of the same type simultaneously, by the existence of the linear isometry. For each of these cases is an appropriate version of the Banach-Stone theorem\textsuperscript{4} which we can apply, as in [Ta1] Proposition 1.3 and Corollary 1.5, to this setting. We thus find that $\gamma_v^{-1}T(1_A) = c$ is invertible and that $\gamma_v^{-1}m_{c^{-1}} \circ T$ is a $K_v$-algebra isomorphism of $A_v$ into $B_v$. Now it is straightforward to verify that $b = T(1_A) = \gamma_v c$ is invertible and that $m_{b^{-1}} \circ T$ is an isomorphism of $K$-algebras. \[\blacksquare\]

To conclude the paper we state, as a corollary, a special case of the above proposition which generalizes Theorem 3.8 of [Ta1] by removing the condition of $A$ being isotypical (i.e., of the form $B^n$ for some simple $K$-algebra $B$).

Corollary 2.6. Let $A$ be a finite étale $K$-algebras. Suppose $T \in \text{End}_K([A])$ is such that $H_A(T(a)) = H_A(a)$ for all $a \in A$, then $b = T(1_A)$ is invertible and $m_{b^{-1}} \circ T$ is a $K$-algebra automorphism.

\textsuperscript{3}In [Ta2] the operator height associated to an adelic norm $F$ was defined including 0 in the set over which we take the superior, see section 3 of [Ta2]. Now it is easy to verify that for every $T \in \text{End}_K(X)$ that is not nilpotent we have $H_F'(T^k) \geq 1$ for all $k \geq \dim_x X$. Therefore the discrepancy between the two definitions does not have any bearing on the limit formula.

\textsuperscript{4}For the precise statement and the proof see [BD] 1.5.8 and 1.13.7 for the case of real or complex Banach Algebras, and [KL] Theorem 5.1.4 for the real function algebras case.
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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI L’AQUILA, VIA VETOIO snc, CUPITO, 67040, ITALY
E-mail address: valerio@dmat.univaq.it, valerio@matm3.mat.uniroma3.it