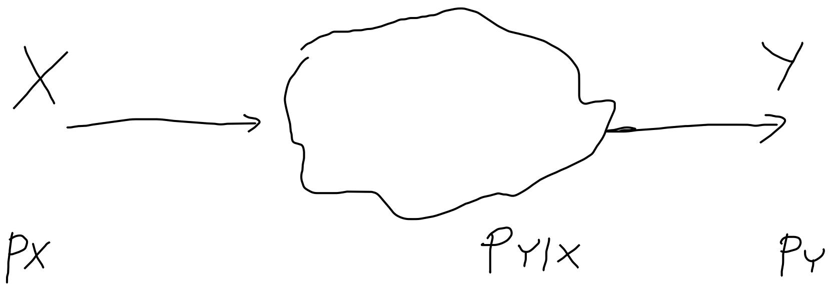
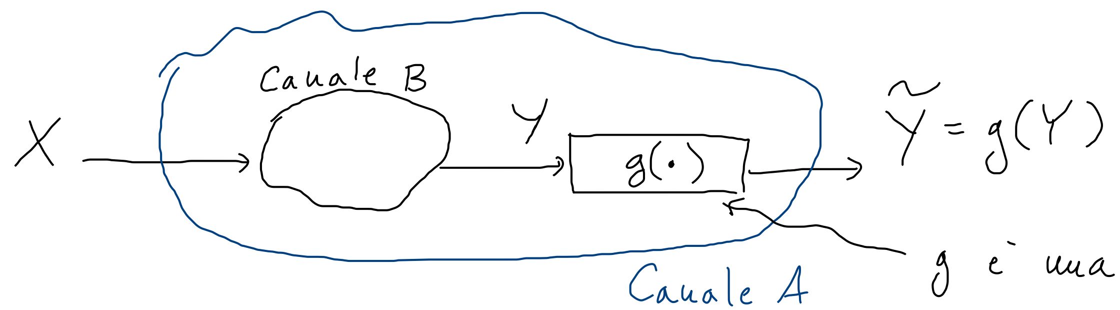
# CALCOLO DELLA CAPACITÀ DI CANALE



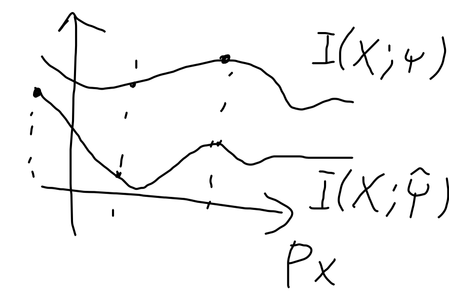

Capacità del canale : $\max_{p_X} I(X;Y) = C$

$p(y) = p(y|x) \cdot p(x) \quad \forall x \in \mathcal{X}, y \in \mathcal{Y}$

$\Gamma_{ij} = p(y_j | x_i)$ matrice

① 


g è una funzione (deterministica)




Dimostrare che $\boxed{C_A \leq C_B}$.

$$C_A = \max_{p_X} I(X;\tilde{Y}) \overset{?}{\leq} \max_{p_X} I(X;Y) = C_B$$

$\boxed{X \to Y \to \tilde{Y}}$

Mostriamo che, qualunque sia $p_X$, ho $I(X;\tilde{Y}) \overset{?}{\leq} I(X;Y)$

2° teorema di elaborazione dati : se vale la catena di Markov $X \to Y \to Z$ allora ho $I(X;Y) \geq I(X;Z)$

(V) Vale $X \to Y \to \tilde{Y}$ se la v.a. $\tilde{Y}$ è indipendente da $X$ data $Y$

ovvero: $\tilde{Y} | Y$ è indipendente da $X$

Nel nostro caso vale perché $\tilde{Y} = g(Y)$

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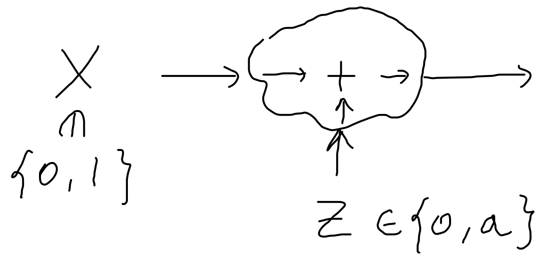
(2)




$a \in \mathbb{R}$ è un parametro

$Pr[Z=0] = 1/2$

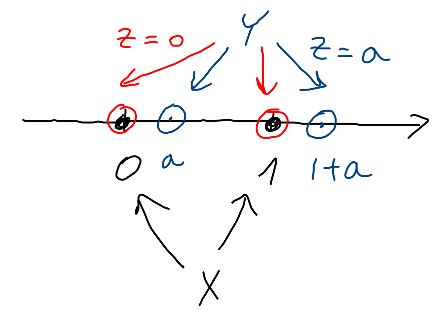
$Pr[Z=a] = 1/2$

| $X \downarrow$ / $Y$, $Z \to$ | 0 | a |
|---|---|---|
| 0 | 0 | a |
| 1 | 1 | 1+a |

$Pr[X=0] =: \alpha$

$Pr[X=1] =: 1-\alpha$

per qualche $\alpha \in [0,1]$.




Svolgimento.

Caso $\boxed{a=0}$: $Y = X + 0 = X$;

capacità del canale: $\max_{p_X} I(X;Y) = \max_{p_X} \overbrace{I(X;X)}^{H(X)} = 1$ bit. $\quad p_X = (1/2, 1/2)$

Caso $\boxed{a \neq 1 \ (e\ a \neq -1)}$

Capacità: $\max_{p_X} I(X;Y) = \max_{p_X} [H(Y) - H(Y|X)]$

$= \max_{p_X} [H(X) - H(X|Y)]$

$a = 1$

| $X \downarrow$ / $Y$, $Z \to$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 2 |

Scenario : $a \neq 1$, $a \neq -1$ $(a \neq 0)$

| $X \backslash Y$, $Z \rightarrow$ | $0$ | $a$ |
|---|---|---|
| $0$ | $0$ | $a$ |
| $1$ | $1$ | $1+a$ |

4 valori <u>tutti distinti</u>

In questo caso,

$H(X|Y) = 0$ perché $X$ è completamente determinata data $Y$.

$\Longrightarrow$ capacità $= \max\limits_{p_X} [H(X) - 0] = 1$ bit.

prendo $p_X = (1/2, 1/2)$

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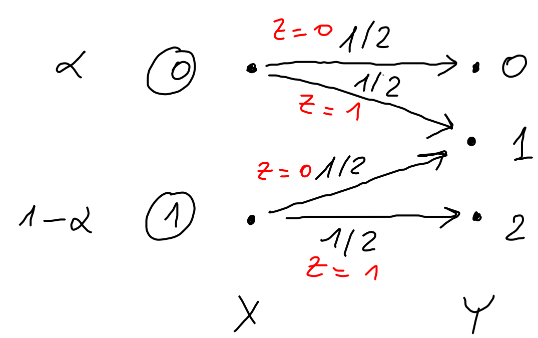
Ultimo caso: $\boxed{a = 1}$ oppure $\boxed{a = -1}$.

| $X \backslash Y$, $Z$ | $0$ | $1$ |
|---|---|---|
| $0$ | $0$ | $1$ |
| $1$ | $1$ | $2$ |

$Pr[Y=0] = Pr[X=0 \wedge Z=0] = Pr[X=0]Pr[Z=0] = \alpha \cdot 1/2$

$Pr[Y=1] = Pr[X=0 \wedge Z=1] + Pr[X=1 \wedge Z=0] = \alpha \cdot 1/2 + (1-\alpha) \cdot 1/2 = 1/2$.

$Pr[Y=2] = Pr[X=1 \wedge Z=1] = (1-\alpha) \cdot 1/2$.




Canale con cancellazione con parametro $\varepsilon = 1/2$

Capacità del canale con cancellazione:

$C = 1 - \varepsilon = 1 - 1/2 = 1/2$ bit.

③ $\mathcal{X} = \{0, 1, \dots, 10\}$

$\mathcal{Z} = \{1, 2, 3\}$

$Y = (X + Z) \bmod 11$

$\mathcal{Y} = \{0, 1, \dots, 10\}$

$$X \longrightarrow \underset{\substack{\uparrow \\ Z}}{\overset{\text{mod } 11}{\oplus}} \longrightarrow Y$$

con: $\Pr[Z=1] = \Pr[Z=2] = \Pr[Z=3] = 1/3$.

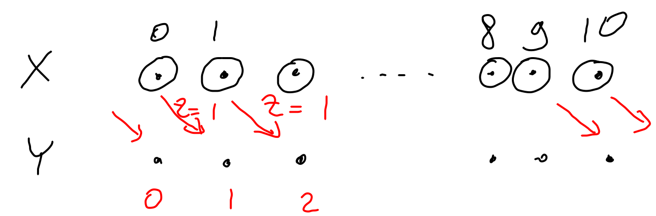
(a) Trovare la capacità del canale

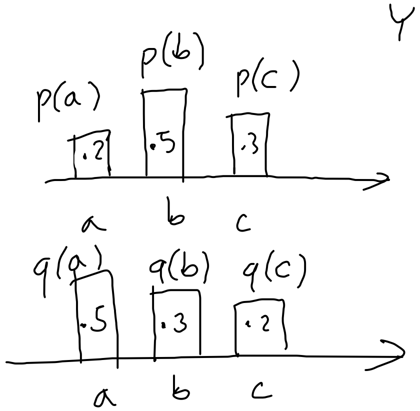
(b) Determinare la $p_X$ corrispondente (quella che massimizza $I(X;Y)$)

Sol.

$p_X$

$$H(Y|X) = H((X+Z) \bmod 11 \mid X)$$
$$= H(Z|X)$$
$$= H(Z) = \log 3 .$$




$$H(p) = H(q)$$

$$I(X;Y) = H(Y) - H(Y|X) =$$
$$= H(Y) - \log 3$$

$$C = \max_{p_X} \left[ H(Y) - \log 3 \right] = \left( \max_{p_X} H(Y) \right) - \log 3$$

ho =

Certamente ho $H(Y) \leq \log |\mathcal{Y}| = \log 11$

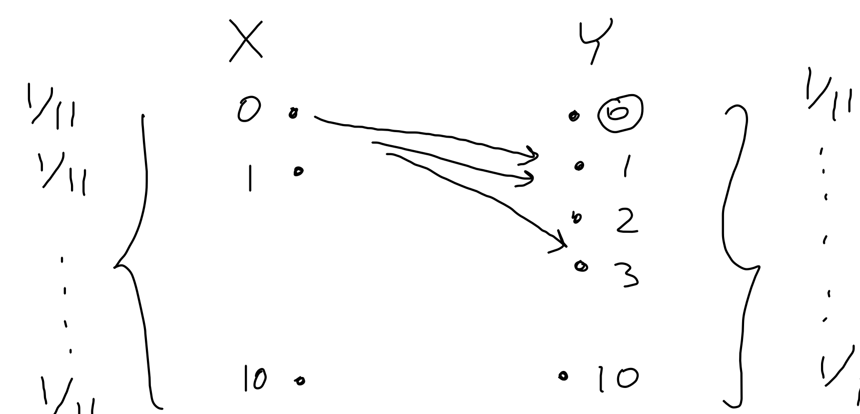
In effetti posso avere $H(Y) = \log 11$ quando $p_Y$ è uniforme

Posso scegliere $p_X$ tale da avere $p_Y$ uniforme? 0 1 .... 10

Sì, basta scegliere $p_X$ essa stessa uniforme: $p_X = (1/11, 1/11, \ldots, 1/11)$

$\rightarrow C = \log 11 - \log 3 = \log 11/3$. $\square$

scelgo $p_X$ uniforme (e quindi $p_Y$ uniforme)

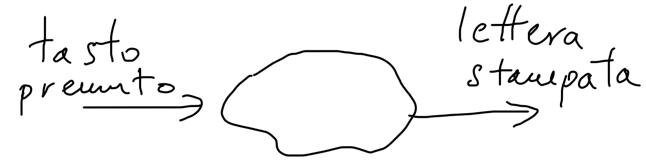



$3 \cdot \left( 1/11 \cdot 1/3 \right) = 1/11$

④ "Macchina da scrivere rumorosa" $\mathcal{Y} = \mathcal{X} = \{x_1, x_2, \ldots\ldots, x_{26}\}$

Consideriamo una macchina da scrivere con 26 tasti 'A', 'B', 'C', ...., 'Z'

→ (a) Se la pressione di ogni tasto produce la corrispondente lettera, quanto vale la capacità?




(b) se invece: 'A' $\xrightarrow{1/2}$ 'A', 'A' $\xrightarrow{1/2}$ 'B'; 'B' $\xrightarrow{1/2}$ 'B', 'B' $\xrightarrow{1/2}$ 'C' .... $\mathcal{Y} = \mathcal{X}$

... 'Z' $\xrightarrow{1/2}$ 'Z', 'Z' $\xrightarrow{1/2}$ 'A', quanto vale la capacità?

Sol. (a)

$$C = \max_{p_X} I(X;Y) = \max_{p_X} \big(H(X) - H(X|Y)\big)$$

$p_X$ uniforme

In questo caso, $H(X|Y) = 0$; quindi $C = \max_{p_X} H(X) = \log|\mathcal{X}| = \log 26$.

(b) $$C = \max_{p_X} I(X;Y) = \max_{p_X} \big(H(Y) - H(Y|X)\big)$$

$$H(Y|X) = \sum_{i=1}^{26} p(x_i) \underbrace{H(Y|X=x_i)}_{=1} = \sum_{i=1}^{26} p(x_i) = 1$$

$H(Y|X=x_i)$ è l'entropia di $P_{Y|X=x_i}$

$X = x_7$ $\quad Y = ?$ $\quad Y \in \{x_7, x_8\}$ (1/2, 1/2)

$P_{Y|X=x_7} = (1/2, 1/2)$

$= (0, 0, 0, \ldots, 1/2, 1/2, 0, \ldots, 0)$

$$C = \max_{p_X} \left( H(Y) - 1 \right) = \left( \max_{p_X} H(Y) \right) - 1$$

Ho $H(Y) \leq \log |\mathcal{Y}| = \log 26$

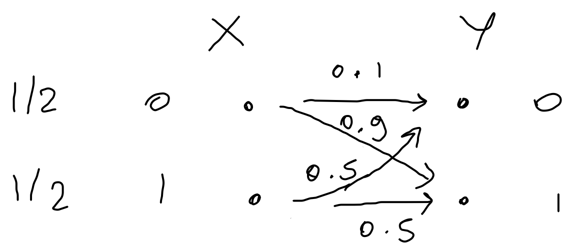
e $H(Y) = \log 26$ quando $p_Y$ è uniforme;

$p_Y$ è uniforme se $p_X$ è uniforme $\longrightarrow \max_{p_X} H(Y) = \log 26$ ← $p_X$ è uniforme

$$\rightarrow \quad C = (\log 26) - 1 = \log 26 - \log 2 = \log 26/2 = \log 13. \quad \square$$

$X \rightarrow \oplus$ ← $Z$

$Y = X + Z$




$$Pr[Y=0] = Pr[X=0] \cdot \underbrace{0.1}_{Pr[Y=0|X=0]} + Pr[X=1] \cdot \overbrace{0.5}^{Pr[Y=0|X=1]}$$

$p_X \in \mathbb{R}^{|\mathcal{X}|}$

$p_Y \in \mathbb{R}^{|\mathcal{Y}|}$

$$\Gamma = \left[ \quad p(y_j|x_i) \quad \right] \in \mathbb{R}^{|\mathcal{X}| \times |\mathcal{Y}|}$$

$$p(y_j) = \sum_{i=1}^{K} p(y_j|x_i) \cdot p(x_i)$$

Se conosco $p_X$
e conosco il canale (quindi conosco $\Gamma$)
come determino $p_Y$?

$$\boxed{p_Y^T = p_X^T \, \Gamma}.$$