① v.a. X = risultato del lancio di un dado a 6 facce $\quad \mathcal{X} = \{1, 2, 3, 4, 5, 6\}$

K

Dado "onesto": $p_X = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$

$\rightarrow H(X) = -6 \cdot 1/6 \log 1/6 = -\log 1/6 = \log 6 = \log |\mathcal{X}| \simeq 2.58$ bit

Dado "truccato": $p_X = (0.001, 0.001, 0.001, 0.001, 0.001, 0.995)$

$\rightarrow H(X) = -\sum_{i=1}^{6} p_i \log p_i \simeq \underline{0.057}$ bit

② Qual è il minimo di $H(p_1, p_2, \ldots, p_K) = H(p)$ quando p varia nell'insieme di tutte le d.d.p. K-dimensionali?

Determinare i vettori p che determinano il minimo di $H(p)$.

$$H(p) = -\sum_{i=1}^{k} p_i \log p_i = \sum_{i=1}^{k} \underbrace{\underbrace{p_i}_{\geq 0} \underbrace{\log(1/p_i)}_{\geq 0}}_{\geq 0} \geq 0$$

$H(p)$ è $= 0$ se e solo se $p_i \log(1/p_i) = 0$ per $i = 1, 2, \ldots, K$.

$\nearrow p_i = 0$

$\rightarrow \log(1/p_i) = 0 \Leftrightarrow p_i = 1$

Quindi p dev'essere della forma:

$$(0, 0, \dots, 0, 1, 0, \dots, 0)$$

Ci sono K vettori di quel tipo:

$$\left.\begin{matrix} (1, 0, \dots, 0) \\ (0, 1, \dots, 0) \\ \dots \\ (0, 0, \dots, 1) \end{matrix}\right\} \rightarrow H(p) = 0$$

③ Entropia vs. varianza

Entropia v.a. $X \in \mathcal{X}$ insieme di simboli ; $H(X) = -\sum_{i=1}^{K} p_i \log p_i$
$= \mathbb{E}[-\log p(X)]$

Varianza v.a. $X \in \mathbb{R}$ insieme di numeri reali ; $\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}X)^2]$

Se $\mathcal{X} \subseteq \mathbb{R}$, sono applicabili entrambi i concetti. Come si differenziano?

Esempio:

$$X = \begin{cases} 0 & \text{con prob. } 1/2 \\ a \ (\neq 0) & \text{con prob. } 1/2 \end{cases} \rightarrow H(X) = -\tfrac{1}{2}\log\tfrac{1}{2} - \tfrac{1}{2}\log\tfrac{1}{2}$$

$$= -\log\tfrac{1}{2} = \log 2 = 1 \text{ bit}$$

non dipende da a

$\mathcal{X} = \{\overset{1/2}{0}, \overset{1/2}{a}\}$

(a è una costante $\neq 0$)

linearità del valore atteso

$\mathrm{Var}(X)$? $\quad \mathbb{E}[X^2 - 2X\cdot\mathbb{E}X + (\mathbb{E}X)^2] = \mathbb{E}[X^2] - 2\,\mathbb{E}[X\cdot\mathbb{E}X] + \cancel{\mathbb{E}}[(\mathbb{E}X)^2]$

$X^2 = \begin{cases} 0^2 & \text{con p. } 1/2 \\ a^2 & \text{con p. } 1/2 \end{cases}$

$$= \mathbb{E}[X^2] - 2\,\underbrace{\mathbb{E}X\cdot\mathbb{E}[X]}_{(\mathbb{E}X)^2} + (\mathbb{E}X)^2 = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = \frac{1}{2}a^2 - \left(\frac{a}{2}\right)^2 = \boxed{\frac{a^2}{4}}.$$

dipende da a

X, Y v.a. v.a. indipendenti : $p_{XY} = p_X \cdot p_Y$ $\quad p(x,y) = p(x)\,p(y) \quad \forall (x,y) \in \mathcal{X} \times \mathcal{Y}$

vs $\Downarrow \quad \not\Uparrow$ $\quad \Longleftrightarrow D(p_{XY} \| p_X \cdot p_Y) = 0 \Longleftrightarrow I(X;Y) = 0$

v.a. scorrelate : $\mathrm{Cov}(X,Y) = 0$

$$\mathrm{Cov}(X,Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]$$

$$(\mathrm{Var}(X) = \mathrm{Cov}(X,X))$$

Esempio in cui X e Y sono scorrelate ma non indipendenti.

p_{XY}	Y: -1	0	$+1$	p_X
X: -1	0	1/4	0	1/4
0	1/4	0	1/4	1/2
$+1$	0	1/4	0	1/4
p_Y	1/4	1/2	1/4	

X e Y non sono indipendenti; per es.

$$\underbrace{p(X=-1, Y=-1)}_{=0} \neq p(\underbrace{X=-1}_{1/4}) \cdot p(\underbrace{Y=-1}_{1/4})$$

$$I(X;Y) = \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} p(x,y) \log \frac{p(x,y)}{p(x)\,p(y)}$$

$$= \cancel{1/4} \cdot \log \frac{\cancel{1/4}}{1/2 \cdot \cancel{1/4}} \cdot \cancel{4} = \log 2 = 1$$

$\mathbb{E}X = \frac{1}{4}(-1) + \cancel{\frac{1}{2} \cdot 0} + \frac{1}{4}(+1) = 0$

$\mathbb{E}Y = 0$

$\Rightarrow \mathrm{Cov}(X,Y) = \mathbb{E}[XY] = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 + \dots + \frac{1}{4} \cdot 0 = 0$

$\Rightarrow$ X e Y sono scorrelate.

---

(4) X, Y v.a.

Calcolare $H(X)$, $H(X|Y)$, e $I(X;Y)$.

p_{XY}	c	d	p_X
a	0	$1/8$	$1/8$
b	$3/4$	$1/8$	$7/8$
p_Y	$3/4$	$1/4$	

$p_X = (1/8, 7/8)$

$\Downarrow$

$H(X) = -\frac{1}{8}\log\frac{1}{8} - \frac{7}{8}\log\frac{7}{8} \approx 0.544$ bit

$H(X|Y) = \sum_{y \in \mathcal{Y}} p(y) H(X|Y=y) =$

$p(x|y) = \dfrac{p(x,y)}{p(y)}$

$= \frac{3}{4} \cdot H(X|Y=c) + \frac{1}{4} H(X|Y=d) =$

$p_{X|Y=c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad p_{X|Y=d} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

$= \cancel{\frac{3}{4}[\underbrace{H((0,1))}_{=0}]} + \frac{1}{4}[\underbrace{H((1/2,1/2))}_{= \frac{1}{4} \cdot 1}] = \frac{1}{4}$

$\log 2 = 1$ bit

$I(X;Y) = H(X) - H(X|Y) \cong 0.544 - 1/4$.

$H(Y) = \log 4 = 2$ bit

⑤ V.a. X, Y

Calcolare $H(X)$, $H(X|Y)$, e $I(X;Y)$.

p_{XY}	Y: 1	2	3	4	p_X
X: 1	$1/8$	$1/16$	$1/16$	$1/4$	$1/2$
2	$1/16$	$1/8$	$1/16$	0	$1/4$
3	$1/32$	$1/32$	$1/16$	0	$1/8$
4	$1/32$	$1/32$	$1/16$	0	$1/8$
p_Y	$1/4$	$1/4$	$1/4$	$1/4$	

$$H(X) = -\tfrac{1}{2}\log\tfrac{1}{2} - \tfrac{1}{4}\log\tfrac{1}{4} - \tfrac{2}{8}\log\tfrac{1}{8} = \tfrac{1}{2} + \tfrac{1}{4}\cdot 2 + \tfrac{2}{8}\cdot 3 = \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{3}{4} = \tfrac{7}{4}.$$

$$H(X|Y) = \sum_{y\in\mathcal{Y}} p(y) H(X|Y=y) = \tfrac{1}{4}H(X|Y=1) + \tfrac{1}{4}H(X|Y=2) + \tfrac{1}{4}H(X|Y=3) + \tfrac{1}{4}H(X|Y=4) = 11/8.$$

$$I(X;Y) = H(X) - H(X|Y) = 7/4 - 11/8 = 14/8 - 11/8 = 3/8.$$

6 SIGSALY

$(X \oplus Y) \oplus Y = X$

È possibile definire tre v.a. X, Z, Y:

$$\begin{cases} I(X;Z|Y) > 0 \quad ✓ \\ I(X;Z) = 0 \quad ✓ \end{cases}$$

$\mathcal{X} = \{0,1\}$

$p_X = (\alpha, 1-\alpha)$

per qualche $\alpha \in [0,1]$

$$X \Rightarrow \oplus \Rightarrow Z \overset{\text{Canale}}{=\!=\!=} Z \oplus \Rightarrow X$$

$$\Uparrow Y \longleftrightarrow \Uparrow Y$$

$Z = X \oplus Y$

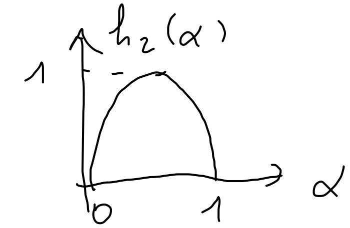
$p_Y = (1/2, 1/2)$

$\mathcal{Y} = \{0,1\}$

(a) Supp. che $p_X = (\alpha, 1-\alpha)$, $p_Y = (1/2, 1/2)$ e che X e Y siano v.a. indipendenti

Dimostrare che $H(X|Z) = H(X)$ ("Z non fornisce alcuna informazione su X")

(b) Supp. che $p_Y = (1/4, 3/4)$. Cosa cambia?



(a) $H(X) = -\alpha \log \alpha - (1-\alpha)\log(1-\alpha) = h_2(\alpha)$

$H(X) - H(X|Z) = I(X;Z)$ → dipende da p_{XZ}

$I(X;Z) = 0$

p_{XY} (X \ Y)	0	1	p_X
0	$\alpha/2$	$\alpha/2$	α
1	$(1-\alpha)/2$	$(1-\alpha)/2$	$1-\alpha$
p_Y	$1/2$	$1/2$	

$Z=0$ $Z=1$

p_{XZ} (X \ Z)	0	1	p_X
0	$\alpha/2$	$\alpha/2$	α
1	$(1-\alpha)/2$	$(1-\alpha)/2$	$1-\alpha$
p_Z	$1/2$	$1/2$	

$p_{XZ} = p_X \cdot p_Z$

$$I(X;Z|Y) = \frac{1}{2} I(X;Z|Y=0) + \frac{1}{2} I(X;Z|Y=1)$$

$$= \frac{1}{2} \overbrace{I(X;X)}^{\geq 0} + \frac{1}{2} \overbrace{I(X;1-X)}^{\geq 0}$$

Se $Y=0$, $X=Z$

Se $Y=1$, $X=1-Z$

$$\geq \frac{1}{2} H(X)$$

Quindi per esempio con $\alpha = 1/2$,

ho $I(X;Z|Y) \geq \frac{1}{2} H(X) = \frac{1}{2} > 0$.