A note on height pairings on polarized abelian varieties

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Abstract. Let A be an abelian variety defined over a number field k. In this short note we give a characterization of the endomorphisms that preserve the height pairing associated to a polarization. We also give a functorial interpretation of this result.

Una Nota sugli accoppiamenti associati alle altezze sulle varietà abeliane polarizzate

Riassunto. Sia A una varietà abeliana definita su un campo di numeri k. In quest breve nota diamo una caratterizzazione degli endomorfismi che lasciano invariata la forma bilineare associata all'altezza canonica definita a partire da una polarizzazione. Diamo, inoltre, un interpretazione funtoriale di questo risultato.

1. Introduction

Let A be an abelian variety defined over a number field k. In [4], A. Néron introduced the canonical height pairing: $<\,,\,>: \hat{A}(\overline{k}) \times A(\overline{k}) \longrightarrow \mathbb{R}$, between A and its dual abelian variety, $\hat{A}$. This pairing satisfies the following fundamental properties:

HP1 The pairing $<\,,\,>$ is bilinear.
HP2 If $f: A \to B$ is a k-homomorphism, then

$$<\hat{f}(b), a> = <b, f(a)>$$

for all $a \in A(\overline{k})$ and $b \in \hat{B}(\overline{k})$.

A *polarization* on A is an isogeny $\lambda : A \to \hat{A}$, such that $\lambda_{\overline{k}} = \varphi_{\mathcal{L}}$ for some ample invertible sheaf on $A_{\overline{k}} = A \times_{\operatorname{spec} k} \operatorname{spec} \overline{k}$ where $\varphi_{\mathcal{L}}(a) = t_a^* \mathcal{L} \otimes \mathcal{L}^{-1}$, and t_a is the traslation by a map (cf. [3]). To any polarization λ we can then associate a symmetric bilinear pairing

$$\begin{aligned} <\,,\,>_\lambda : A(\overline{k}) \times A(\overline{k}) &\longrightarrow \mathbb{R}, \\ (a,b) &\longmapsto <\lambda(a), b>, \end{aligned}$$

which is called the *height pairing associated to* λ.

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In this short note we give a characterization of the endomorphisms that preserve the height pairing associated to a polarization (actually, we prove a slightly more general result see the proposition below). In order to state our functiorial interpretation of this result we need to define the category $\mathfrak{hgm}$, of height Galois modules. This is done as follows:

objects: pairs $(G, <\, ,\, >_G)$, where G is an abelian group endowed with an action of $\Gamma = \mathrm{Gal}\left(\overline{k}\,/\,k\right)$; $<\, ,\, >_G$ is a symmetric bilinear real valued pairing on G, which is Γ-equivariant if we let Γ act trivially on $\mathbb{R}$.

morphisms: Γ-equivariant homomorphisms $f : G \to H$ such that

$$<f(a), f(a')>_H = <a, a'>_G \qquad (*)$$

for all $a\, a' \in G$.

Theorem. *Let k be a number field and $\Gamma = \mathrm{Gal}\left(\overline{k}\,/\,k\right)$. Let $\mathfrak{pav}_k$ denote the category of polarized abelian varieties defined over k Then, the functor $\mathcal{F} : \mathfrak{pav}_k \to \mathfrak{hgm}$, which assign to (A, λ) the height Galois module $\mathcal{F}(A, \lambda) = (A(\overline{k}), <\, ,\, >_\lambda)$, and to any morphism $f : A \longrightarrow B$ the induced morphism $f : A(\overline{k}) \longrightarrow B(\overline{k})$, is fully faithful.*

2. Height Pairings and Homomorphisms

Let $(A, \lambda), (B, \eta)$, be two polarized abelian varieties defined over k. An homomorphism of polarized abelian varieties is a homomorphism $f : A \longrightarrow B$ such that $\lambda = \hat{f}\circ\eta\circ f$. We denote by $\mathrm{Hom}_k\big((A, \lambda), (B, \eta)\big)$ the set formed by those homomorphisms from (A, λ) to (B, η) that are defined over k.

Our aim is the following:

Proposition. *Let $g : A \to B$ be a morphism defined over k. Suppose that λ, η are polarizations on A and B respectively. Let $g = t_u \circ f$, where $u \in B(k)$, and $f : A \longrightarrow B$ is a homomorphism. Then*

$$<g(a), g(a')>_\eta = <a, a'>_\lambda \qquad \forall a, a' \in A(\overline{k}) \tag{2.1}$$

if and only if $f \in \mathrm{Hom}_k\big((A, \lambda), (B, \eta)\big)$ and u is torsion point.

Corollary. *Let (A, λ) be a polarized abelian variety and f an endomorphism of A. Then*

$$<f(a), f(a')>_\lambda = <a, a'>_\lambda \qquad \forall a, a' \in A(\overline{k})$$

if and only if $\lambda = \hat{f}\circ\lambda\circ f$.

We need a preliminary lemma.

Lemma. *Let λ and η be two polarizations on A. Then $<\, ,\, >_\lambda = <\, ,\, >_\eta \iff \lambda = \eta$.*

Proof. Let λ and η be two polarizations on A such that $<\, ,\, >_\lambda = <\, ,\, >_\eta$. Then

$$<\lambda(a) - \eta(a), a'> = <a, a'>_\lambda - <a, a'>_\eta = 0$$

for all a' in A. Since the kernel on each side of the Néron pairing is the torsion subgroup of $A(\overline{k})$ (see, e.g. [2, Theorem 5.6.3]) we find that for every $a \in A(\overline{k})$ there exists $n \in \mathbb{Z}$, depending on a, such that $[n]\lambda(a) = [n]\eta(a)$ Let C a simple abelian subvariety of A. Then $\lambda(a) = \eta(a)$ for infinitely many $a \in C(\overline{k})$, and thus λ and η coincide when restricted to C. The Poincaré reducibility theorem yields the lemma. ∎

Proof of the Proposition. We start by proving the proposition for homomorphisms. If $a, a' \in A(\overline{k})$, then

$$<f(a), f(a')>_\eta = <\eta\big(f(a)\big), f(a')> = <(\hat{f}\circ\eta\circ f)(a), a'> = <a, a'>_{\hat{f}\circ\eta\circ f} \tag{2.2}$$

If $f \in \mathrm{Hom}\big((A, \lambda), (B, \eta)\big)$, then $\hat{f}\circ\eta\circ f = \lambda$, and hence f has the desired property. Conversely, suppose that $<f(a), f(a')>_\eta = <a, a'>_\lambda$ for all $a, a' \in A(\overline{k})$. Then, by (2.2), we have $<\, ,\, >_{\hat{f}\circ\eta\circ f} = <\, ,\, >_\lambda$. Therefore, $\hat{f}\circ\eta\circ f = \lambda$ by the above lemma. Now we deal with the general case. Suppose $g = t_u \circ f$, where

$f\in \mathrm{Hom}\big((A,\,\lambda),\,(B,\,\lambda)\big)$, and $u\in B(k)$ a torsion point. The bilinearity of $<g(a), g(b)>_\eta$ combined with (2.2), gives

$$<g(a), g(b)>_\eta = <f(a), f(a')>_\eta = <a, a'>_\lambda.$$

Finally, suppose that $g = t_u \circ f$ satisfies (2.1). Then $<g(a), u>_\eta = 0$ for all $a\in A(\overline{k})$. Using the bilinearity of $<\,,\,>_\eta$ and (2.2), we find

$$<a, a'>_\lambda = <f(a), f(a')>_\eta = <a, a'>_{\hat{f}\circ\eta\circ f}.$$

It then follows form the lemma above that $\hat{f}\circ\eta\circ f = \lambda$. It remains to show that u is a torsion point. Let $\mathcal{L}$ be an ample invertible sheaf on such that $\eta_{\overline{k}} = \varphi_{\mathcal{L}}$. Then $\mathcal{L}_0 = \mathcal{L} \otimes \mathcal{L}^-$ (where $\mathcal{L}^- = [-1]^*\mathcal{L}$) is ample and symmetric. Since $\mathcal{L} \otimes (\mathcal{L}^-)^{-1}$ is algebraically equivalent to zero, we have that $\varphi_{\mathcal{L}} = \varphi_{\mathcal{L}^-}$, so

$$<u, u>_{\varphi_{\mathcal{L}_0}} = 2<u, u>_{\varphi_{\mathcal{L}}} = 2<u, u>_\eta = <0, 0>_\lambda = 0.$$

But, being $\mathcal{L}_0$ symmetric, $<u, u>_{\varphi_{\mathcal{L}_0}}$ is proportional to the canonical height associated to $\mathcal{L}_0$, which, for an ample symmetric divisor, vanishes only on torsion points. ∎

A Functorial Interpretation

The above proposition has a functorial interpretation as we shall now show. Let $\mathfrak{pav}_k$ denote the category whose objects are polarized abelian varieties defined over k, and let $\mathfrak{hgm}$ be the category of height Galois modules, which we defined in the introduction. Given $(G, <\,,\,>_G)$ and $(H, <\,,\,>_H)$, we denote by $\mathrm{Hom}_{\mathfrak{h}}(G, H)$ the set of morphisms (in the category of height Galois modules) from $(G, <\,,\,>_G)$ to $(H, <\,,\,>_H)$. We define a functor $\mathcal{F} : \mathfrak{pav}_k \to \mathfrak{hgm}$ as follows: given a polarized abelian variety $(A,\,\lambda)$ defined over k we let $\mathcal{F}(A,\,\lambda) = (A(\overline{k}), <\,,\,>_\lambda)$. Given a k-morphism $f : (A,\,\lambda) \to (B,\,\eta)$ then $\mathcal{F}(f)$ is just the restriction of f to $A(\overline{k})$, which is Γ-equivariant because f is defined over k. Moreover $\mathcal{F}(f)$ satisfies $(*)$ by the above proposition. The only thing that remains to be verified is that $(A(\overline{k}), <\,,\,>_\lambda)$ is an object of $\mathfrak{hgm}$, i.e. that $<\,,\,>_\lambda$ is invariant under the action of Γ. Recall that the canonical height pairing coincides with the canonical height associated to the Poincaré bundle on $\hat{A} \times A$ (this can be seen by comparing [4, section 14] and [5, section 3.4], also cf. [7, Appendix A1 Proposition 6]). But the Poincaré bundle is defined over the ground field, and absolute projective heights are invariant under the action of Γ (see. [6, lemma 5.10]). Therefore, $<\,,\,>$ is invariant under the action of Γ. Thus

$$<\sigma(a), \sigma(a')>_\lambda = <\lambda\big(\sigma(a)\big), \big(\sigma(a')\big)> = <\sigma\big(\lambda(a)\big), \sigma(a')\big)>_= <\lambda(a), a'> = <a, a'>_\lambda.$$

where we used that λ is defined over k.

Theorem. *Let k be a number field. Then the functor $\mathcal{F} : \mathfrak{pav}_k \longrightarrow \mathfrak{hgm}$ is fully faithful.*

Proof. The faithfulness of $\mathcal{F}$ follows directly from the above proposition. To prove that $\mathcal{F}$ is full let A and B be two abelian varieties defined over k, and suppose that $\tilde{f}$ is in $\mathrm{Hom}_{\mathfrak{h}}\big(A(\overline{k}), B(\overline{k})\big)$. In particular $\tilde{f}$ is a Γ-equivariant homomorphism from $A(\overline{k})$ to $B(\overline{k})$. It was shown by Faltings (as a non-trivial consequence of the Tate's conjecture) that the natural injection $\mathrm{End}_k(A) \hookrightarrow \mathrm{End}_\Gamma\big(A(\overline{k})\big)$ is an isomorphism (see [1, theorem 5 on page 205]). Applying this result to $A \times B$, we find that there exists an f belonging to $\mathrm{Hom}_k(A,\,B)$ such that $\mathcal{F} = \tilde{f}$. By assumption $\tilde{f}$ belongs to $\mathrm{Hom}_{\mathfrak{h}}\big(A(\overline{k}), B(\overline{k})\big)$, thus $<a, a'>_\lambda = <f(a), f(a')>_\eta$ for all $a\, a' \in A(\overline{k})$. By the above proposition f belongs to $\mathrm{Hom}_k\big((A,\,\lambda),\,(B,\,\eta)\big)$. ∎

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