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On canonical heights on Endomorphism rings over global function fields

Abstract. We present a construction of a canonical height on the endomorphisms ring of a finite dimensional vector space over a global function field. We also prove a limit formula analogous to the Tate's formula defining the canonical heights on abelian varieties.

Keywords. Heights, adelic vector bundles, endomorphism rings.

Mathematics Subject Classification (2010): 11G.

1 - Introduction

Canonical heights appeared in the mid 50's independently in the work of A. Néron (cf. [6]) and J. Tate (unpublished, see [7] where Tate's method first appeared in print) on abelian varieties. Since then canonical heights have been constructed and studied in several settings, such as Drinfeld modules (cf. [3]), varieties with morphism (cf. [2]), endomorphism rings of vector space (cf. [9]) and finite sets of matrices in $\mathrm{GL}_d(\overline{\mathbb{Q}})$ (cf. [1]), just to mention a few. While Néron's construction is local, i.e., he defines the canonical height as a product of local factors, Tate's method is global and is more suited to be applied in different contexts. We briefly recall Tate's method. Let A be an abelian variety defined over a number field $\mathbf{F}$. Let $[2] : A \to A$ denote the duplication map on A. Let c be a linear equivalence class of divisors on A containing a symmetric ample divisor D and choose $\phi : A \to \mathbb{P}_N$ so that D is the pull back of a hyperplane

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via ϕ. Let $h_{nw} : \mathbb{P}_N(\overline{\mathbb{Q}}) \to \mathbb{R}$ denote the absolute logarithmic Northcott-Weil height and set $h_\phi(P) := h_{nw}(P)$. Then the limit

$$\widehat{h}_c(P) := \lim_{m\to\infty} \frac{1}{4^m} h_\phi([2^m]P) \tag{1}$$

exists for each $P \in A(\overline{\mathbb{Q}})$ and is independent of ϕ. The function $\widehat{h}_c$ is called the *canonical height* associated to c. Not only is $\widehat{h}_c$ independent of ϕ but also of the choice of D. Moreover, if we perform the limit in (1) using any other multiplication map we obtain the same function (cf. [8, Lemma 3.1.]). The equality between Néron's and Tate's definitions is achieved as follows: first it is shown that there is a unique function $\tilde{h}$ satisfying the following two properties:

a) h_ϕ and $\tilde{h}$ are in the same class modulo bounded functions;

b) $\tilde{h}([2]P) = 4\tilde{h}(P)$;

then, it is shown that both Neron's and Tate's height satisfy both a) and b).

Let E be a finite dimensional vector space over a global function field $\mathbf{k}$. In this paper we present a construction of a canonical height (called the *spectral height*) on $\mathrm{End}_{\mathbf{k}}(E)$. Our construction is closer to Neron's one (albeit more elementary) as we define the spectral height as a product of local factors (the local spectral radii, cf. section 5). We then prove a limit formula (cf. Theorem 5.1), analogous to (1), relating the spectral height and heights on $\mathrm{End}_{\mathbf{k}}(E)$ associated to adelic vector bundle over $\mathbf{k}$.

The paper is organised as follows: in section 2 we introduce adelic vector bundles and their associated heights. In section 3 we prove an interesting auxiliary inequality between the height relative to a sub-bundle $\overline{D}$ and the minimum of the height on a coset of D on E, cf. Proposition 3.1. Section 4 deals with heights on $\mathrm{End}_{\mathbf{k}}(E)$, while the spectral height is introduced in section 5 where we prove our main theorem (cf. Theorem 5.1).

Notation Throughout this paper $\mathbf{k} \supset \mathbb{F}_p(t)$ is a global function field of characteristic $p > 0$. We let $\mathcal{M}_{\mathbf{k}}$ be the set of places of $\mathbf{k}$. Given $v \in \mathcal{M}_{\mathbf{k}}$ we denote by $\mathbf{k}_v$ the completion of $\mathbf{k}$ with respect to v and by $\mathbf{C}_v$ the completion of the algebraic closure of $\mathbf{k}_v$. The maximal compact subring of $\mathbf{k}_v$ is denoted by $\mathcal{O}_v$ and we let $n_v = [\mathbf{k}_v : \mathbb{F}_p(t)_\omega]$, where ω is the restrictionof v to $\mathbb{F}_p(t)$. For each $v \in \mathcal{M}_{\mathbf{k}}$ we fix an absolute value $|\cdot|_v$, in the class of v, by requiring that $|a|_v^{n_v}$ coincides with the modulus, with respect to the Haar measure on the locally compact group $\mathbf{k}_v$, of the automorphism $x \mapsto ax$. With these normalizations the product formula reads: $\prod_{v\in\mathcal{M}_{\mathbf{k}}} |a|_v^{n_v} = 1$. Regarding vector space we will employ the following notation: given a vector space E, $E^\times$ denotes the set of non-zero vectors in E and $< \mathbf{e}_1, \ldots, \mathbf{e}_r >$ denotes the subspace generated by $\mathbf{e}_1, \ldots, \mathbf{e}_r \in E$. Lastly if E is a vector space over $\mathbf{k}$, we set $E_v = E \otimes \mathbf{k}_v$

2 - Adelic vector bundles

In this paper we use heights associated to adelic vector bundles. Adelic vector bundles have been recently introduced and studied by É. Gaudron (see

[4] and [5]). An adelic vector bundle $\overline{E}=(E,\{\|\cdot\|_{\overline{E},v}\}_{v\in\mathcal{M}_{\mathbf{k}}})$ (over spec $\mathbf{k}$ or over $\mathbf{k}$ for short) of dimension n consists of the following data (cf. [4, Definition 2.1]): a $\mathbf{k}$-vector space E of dimension n (called the *support* of $\overline{E}$) and a family of ultrametric norms $\|\cdot\|_{\overline{E},v}:E\otimes_{\mathbf{k}}\mathbf{C}_v\to\mathbb{R}$, satisfying the following conditions:

1) There exists a $\mathbf{k}$-basis $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ of E over $\mathbf{k}$, such that for all but finitely many $v\in\mathcal{M}_{\mathbf{k}}$ we have

$$\left\|\sum_{i=1}^n \alpha_i\mathbf{e}_i\right\|_{\overline{E},v}=\max_{1\le i\le n}\{|\alpha_i|_{\mathbf{C}_v}\}\quad\forall(\alpha_1,\ldots\alpha_n)\in\mathbf{C}_v$$

where $|\cdot|_{\mathbf{C}_v}$ is the unique extension of $|\cdot|_v$ to $\mathbf{C}_v$;

2) let $\mathrm{Gal}(\mathbf{C}_v/\mathbf{k}_v)$ denote the set of continuous automorphism of $\mathbf{C}_v$ which leaves the elements of $\mathbf{k}_v$ fixed. Then $\|\cdot\|_{\overline{E},v}$ is invariant under the standard action of $\mathrm{Gal}(\mathbf{C}_v/\mathbf{k}_v)$ on $E\otimes_{\mathbf{k}}\mathbf{C}_v$.

An adelic vector bundle is called *v-pure* if $\|x\|_{\overline{E},v}$ belongs to the value set of $|\cdot|_v$ for all $x\in E$ and it is called *pure*[1] if it is v-pure for all $v\in\mathcal{M}_{\mathbf{k}}$. Let $\overline{E}=(E,\{\|\cdot\|_{\overline{E},v}\}_{v\in\mathcal{M}_{\mathbf{k}}})$ be a pure adelic vector bundle over $\mathbf{k}$. It is possible to perform several algebraic constructions with adelic vector bundles, such as exterior powers, symmetric powers and so on. We refer the reader to [4, Section 3.3] for details and briefly recall the few that we need. The absence of archimedean places simplifies some definitions. We say that $\overline{D}$ is an adelic sub-bundle of $\overline{E}$ if $D\subset E$, and for every v the norms of $\overline{D}$ are the restriction of those of $\overline{E}$. If $\overline{D}\subset\overline{E}$ is a sub-bundle then E/D inherits an adelic vector bundle structure (denoted by $\overline{E}/\overline{D}$) where the norms are the quotient norms of those of $\overline{E}$. If $\overline{F}=(F,\{\|\cdot\|_{\overline{F},v}\}_{v\in\mathcal{M}_{\mathbf{k}}})$ is another adelic vector bundle over $\mathbf{k}$, we set

$$\|T\|_{\overline{E},\overline{F},v}:=\sup_{\mathbf{e}\in E_v^\times}\frac{\|T(\mathbf{e})\|_{\overline{F},v}}{\|\mathbf{e}\|_{\overline{E},v}}$$

for all $T\in Hom_{\mathbf{k}}(E,\,F)\otimes_{\mathbf{k}}\mathbf{C}_v$ and all $v\in\mathcal{M}_{\mathbf{k}}$. It is straightforward to verify that $Hom_{\mathbf{k}}(\overline{E},\,\overline{F})=(Hom_{\mathbf{k}}(E,\,F),\{\|\cdot\|_{\overline{E},\overline{F},v}\}_{v\in\mathcal{M}_{\mathbf{k}}})$ is an adelic vector bundle having $Hom_{\mathbf{k}}(E,\,F)$ as support. Note that if $\overline{F}$ is the trivial bundle this gives the structure of adelic vector bundle to E^* the dual of E. Next $\overline{E}\otimes_{\mathbf{k}}\overline{F}$ is the adelic vector bundle having support $E\otimes_{\mathbf{k}}F$ and norms induced by the isomorphism $E\otimes_{\mathbf{k}}F\simeq Hom_{\mathbf{k}}(E^*,\,F)$. Lastly we denote by $\overline{\bigwedge^m E}=(\bigwedge^m E,\|\cdot\|_{\overline{\wedge^m E},v})$ the adelic vector bundle having $\bigwedge^m E$ as support and whose norms are the quotient norms of $\overline{E}^{\otimes m}$.

Let $\overline{E}=(E,\{\|\cdot\|_{\overline{E},v}\}_{v\in\mathcal{M}_{\mathbf{k}}})$ be an adelic vector bundle over spec $\mathbf{k}$. The height function $H_{\overline{E}}:E\to\mathbb{R}$, relative to $\overline{E}$ is defined by setting:

[1] It is not difficult to prove that there is a one to one correspondence between pure adelic vector bundles over $\mathbf{k}$ having E as support and coherent systems of k_v-lattices belonging to E as defined by A. Weil in [11].

$$(2) \qquad H_{\overline{E}}(\mathbf{e}) = \prod_{v\in\mathcal{M}_{\mathbf{k}}} \|\mathbf{e}\|_{\overline{E},v}^{n_v}$$

for all $\mathbf{0} \neq \mathbf{e}\in E$. As usual we set $H_{\overline{E}}(\mathbf{0}) = 1$. It follows from the product formula that $H_{\overline{E}}$ is constant on one dimensional subspaces of E. The height of a subspace $D \subset E$, is defined as follows: choose a basis $\mathbf{d}_1, \ldots, \mathbf{d}_m$ of D over $\mathbf{k}$ and set $H_{\overline{E}}(D) = H_{\overline{\bigwedge^m E}}(\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_m)$, which does not depend on the choice of the basis by the product formula (see [5, Introduction]). Lastly if B is a subset of E we set

$$\lambda_1^{\overline{E}}(B) = \inf_{\mathbf{x}\in B} H_{\overline{E}}(\mathbf{x}).$$

3 - Comparison between $H_{\overline{E}/\overline{D}}([\mathbf{e}]_D)$ and $\lambda_1^{\overline{E}}([\mathbf{e}]_D)$

Let D be a subspace of E. If $\mathbf{e}\in E - D$ we denote by $< D, \mathbf{e} >$ the subspace generated by D and $\mathbf{e}$, and by $[\mathbf{e}]_D$ the coset of $\mathbf{e}$ modulo D. In this section we obtain a comparison result (Proposition 3.1 below) for $H_{\overline{E}/\overline{D}}\big([\mathbf{e}]_D\big)$ and $\lambda_1^{\overline{E}}([\mathbf{e}]_D)$. One of the main consituents of the proof of Proposition 3.1 is the uniform Sigel's lemma recently proved by É. Gaudron in [5]. The lower bound obtained in Proposition 3.1 is a key ingredient for the results of the next section. The quotient height $H_{\overline{E}/\overline{D}}$ is defined as

$$\begin{aligned} H_{\overline{E}/\overline{D}} &: E/D \longrightarrow \mathbb{R} \\ &[\mathbf{e}]_D \longmapsto H_{\overline{E}/\overline{D}}([\mathbf{e}]_D) = \prod_{v\in\mathcal{M}_{\mathbf{k}}} \inf_{\mathbf{e}'\in[\mathbf{e}]_{D_v}} \|\mathbf{e}'\|_{\overline{E},v}^{n_v} \end{aligned}$$

if $[\mathbf{e}]_D \neq [0]_D$, as usual we set $H_{\overline{E}/\overline{D}}([0]_D) = 1$.

Proposition 3.1. *Let $\overline{E}$ be a pure adelic vector bundle over $\mathbf{k}$. Let D be a subspace of dimension d and suppose $\mathbf{e}\in E - D$. Then*

$$(3) \qquad \frac{\lambda_1^{\overline{E}}(< D, \mathbf{e} >)^d}{q^{2(d+1)g(\mathbf{k})} H_{\overline{E}}(D)} \lambda_1^{\overline{E}}([\mathbf{e}]_D) \leq H_{\overline{E}/\overline{D}}([\mathbf{e}]_D) \leq \lambda_1^{\overline{E}}([\mathbf{e}]_D),$$

where q is the cardinality of the constant field of $\mathbf{k}$ and $g(\mathbf{k})$ is the genus of $\mathbf{k}$.

Proof. The inequality $H_{\overline{E}/\overline{D}}([\mathbf{e}]_D) \leq \lambda_1^{\overline{E}}([\mathbf{e}]_D)$ follows immediately from the definitions. To prove the other inequality we need the following lemma which gives a decomposition for the heights of a subspace[2].

Lemma 3.1. *Let $\overline{E} = (E, \{\|\cdot\|_{\overline{E},v}\}_{v\in\mathcal{M}_{\mathbf{k}}})$ be a pure adelic vector bundle. Let $\overline{D} \subset \overline{E}$ be a sub-bundle of dimension d. Then*

$$H_{\overline{E}}(< D, \mathbf{e} >) = H_{\overline{E}}(D) H_{\overline{E}/\overline{D}}([\mathbf{e}]_D)$$

[2]The same result, although stated in terms of orthogonal projection was first proven over number fields for the standard L^2-height by J. Vaaler , see [10, Lemma 4]

P r o o f. Let $\mathbf{d}_1, \ldots, \mathbf{d}_d$ be a basis for D. Clearly it suffices to show that

$$\|\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_d \wedge \mathbf{e}\|_{\overline{\wedge^{d+1} E}, v} = \|\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_d\|_{\overline{\wedge^d E}, v} \inf_{\mathbf{e}' \in [\mathbf{e}]} \|\mathbf{e}'\|_{\overline{E}, v}$$

for all $v \in \mathcal{M}_{\mathbf{k}}$. Fix $v \in \mathcal{M}_{\mathbf{k}}$. Since $\overline{E}$ is pure we can find, by [11, Ch.II-2 Thm.1], a basis $\mathbf{f}_1, \ldots, \mathbf{f}_n$ of E_v such that

(i) $\|\gamma_1 \mathbf{f}_1 + \cdots + \gamma_n \mathbf{f}_n\|_{\overline{E}, v} = \sup_{1 \leq i \leq n} |\gamma_i|_v$, for all $\gamma_1, \ldots, \gamma_n \in \mathbf{k}_v$

(ii) $\mathbf{d}_{k+1} \in < \mathbf{f}_n, \ldots, \mathbf{f}_{n-k} >$ for all $k = 0 \ldots, d-1$ and $\mathbf{e} \in < \mathbf{f}_n, \ldots, \mathbf{f}_{n-d} >$.

Write $\mathbf{d}_k = \sum_{i=1}^{k} \alpha_{ki} \mathbf{f}_{n-i+1}$ for $k = 1, \ldots, d$ and $\mathbf{e} = \sum_{i=1}^{d+1} \beta_i \mathbf{f}_{n-i+1}$, then an easy calculation shows that (ii) implies that

$$\|\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_d\|_{\overline{\wedge^d E}, v} = |\alpha_{11} \alpha_{22} \cdots \alpha_{dd}|_v$$

and

$$\|\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_d \wedge \mathbf{e}\|_{\overline{\wedge^{d+1} E}, v} = |\alpha_{11} \alpha_{22} \cdots \alpha_{dd} \beta_{d+1}|_v.$$

It remains to show that $|\beta_{d+1}|_v = \inf_{\mathbf{e}' \in [\mathbf{e}]} \|\mathbf{e}'\|_{\overline{E}, v}$. To this end, note that by construction $[\mathbf{e}]_{D_v} = [\beta_{d+1} \mathbf{f}_{n-d}]_{F_v}$ and hence $\inf_{\mathbf{e}' \in [\mathbf{e}]} \|\mathbf{e}'\|_{\overline{E}, v} \leq |\beta_{d+1}|_v$. On the other hand any $\mathbf{d} \in D_v$ can be written as $\sum_{i=1}^{d} \gamma_i \mathbf{f}_{n-i+1}$ and so

$$\|\mathbf{e} - \mathbf{d}\|_{\overline{E}, v} = \|\sum_{i=1}^{d+1} \beta_i \mathbf{f}_{n-i+1} - \sum_{i=1}^{d} \gamma_i \mathbf{f}_{n-i+1}\|_{\overline{E}, v} \geq |\beta_{d+1}|_v.$$

□

Now we can quickly finish the proof of Proposition 3.1. By the uniform Sigel's lemma for global function fields, see [5, Cor. 3.3], there exists $\mathbf{f}$ belonging to $< D, \mathbf{e} >$ but not belonging to D such that

$$H_{\overline{E}}(\mathbf{f}) \leq \frac{q^{2(d+1)g(\mathbf{k})} H_{\overline{E}}(< D, \mathbf{e} >)}{\lambda_1^{\overline{E}}(< D, \mathbf{e} >)^d}.$$

By definition $\lambda_1^{\overline{E}}([\mathbf{e}]_D) \leq H_{\overline{E}}(\mathbf{f})$ and hence Lemma 3.1, yields

$$\lambda_1^{\overline{E}}([\mathbf{e}]_D) \leq \frac{q^{2(d+1)g(\mathbf{k})} H_{\overline{E}}(D) H_{\overline{E/D}}([\mathbf{e}]_D))}{\lambda_1^{\overline{E}}(< D, \mathbf{e} >)^d}$$

□

4 - Heights of linear transformations

Let us start by recalling the definition of the operator height for linear transformations and compare it with $H_{\overline{E}, \overline{F}} := H_{Hom_{\mathbf{k}}(\overline{E}, \overline{F})}$. So let $\overline{E}$ and $\overline{F}$ be two adelic vector bundles over $\mathbf{k}$-vector spaces. Given $T\in Hom_{\mathbf{k}}(F,\, E)$, set :

$$H^{op}_{\overline{E},\overline{F}}(T) := \sup_{\mathbf{e}\in E}\frac{H_{\overline{F}}\big(T(\mathbf{e})\big)}{H_{\overline{E}}(\mathbf{e})} = \sup_{[\mathbf{e}]_D\in E/D}\frac{H_{\overline{F}}\big(T(\mathbf{e})\big)}{\lambda_1^{\overline{E}}(D+\mathbf{e})}.$$

where $D=\ker(T)$. The function $H^{op}_{\overline{E},\overline{F}}$ is called the *operator height* on $Hom_{\mathbf{k}}(E,\, F)$ associated to $\overline{E}$ and $\overline{F}$. If $\overline{E}=\overline{F}$ we will use $H^{op}_{\overline{E}}$ (respectively $H_{\overline{E}}$) instead of $H^{op}_{\overline{E},\overline{E}}$ (respectively $H_{\overline{E},\overline{E}}$). The main goal of this section is to prove a comparison result between $H^{op}_{\overline{E},\overline{F}}$ and $H_{\overline{E},\,\overline{F}}$, which will be used in the proof of Theorem 5.1. Clearly $H^{op}_{\overline{E},\overline{F}}(T)\leq H_{\overline{E},\,\overline{F}}(T)$, so our next objective is to prove a reverse inequality where, for non invertible linear transformations, some arithmetic constants, such as the height of the kernel, will appear, see Proposition 4.2. We start with a preparatory result that not only establishes a useful alternative description for $H_{\overline{E},\,\overline{F}}$ but also proves that $H_{\overline{E},\,\overline{F}}(T)=H^{op}_{\overline{E},\overline{F}}(T)$ if T is an injective linear transformation.

Proposition 4.1. *Let $\overline{E}$ and $\overline{F}$ be pure adelic vector bundles over $\mathbf{k}$. Given T in $Hom_{\mathbf{k}}(E,\, F)$, set $D=\ker T\subset E$. Then:*

$$H_{\overline{E},\,\overline{F}}(T)=\sup_{[\mathbf{e}]\in E/D}\frac{H_{\overline{F}}\big(T(\mathbf{e})\big)}{H_{\overline{E}/\overline{D}}([\mathbf{e}]_D)}.$$

In particular if T is injective we have $H_{\overline{E},\,\overline{F}}(T)=H^{op}_{\overline{E},\overline{F}}(T)$.

Proof. Clearly

$$H_{\overline{E},\,\overline{F}}(T)=\prod_{v\in\mathcal{M}_{\mathbf{k}}}\sup_{\mathbf{e}\in E_v^{\times}}\frac{\|T(\mathbf{e})\|^{n_v}_{\overline{F},v}}{\|\mathbf{e}\|^{n_v}_{E,v}}\geq\sup_{[\mathbf{e}]_D\in\overline{E}/\overline{D}}\frac{H_{\overline{F}}\big(T(\mathbf{e})\big)}{H_{\overline{E}/\overline{D}}([\mathbf{e}]_D)}.$$

To prove the reverse inequality we need the following:

Lemma 4.1. *Under the hypotheses of Proposition 4.1 there exists a finite set of places $\mathcal{S}\subset\mathcal{M}_{\mathbf{k}}$ and a subspace $G\subset E$ of dimension equal to the rank of T such that for all $v\notin\mathcal{S}$ we have*

(a) $\inf_{\mathbf{g}'\in[\mathbf{g}]_{D_v}}\|\mathbf{g}'\|_{\overline{E},v}=\|\mathbf{g}\|_{\overline{E},v}$ *for all* $\mathbf{g}\in G_v$.

(b) $\|T(\mathbf{g})\|_{\overline{F},v}=\|\mathbf{g}\|_{E,v}$ *for all* $\mathbf{g}\in G_v$.

Proof. From the definition of an adelic vector bundle and the fact that we are proving a statement for all but finitely many places it follows that we can assume that $\overline{E}=(\mathbf{k}^n,\{\|\cdot\|_v\}_{v\in\mathcal{M}_{\mathbf{k}}})$, $\overline{F}=(\mathbf{k}^m,\{\|\cdot\|_v\}_{v\in\mathcal{M}_{\mathbf{k}}})$, where $\|\cdot\|_v$ is the sup norm on $\mathbf{k}_v^n$ and $\mathbf{k}_v^m$. If $m=n$ and T is invertible (a) is trivial and (b) is equivalent to say that an invertible $n\times n$ matrix with coefficients in $\mathbf{k}$ actually belongs to $\mathrm{GL}_n(\mathcal{O}_v)$ for all but finitely many $v\in\mathcal{M}_{\mathbf{k}}$. In general let $r=\mathrm{rank}(T)$, and choose ϕ to be an automorphism of $\mathbf{k}^n$ such that $\ker T=\phi(U)$ where U is the subspace generated the last $n-r$ vectors of the standard basis of $\mathbf{k}^n$. Moreover choose ψ to be an automorphism of $\mathbf{k}^m$ mapping $W = \mathrm{Im}(T)$ onto the subspace generated by the first r vectors of the standard basis of $\mathbf{k}^m$. Finally we let G be the image via ϕ of the subspace generated by the first r vectors of the standard basis of $\mathbf{k}^n$. Since ϕ is invertible there exists a finite set $\mathcal{S}_\phi \subset \mathcal{M}_\mathbf{k}$ such that ϕ preserves $\|\cdot\|_v$ for all $v \notin \mathcal{S}_\phi$. For $\mathbf{g} \in G_v$ and $v \notin \mathcal{S}_\phi$, we have:

$$\inf_{\mathbf{g}' \in [\mathbf{g}]_{D_v}} \|\mathbf{g}'\|_v = \inf_{\mathbf{d} \in D_v} \|\phi^{-1}(\mathbf{g}) - \phi^{-1}(\mathbf{d})\|_v = \inf_{\mathbf{u} \in U_v} \|\phi^{-1}(\mathbf{g}) - \mathbf{u}\|_v = \|(\mathbf{g})\|_v$$

proving (a). To prove (b) let $\mathcal{S}_\psi \subset \mathcal{M}_\mathbf{k}$ be the finite subset such that ψ preserves $\|\cdot\|_v$ for all $v \notin \mathcal{S}_\psi$, and set $\mathcal{S} = \mathcal{S}_\phi \cup \mathcal{S}_\psi$. Given $\mathbf{g} \in G$ and $v \in \mathcal{M}_\mathbf{k} - \mathcal{S}$, we have:

$$\|\mathbf{g}\|_= \|T(\mathbf{g})\|_{M_v} \iff \|\phi^{-1}(\mathbf{g})\|_v = \|(\psi_{|_{T(G)}} \circ T \circ \phi_{|_{\phi^{-1}(G)}} \left(\phi^{-1}(\mathbf{g})\right) \|_v.$$

But $\psi_{|_{T(G)}} \circ T \circ \phi_{|_{\phi^{-1}(G)}} : \phi^{-1}(G) \to \psi(W)$ is an invertible linear transformation between vector spaces of the same dimension, and so (b) follows. $\square$

Let $G \subset E$ and $\mathcal{S} \subset \mathcal{M}_\mathbf{k}$ be as in the conclusion of Lemma 4.1. Given $\mathbf{e} \in E$ write $\mathbf{e} = \mathbf{d} + \mathbf{g}$ with $\mathbf{g} \in G$ and $\mathbf{d} \in D$. By Lemma 4.1 we have that

$$\frac{\|T(\mathbf{e})\|_{\overline{F},v}}{\|\mathbf{e}\|_{E,v}} = \frac{\|T(\mathbf{g})\|_{\overline{F},v}}{\|\mathbf{e}\|_{E,v}} \leq \frac{\|T(\mathbf{g})\|_{\overline{F},v}}{\inf_{\mathbf{g}' \in [\mathbf{g}]_{D_v}} \|\mathbf{g}'\|_{\overline{E},v}} = \frac{\|T(\mathbf{g})\|_{\overline{F},v}}{\|\mathbf{g}\|_{\overline{E},v}} = 1$$

for all $v \notin \mathcal{S}$. Hence $\|T\|_{\overline{E},\overline{F},v} = 1$ for all $v \notin \mathcal{S}$, and so

$$H_{\overline{E},\,\overline{F}}(T) = \prod_{v \in \mathcal{S}} \|T\|^{n_v}_{\overline{E},\overline{F},v}. \tag{4}$$

A second consequence of Lemma 4.1 is that for all $\mathbf{e} \notin D$ we have

$$\frac{H_{\overline{F}}\big(T(\mathbf{e})\big)}{H_{\overline{E}/\overline{D}}([\mathbf{e}]_D)} = \prod_{v \in \mathcal{S}} \frac{\|T(\mathbf{e})\|^{n_v}_{\overline{F},v}}{\inf_{\mathbf{d} \in D_v} \|\mathbf{e} + \mathbf{d}\|^{n_v}_{\overline{E},v}}. \tag{5}$$

Now given $\epsilon > 0$ choose $\delta > 0$ so that $\prod_{v \in \mathcal{S}} \|T\|^{n_v}_{\overline{E},\overline{F},v} < \epsilon + \prod_{v \in \mathcal{S}} \left(\|T\|^{n_v}_{\overline{E},\overline{F},v} - \delta\right)$. By the strong approximation theorem we can find $\mathbf{e} \in E$ such that

$$\|T\|^{n_v}_{\overline{E},\overline{F},v} - \delta \leq \frac{\|T(\mathbf{e})\|^{n_v}_{\overline{F},v}}{\|\mathbf{e}\|^{n_v}_{\overline{E},v}} \leq \frac{\|T(\mathbf{e})\|^{n_v}_{\overline{F},v}}{\inf_{\mathbf{d} \in D_v} \|\mathbf{e} + \mathbf{d}\|^{n_v}_{\overline{E},v}}.$$

Taking the product over $v \in \mathcal{S}$ and using (4) and (5) yields

$$H_{\overline{E},\,\overline{F}}(T) = \prod_{v \in \mathcal{S}} \|T\|^{n_v}_{\overline{E},\overline{F},v} < \epsilon + \prod_{v \in \mathcal{S}} \left(\|T\|^{n_v}_{\overline{E},\overline{F},v} - \delta\right) \leq \epsilon + \frac{H_{\overline{F}}\big(T(\mathbf{e})\big)}{H_{\overline{E}/\overline{D}}([\mathbf{e}]_{\overline{D}})}$$

completing the proof of the proposition. $\square$

Corollary 4.1. *Let $\overline{E}$ and $\overline{F}$ be pure adelic vector bundles over $\mathbf{k}$. Suppose T in $Hom_\mathbf{k}(E,\,F)$ is injective. Then $H_{\overline{E},\,\overline{F}}(T) = H^{op}_{\overline{E},\overline{F}}(T)$.*

We are now in the position to prove the main result of this section.

P r o p o s i t i o n 4.2. *Let $\overline{E}$ and $\overline{F}$ be pure adelic vector bundles over* $\mathbf{k}$. *Given* $T\in Hom_{\mathbf{k}}(E,\,F)$ *let* $D=\ker T$ *and* $d=\dim_{\mathbf{k}} D$. *Then:*

(a) If $1\leq d< n-1$. *Then*

$$H^{op}_{\overline{E},\overline{F}}(T)\leq H_{\overline{E},\,\overline{F}}(T)\leq \frac{q^{2(d+1)g(\mathbf{k})}H_{\overline{E}}(D)}{\lambda_1^{\overline{E}}(E)^d}H^{op}_{\overline{E},\overline{F}}(T).$$

(b) If $d=n-1$. *Then* $H_{\overline{E},\,\overline{F}}(T)=\frac{\lambda_1^{\overline{E}}(E-D)H_{\overline{E}}(D)}{H_{\overline{E}}(E)}H^{op}_{\overline{E},\overline{F}}(T)$.

P r o o f. (a) We have:

$$\begin{aligned}
H_{\overline{E},\,\overline{F}}(T)&=\sup_{[\mathbf{e}]_D\in E/D}\frac{H_{\overline{F}}\big(T(\mathbf{e})\big)}{H_{\overline{E}/\overline{D}}([\mathbf{e}]_D)} && \text{by Proposition 4.1}\\
&\leq \sup_{[\mathbf{e}]_D\in E/D}\frac{q^{2(d+1)g(\mathbf{k})}H_{\overline{E}}(D)}{\lambda_1^{\overline{E}}(<D,\,\mathbf{e}>)^d}\frac{H_{\overline{F}}\big(T(\mathbf{e})\big)}{\lambda_1^{\overline{E}}([\mathbf{e}]_D)} && \text{by Proposition 3.1}\\
&\leq \frac{q^{2(d+1)g(\mathbf{k})}H_{\overline{E}}(D)}{\lambda_1^{\overline{E}}(< E >)^d}\sup_{[\mathbf{e}]_D\in E/D}\frac{H_{\overline{F}}\big(T(\mathbf{e})\big)}{\lambda_1^{\overline{E}}([\mathbf{e}]_D)} && \text{for } \lambda_1^{\overline{E}}(E)\leq\lambda_1^{\overline{E}}(<D,\,\mathbf{e}>)\\
&=\frac{q^{2(d+1)g(\mathbf{k})}H_{\overline{E}}(D)}{\lambda_1^{\overline{E}}(E)^d}H^{op}_{\overline{E},\overline{F}}(T)
\end{aligned}$$

proving (a). To prove (b) note that since $\dim_{\mathbf{k}}(D)=n-1$ we have $<D,\,\mathbf{e}>=E$ for any $\mathbf{e}\notin D$. It follows from Proposition 3.1 and Proposition 4.1 that for any $\mathbf{e}\in E-D$ we have $H_{\overline{E}}(E)H_{\overline{E},\,\overline{F}}(T)=H_{\overline{F}}\big(T(\mathbf{e})\big)H_{\overline{E}}(D)$. On the other hand $\lambda_1^{\overline{E}}(E-D)H^{op}_{\overline{E},\overline{F}}(T)=H_{\overline{F}}\big(T(\mathbf{e})\big)$, proving (b). □

5 - The spectral height

The goal of this section is to define the spectral height on the endomorphism ring of a $\mathbf{k}$-vector space E and prove the analogue of the spectral radius formula for operator heights associated to adelic vector bundles over spec $\mathbf{k}$ having E as support. Let $(X,\|\cdot\|)$ be a finite dimensional normed space over $\mathbf{C}_v$. Given $T\in \mathrm{End}_{\mathbf{C}_v}(X)$, the *spectral radius of* T is:

$$\rho_v(T)=\sup\nolimits_{\lambda\in\,\mathrm{sp}(T)}|\lambda|_{\mathbf{C}_v}$$

where sp(T) denotes the set of roots of the minimal polynomial of T.

P r o p o s i t i o n 5.1 (Local spectral formula). *Let* $(X,\|\cdot\|)$ *be a finite dimensional normed space over* $\mathbf{C}_v$. *For all* $T\in\mathrm{End}_{\mathbf{C}_v}(X)$ *we have:*

$$\lim_{m\to\infty}\|T^m\|^{1/m}=\rho_v(T). \tag{6}$$

P r o o f. This result should be well known, but since we could not find a reference for it, we provide a sketch of its proof. First, note that if T is nilpotent both sides are 0, and so there is nothing to prove. Hence we may assume that T is not nilpotent. Since all norms on X are equivalent we only have to prove the limit formula for one norm. We are going to use the operator norm relative to the sup norm attached to a basis of X. The key point being that for such a norm the corresponding operator norm of T is simply the maximum of the absolute value of the entries of the matrix representing T with respect to the chosen basis. If $\rho_v(T) > 1$ we choose a basis $\mathcal{B}$ having the property that the matrix of T with respect to $\mathcal{B}$ is the Jordan normal form. Clearly (6) follows. If $\rho_v(T) < 1$, we choose a basis in such a way that non-zero entries not on the diagonal have absolute value strictly smaller than $\rho_v(T)$. Again (6) follows at once. $\square$

Now we go back to global function fields and define the spectral height:

D e f i n i t i o n 5.1. Let $T \in \mathrm{End}_{\mathbf{k}}(E)$, where E is a finite dimensional $\mathbf{k}$-vector space. Let T_v denote the linear transformation induced by T on $E \otimes_{\mathbf{k}} \mathbf{C}_v$. If T is not nilpotent then the *spectral height* of T is

$$H_s(T) = \prod_{v \in \mathcal{M}_{\mathbf{k}}} \rho_v(T_v)^{n_v} \tag{7}$$

while if T is nilpotent we set $H_s(T) = 0$.

It is straightforward to verify that the spectral height enjoys the following properties, as their proof follows directly from the analogous properties of the spectrum of linear transformations:
(S1) $H_s(\lambda T) = H_s(T)$, for all $\lambda \in \mathbf{k}^{\times}$;
(S2) $H_s(T) \geq 1$;
(S3) $H_s(T^m) = H_s(T)^m$, for all $m \geq 1$;
(S4) If $T, T' \in \mathrm{End}_{\mathbf{k}}(X)$ commute, $H_s(TT') \leq H_s(T)H_s(T')$;
(S5) H_s is invariant under conjugation.

As it is apparent from (S5) the Northcott finiteness theorem does not hold for H_s. The main result of this section is the following:

T h e o r e m 5.1. *Let* $\mathbf{k}$ *be a global function field, and* $\overline{E} = (E, \{\|\cdot\|_{\overline{E},v}\}_{v \in \mathcal{M}_{\mathbf{k}}})$ *be an adelic vector bundle over* $\mathbf{k}$. *Let* T *belong to* $\mathrm{End}_{\mathbf{k}}(E)$, *then*

(a) $\lim_{m \to \infty} H_{\overline{E}}(T^m)^{1/m} = H_s(T)$

(b) $\lim_{m \to \infty} H^{op}_{\overline{E}}(T^m)^{1/m} = H_s(T)$

P r o o f. First of all note that (a) follows directly from the local spectral formula. (b) If T is nilpotent there is nothing to prove since both sides are zero. Let $D_m = \ker T^m$ and $d_m = \dim_{\mathbf{k}} D_m$. If $d_1 = 0$ then $H^{op}_{\overline{E}}(T^m) = H_{\overline{E}}(T^m)$ for all m, and so (b) follows. If $d_1 = n - 1$ then by Proposition 4.2.(b) we have

$$\frac{H_{\overline{E}}(T^m) H_{\overline{E}}(E)}{\lambda_1^{\overline{E}}(E - D_m) H_{\overline{E}}(D_m)} = H^{op}_{\overline{E}}(T^m)$$

for all $m \geq 1$. Since $D_k = D_h$ for $h, k \geq n$ we have that (b) follows from (a). Lastly suppose that $0 < d < n-1$. By Proposition 4.2.(a) we have

$$H_{\overline{E}}(T^m)\frac{\lambda_1^{\overline{E}}(E)^d}{q^{2(d_m+1)g}H_{\overline{E}}(D_m)} \leq H^{op}_{\overline{E}}(T^m) \leq H_{\overline{E}}(T^m)$$

for all $s \geq 1$. As we noted before $D_h = D_k$ for all $h, k \geq n$ and hence

$$\lim_{m\to\infty}\left(\frac{\lambda_1^{\overline{E}}(E)^d}{q^{2(d_m+1)g}H_{\overline{E}}(D_m)}\right)^{1/m} = 1.$$

Again (b) follows from (a). □

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