On canonical heights on Endomorphism rings over global function fields

Valerio Talamanca

1 - Introduction

Canonical heights appeared in the mid 50’s independently in the work of A. Néron (cf. [6]) and J. Tate (unpublished, see [7] where Tate’s method first appeared in print) on abelian varieties. Since then canonical heights have been constructed and studied in several settings, such as Drinfeld modules (cf. [3]), varieties with morphism (cf. [2]), endomorphism rings of vector space (cf. [9]) and finite sets of matrices in \( \text{GL}_d(\mathbb{Q}) \) (cf. [1]), just to mention a few. While Néron’s construction is local, i.e., he defines the canonical height as a product of local factors, Tate’s method is global and is more suited to be applied in different contexts. We briefly recall Tate’s method. Let \( A \) be an abelian variety defined over a number field \( \mathbb{F} \). Let \( [2] : A \to A \) denote the duplication map on \( A \). Let \( c \) be a linear equivalence class of divisors on \( A \) containing a symmetric ample divisor \( D \) and choose \( \phi : A \to \mathbb{P}_N \) so that \( D \) is the pull back of a hyperplane

This research was partially supported by G.N.S.A.G.A of Istituto Nazionale di Alta Matematica and Prin2011 Geometria delle Varietà Algebriche
via $\phi$. Let $h_{nw} : \mathbb{P}_N(\mathbb{Q}) \to \mathbb{R}$ denote the absolute logarithmic Northcott-Weil height and set $h_\phi(P) := h_{nw}(P)$. Then the limit

\[
(1) \quad \hat{h}_c(P) := \lim_{m \to \infty} \frac{1}{4^m} h_\phi([2^m]P)
\]

exists for each $P \in A(\mathbb{Q})$ and is independent of $\phi$. The function $\hat{h}_c$ is called the canonical height associated to $c$. Not only is $\hat{h}_c$ independent of $\phi$ but also of the choice of $D$. Moreover, if we perform the limit in (1) using any other multiplication map we obtain the same function (cf. [8, Lemma 3.1.]). The equality between Néron’s and Tate’s definitions is achieved as follows: first it is shown that there is a unique function $\tilde{h}$ satisfying the following two properties:

a) $h_\phi$ and $\tilde{h}$ are in the same class modulo bounded functions;

b) $\tilde{h}([2]P) = 4\tilde{h}(P)$;

then, it is shown that both Neron’s and Tate’s height satisfy both a) and b).

Let $E$ be a finite dimensional vector space over a global function field $k$. In this paper we present a construction of a canonical height (called the spectral height) on $\text{End}_k(E)$. Our construction is closer to Neron’s one (albeit more elementary) as we define the spectral height as a product of local factors (the local spectral radii, cf. section 5). We then prove a limit formula (cf. Theorem 5.1), analogous to (1), relating the spectral height and heights on $\text{End}_k(E)$ associated to adelic vector bundle over $k$.

The paper is organised as follows: in section 2 we introduce adelic vector bundles and their associated heights. In section 3 we prove an interesting auxiliary inequality between the height relative to a sub-bundle and the minimum of the height on a coset of $D$ on $E$, cf. Proposition 3.1. Section 4 deals with heights on $\text{End}_k(E)$, while the spectral height is introduced in section 5 where we prove our main theorem (cf. Theorem 5.1).

Notation Throughout this paper $k \supset \mathbb{F}_p(t)$ is a global function field of characteristic $p > 0$. We let $\mathcal{M}_k$ be the set of places of $k$. Given $v \in \mathcal{M}_k$ we denote by $k_v$ the completion of $k$ with respect to $v$ and by $\mathcal{O}_v$ the completion of the algebraic closure of $k_v$. The maximal compact subring of $k_v$ is denoted by $\mathcal{O}_v$ and we let $n_v = [k_v : \mathbb{F}_p(t)_v]$, where $\omega$ is the restriction of $v$ to $\mathbb{F}_p(t)$. For each $v \in \mathcal{M}_k$ we fix an absolute value $|\cdot|_v$, in the class of $v$, by requiring that $|a|_v^{n_v} \equiv 1$. Regarding vector space we will employ the following notation: given a vector space $E$, $E^\times$ denotes the set of non-zero vectors in $E$ and $\langle e_1, \ldots, e_r \rangle$ denotes the subspace generated by $e_1, \ldots, e_r \in E$. Lastly if $E$ is a vector space over $k$, we set $E_v = E \otimes k_v$.

2 - Adelic vector bundles

In this paper we use heights associated to adelic vector bundles. Adelic vector bundles have been recently introduced and studied by É. Gaudron (see
An adelic vector bundle \( E = (E, \{ \| \cdot \|_{E,v} \}_{v \in \mathcal{M}_k}) \) (over \( \mathop{\text{spec}} k \) or over \( k \) for short) of dimension \( n \) consists of the following data (cf. [4, Definition 2.1]): a \( k \)-vector space \( E \) of dimension \( n \) (called the support of \( E \)) and a family of ultrametric norms \( \| \cdot \|_{E,v} : E \otimes_k \mathbb{C}_v \to \mathbb{R} \), satisfying the following conditions:

1) There exists a \( k \)-basis \( \{e_1, \ldots, e_n\} \) of \( E \) over \( k \), such that for all but finitely many \( v \in \mathcal{M}_k \) we have

\[
\left\| \sum_{i=1}^{n} \alpha_i e_i \right\|_{E,v} = \max_{1 \leq i \leq n} \{ |\alpha_i|_{\mathbb{C}_v} \} \quad \forall (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}_v
\]

where \( | \cdot |_{\mathbb{C}_v} \) is the unique extension of \( | \cdot |_v \) to \( \mathbb{C}_v \);

2) let \( \text{Gal}(\mathbb{C}_v/k_v) \) denote the set of continuous automorphism of \( \mathbb{C}_v \) which leaves the elements of \( k_v \) fixed. Then \( \| \|_{E,v} \) is invariant under the standard action of \( \text{Gal}(\mathbb{C}_v/k_v) \) on \( E \otimes_k \mathbb{C}_v \).

An adelic vector bundle is called \( v \)-pure if \( \| x \|_{E,v} \) belongs to the value set of \( | \cdot |_v \) for all \( x \in E \) and it is called pure\(^1\) if it is \( v \)-pure for all \( v \in \mathcal{M}_k \). Let \( \overline{E} = (E, \{ \| \cdot \|_{\overline{E},v} \}_{v \in \mathcal{M}_k}) \) be a pure adelic vector bundle over \( k \). It is possible to perform several algebraic constructions with adelic vector bundles, such as exterior powers, symmetric powers and so on. We refer the reader to [4, Section 3.3] for details and briefly recall the few that we need. The absence of archimedean places simplifies some definitions. We say that \( D \) is an adelic sub-bundle of \( E \) if \( D \subset E \), and for every \( v \) the norms of \( D \) are the restriction of those of \( E \). If \( D \subset E \) is a sub-bundle then \( E/D \) inherits an adelic vector bundle structure (denoted by \( E/\overline{D} \)) where the norms are the quotient norms of those of \( E \). If \( F = (F, \{ \| \cdot \|_{F,v} \}_{v \in \mathcal{M}_k}) \) is another adelic vector bundle over \( k \), we set

\[
\| T \|_{\overline{E},F,v} := \sup_{e \in E,v} \frac{\| T(e) \|_{\overline{E},v}}{\| e \|_{\overline{E},v}}
\]

for all \( T \in \text{Hom}_k(E, F) \otimes_k \mathbb{C}_v \) and all \( v \in \mathcal{M}_k \). It is straightforward to verify that \( \text{Hom}_k(\overline{E}, F) = (\text{Hom}_k(E, F), \{ \| \cdot \|_{\overline{E},F,v} \}_{v \in \mathcal{M}_k}) \) is an adelic vector bundle having \( \text{Hom}_k(E, F) \) as support. Note that if \( \overline{F} \) is the trivial bundle this gives the structure of adelic vector bundle to \( E^* \) the dual of \( E \). Next \( \overline{E} \otimes_k F \) is the adelic vector bundle having support \( E \otimes_k F \) and norms induced by the isomorphism \( E \otimes_k F \cong \text{Hom}_k(E^*, F) \). Lastly we denote by \( \bigwedge^m E = (\bigwedge^m E, \{ \| \cdot \|_{\bigwedge^m_{E,v}} \}_{v \in \mathcal{M}_k}) \) the adelic vector bundle having \( \bigwedge^m E \) as support and whose norms are the quotient norms of \( \overline{E} \otimes^m \).

Let \( \overline{E} = (E, \{ \| \cdot \|_{\overline{E},v} \}_{v \in \mathcal{M}_k}) \) be an adelic vector bundle over \( \mathop{\text{spec}} k \). The height function \( H_{\overline{E}} : E \to \mathbb{R} \), relative to \( \overline{E} \) is defined by setting:

\(^1\)It is not difficult to prove that there is a one to one correspondence between pure adelic vector bundles over \( k \) having \( E \) as support and coherent systems of \( k_v \)-lattices belonging to \( E \) as defined by A. Weil in [11].
\[ H_{E}(e) = \prod_{v \in \mathcal{M}_k} \|e\|^{n_v}_{E,v} \]

for all \(0 \neq e \in E\). As usual we set \(H_{E}(0) = 1\). It follows from the product formula that \(H_{E}\) is constant on one dimensional subspaces of \(E\). The height of a subspace \(D \subset E\), is defined as follows: choose a basis \(d_1, \ldots, d_m\) of \(D\) over \(k\) and set \(H_{E}(D) = H_{E}(d_1 \wedge \cdots \wedge d_m)\), which does not depend on the choice of the basis by the product formula (see [5, Introduction]). Lastly if \(B\) is a subset of \(E\) we set

\[ \lambda_{E}^{P}(B) = \inf_{x \in B} H_{E}(x). \]

### 3 - Comparison between \(H_{E/D}(\langle e \rangle_{D})\) and \(\lambda_{E}^{P}(\langle e \rangle_{D})\)

Let \(D\) be a subspace of \(E\). If \(e \in E - D\) we denote by \(\langle D, e \rangle\) the subspace generated by \(D\) and \(e\), and by \(\langle e \rangle_{D}\) the coset of \(e\) modulo \(D\). In this section we obtain a comparison result (Proposition 3.1 below) for \(H_{E/D}(\langle e \rangle_{D})\) and \(\lambda_{E}^{P}(\langle e \rangle_{D})\). One of the main constituents of the proof of Proposition 3.1 is the uniform Sigel’s lemma recently proved by É. Gaudron in [5]. The lower bound obtained in Proposition 3.1 is a key ingredient for the results of the next section.

The quotient height \(H_{E/D}\) is defined as

\[ H_{E/D} : E/D \rightarrow \mathbb{R} \]

\[ [e]_D \rightarrow H_{E/D}(\langle e \rangle_{D}) = \prod_{v \in \mathcal{M}_k} \inf_{e' \in [e]_{D,v}} \|e'\|^{n_v}_{E,v} \]

if \(\langle e \rangle_{D} \neq [0]_{D}\), as usual we set \(H_{E/D}([0]_{D}) = 1\).

Proposition 3.1. Let \(E\) be a pure adelic vector bundle over \(k\). Let \(D\) be a subspace of dimension \(d\) and suppose \(e \in E - D\). Then

\[ \lambda_{E}^{P}(\langle D, e \rangle)^{d} \cdot \left( \frac{q}{g^{2(d+1)}(k)^{2}} \right) H_{E}(D) \lambda_{E}^{P}(\langle e \rangle_{D}) \leq H_{E/D}(\langle e \rangle_{D}) \leq \lambda_{E}^{P}(\langle e \rangle_{D}), \]

where \(q\) is the cardinality of the constant field of \(k\) and \(g(k)\) is the genus of \(k\).

Proof. The inequality \(H_{E/D}(\langle e \rangle_{D}) \leq \lambda_{E}^{P}(\langle e \rangle_{D})\) follows immediately from the definitions. To prove the other inequality we need the following lemma which gives a decomposition for the heights of a subspace\(^2\).

Lemma 3.1. Let \(E = (E, \{\|\cdot\|_{E,v}\}_{v \in \mathcal{M}_k})\) be a pure adelic vector bundle. Let \(D \subset E\) be a sub-bundle of dimension \(d\). Then

\[ H_{E}(\langle D, e \rangle) = H_{E}(D) H_{E/D}(\langle e \rangle_{D}) \]

\(^2\)The same result, although stated in terms of orthogonal projection was first proven over number fields for the standard \(L^2\)-height by J. Vaaler, see [10, Lemma 4].
Proof. Let \( d_1, \ldots, d_d \) be a basis for \( D \). Clearly it suffices to show that

\[
\|d_1 \wedge \cdots \wedge d_d \wedge e\|_{\mathcal{E},v} = \|d_1 \wedge \cdots \wedge d_d\|_{\mathcal{E},v} \inf_{e' \in [e]} \|e'\|_{\mathcal{E},v}
\]

for all \( v \in \mathcal{M}_k \). Fix \( v \in \mathcal{M}_k \). Since \( \mathcal{E} \) is pure we can find, by [11, Ch.II-2 Thm.1], a basis \( f_1, \ldots, f_n \) of \( E_v \) such that

(i) \( \|\gamma_1 f_1 + \cdots + \gamma_n f_n\|_{\mathcal{E},v} = \sup_{1 \leq i \leq n} |\gamma_i|_v \) for all \( \gamma_1, \ldots, \gamma_n \in k_v \)

(ii) \( d_{k+1} \in \langle f_n, \ldots, f_{n-k} \rangle \) for all \( k = 0, \ldots, d-1 \) and \( e \in \langle f_n, \ldots, f_{n-d} \rangle \).

Write \( d_k = \sum_{i=1}^{k} \alpha_{ki} f_{n-i+1} \) for \( k = 1, \ldots, d \) and \( e = \sum_{i=1}^{d+1} \beta_i f_{n-i+1} \), then an easy calculation shows that (ii) implies that

\[
\|d_1 \wedge \cdots \wedge d_d \|_{\mathcal{E},v} = |\alpha_{11} \alpha_{22} \cdots \alpha_{dd}|_v
\]

and

\[
\|d_1 \wedge \cdots \wedge d_d \wedge e\|_{\mathcal{E},v} = |\alpha_{11} \alpha_{22} \cdots \alpha_{dd} \beta_{d+1}|_v.
\]

It remains to show that \( |\beta_{d+1}|_v = \inf_{e' \in [e]} \|e'\|_{\mathcal{E},v} \). To this end, note that by construction \( [e]_{\mathcal{E},v} = [\beta_{d+1} f_{n-d}]_{\mathcal{F},v} \) and hence \( \inf_{e' \in [e]} \|e'\|_{\mathcal{E},v} \leq |\beta_{d+1}|_v \). On the other hand any \( d \in D_v \) can be written as \( \sum_{i=1}^{d} \gamma_i f_{n-i+1} \) and so

\[
\|e - d\|_{\mathcal{E},v} = \|\sum_{i=1}^{d+1} \beta_i f_{n-i+1} - \sum_{i=1}^{d} \gamma_i f_{n-i+1}\|_{\mathcal{E},v} \geq |\beta_{d+1}|_v.
\]

Now we can quickly finish the proof of Proposition 3.1. By the uniform Sigel’s lemma for global function fields, see [5, Cor. 3.3], there exists \( f \) belonging to \( <D, e> \) but not belonging to \( D \) such that

\[
H_{\mathcal{E}}(f) \leq \frac{q^{2(d+1)g(k)} H_{\mathcal{E}}(<D, e>)}{\lambda_{\mathcal{E}}^{(2)}(<D, e>)^d}.
\]

By definition \( \lambda_{\mathcal{E}}^{(2)}([e]_D) \leq H_{\mathcal{E}}(f) \) and hence Lemma 3.1, yields

\[
\lambda_1([e]_D) \leq \frac{q^{2(d+1)g(k)} H_{\mathcal{E}}(D) H_{\mathcal{E},\mathcal{F}}([e]_D))}{\lambda_{\mathcal{E}}^{(2)}(<D, e>)^d}
\]

\( \square \)

4 - Heights of linear transformations

Let us start by recalling the definition of the operator height for linear transformations and compare it with \( H_{\mathcal{E},\mathcal{F}} : = H_{\text{Hom}_k(\mathcal{E}, \mathcal{F})} \). So let \( \mathcal{E} \) and \( \mathcal{F} \) be two
adelic vector bundles over $k$-vector spaces. Given $T \in \text{Hom}_k(F, E)$, set:

$$H^{op}_{E, \overline{F}}(T) := \sup_{e \in E} \frac{H_T(T(e))}{H_{\overline{F}}(e)} = \sup_{|e| \in F/D} \frac{H_T(T(e))}{H_{\overline{F}}(e)}.$$ 

where $D = \ker(T)$. The function $H^{op}_{E, \overline{F}}$ is called the operator height on $\text{Hom}_k(E, F)$ associated to $E$ and $\overline{F}$. If $E = F$ we will use $H^{op}_{E}$ (respectively $H_{\overline{F}}$) instead of $H^{op}_{E, \overline{F}}$ (respectively $H_{E, \overline{F}}$). The main goal of this section is to prove a comparison result between $H^{op}_{E, \overline{F}}$ and $H_{E, \overline{F}}$, which will be used in the proof of Theorem 5.1. Clearly $H^{op}_{E, \overline{F}}(T) \leq H_{E, \overline{F}}(T)$, so our next objective is to prove a reverse inequality where, for non-invertible linear transformations, some arithmetic constants, such as the height of the kernel, will appear, see Proposition 4.2. We start with a preparatory result that not only establishes a useful alternative description for $H_{E, \overline{F}}$ but also proves that $H_{E, \overline{F}}(T) = H^{op}_{E, \overline{F}}(T)$ if $T$ is an injective linear transformation.

**Proposition 4.1.** Let $E$ and $\overline{F}$ be pure adelic vector bundles over $k$. Given $T$ in $\text{Hom}_k(E, F)$, set $D = \ker T \subset E$. Then:

$$H_{E, \overline{F}}(T) = \sup_{|e| \in F/D} \frac{H_T(T(e))}{H_{\overline{F}}(e)}.$$ 

In particular if $T$ is injective we have $H_{E, \overline{F}}(T) = H^{op}_{E, \overline{F}}(T)$.

**Proof.** Clearly

$$H_{E, \overline{F}}(T) = \prod_{v \in M_k} \sup_{e \in E_{\nu}} \|T(e)\|^{v}_{E, v} \geq \sup_{|e| \in F/D} \frac{H_T(T(e))}{H_{\overline{F}}(e)}.$$ 

To prove the reverse inequality we need the following:

**Lemma 4.1.** Under the hypotheses of Proposition 4.1 there exists a finite set of places $S \subset M_k$ and a subspace $G \subset E$ of dimension equal to the rank of $T$ such that for all $v \in S$ we have

(a) $\inf g \in |g|_{\nu_0} \|g\|_{E, v} = \|g\|_{E, v}$ for all $g \in G_v$,

(b) $\|T(g)\|_{F, v} = \|g\|_{E, v}$ for all $g \in G_v$.

**Proof.** From the definition of an adelic vector bundle and the fact that we are proving a statement for all but finitely many places it follows that we can assume that $E = (k^n, \{\| \cdot \|_v\}_{v \in M_k}), \overline{F} = (k^m, \{\| \cdot \|_v\}_{v \in M_k})$, where $\| \cdot \|_v$ is the sup norm on $k^n_v$ and $k^m_v$. If $m = n$ and $T$ is invertible (a) is trivial and (b) is equivalent to say that an invertible $n \times n$ matrix with coefficients in $k$ actually belongs to $\text{GL}_n(\mathcal{O}_v)$ for all but finitely many $v \in M_k$. In general let $r = \text{rank}(T)$, and choose $\phi$ to be an automorphism of $k^n$ such that $\ker T = \phi(U)$ where $U$ is the subspace generated the last $n - r$ vectors of the standard basis.
of $k^n$. Moreover choose $\psi$ to be an automorphism of $k^n$ mapping $W = \text{Im}(T)$ onto the subspace generated by the first $r$ vectors of the standard basis of $k^n$.

Finally we let $G$ be the image via $\phi$ of the subspace generated by the first $r$ vectors of the standard basis of $k^n$. Since $\phi$ is invertible there exists a finite set $S_\phi \subset M_k$ such that $\phi$ preserves $\| \cdot \|_v$ for all $v \in S_\phi$. For $g \in G$ and $v \not\in S_\phi$, we have:

$$\inf_{g' \in [g]_D} \|g'\|_v = \inf_{d \in D} \|\phi^{-1}(g) - \phi^{-1}(d)\|_v = \inf_{u \in U} \|\phi^{-1}(g) - u\|_v = \|g\|_v$$

proving (a). To prove (b) let $S_\psi \subset M_k$ be the finite subset such that $\psi$ preserves $\| \cdot \|_v$ for all $v \in S_\psi$, and set $S = S_\phi \cup S_\psi$. Given $g \in G$ and $v \not\in M_k - S$, we have:

$$\|g\|_v = \|T(g)\|_M \iff \|\phi^{-1}(g)\|_v = \|\psi_\tau \circ T \circ \phi_{\psi^{-1}(\tau)}(\phi^{-1}(g))\|_v.$$  

But $\psi_\tau \circ T \circ \phi_{\psi^{-1}(\tau)} : \phi^{-1}(G) \rightarrow \psi(W)$ is an invertible linear transformation between vector spaces of the same dimension, and so (b) follows.

Let $G \subset E$ and $S \subset M_k$ be as in the conclusion of Lemma 4.1. Given $e \in E$ write $e = d + g$ with $g \in G$ and $d \in D$. By Lemma 4.1 we have that

$$\frac{\|T(e)\|_\mathcal{F}, \mathcal{F}, v}{\|e\|_E, v} = \frac{\|T(e)\|_\mathcal{F}, \mathcal{F}, v}{\|e\|_E, v} \leq \frac{\|T(g)\|_\mathcal{F}, \mathcal{F}, v}{\inf_{g' \in [g]_D} \|g'\|_\mathcal{F}, \mathcal{F}, v} = \frac{\|T(g)\|_\mathcal{F}, \mathcal{F}, v}{\|g\|_\mathcal{F}, \mathcal{F}, v} = 1$$

for all $v \not\in S$. Hence $\|T\|_{\mathcal{F}, \mathcal{F}, v} = 1$ for all $v \not\in S$, and so

$$H_{\mathcal{F}, \mathcal{F}}(T) = \prod_{v \in S} \|T\|_{\mathcal{F}, \mathcal{F}, v}.$$  

A second consequence of Lemma 4.1 is that for all $e \in D$ we have

$$\frac{H_{\mathcal{F}}(T(e))}{H_{\mathcal{F}/\mathcal{F}}(e)} = \prod_{v \in S} \|T(e)\|_{\mathcal{F}, \mathcal{F}, v} \inf_{d \in D} \|e + d\|_{\mathcal{F}, \mathcal{F}, v}.$$  

Now given $\epsilon > 0$ choose $\delta > 0$ so that $\prod_{v \in S} \|T\|_{\mathcal{F}, \mathcal{F}, v} < \epsilon + \prod_{v \in S} \left(\|T\|_{\mathcal{F}, \mathcal{F}, v} - \delta\right)$.

By the strong approximation theorem we can find $e \in E$ such that

$$\|T\|_{\mathcal{F}, \mathcal{F}, v} - \delta \leq \frac{\|T(e)\|_{\mathcal{F}, \mathcal{F}, v}^m}{\|e\|_{E, v}^m} \leq \frac{\|T(e)\|_{\mathcal{F}, \mathcal{F}, v}^m}{\inf_{d \in D} \|e + d\|_{\mathcal{F}, \mathcal{F}, v}^m}.$$  

Taking the product over $v \in S$ and using (4) and (5) yields

$$H_{\mathcal{F}, \mathcal{F}}(T) = \prod_{v \in S} \|T\|_{\mathcal{F}, \mathcal{F}, v} < \epsilon + \prod_{v \in S} \left(\|T\|_{\mathcal{F}, \mathcal{F}, v} - \delta\right) \leq \epsilon + \frac{H_{\mathcal{F}}(T(e))}{H_{\mathcal{F}/\mathcal{F}}(e)}$$

completing the proof of the proposition.

\[\square\]

Corollary 4.1. Let $E$ and $F$ be pure adelic vector bundles over $k$. Suppose $T$ in $\text{Hom}_k(E, F)$ is injective. Then $H_{\mathcal{F}, \mathcal{F}}(T) = H_{\mathcal{F}, \mathcal{F}}^\text{op}(T)$.  

7
We are now in the position to prove the main result of this section.

Proposition 4.2. Let $\mathcal{E}$ and $\mathcal{F}$ be pure adelic vector bundles over $k$. Given $T \in \text{Hom}_{k}(E, F)$ let $D = \ker T$ and $d = \dim_k D$. Then:

(a) If $1 \leq d < n - 1$. Then

$$H^\text{op}_{E, F}(T) \leq q^{2(d+1)g(k)} H_{\mathcal{F}}(D) \frac{\lambda^\mathcal{F}(E-D) H_{\mathcal{F}}(D)}{\lambda^\mathcal{F}(E)} H^\text{op}_{E, F}(T).$$

(b) If $d = n - 1$. Then

$$H_{E, F}(T) = \frac{\lambda^\mathcal{F}(E-D) H_{\mathcal{F}}(D)}{\lambda^\mathcal{F}(E)} H^\text{op}_{E, F}(T).$$

Proof. (a) We have:

$$H_{E, F}(T) = \sup_{[e]_D \in E/D} H_{\mathcal{F}}(T([e]_D))$$

by Proposition 4.1

$$\leq \sup_{[e]_D \in E/D} q^{2(d+1)g(k)} H_{\mathcal{F}}(D) \frac{\lambda^\mathcal{F}(E-D) H_{\mathcal{F}}(D)}{\lambda^\mathcal{F}(E)} H_{\mathcal{F}}(T([e]_D))$$

by Proposition 3.1

$$\leq q^{2(d+1)g(k)} H_{\mathcal{F}}(D) \sup_{[e]_D \in E/D} \frac{\lambda^\mathcal{F}(E-D) H_{\mathcal{F}}(D)}{\lambda^\mathcal{F}(E)} H_{\mathcal{F}}(T([e]_D))$$

for $\lambda^\mathcal{F}(E) \leq \lambda^\mathcal{F}(E-D)$.

proving (a). To prove (b) note that since $\dim_k(D) = n - 1$ we have $<D, e> = E$ for any $e \notin D$. It follows from Proposition 3.1 and Proposition 4.1 that for any $e \in E - D$ we have $H_{\mathcal{F}}(E) H_{E, F}(T) = H_{\mathcal{F}}(T([e]_D))$. On the other hand

$$\lambda^\mathcal{F}(E-D) H^\text{op}_{E, F}(T) = H_{\mathcal{F}}(T([e]_D))$$

proving (b). \qed

5 - The spectral height

The goal of this section is to define the spectral height on the endomorphism ring of a $k$-vector space $E$ and prove the analogue of the spectral radius formula for operator heights associated to adelic vector bundles over $\text{Spec } k$ having $E$ as support. Let $(X, \|\cdot\|)$ be a finite dimensional normed space over $C_v$. Given $T \in \text{End}_{C_v}(X)$, the spectral radius of $T$ is:

$$\rho_v(T) = \sup_{\lambda \in \text{sp}(T)} |\lambda|_{C_v}$$

where $\text{sp}(T)$ denotes the set of roots of the minimal polynomial of $T$.

Proposition 5.1 (Local spectral formula). Let $(X, \|\cdot\|)$ be a finite dimensional normed space over $C_v$. For all $T \in \text{End}_{C_v}(X)$ we have:

$$\lim_{m \to \infty} \|T^m\|^{1/m} = \rho_v(T).$$
Proof. This result should be well known, but since we could not find a reference for it, we provide a sketch of its proof. First, note that if $T$ is nilpotent both sides are 0, and so there is nothing to prove. Hence we may assume that $T$ is not nilpotent. Since all norms on $X$ are equivalent we only have to prove the limit formula for one norm. We are going to use the operator norm relative to the sup norm attached to a basis of $X$. The key point being that for such a norm the corresponding operator norm of $T$ is simply the maximum of the absolute value of the entries of the matrix representing $T$ with respect to the chosen basis. If $\rho_v(T) > 1$ we choose a basis $B$ having the property that the matrix of $T$ with respect to $B$ is the Jordan normal form. Clearly (6) follows. If $\rho_v(T) < 1$, we choose a basis in such a way that non-zero entries not on the diagonal have absolute value strictly smaller than $\rho_v(T)$. Again (6) follows at once. □

Now we go back to global function fields and define the spectral height:

Definition 5.1. Let $T \in \text{End}_k(E)$, where $E$ is a finite dimensional $k$-vector space. Let $T_v$ denote the linear transformation induced by $T$ on $E \otimes_k C_v$. If $T$ is not nilpotent then the spectral height of $T$ is

$$H_s(T) = \prod_{v \in M_k} \rho_v(T_v)^{n_v}$$

while if $T$ is nilpotent we set $H_s(T) = 0$.

It is straightforward to verify that the spectral height enjoys the following properties, as their proof follows directly from the analogous properties of the spectrum of linear transformations:

(S1) $H_s(\lambda T) = H_s(T)$, for all $\lambda \in k^\times$;
(S2) $H_s(T) \geq 1$;
(S3) $H_s(T^m) = H_s(T)^m$, for all $m \geq 1$;
(S4) If $T,T' \in \text{End}_k(X)$ commute, $H_s(TT') \leq H_s(T)H_s(T')$;
(S5) $H_s$ is invariant under conjugation.

As it is apparent from (S5) the Northcott finiteness theorem does not hold for $H_s$. The main result of this section is the following:

Theorem 5.1. Let $k$ be a global function field, and $\mathcal{E} = (E, \{\|\cdot\|_v\}_v \backslash \{1\})$ be an adelic vector bundle over $k$. Let $T$ belong to $\text{End}_k(E)$, then

(a) $\lim_{m \to \infty} H^{\mathcal{E}}(T^m)^{1/m} = H_s(T)$

(b) $\lim_{m \to \infty} H^{\mathcal{E}}_\text{op}(T^m)^{1/m} = H_s(T)$

Proof. First of all note that (a) follows directly from the local spectral formula. (b) If $T$ is nilpotent there is nothing to prove since both sides are zero. Let $D_m = \ker T^m$ and $d_m = \dim_k D_m$. If $d_1 = 0$ then $H^{\mathcal{E}}_\text{op}(T^m) = H^{\mathcal{E}}(T^m)$ for all $m$, and so (b) follows. If $d_1 = n - 1$ then by Proposition 4.2.(b) we have

$$\frac{H^{\mathcal{E}}(T^m)H^{\mathcal{E}}(E)}{\lambda^{\mathcal{E}}_1(E - D_m)H^{\mathcal{E}}(D_m)} = H^{\mathcal{E}}_\text{op}(T^m)$$
for all \( m \geq 1 \). Since \( D_k = D_h \) for \( h, k \geq n \) we have that (b) follows from (a).

Lastly suppose that \( 0 < d < n - 1 \). By Proposition 4.2.(a) we have

\[
H_{\mathbb{E}}(T^m) \frac{\lambda_1^\mathbb{E}(E)^d}{q^{2(d_m+1)g}H_{\mathbb{E}}(D_m)} \leq H_{\mathbb{E}}^{op}(T^m) \leq H_{\mathbb{E}}(T^m)
\]

for all \( s \geq 1 \). As we noted before \( D_h = D_k \) for all \( h, k \geq n \) and hence

\[
\lim_{m \to \infty} \left( \frac{\lambda_1^\mathbb{E}(E)^d}{q^{2(d_m+1)g}H_{\mathbb{E}}(D_m)} \right)^{1/m} = 1.
\]

Again (b) follows from (a).

\[\square\]

References


Valerio Talamanca  
Università degli Studi Roma Tre  
Largo San Leonardo Murialdo 1  
Roma, 00185, Italy  
e-mail: valerio@mat.uniroma3.it