AN INTRODUCTION TO THE THEORY OF HEIGHT FUNCTIONS

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Abstract. We give an introduction to the theory of height functions. The following topics are discussed: heights on projective spaces, heights on projective varieties, Néron-Tate heights on abelian varieties, heights on curves.

Introduction

Heights have been used as a tool in arithmetic algebraic geometry for quite a long time. Their first appereance dates back to the latter third of last century in a work by G. Cantor [5] where he defined what today we call the inhomogeneous height of an algebraic number. It is interesting to note that Cantor used heights to prove that the cardinality of the set of real algebraic numbers is smaller than the cardinality of the real numbers. At the beginning of this century E. Borel in [3] gave the first definition of heights for "système "of rational numbers. However the systematic use of heights in arithmetic algebraic geometry started only with D.G. Northocott ([18] and [19]) and A. Weil [27] in the late forties. Since then Northcott-Weil heigths and their generalization (Arakelov heights and Faltings or modular heights cf.[12]) have proven to be a key tool (especially Northcott's finiteness theorem) for proving major results as the Mordell Conjecture (see [2], [6], [26]). It has to be noticed that heights also play an important role in various conjectures e.g. Vojta conjecture [25], the *abc* conjecture [13] and the Birch and Swinnerton-Dyer conjecture [1].

In this note we give an introduction to the theory of height functions with special regard toward its applications in arithmetical algebraic geometry. As Schimdt pointed out in [23] one can define the height associated to any distance function on R^n. On the other hand the heights that are actually used in the literature are those arising from the ℓ^p norms for $p = 1, 2$ and ∞. In this note we will use $p = \infty$ in order to simplify some of the arguments but we remark that is possible to treat simultaneously all the heights arising form the ℓ^p-norms cf. [24].

We assume familiarity with algebraic geometry and especially with the theory of abelian varieties. We refer the reader to [10], [14], [15], [16].

Conventions and Notations. We fix once and for all an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$. All the algebro-geometric objects that we will be dealing with (varieties, divisors, morphisms, etc.) are assumed to be defined over $\overline{\mathbf{Q}}$.

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2. Heights on Projective Spaces

The idea behind the definition of heights is to measure the "arithmetic size" of a point. Let us start with a simple case. Consider $\mathbf{P}^1(\mathbf{Q})$ as the space of lines in the plane $\mathbf{Q}^2$. A good measure of the arithmetic complexity of a straight line V in Q^2 is the distance between integral points, which is nothing else than the distance from the origin of a primitive integral point $P=(z_0,z_1)\in V$ (primitive means that $\gcd(z_0,z_1)=1$). Suppose that the line V is defined by the equation $ax+by=0$ with a,b positive integers. Then $P=\left(\frac{-b}{\gcd(a,b)},\frac{a}{\gcd(a,b)}\right)$ is a primitive integral point of V and so the height of V is $\sup\{\frac{|b|}{|\gcd(a,b)|},\frac{|a|}{|\gcd(a,b)|}\}$, where $|\cdot|$ denotes the ordinary archimedean absolute value on $\mathbf{Q}$. More generally given $P\in\mathbf{P}^n(\mathbf{Q})$ choose integral coordinates $P=[x_0:\ldots:x_n]$ and set

$$H(P)=\frac{\sup_{0\le i\le n}|x_i|}{\gcd(x_0,\ldots x_n)}.$$

Note that H is independent from the particular choice of integral coordinates for P. How can we generalize this definition to $\mathbf{P}^n(K)$, with K a number field? First of all note that $\gcd(x_0,\ldots x_n)=\mathbf{N}\mathfrak{a}_x=\#\,\mathbf{Z}/\mathfrak{a}_x$, where $\mathfrak{a}_x$ denotes the ideal generated by $x_0,\ldots x_n$. Thus the greatest common divisor ought to be replaced by the absolute norm. Secondly since there is no way to pick a particular archimedean absolute value on K we have to use all of them. Precisely let $\mathcal{M}_K^\infty$ denote the set of equivalence classes of archimedean absolute values on K (for generalities about number fields and absolute values see [8]). For any $v\in\mathcal{M}_K^\infty$ we denote by $|\cdot|_v$ the representative of v normalized by requiring that its restriction to $\mathbf{Q}$ coincides with the ordinary archimedean absolute value. Then we define the *(homogeneous)* ℓ^∞-height of $P=[x_0:\ldots:x_n]\in\mathbf{P}^n(K)$ to be

$$H_K(P)=\frac{1}{\mathbf{N}\mathfrak{a}}\prod_{v\in\mathcal{M}_K^\infty}\sup_{1\le i\le n}|x_i|_v$$

where $\mathfrak{a}$ is the fractional ideal generated by $x_0,\ldots,x_n$ and $n_v=[K_v:\mathbf{R}]$, K_v being the completion of K with respect to $|\cdot|_v$. Note that H_K is well defined (i.e. independent from the choice of coordinates for P) because $\prod_{v\in\mathcal{M}_K^\infty}|\lambda|_v^{n_v}=\mathbf{N}\mathfrak{a}_\lambda$, where $\mathfrak{a}_\lambda$ is the fractional ideal generated by λ. To be able to define an height function on the whole $\mathbf{P}^n(\overline{\mathbf{Q}})$, we need to study its behaviour under field extension. So let $K\subset L$ be an extension of number fields. Given $v\in\mathcal{M}_K^\infty$ set $\mathcal{M}_L^v=\{w\in\mathcal{M}_L \mid |\cdot|_w\big|_K=|\cdot|_v\}$, then $\mathcal{M}_L^\infty=\coprod_{v\in\mathcal{M}_K^\infty}\mathcal{M}_L^v$ and $\sum_{w\in\mathcal{M}_L^v}n_w=[L:K]n_v$. Therefore for any $v\in\mathcal{M}_K^\infty$ we have:

$$\prod_{w\in\mathcal{M}_L(v)}\left(\sup_{1\le i\le n}|x_i|_w\right)^{n_w}=\left(\sup_{0\le i\le n}|x_i|_v\right)^{\sum_{w\in\mathcal{M}_L(v)}n_w}=\left(\sup_{0\le i\le n}|x_i|_v\right)^{[L:K]n_v}.$$

On the other hand for a fractional ideal $\mathfrak{a}\subset K$ we have $\mathbf{N}(\mathfrak{a}\cdot\mathcal{O}_L)=(\mathbf{N}\mathfrak{a})^{[L:K]}$, so that $H_L(x)=H_K(x)^{[L:K]}$. It follows that the function

$$\begin{aligned}H:\mathbf{P}^n(\overline{\mathbf{Q}})&\longrightarrow\mathbf{R}\\ P&\longmapsto\left(H_K(P)\right)^{\frac{1}{[K:\mathbf{Q}]}}\end{aligned}$$

where K is any finite extension of $\mathbf{Q}$ such that $P\in\mathbf{P}^n\big(K\big)$ is well defined, and will be called the *absolute homogeneous ℓ^∞-height* on $\mathbf{P}^n\big(\overline{\mathbf{Q}}\big)$. The above formulation of the definition of heights is the one used by Northcott. We shall now give Weil's formulation. Let $\mathcal{M}_K^0$ denote the set of equivalence classes of non-archimedean absolute values of K. Since $\mathcal{M}_K^0$ is in one to one correspondence with the set of prime ideals of the ring of integers of K, for each $v\in\mathcal{M}_K^0$ we can choose a representative $|\cdot|_v$ for v by requiring that $|\cdot|_v$ restricted to $\mathbf{Q}$ coincide with the p-adic absolut value, where p is the prime lying below $\mathfrak{p}$, $\mathfrak{p}$ being the prime ideal corresponding to v. Explicitly given $v\in\mathcal{M}_K^0$ we set $|a|_v = (1/p)^{(\mathrm{ord}_\mathfrak{p}(a)/e_\mathfrak{p})}$ where $\mathrm{ord}_\mathfrak{p}$ is the discrete valuation associated to $\mathfrak{p}$ and $e_\mathfrak{p} = \mathrm{ord}_\mathfrak{p}(p\mathcal{O}_K)$. With this normalization the absolute norm of the fractional ideal $\mathfrak{a}$ generated by $x_0\dots x_n\in K$ is $(\mathbf{N}\mathfrak{a}) = \prod_{v\in\mathcal{M}_K^0}\sup_{0\leq i\leq n}|x_i|_v^{-n_v}$, where $n_v = [K_v : \mathbf{Q}_v]$. Therefore the absolute ℓ^∞-height of $P = [x_0:\ldots:x_n]\in\mathbf{P}^n\big(K\big)$ can be expressed as

$$H(P) = \prod_{v\in\mathcal{M}_K}\sup_{1\leq i\leq n}|x_i|_v^{d_v}$$

where $d_v = n_v/[K:\mathbf{Q}]$ and $\mathcal{M}_K = \mathcal{M}_K^0\bigcup\mathcal{M}_K^\infty$. As we shall see the above formulation is very convenient when proving inequalities about heights. For example it is now clear that $H(P)\geq 1$ for all $P\in\mathbf{P}^n\big(\overline{\mathbf{Q}}\big)$ and that H is invariant under the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.

As we said in the introduction one of the most important results about heights is Northcott's theorem which we shall now prove. Given $P = [x_0:\ldots:x_n]\in\mathbf{P}^n\big(\overline{\mathbf{Q}}\big)$ we denote by $d(P)$ the degree of the number field generated by the ratios x_i/x_j (with $x_j\neq 0$).

Theorem 2.1.(Northcott's finiteness theorem). *The set*

$$\mathcal{N}\big(\mathbf{P}^n,\, C,\, d\big) = \big\{P\in\mathbf{P}^n\big(\overline{\mathbf{Q}}\big)\mid H(P)\leq C \text{ and } d(P)\leq d\big\}$$

is finite for any choice of the constants C and d.

Proof. The case $d=1$ is immediate, for the points with height less then C are in one to one correspondence with prmitive points in Z^{n+1} with bounded ℓ^∞-norm. To reduce the general case to the case $d=1$ we need the following:

Lemma 2.2. *If $F(T) = T^d+\cdots+a_d = (T-b_1)\cdots(T-b_d)\in\overline{\mathbf{Q}}\big[T\big]$, then*

$$H\big([1:a_0:\ldots:a_d]\big)\leq 2^d\prod_{j=1}^d H\big([1:b_j]\big).$$

Proof. Let K be a number field containing all the roots and all the coefficients of $F(T)$. Denote by $S_1,\ldots,S_d$ the elementary symmetric functions in d variables and by S_0 the costant function with value 1. Then $a_i = S_i(b_1,\ldots,b_d)$ for all $i=1,\ldots,d$. Note that standard inequalities of real numbers (see [9]) yield:

(a) If $v\in\mathcal{M}_K^0$, then $\sup_{0\leq h\leq d}|S_h(b_1,\ldots,b_d)|_v^{d_v} = \prod_{j=1}^d\sup\big\{1,\,|b_j|_v^{d_v}\big\}$.

(b) If $v\in\mathcal{M}_K^\infty$, then $\sup_{h=0}^d|S_h(b_1,\ldots,b_d)|_v^{d_v}\leq 2^{dq}\prod_{j=1}^d(\sup\big\{1,\,|b_j|_v^{d_v}\big\})$.

For (a) one also need to use the ultrametric property of $|\cdot|_v$. Then

$$\begin{aligned} H\big([1:a_0:\ldots:a_d]\big) &= \prod_{v\in\mathcal{M}_K} \sup_{0\le i\le d} |S_i(b_1,\ldots,b_d)|_v^{d_v} \\ &\le \prod_{v\in\mathcal{M}_K^0}\prod_{j=1}^{d}\sup\{1,\,|b_j|_v^{d_v}\}\prod_{v\in\mathcal{M}_K^\infty} 2^{dn_v}\prod_{j=1}^{d}\sup\{1,\,|b_j|_v^{d_v}\} \\ &=2^d\prod_{j=1}^{d} H\big([1:b_j]\big). \end{aligned}$$

Now we can readly prove Northcott' theorem. Suppose $P\in\mathcal{N}\big(\mathbf{P}^n, C, d\big)$. Choose homogeneous coordinates $P=[x_0:\ldots:x_n]$ with $x_j=1$ for some j. Let

$$P_i(T)=T^{d_i}+a_{i,1}T^{d_i-1}+\ldots+a_{i,d_i}\in\mathbf{Q}[T]$$

be the minimal polynomial of x_i. Then $\deg(P_i(T))=[\mathbf{Q}(x_i):\mathbf{Q}]\le d$. Since $H\big([1:x_i]\big)\le H(P)$ the invariance of H under the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ combined with the above lemma yields $H\big([1:a_1:\ldots:a_{d_i}]\big)\le 2^{d_i}C^{d_i}$. Set $\mathcal{Q}=\coprod_{l/d}\mathcal{N}\big(\mathbf{P}^l,\,2^lC^l,\,1\big)$ then what the above argument is sayng is that to any $P\in\mathcal{N}\big(\mathbf{P}^n, C, d\big)$ we can associate a point of $\mathcal{Q}^n$, the cartesian product of $\mathcal{Q}$ with itself n-times, which is a finite set. Since the number of points of $P\in\mathcal{N}\big(\mathbf{P}^n, C, d\big)$ mapping to the same point is at most $n!\prod_{l/d} l$ the theorem follows. ∎

Another important aspect of the theory of heights is the study of the behaviour of H under morphisms. Let us start with the Segre

$$\begin{aligned} \sigma_{n,m}:\mathbf{P}^n\times\mathbf{P}^m &\longrightarrow \mathbf{P}^{(n+1)(m+1)-1} \\ \big([x_0:\ldots:x_n],[y_0:\ldots:y_m]\big) &\mapsto [x_0y_0:\ldots x_iy_j:\ldots x_ny_m] \end{aligned}$$

and the d^{th} Veronese embedding

$$\begin{aligned} \nu_d:\mathbf{P}^n &\longrightarrow \mathbf{P}^{N-1} \\ [x_0:\ldots:x_n] &\mapsto [F_0(x_0,\ldots x_n):\ldots:F_N(x_0,\ldots x_n)] \end{aligned}$$

where $F_0,\ldots F_N$ are all the monomials of degree d in n variables so $N=\binom{n+d}{d}-1$.

Proposition 2.3. *(a)* $H\big(\sigma(P,\,Q)\big)=H(P)H(Q)$.
(b) $H\big(\nu_d(P)\big)=H(P)^d$.

Proof. Let K be a field of definition for $P=[x_0:\ldots:x_n]$ and $Q=[y_0:\ldots:y_m]$. Then for every $v\in\mathcal{M}_K$ we have $\sup_{0\le i,j\le n}|x_iy_j|_v=\sup_{0\le i\le n}|x_i|_v\sup_{0\le j\le n}|y_j|_v$, from which (a) follows.
To prove (b) it suffices to note that for $v\in\mathcal{M}_K$, we have $\sup_{0\le i\le N}|F_i(x_0,\ldots x_n)|_v=|x_{j_v}|_v^d=\big(\sup_{0\le i\le n}|x_i|_v\big)^d$. ∎

Now we pass to the study of the behaviour of H under arbitrary morphisms between projective spaces. Any non constant morphism $\varphi:\mathbf{P}^n\to\mathbf{P}^m$ can be obtained as the composition of the d^{th} Veronese embedding (where d is the unique positive integer such that $\varphi^*\mathcal{O}_{\mathbf{P}^m}(1)\simeq\mathcal{O}_{\mathbf{P}^n}(d)\big)$, a linear projection $\mathbf{P}^N-\Lambda\to\mathbf{P}^m$ and an automorphism of $\mathbf{P}^m$. The integer d will be called the degree of φ. We have alredy seen how H behaves under the Veronese embedding so we now have to study its behaviour under linear projections and automorphisms.

Lemma 2.4. *Let $\varphi : \mathbf{P}^n \to \mathbf{P}^m$ be a morphism defined by linear polynomials. Then*

$$H\big(\varphi(P)\big) \leq CH(P)$$

for some positive constant C. In particular if $\alpha : \mathbf{P}^n \to \mathbf{P}^m$ is an automorphism then $C'H(P) \leq H\big(\varphi(P)\big) \leq CH(P)$ for two constants $C, C' > 0$.

Proof. Let $f_0, \dots f_m$ be the linear polynomials defining φ. Given $P = [x_0 : \dots : x_n]$ in $\mathbf{P}^n\big(\overline{\mathbf{Q}}\big)$, choose a finite extension K of $\mathbf{Q}$ so that $P \in \mathbf{P}^n\big(K\big)$ and K contains all the coefficients of the f_i's. Denote by $\Upsilon_f \subset K$ the set of the coefficients of the f_i's. For $v \in \mathcal{M}_K$ set $c_v(f) = \sup_{a \in \Upsilon_f} |a|_v$. If $v \in \mathcal{M}_K^0$ then

$$|f_i(x_0, \dots x_n)|_v \leq c_v(f) \sup_{0 \leq i \leq n} |x_i|_v,$$

while if $v \in \mathcal{M}_K^\infty$, then $|f_j(x_0, \dots x_n)|_v \leq (n+1) c_v(f) \sup_{0 \leq i \leq n} |x_i|_v$. Therefore $H\big(f(P)\big) \leq (n+1) c(f) N H(P)^d$ where $c(f) = \Big(\prod_{v \in \mathcal{M}_K} c_v(f)^{n_v}\Big)^{\frac{1}{[K:\mathbf{Q}]}}$ is independent of the field K. ■

Lemma 2.5. *Let $\Lambda \subset \mathbf{P}^n$ be a linear subvariety of dimension d. Let $\pi : \mathbf{P}^n - \Lambda \to \mathbf{P}^{n-d-1}$ be the linear projection defined by Λ, $X \subset \mathbf{P}^n$ be a closed projective subvariety not intersecting Λ. Then there exist two positive constants C_1, C_2 such that*

$$C_1 H(P) \leq H\big(\pi(P)\big) \leq C_2 H(P)$$

for all $P \in X\big(\overline{\mathbf{Q}}\big)$.

Proof. Note that it suffices to prove the case $d = 0$. For the image of a closed sub-variety not meeting the subspace from which we are projecting is again closed. Suppose first that $d = 0$ hence Λ is a point. By lemma 2.4 we can assume that the point is $E_n = [0 : \dots : 0 : 1]$. Moreover we can replace X by a hypersurface that contains it and which does not contain E_N. So we assume that X is the zero locus of a homogeneous polynomial $F(T_0, \dots, T_n)$ of degree m not vanishing at E_n. Then the coefficient of T_n^m is not zero and so T_n^m is a linear combination of the other monomials. In other words if we let $\mu : \mathbf{P}^n \to \mathbf{P}^{N-1}$ be the map defined by all the monomials of degree m with exception of T_n^m, there exists a morphism $\phi : \mathbf{P}^{N-1} \to \mathbf{P}^N$, defined by linear polynomials, such that $\nu_m(P) = \phi\big(\mu(P)\big)$. By proposition 2.4 there exists $C' > 0$ such that

$$H\big(\nu_m(P)\big) \leq C' H\big(\mu(P)\big) \qquad \text{for all } P \in X\big(\overline{\mathbf{Q}}\big).$$

On the other hand the coordinates of $g(P) = \sigma_{r,n-1}\big(\nu_{m-1}(P), \pi(P)\big)$ are exactly, up to repetitions, the coordinates of $\mu(P)$. Then by propositon 2.3

$$H\big(\mu(P)\big) \leq CH\big(\nu_{m-1}(P)\big) H\big(\pi(P)\big).$$

Combining these inequalities with the trasformation property of H with respect to the Veronese embeddings yields the lower bound for $H\big(\pi(P)\big)$. The upper bound being clear the proof is completed. ■

As consequence of the above results we have

Proposition 2.6. *Let $f : \mathbf{P}^n \longrightarrow \mathbf{P}^m$ be a morphism of degree d defined over $\overline{\mathbf{Q}}$. Then there are two constants C_1, $C_2 \in \mathbf{R}^+$ such that*

$$C_1 H(P)^d \leq H\big(f(P)\big) \leq C_2 H(P)^d.$$

3. Heights on Projective Varieties

Let X be a smooth projective variety and $\mathcal{L}$ an invertible sheaf on X which is generated by global sections. Recall that the morphisms $\varphi : X \to \mathbf{P}^n$ such that $\varphi^* \mathcal{O}_{\mathbf{P}^n}(1) \simeq \mathcal{L}$, correspond to sets of global sections which generate $\mathcal{L}$. Moreover any two morphisms $\varphi : X \to \mathbf{P}^n$, $\psi : X \to \mathbf{P}^m$, $(m \geq n)$ with the above property differ by a suitable linear projection $\mathbf{P}^m - \Lambda \to \mathbf{P}^n$ and an automorphism of $\mathbf{P}^n$, where Λ has dimension $m-n-1$. In particular the quotient $(H \circ \varphi)/(H \circ \psi)$ is a positive bounded function on $X(\overline{\mathbf{Q}})$ by lemmata 2.4 and 2.5.

The above argument can be rephrased as saying that we can assign to any invertible sheaf $\mathcal{L}$ an element $\mathfrak{H}_{\mathcal{L}} \in \mathcal{H}(X(\overline{\mathbf{Q}})) = \mathcal{F}(X(\overline{\mathbf{Q}}), \mathbf{R}^+)/\mathcal{BF}(X(\overline{\mathbf{Q}}), \mathbf{R}^+)$, where $\mathcal{F}(X(\overline{\mathbf{Q}}), \mathbf{R}^+)$ is the multiplicative group of functions on $X(\overline{\mathbf{Q}})$ with values in the positive real numbers and $\mathcal{BF}(X(\overline{\mathbf{Q}}), \mathbf{R}^+) \subset \mathcal{F}(X(\overline{\mathbf{Q}}), \mathbf{R}^+)$ is the subgroup of bounded functions. Note also that $\mathfrak{H}_{\mathcal{L}}$ depends only on the isomorphism class of $\mathcal{L}$.

Lemma 3.1. *Suppose that $\mathcal{L}$ and $\mathcal{M}$ are two invertible sheaves generated by global sections. Then $\mathfrak{H}_{\mathcal{L} \otimes \mathcal{M}} = \mathfrak{H}_{\mathcal{L}} \mathfrak{H}_{\mathcal{M}}$.*

Proof. Choose bases for $\Gamma(X, \mathcal{L})$, $\Gamma(X, \mathcal{M})$, and let φ and ψ be the associated morphisms to the projective space. Then the morphism

$$X \xrightarrow{\varphi \times \psi} \mathbf{P}^n \times \mathbf{P}^m \xrightarrow{\sigma_{n,m}} \mathbf{P}^{(n+1)(m+1)-1}$$

corresponds to the basis for $\Gamma(X, \mathcal{L} \otimes \mathcal{M})$ given by the tensor product of the bases chosen for $\Gamma(X, \mathcal{L})$ and $\Gamma(X, \mathcal{M})$. The result then follows from the fact that height behaves multipicatively with respect to the Segre product (cf. proposition 2.3). ∎

The above lemma enables us to define $\mathfrak{H}_{\mathcal{L}}$ even for invertible sheaves that are not generated by global sections. In fact given any $\mathcal{L}$ there exist $\mathcal{M}_0$ and $\mathcal{M}_1$ generated by global sections, such that $\mathcal{L} = \mathcal{M}_0 \otimes \mathcal{M}_1^{-1}$. It is then natural to set $\mathfrak{H}_{\mathcal{L}} = \mathfrak{H}_{\mathcal{M}_0}/\mathfrak{H}_{\mathcal{M}_1}$.

Before stating the next proposition, which is essentially due to Weil (cf. [27]) and summarizes the properties of this construction, we need three more pieces of notation. Given $F \in \mathcal{F}(X(\overline{\mathbf{Q}}), \mathbf{R}^+)$ we denote by $\mathrm{cl}(F)$ its class in $\mathcal{H}(X(\overline{\mathbf{Q}}))$. Given $\mathfrak{H} \in \mathcal{H}(X(\overline{\mathbf{Q}}))$ we say that $\mathfrak{H}$ satisfies the Northcott's finiteness property if for one (and hence for all) representative $F_{\mathfrak{H}}$ of $\mathfrak{H}$ in $\mathcal{F}(X(\overline{\mathbf{Q}}), \mathbf{R}^+)$ the following holds: for any constant $C > 0$ and any number field K the set $\mathcal{N}(X(K), C, H) = \{P \in X(K) \mid H(P) \leq C\}$ is finite. Finally we remark that any map $\phi : Y(\overline{\mathbf{Q}}) \to X(\overline{\mathbf{Q}})$ gives rise to a homomorphism $\mathcal{H}(X(\overline{\mathbf{Q}})) \to \mathcal{H}(Y(\overline{\mathbf{Q}}))$ which, by an abuse of notation, will also be denoted by ϕ.

Theorem 3.2. *The assignment $\mathcal{L} \longmapsto \mathfrak{H}_{\mathcal{L}}$ defines a homomorphism:*

$$\Phi : \mathrm{Pic}(X) \longrightarrow \mathcal{H}(X(\overline{\mathbf{Q}}))$$

which has the following properties:
(a) If $\mathcal{L}$ is generated by global sections and $\varphi : X \to \mathbf{P}^n$ is a morphism associated to some basis of $\Gamma(X, \mathcal{L})$, then $\mathrm{cl}(H_{\varphi,q}) = \mathfrak{H}_{\mathcal{L}}$, for all $1 \leq q \leq \infty$. Moreover Φ is uniquely charcterized by this property.
(b) If $\psi : Y \to X$ is a morphism of smooth varieties, and $\mathcal{L}$ is an invertible sheaf

on X, then $\phi(\mathfrak{H}_{\mathcal{L}}) = \mathfrak{H}_{\phi^*\mathcal{L}}$.
(c) If $\mathcal{L}$ is ample, then $\mathfrak{H}_{\mathcal{L}}$ satisfies the Northcott'finiteness property.
(d) If $s\in\Gamma(X, \mathcal{L})$, there exists $H\in\mathfrak{H}_{\mathcal{L}}$ such that $H \geq 1$ outside the (divisor of) zeroes of s.

Proof. We need only to prove (c) and (d), since (a) and (b) follow directely from the construction of Φ.
(c) Note that if $\mathcal{L}$ is ample then $\mathcal{L}^m$ is very ample for some m large enough. Thus there exists a basis of $\Gamma(X, \mathcal{L})$ such that the associated morphism $\varphi : X \to \mathbf{P}^n$ is a closed immersion. Therefore $\mathfrak{H}_{\mathcal{L}^m}$ satisfeis the Northcott's finiteness property and since $\mathfrak{H}_{\mathcal{L}^m} = \mathfrak{H}_{\mathcal{L}}^m$ so does $\mathfrak{H}_{\mathcal{L}}$.
(d) Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be two very ample invertible sheaves such that $\mathcal{L} \simeq \mathcal{M}_0 \otimes \mathcal{M}_1^{-1}$. By definition $\mathfrak{H}_{\mathcal{L}} = \mathfrak{H}_{\mathcal{M}_0}/\mathfrak{H}_{\mathcal{M}_1}$ thus to prove (d) we need to show that we can find representatives F_0 and F_1 for $\mathfrak{H}_{\mathcal{M}_0}$ and $\mathfrak{H}_{\mathcal{M}_1}$ such that $F_1 \leq CF_0$, with C a positive constant. Let $s_0, \dots s_n$ be a basis for $\Gamma(X, \mathcal{M}_1)$, φ_1 the associated morphism to $\mathbf{P}^n$. By construction ss_i is a global section of $\mathcal{M}_0$, so we can find a basis of $\Gamma(X, \mathcal{M}_0)$ of the form $\{ss_0, \dots, ss_n, r_1, \dots r_m\}$. Let φ_0 be the associated morphism to $\mathbf{P}^{n+m}$. Then if we set $F_0 = H\circ\varphi_0$ and $F_1 = H\circ\varphi_1$ the desired inequality is satisfeid. ∎

Note that the homomorphism Φ extends formally to a homomorphism from $\mathrm{Pic}(X)_{\mathbf{R}} = \mathrm{Pic}(X) \otimes \mathbf{R}$ to $\mathcal{H}(X(\overline{\mathbf{Q}}))$. One would like to attach to any $c\in \mathrm{Pic}(X)$ a function on $X(\overline{\mathbf{Q}})$ and not just a class modulo bounded functions. The next lemma, due to Tate (cf. [22, lemma 3.1] or [4, theorem 1.1.1]), gives a sufficient condition for this to happen. In this contest it is convenient to use the "O(1)" notation: given two real valued functions f and g on a set S we write $f = g + \mathrm{O}(1)$ if $|f - g|$ is bounded on S.

Lemma 3.3. *Let S be a set, $\pi : S \to S$ a map and f a real valued function on S such that $f\circ\pi = df + \mathrm{O}(1)$ for some real number $d > 1$. Then*
(a) There exists a unique real valued function $\hat{f}$ such that
- *(1)* $\hat{f} = f + \mathrm{O}(1)$
- *(2)* $\hat{f}\circ\pi = d\hat{f}$
- *(3)* $\hat{f}(x) = \lim_{n\to\infty} \frac{1}{d^n} f(\pi^n(x)) \quad \forall x\in S.$

(b) Let $\sigma : S \to S'$ be a map and let $\pi' : S' \to S'$ be such that $\pi'\circ\sigma = \sigma\circ\pi$. Suppose that f' is a real valued function on S' such that $f'\circ\pi' = df' + \mathrm{O}(1)$ and set $f = f'\circ\sigma$. Then $f\circ\pi = df + \mathrm{O}(1)$ and $\widehat{f} = \widehat{f'}\circ\sigma$.
(c) Let $\{\pi_i\}$ be a family of endomorphisms of S commuting with π, and suppose that $f\circ\pi_i = d_i f + \mathrm{O}_i(1)$ where $d_i\in\mathbf{R}$ and $\mathrm{O}_i(1)$ means that the bound depends on i. Then $\hat{f}\circ\pi_i = d_i\hat{f}$.

Proof. Let B be the Banach algebra of bounded real valued functions on S equipped with the sup norm. $T : B \to B$, $T(f) = f\circ\pi$ is a contraction operator (i.e. $\|T\| \leq 1$), therefore $1 - \frac{1}{\lambda}T$ is invertible and its inverse is given by $(1 - \frac{1}{\lambda}T)^{-1} = \sum_{k=0}^{\infty}(\frac{1}{\lambda}T)^k$. By hypothesis $(\frac{1}{d})f\circ\pi - f$ is a bounded function, hence there exits $g\in B$ such that

$$(1 - \frac{1}{\lambda}T)g = (\frac{1}{d})f\circ\pi - f.$$

It is immediate to verify that $\widehat{f} = f + g$ has the required properties proving (a). (b) follows from the uniqueness of $\widehat{f}$, while (c) follows from (b) applied to $S = S'$, $\sigma = \pi_i$ and $\pi' = \pi$. ∎

Corollary 3.4. *Let X be a smooth projective variety defined over $\overline{\mathbf{Q}}$ and $\pi : X \to X$ a morphism. Suppose that $\mathcal{L} \in \mathrm{Pic}(X)_{\mathbf{R}}$ is such that $\pi^*\mathcal{L}$ is isomorphic to $\mathcal{L}^d$ for some $d > 1$. Then there exists a unique function $H_{\mathcal{L}}$ such that*
(a) $\mathrm{cl}(H_{\mathcal{L}}) = \mathfrak{H}_{\mathcal{L}}$.
(b) $H_{\mathcal{L}} \circ \pi = H_{\mathcal{L}}^d$.
$H_{\mathcal{L}}$ is called the normalized (or canonical) height associated to $\mathcal{L}$ with respect to π.

Unfortunately there are very few examples of triples $(X, \mathcal{L}, \pi)$ satisfying the hypothesis of the corollary. As far as we know they are:
- $(\mathbf{P}^n, \mathcal{L}, [n])$ where φ is any morphism defined by homogeneous polynomials (with no common zeroes) of degree $d \geq 2$ and $\mathcal{L}$ is any positive invertible sheaf.
- $(S, \mathcal{L}, \pi)$ where $S \subset \mathbf{P}^2 \times \mathbf{P}^2$ is a K3 surfaces given by the intersection of (2,2)-form and a (1,1)-form, $\pi = \sigma_1 \circ \sigma_2$ where σ_1, σ_2 are the two involutions of S induced by the two double covers $p_i S : \to \mathbf{P}^2$, and finally $\mathcal{L} = \left(p_1^*\mathcal{O}_{\mathbf{P}^2}(1) + p_2^*\mathcal{O}_{\mathbf{P}^2}(1)\right)^{\beta - 1}$ with $\beta = 2 + \sqrt{3}$. For more information on this example see [21].
- $(A, \mathcal{L}, [n])$ where A is an abelian variety, $\mathcal{L}$ is a simmetric or anti-symmetric invertible sheaf and $[n]$ is the multiplication by n map.

In the last example we have a rather large group of commuting endomorphisms and so it is possible to defined a normalized height independently of the morphism chosen, see proposition 3.5 below. There is also another approch to the construction of heights on abelian varieties which is due to Néron and which will be treated in the next section.

Proposition 3.5. *Let A be an abelian variety. There exists a unique homomorphism $c \mapsto H_c$ from $\mathrm{Pic}(A)$ to $\mathcal{F}\left(A(\overline{\mathbf{Q}}), \mathbf{R}^+\right)$ such that*
(a) $\mathrm{cl}(H_c) = \mathfrak{H}_c$.
(b) For all endomorphisms $\varphi : A \to A$ we have, $H_{\varphi^* c} = H_c \circ \varphi$. In particular

$$H_c\left([n]a\right) = H_c^{\frac{n(n+1)}{2}}(a) H_{c^-}^{\frac{n(n-1)}{2}}(a) \quad \forall n \in \mathbf{Z}.$$

Moreover if $\psi : B \to A$ is a homomorphism , then $H_{\psi^* c} = H_c \circ \psi$ for all $c \in \mathrm{Pic}(A)$.

Proof. Given $c \in \mathrm{Pic}(A)$, we have $2c = (c + c^-) + (c - c^-)$. Since $c + c^-$ and $c - c^-$ are respectively symmetric and anti-symmetric, it suffices to establish the proposition for symmetric and anti-symmetric class. By the theorem of the cube $[n]^*c = n^2 c$ for c symmetric and $[n]^*c = nc$ for c anti-symmetric. Since [2] commutes with all endomorphisms of A the proposition follows from lemma 3.4. ∎

4. Nèron-Tate Heights on Abelian Varieties

Let A be an abelian variety. We have seen in the previous section how to construct a homomorphism from $\mathrm{Pic}(A)$ to $\mathcal{F}\left(A(\overline{\mathbf{Q}}), \mathbf{R}^+\right)$ using Tate's method for defining normalized heights. Now we want to examine Néron construction of normalized heights .
Néron approch (cf. [18]) is based on the costruction of a symbol (which since then has been called the Néron symbol)

$$\langle\, , \,\rangle_N : \mathrm{Div}(A) \times \mathcal{Z}_0(A) \to \mathbf{R}$$

where $\mathcal{Z}_0(A)$ is the group of zero cycles on A and is the group of divisors on A. The Nèron symbol is defined as a sum over all $v \in \mathcal{M}_K$ (K being a number field where the divisor and zero-cycle, of which we are computing the symbol, are defined) of local symbols, which in turns are defined in term of Néron's quasi-functions. We will not dwell here on the definiton of the local symbols, on the contrary we take as a starting poin of our discussion the global symbol.

The Néron symbol enjoys the following interesting and useful properties (see [18, Proposition 6]):

NS1 The symbol $\langle\, ,\, \rangle_N$ is bi-additive.

NS2 If D is linearly equivalent to zero, then $\langle D, \mathfrak{a}\rangle_N = 0$ for all $\mathfrak{a}\in\mathcal{Z}_0(A)$.

NS3 If $g : A \to B$ is a morphism, then $\langle g^*D, \mathfrak{a}\rangle_N = \langle D, g(\mathfrak{a})\rangle_N$.

Following Néron we will construct the canonical height pairing, the height pairing associated to an element of the Néron-Severi group of A and the canonical height associated to a divisor from the Néron symbol. First of all we need two definitions and a lemma. The kernel of the homomorphism $S : \mathcal{Z}_0(A) \to A(\overline{\mathbf{Q}})$, defined by $\mathfrak{a} = \sum m_i(a_i) \mapsto \sum m_i a_i$, where on the right we are summing on A is denoted by $\mathcal{Z}_\iota(A)$. A divisor D is algebraically equivalent to zero if $D + a$ is linearly equivalent to D for all $a\in A(\overline{\mathbf{Q}})$. The set such divisors is denoted by $\mathrm{Div}_a(A)$ and the quotient of $\mathrm{Div}_a(A)$ modulo linear equivalence by $\mathrm{Pic}^0(A)$.

Lemma 4.1. *Let $D\in\mathrm{Div}_a(A)$. Then $\langle D, \mathfrak{a}\rangle_N = 0$. for all $\mathfrak{a}\in\mathcal{Z}_\iota(A)$.*

Proof. Suppose first that $\mathfrak{a} = (a + b) - (a) - (b) + (0)$. Then

$$\begin{aligned}
\langle D, \mathfrak{a}\rangle_N &= \langle D, (a + b) - (b)\rangle_N - \langle D, (a) - (0)\rangle_N && \text{(by bi-additivity)}\\
&= \langle D_{-b}, (a) - (0)\rangle_N - \langle D, (a) - (0)\rangle_N && \text{(by NS4)}\\
&= \langle D_{-b} - D, (a) - (0)\rangle_N && \text{(by bi-additivity)}\\
&= 0. && \text{(by NS2)}
\end{aligned}$$

The general case follows from the bi-additivity of the Néron symbol. ∎

Let $\widehat{A}$ be the dual abelian variety of A, which, when convenient, will be identified with $\mathrm{Pic}^0(A)$. The pairing $<\, ,\, > : \widehat{A}(\overline{\mathbf{Q}}) \times A(\overline{\mathbf{Q}}) \to \mathbf{R}$ defined by setting

$$<c, a> = \langle c, (a) - (0)\rangle_N$$

is well defined (by lemma 3.1 and NS2) and is called the *canonical height pairing* on $\widehat{A}(\overline{\mathbf{Q}}) \times A(\overline{\mathbf{Q}})$. By NS1 and NS3 the canonical height pairing satisfies the following properties:

HP1 The pairing $<\, ,\, >$ is bi-additive.

HP2 If $f : A \to B$ is a homomorphism, then $<\hat{f}(c), a> = <c, f(a)>\, \forall\, a\in A(\overline{\mathbf{Q}})$ and $\forall\, c\in\widehat{B}(\overline{\mathbf{Q}})$.

Next we want to define the height pairing associated to an element ξ of $\mathrm{NS}(A) = \mathrm{Pic}(A)/\mathrm{Pic}^0(A)$. Recall that to any $\xi\in\mathrm{NS}(A)$ we can associate a homomorphism $\varphi_\xi : A \to \widehat{A}$, by setting $\varphi_\xi(a) = \mathrm{cl}(D_a - D)$, where $D\in\mathrm{Div}(A)$ is any element of ξ. By the theorem of the square φ_ξ is a homomorphism. The symmetric bi-additive pairing $<\, ,\, >_\xi : A(\overline{\mathbf{Q}}) \times A(\overline{\mathbf{Q}}) \longrightarrow \mathbf{R}$, defined by

$$<a, b>_\xi = <\varphi_\xi(a), b> = -\langle D, (b + a) - (b) - (a) + (0)\rangle_N$$

is called the *height pairing associated to ξ*.

Finally the *Néron-Tate height* associated to a divisor $D\in\mathrm{Div}(A)$, is

$$\begin{aligned}
h_D : A(\overline{\mathbf{Q}}) &\to \mathbf{R}\\
a &\mapsto h_D(a) = -\langle D, (a) - (0)\rangle_N
\end{aligned}$$

By NS2 h_D depends only on the linear equivalence class of D and NS3 implies that given a homomorphism $f : B \to A$, then $h_{f^*(D)} = h_D \circ f$. Let us check which are the relations among the various height pairings and height functions we just defined

Proposition 4.2. *Let $D \in \mathrm{Div}(A)$ and ξ be the class of D in $\mathrm{NS}(A)$. Then*

$$h_D(a) = -\frac{1}{2} <a, a>_\xi + \frac{1}{2} <D^- - D, a>.$$

Proof. If $a \in A(\overline{\mathbf{Q}})$, then $h_D(a) + h_D(-a) = -<a, a>_\xi$ by the bi-additivity of $\langle \, , \, \rangle_N$. On the other hand by NS3 $h_D(-a) = -\langle D, (-a) - (0) \rangle_N = -\langle D^-, (a) - (0) \rangle_N$ hence

$$h_D(a) - h_D(-a) = \langle D^- - D, (a) - (0) \rangle_N = <D - D^-, a>.$$

Summing the two relations yield the proposition. ∎

Recall that $\widehat{A} \times A$ comes equipped with a divisor class $\mathcal{P}$, the Poincaré class, which is defined by requiring that $\mathcal{P}|_{\{a\} \times A}$ represents the element of $\mathrm{Pic}^0(A)$ determined by $\hat{a}$ under the identification of $\mathrm{Pic}^0(A)$ with $\widehat{A}$. One also require that $\mathcal{P}|_{\{0\} \times \widehat{A}}$ is trivial.

Proposition 4.3. *Let $\mathcal{P}$ be the Poincaré divisor class on $\widehat{A} \times A$. Then*

$$h_{\mathcal{P}}(c, a) = -<c, a> = h_c(a)$$

for all $a \in A(\overline{\mathbf{Q}})$ and $c \in \widehat{A}(\overline{\mathbf{Q}})$.

Proof. Since $\mathcal{P}$ is symmetric

$$h_{\mathcal{P}}(c, a) = -\frac{1}{2} <(c, a), (c, a)>_{\mathcal{P}}.$$

Using the defining properties of the Poincaré divisor class one immediately verifies that

$$<(0, a), (0, a)>_{\mathcal{P}} = 0 = <(c, 0), (c', 0)>_{\mathcal{P}}.$$

The proposition follows from

$$<(0, a), (c, 0)>_{\mathcal{P}} = <c, a> = <(c, 0), (0, a)>_{\mathcal{P}} \qquad (*)$$

in fact if $(*)$ holds we have

$$h_{\mathcal{P}}(c, a) = -\frac{1}{2} <(c, a), (c, a)>_{\mathcal{P}} = -\frac{1}{2} (<(0, a), (c, 0>_{\mathcal{P}} + <(c, 0), (0, a)>_{\mathcal{P}}) = <c, a>.$$

It remains to prove $(*)$. By the bi-additivity of $\langle \, , \, \rangle_N$ we have

$$<(c,\ 0), (0,\ a)>_{\mathcal{P}} = -\langle \mathcal{P}, (c,\ a) - (c,\ 0) \rangle_N - \langle \mathcal{P}, (0,\ a) + (0,\ 0) \rangle_N.$$

Given $c \in \widehat{A}$ set $f_c : A \to \widehat{A} \times A,\ a \mapsto (c,\ a)$, then, $f_c^*(\mathcal{P}) = c \in \mathrm{Pic}^0(A)$. By NS3

$$-\langle \mathcal{P}, (c,\ a) - (c,\ 0) \rangle_N = \langle f_c^*(\mathcal{P}), (a) - (0) \rangle_N = -\langle c, (a) - (0) \rangle_N = <c, a>.$$

For $a \in A$ set $f_a : \widehat{A} \to \widehat{A} \times A,\ c \mapsto (c,\ a)$. Since $\mathcal{P}$ induces 0 on $\widehat{A} \times 0$ we have

$$\langle \mathcal{P}, (0,\ a) + (0,\ 0) \rangle_N = \langle f_0^*(\mathcal{P}), (a) - (0) \rangle_N = \langle 0, (c) - (0) \rangle_N = 0$$

completing the proof. ∎

We now come to the relation between Néron-Tate height and normalized height.

Theorem 4.4. *Let* $c \in \mathrm{Pic}(A)$*. Then* $\exp(h_c) = H_c$*.*

Proof. The map $c \mapsto \exp(h_c)$ from $\mathrm{Pic}(A)$ to $\mathcal{F}\big(A(\overline{\mathbf{Q}}), \mathbf{R}^+\big)$ is a homomorphism by the bi-additivity of the Néron simbol and by NH 2 it satisfies property (b) of proposition 3.5. Therefore to prove the theorem it suffices to show that $\mathrm{cl}\big(\exp(h_c)\big) = \mathfrak{H}_c$. Since the proof of this uses properties of the local symbol's (and we have not even defined them) it will not be given here. The interested reader is referred to Néron's article ([18]) for the proof. ∎

We now derive some consequence of the above theorem

Proposition 4.5. *(a) If D is ample and symmetric then $h_D(a) \geq 0$ with equality if and only if $a \in A_{\mathrm{Tors}}$.*
(b) If ξ is a polarization on A, then for every number field K and every constant $C > 0$ the set $\big\{a \in A(K) \;\big|\; -{<}a, a{>}_\xi \leq C\big\}$ is finite.

Proof. (a) By the above theorem there exists a constant C such that $h_D(a) \geq C$ for all $a \in A\big(\overline{\mathbf{Q}}\big)$. Since D is symmetric $n^2 h_D(a) = h_D([n]a) \geq C$. Therfore $h_D(a) \geq C/n^2$ for all n, proving $h_D(a) \geq 0$. If $h_D(a) = 0$, then $h_D([n]a) = 0$ for all n. Thus, by Northcott's finiteness theorem the set $\{[n]a, n \in \mathbf{Z}\}$ is finite, i.e. a is a torsion point.
(b) Let D be an ample divisor such that $[D] = \xi$. Then $-{<},{>}_\xi = h_{D+D^-}$ and since $D + D^-$ is ample and symmetric (b) follows by combining theorem 4.4 with Northcott's finiteness theorem. ∎

Another applicaton of theorem 4.5 regards the non-degenarancy of the height pairing associated to a polarization. Let V be a vector space over $\mathbf{Q}$ and F a real valued symmetric bilinear form on V. Then F is said to be positive definte if for all finite dimensional subspaces W of V, F considered as symmetric bilinear form on $W \otimes \mathbf{R}$ is positive definite. The following criterion will be used.

Lemma 4.6. *Let V be a $\mathbf{Q}$ vector space and F a real valued symmetric bilinear form on V. If for any finitely generated subgroup Γ of V and for any $C \in \mathbf{R}$ there are only finitely many $\gamma \in \Gamma$ such that $F(\gamma, \gamma) \leq C$, then F is positive definite.*

Proof. Let Γ be a finitely generated subgroup of V. The assumption implies that F is positive on Γ and hence on $W_\Gamma = \Gamma \otimes \mathbf{R}$. Suppose that F is not definite. Then F would arise from a form on a proper subspace U of W_Γ by means of some projection $p : W_\Gamma \to U$. But $p\big|_\Gamma$ is injective by the finiteness hypothesis. It follows that $p(\Gamma)$ is not discrete in U and so there is a sequence γ_i of distinct elements of Γ such that $p(\gamma_i)$ approaches 0. But then $F(\gamma_i, \gamma_i) \leq 1$ for all i's sufficentely large, contraddicting our hypothesis. ∎

Theorem 4.7. *Let ξ be a polarization on A. Then $-{<}\,,{>}_\xi$ induces a positive definite symmetric bilinear form on $V = A\big(\overline{\mathbf{Q}}\big)/A\big(\overline{\mathbf{Q}}\big)_{\mathrm{Tors}}$.*

Proof. Let $\Gamma \subset V$ be a finitely generated subgroup. Let K be a number field such that $A\big(\overline{\mathbf{Q}}\big)$ contains a set of representatives $P_1, \ldots P_n$ for the generators of Γ. By proposition 4.5 there are only finitely many points P in the subgroup generated by $P_1, \ldots P_n$ such that $-{<}P, P{>}_\xi \leq C$ for any given constant C. Thus the symmetric bilinear pairing induced by $-{<}\,,{>}_\xi$ on V satisfies the hypothesis of lemma 4.6. and hence is positive definite. ∎

5. Heights on Curves

In this section C will denote a smooth curve of genus $g \geq 2$ defined over a number field K. We fix a divisor O of degree 1 on C with the property that $(2g-2)O$ is a canonical divisor on C. Since abelian varieties are divisible it is possible to find such a divisor at the cost of a finite extension of the base field. Now we embed C into its Jacobian $J = J(C) = \mathrm{Pic}^0(C)$ by mapping P to $\mathrm{cl}\big((P) - O\big) \in \mathrm{Pic}^0(C)$. From now on we consider C as a subvariety of J and we denote by ι the inclusion. Let C_d be the d-fold symmetric power of C. The embedding of C into its Jacobian induces birational maps of C_d to J. In particular the image of C_{g-1} is a divisor Θ on J, which can also be described as

$$\Theta = \big\{ a \in J \;\big|\; a = a_1 + \dots a_{g-1}, a_i \in C \big\}.$$

Note that in general $\Theta^- = \Theta + O'$ where $O' = -\mathcal{K} + (2g-2)O$ ($\mathcal{K}$ being a canonical divisor on C). The choice we have made for O is justified by observing that with this choice Θ^- turns out to be linearly equivalent to Θ. If we denote by θ the divisor class of Θ, we then have $\theta^- = \theta$. It can be shown that $\varphi_\Theta : J \to \widehat{J}$ is an isomorphism and thus J is selfdual. We identify J with $\widehat{J}$ via φ_Θ and we want to determine the Poincaré bundle on $J \times J$. Let $p_i : J \times J \to J$ be the projection onto the i^{th} factor and $s : J \times J \to J$ the morphism giving the group structure on J. Since

$$p_1^*\Theta\big|_{J\times a} + p_2^*\Theta\big|_{J\times a} - s^*\Theta\big|_{J\times a} \equiv \Theta + 0 - \tau_a\Theta \equiv \varphi_\Theta(a)$$

we have $\mathcal{P} = p_1^*\theta + p_2^*\theta - s^*\theta$. As consquence we get

Theorem 5.1. *The height pairing on a Jacobian variety is positive. Moreover the kernel, on each side, is J_{Tors}.*

Proof. Let $\delta : J \to J \times J$ denote the diagonal embedding. Then

$$-d^*\mathcal{P} = d^*(p_1^*\theta + p_2^*\theta - s^*\theta = [2]^*\theta - 2\theta = 3\theta + \theta^- - 2\theta = 2\theta.$$

By prop. 4.3. $<a, b> = -h_{\mathcal{P}}(a, b)$, therefore

$$<a, a> = -h_{\mathcal{P}}(a, a) = (-h_{\mathcal{P}} \circ d)(a) = h_{-d^*\mathcal{P}}(a) = h_{2\theta}(a) = 2h_\theta(a).$$

Since θ is ample and symmetric, proposition 4.6 applies proving the theorem. ∎

Corollary 5.2. *Let A be an abelian variety. The left and the right kernels of the height pairing are the torsion subgroups of $\widehat{A}$ and A respectively.*

Proof. By symmetry it suffices to prove the corollary for the kernel of one of the two sides only. Let J be a Jacobian variety which admits a surjective homomorphism f to A. Then $\widehat{f} : \widehat{A} \to \widehat{J}$ has finite kernel. Suppose that $b \in \widehat{A}(\overline{\mathbf{Q}})$ is such that $<b, a> = 0$ for all $a \in A(\overline{\mathbf{Q}})$. Then $0 = <b, f(c)> = <\widehat{f}(b), c>$ for all $c \in J(\overline{\mathbf{Q}})$. Therefore $\widehat{f}(b)$ belongs to $\widehat{J}_{\mathrm{Tors}}$. But $\widehat{f}$ has finite kernel so $b \in \widehat{A}_{\mathrm{Tors}}$. ∎

We will now discuss Mumford's inequality. Let Δ be the diagonal of $C \times C$. The basic relations among divisor that we will use are (cf. [17]):

M1 $\iota^*\Theta = gO$

M2 $(\iota \times \iota)^*(\mathcal{P}) \equiv \Delta - C \times O - O \times C$.

The translation into a statement about height functions is the following

Theorem 5.3 (Mumford's inequalty). *There exists a constant M_C depending only on C such that*

$$-<\iota(P), \iota(Q)>_\Theta \leq \frac{1}{2g}\left(h_\Theta(\iota(P)) + h_\Theta(\iota(Q))\right) + M_C\right)$$

for all $P \neq Q \in C(\overline{\mathbf{Q}})$.

Proof. Let f be the real valued function on the $\overline{\mathbf{Q}}$-valued points of $C \times C$ defined by

$$f(P,Q) = \exp\left(\frac{1}{g}\left(h_\Theta(\iota(P)) + h_\Theta(\iota(Q))\right) + 2<\iota(P), \iota(Q)>_\Theta\right).$$

To prove the proposition it then suffices to show that there exists a constant $M \geq 1$ such that $Mf(P,Q) \geq 1$. First of all note that (M1) yields

$$\mathfrak{H}_{(\iota\times\iota)^*(\mathcal{P})}) = \mathfrak{H}_\Delta \mathfrak{H}^{-1}_{C\times O}\mathfrak{H}^{-1}_{C\times O}.$$

On the other hand by theorem 4.3 $h_\mathcal{P}(a,b) = -<a,b>_\Theta$, thus (M2) implies

$$\mathrm{cl}(f) = \mathfrak{H}_{C\times O}\mathfrak{H}_{C\times O}\mathfrak{H}_{(\iota\times\iota)^*(\mathcal{P})} = \mathfrak{H}_\Delta.$$

We know by proposition 3.2 that there exists a representative H for $\mathfrak{H}_\Delta$ such that $H \geq 1$ away from Δ, proving the theorem. ∎

To conclude this short introduction to the theory of heights we want to very briefly examine the role of heights in Bombieri's and Vojta's proof of Mordell's conjecture. Vojta's main result [26], from which an explicit bound for the number of K-rational points is readly obtained, is

Theorem 5.4 (Vojta's inequality). *Let C be defined over a number field K. There exists an effectively determined constant $\gamma(C) > 1$, with the following property: for every pair of points $P, Q \in C(\overline{K})$ satisfying*

$$\gamma(C) \leq \sqrt{\tfrac{1}{2}h_\theta(P)}, \qquad \gamma(C)\sqrt{\tfrac{1}{2}h_\theta(P)} \leq \sqrt{\tfrac{1}{2}h_\theta(Q)}$$

we have

$$<P,Q>_\theta \leq \frac{3}{4}\sqrt{\tfrac{1}{2}h_\theta(P)}\sqrt{\tfrac{1}{2}h_\theta(Q)}.$$

We like to point out that while Vojta's proof of theorem 5.4 uses arithmetic intersection theory as developed by Gillet and Soule, Bombieri's one [2] uses tools that were all avaliable by 1965. We conclude by sketching how one deduces a bound for the number of K-rational points of C from Vojta's inequality, since it is a good example of the versatility of heights.

Theorem (Mordell's Conjecture). *The set $C(K)$ is finite. In particular we have*

$$\#\left\{P \in C(K) \;\middle|\; \sqrt{\tfrac{1}{2}h_\theta(P)} \geq \gamma(C)\right\} \leq \#J(K)_{\mathrm{Tors}} 10^\rho (1 \log\gamma/\log 2)$$

where ρ is the rank of $J(K)$.

Proof. As we have seen in proposition 4.5 $-<,>_\theta$ gives a structure of euclidean space to $\mathbf{R}^\rho=J(K)/J(K)_{\mathrm{Tors}}\bigotimes\mathbf{R}$. It is possible to cover the unit sphere $S^{\rho-1}$ with less than 10^ρ spherical caps B_j such that $<z_1,z_2>_\theta>3/4$ for any $z_1,z_2\in B_j$. Let Γ_j be the cone over B_j. Let $P_0,\dots P_m\in C(K)$ be such that the image of P_i lies in Γ_j, $\sqrt{\frac{1}{2}h_\theta(P_i)}\geq\gamma(C)$ and $\sqrt{\frac{1}{2}h_\theta(P_k)}\geq\sqrt{\frac{1}{2}h_\theta(P_i)}$ if $k\geq i$. By Mumford inequality $2\sqrt{<P_{i-1},P_{i-1}>_\theta}\leq\sqrt{\frac{1}{2}h_\theta(P_i)}$. In particular starting from P_m and iterating we get $2^m\sqrt{\frac{1}{2}h_\theta(P_0)}\leq\sqrt{\frac{1}{2}h_\theta(P_m)}$. On the other hand the images of the P_i's are contained in Γ_j and so $\sqrt{\frac{1}{2}h_\theta(P_i)}\geq\sqrt{<P_0,P_0>_\theta}\geq\gamma(C)$, so Vojta inequality implies

$$\sqrt{\tfrac{1}{2}h_\theta(P_i)}\leq\gamma(C)\sqrt{\tfrac{1}{2}h_\theta(P_0)}.$$

But then we must have $2^m\leq\gamma(C)$, i.e. $m\leq\log\gamma(C)/\log 2$. Thus each cone contains at most $1+\log\gamma(C)/\log 2$ points with height larger than $\gamma(C)$. The number of points of $C(K)$ with height less than $\gamma(C)$ is finite by propositions 3.4 and 4.4 and so the Mordell's conjecture follows. ∎

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