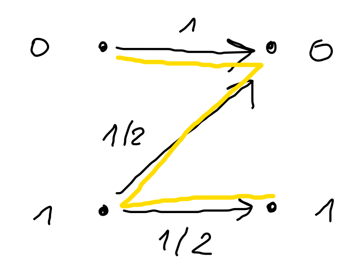
⓪ Canale Z : Matrice di transizione $\Gamma = \begin{array}{cc} \text{"0"} & \text{"1"} \\ \end{array}$

$$\begin{array}{cc|l} \text{"0"} & \text{"1"} & \\ \hline 1 & 0 & \leftarrow \text{"0"} \\ 1/2 & 1/2 & \leftarrow \text{"1"} \end{array} \qquad \Gamma = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$$



(Se $p_X = (1/2, 1/2)$, $p_Y = p_X \Gamma = \frac{1}{2}(1,0) + \frac{1}{2}(1/2, 1/2)$
$= (3/4, 1/4)$)

$$C = \max_{p_X} I(X;Y) = \max_{p_X} [H(Y) - H(Y|X)]$$

$p_X = (1-\alpha, \alpha)$ (con componenti 0, 1) con $\alpha \in [0,1]$

$\rightarrow$ $H(Y|X) = Pr[X=0]\, H(Y|X=0) + Pr[X=1]\, H(Y|X=1)$
$= (1-\alpha)\, H(Y|X=0) + \alpha\, H(Y|X=1)$
$= (1-\alpha) \cdot 0 + \alpha \cdot \underbrace{H((1/2, 1/2))}_{=1} = \alpha$

$p_Y = p_X \Gamma =$
$= (1-\alpha)(1, 0) +$
$\alpha\,(1/2, 1/2)$
$= (1 - \alpha + \alpha/2,\ \alpha/2)$
$= (1 - \alpha/2,\ \alpha/2)$

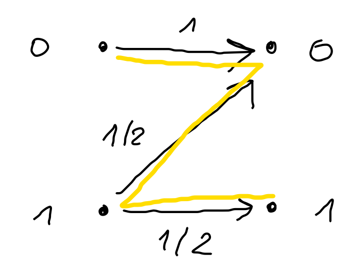
$\rightarrow$ $H(Y) = h_2(\alpha/2)$

$$C = \max_{p_X} [h_2(\alpha/2) - \alpha] = \max_{0 \le \alpha \le 1} [h_2(\alpha/2) - \alpha]$$

$H((1-\varepsilon, \varepsilon))$
$= h_2(\varepsilon)$

⓪ Canale Z : Matrice di transizione

$$\Gamma = \begin{array}{c c} \text{"0"} & \text{"1"} \\ \end{array}$$
$$\begin{array}{c} \\ \end{array}\Gamma = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} \begin{array}{l} \leftarrow \text{"0"} \\ \leftarrow \text{"1"} \end{array}$$



$$\left(\text{Se } p_X = (1/2, 1/2), \quad p_Y = p_X \Gamma = \frac{1}{2}(1,0) + \frac{1}{2}(1/2, 1/2) = (3/4, 1/4) \right)$$

$$C = \max_{p_X} I(X;Y) = \max_{p_X} \left[H(Y) - H(Y|X) \right]$$

$$p_X = \overset{0}{(1-\alpha}, \overset{1}{\alpha)} \quad \text{con } \alpha \in [0,1]$$

$$\rightarrow H(Y|X) = \Pr[X=0]\, H(Y|X=0) + \Pr[X=1]\, H(Y|X=1)$$
$$= (1-\alpha)\, H(Y|X=0) + \alpha\, H(Y|X=1)$$
$$= (1-\alpha) \cdot 0 + \alpha \cdot \underbrace{H((1/2, 1/2))}_{=1} = \alpha$$

$$p_Y = p_X \Gamma = (1-\alpha)(1,0) + \alpha(1/2, 1/2)$$
$$= (1-\alpha+\alpha/2, \alpha/2)$$
$$= (1-\alpha/2, \alpha/2)$$

$$\rightarrow H(Y) = h_2(\alpha/2)$$
$$C = \max_{p_X} \left[h_2(\alpha/2) - \alpha \right] = \max_{0 \leq \alpha \leq 1} \left[h_2(\alpha/2) - \alpha \right]$$

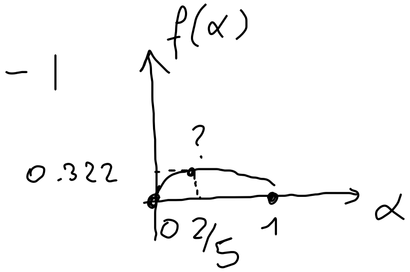
$$H((1-\varepsilon, \varepsilon)) = h_2(\varepsilon)$$

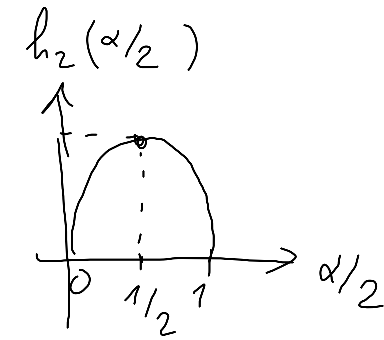
$$C = \max_{0 \le \alpha \le 1} \left[h_2(\alpha/2) - \alpha \right] = \max_{0 \le \alpha \le 1} f(\alpha)$$

$$f(\alpha) = h_2(\alpha/2) - \alpha$$

$$f(0) = 0 - 0 = 0$$

$$f(1) = 1 - 1 = 0$$





$$f'(\alpha) = \left[-\alpha/2 \log \alpha/2 - (1-\alpha/2) \log (1-\alpha/2) - \alpha \right]'$$

$$= -1/2 \log \alpha/2 - \cancel{\alpha/2} \cdot \cancel{2/\alpha} \cdot \cancel{1/2} + 1/2 \log(1-\alpha/2) + \cancel{(1-\alpha/2)} \frac{1}{\cancel{1-\alpha/2}} \cdot \cancel{1/2} - 1$$

$$= -1/2 \log \frac{\alpha}{2} + 1/2 \log (1-\alpha/2) - 1$$

$$= 1/2 \log \frac{1-\alpha/2}{\alpha/2} - 1$$

$$f'(\alpha) = 0 \iff \log_2 \frac{1-\alpha/2}{\alpha/2} = 2$$

$$\iff \frac{1-\alpha/2}{\alpha/2} = 4$$

$$1 - \alpha/2 = 4 \cdot \alpha/2$$

$$5\,\alpha/2 = 1$$

$$\alpha = 2/5$$

$$f(2/5) = h_2(1/5) - 2/5$$

$$\approx 0.322 \text{ bit}. \quad \square$$

① Distanza di Hamming

n-ple (sequenze di lungh. n su uno stesso alfabeto A)

$x = (x_1, x_2, \dots, x_n) \in A^n$

$\quad \overset{=?}{\}} \quad \overset{=?}{\}} \qquad \overset{=?}{\}}$

$y = (y_1, y_2, \dots, y_n) \in A^n$

Distanza di Hamming : $d_H(x,y) = \#\{\text{posizioni } i \ (1 \le i \le n) \text{ tale che } x_i \neq y_i\}$

Sfera di Hamming : $S_\rho(x)$ sfera di Hamming di raggio ρ e centro x

Def : $S_\rho(x) = \{ y \in A^n : d_H(x,y) \le \rho \}$.

Esempio. $A = \{0,1\}$, $n = 3$

$A^n = \{000, 001, 010, 100, 101, 110, 011, 111\}$

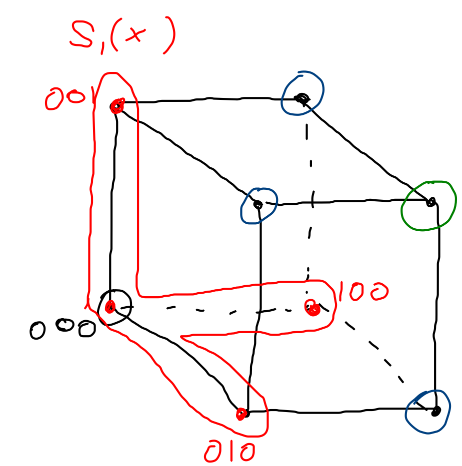
$x = 000$

$S_0(x) = \{x\} = \{000\}$

$S_1(x) = \{000\} \cup \{001, 010, 100\}$

$S_2(x) = S_1(x) \cup \{011, 101, 110\}$

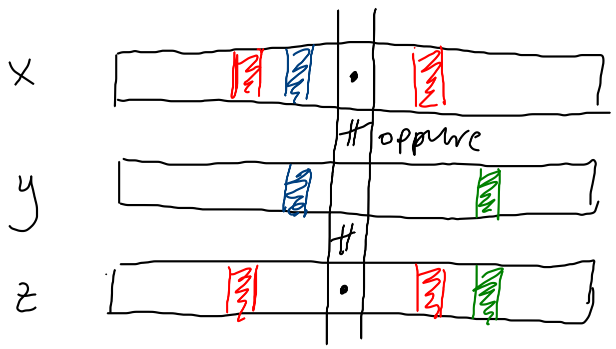
$S_3(x) = A^3$



(Iper)cubo di Hamming A^n

$d_H(\cdot,\cdot)$ è una distanza:

1. $d_H(x,y) \geq 0$ e $d_H(x,y)=0 \iff x=y$ $(\forall x \forall y)$ (definita positiva) ✓
2. $d_H(x,y) = d_H(y,x)$ $(\forall x \forall y)$ (simmetrica) ✓
3. $d_H(x,z) \leq d_H(x,y) + d_H(y,z)$ $(\forall x \forall y \forall z \in \mathcal{A}^n)$ (disuguaglianza triangolare) ?



Proposizione. Per ogni n-pla $x \in \{0,1\}^n$ e ogni $0 \leq p \leq 1/2$, ho

$$|S_{pn}(x)| \leq 2^{h_2(p)\cdot n}.$$

$\rho = p \cdot n$

$S_{pn}(x) \subseteq \mathcal{A}^n$

$|S_{\rho n}(x)| \leq |\mathcal{A}^n| = 2^n$

$(A=\{0,1\})$

$$|S_{pn}(x)| \le 2^{h_2(p)\cdot n} \quad \forall n\in\mathbb{N} \quad \forall x\in A^n \quad \forall p\in[0,1/2].$$

Dim.

$$1 = (p+(1-p))^n =$$

$$= \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} \geq$$

$$\geq \sum_{i=0}^{pn} \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=0}^{pn} \binom{n}{i} (1-p)^n \left(\frac{p}{(1-p)}\right)^i \geq$$

$$\geq \sum_{i=0}^{pn} \binom{n}{i} (1-p)^n \left(\frac{p}{(1-p)}\right)^{pn}$$

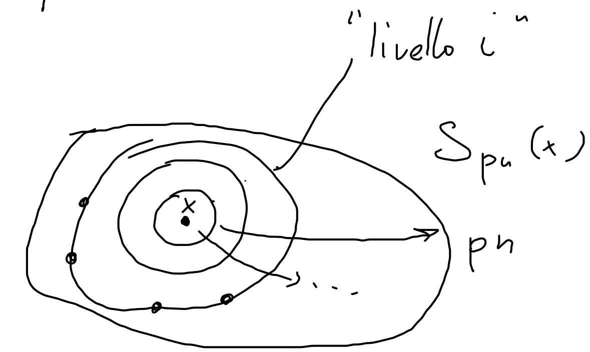
$$= (1-p)^n \sum_{i=0}^{pn} \binom{n}{i} p^{pn} (1-p)^{-pn}$$

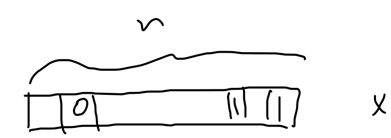
$$= \sum_{i=0}^{pn} \binom{n}{i} p^{pn} (1-p)^{(1-p)n}$$

$\beta^j \geq \beta^k$ se $\beta \le 1$ e $k \geq j$

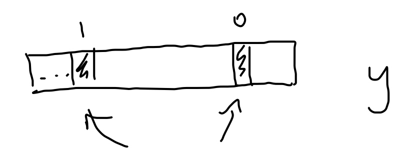
$p\in[0,1/2] \Rightarrow$

$p \le 1/2$, $1-p \geq 1/2$ $\Rightarrow$ $\boxed{p/(1-p) \le 1}$





$$|S_{pn}(x)| = \sum_{i=0}^{pn} |\text{livello } i|$$



Se $y \in$ livello i di $S_{pn}(x)$

i posizioni in cui y differisce da x

Ci sono $\binom{n}{i}$ stringhe y nel livello i.

$$1 \geq \sum_{i=0}^{pn} \binom{n}{i} \underbrace{p^{pn} \quad (1-p)^{(1-p)n}}_{\text{indipendenti dall'indice } i}$$

$$\binom{n}{i} = |\text{livello } i|$$

$$= p^{pn} (1-p)^{(1-p)n} \sum_{i=0}^{pn} \binom{n}{i}$$

= somma delle cardinalità dei primi pn livelli della sfera di Hamming $S_{pn}(x)$

$$= |S_{pn}(x)|$$

$$= 2^{pn \log p} \cdot 2^{(1-p)n \log(1-p)} \cdot |S_{pn}(x)|$$

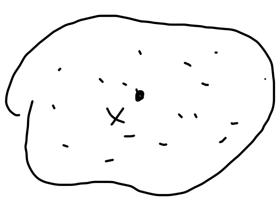
$$= 2^{n[p \log p + (1-p)\log(1-p)]} \cdot |S_{pn}(x)|$$

$$p = 2^{\log p}$$

$$(1-p) = 2^{\log(1-p)}$$

$$= 2^{-n h_2(p)} \, |S_{pn}(x)|$$

$\Longrightarrow$ Moltiplicando per $2^{n h_2(p)}$, ottengo $|S_{pn}(x)| \leq 2^{n h_2(p)}$.



$S_{pn}(x)$

$\forall p \in [0, 1/2]$, $x \in \{0,1\}^n$.

Per sequenze non necessariamente binarie, ($\mathcal{A}$ non necess. $\{0,1\}$)

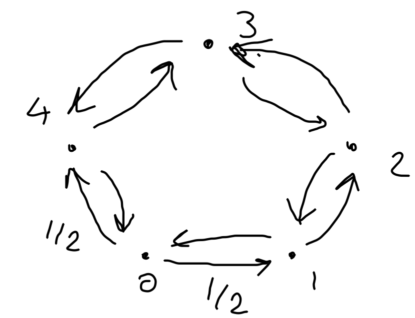
vale comunque : $|S_{pn}(x)| \leq \boxed{(n+1)}\, 2^{n h_2(p)}$

Equivalentemente, per $\rho = pn$, $p = n/\rho$, $|S_\rho(x)| \leq \boxed{(n+1)}\, 2^{n h_2(n/\rho)}$

—

(2) Canale con alfabeto di ingresso e di uscita $\mathcal{A} = \{0,1,2,3,4\} = \mathcal{X} = \mathcal{Y}$

$$p(y|x) = \begin{cases} 1/2 & \text{se } y = x \pm 1 \mod 5 \\ 0 & \text{altrimenti} \end{cases}$$



Calcolare la capacità del canale.

Matrice di transizione :

$$\Gamma = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 2 & 0 & 1/2 & 0 & 1/2 & 0 \\ 3 & 0 & 0 & 1/2 & 0 & 1/2 \\ 4 & 1/2 & 0 & 0 & 1/2 & 0 \end{array}$$

È un canale simmetrico.

$\Rightarrow$ la capacità $C = \max_{p_X} \Big[H(Y) - \underbrace{H(Y|X=0)}_{\text{non dipende da } p_X} \Big] = \left(\max_{p_X} H(Y) \right) - H(Y|X=0)$

$$H(Y|X=0) = H\left(\left(0, \tfrac{1}{2}, 0, 0, \tfrac{1}{2}\right) \right) = h_2\left(\tfrac{1}{2}\right) = 1 .$$

$\max_{p_X} H(Y) \leq \log |\mathcal{Y}| = \log 5$

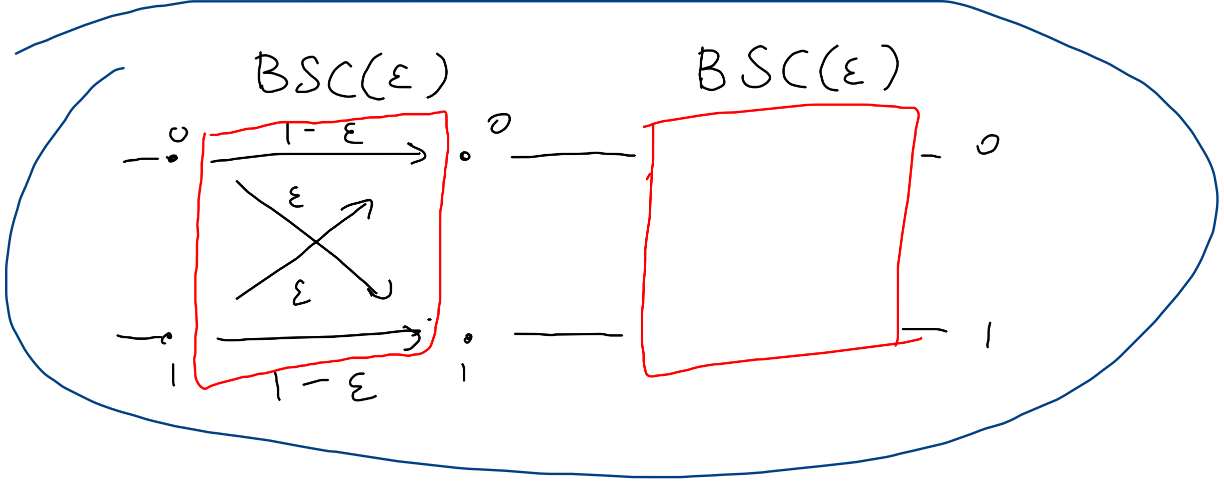
Ho uguaglianza se e solo la p_Y è uniforme su $\mathcal{Y}$

Ho p_Y uniforme se la p_X è uniforme :

$$\underbrace{\left(\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}\right)}_{p_X} \Gamma = \underbrace{\left(\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}\right)}_{p_Y}$$

$\Rightarrow \max_{p_X} H(Y) = H\left(\left(\tfrac{1}{5}, \tfrac{1}{5}, \dots, \tfrac{1}{5} \right) \right) = \log 5 .$ $\quad \Rightarrow C = (\log 5) - 1 .$

$\square$



Qual è la capacità della cascata (concatenazione) di 2 (o di $n \geq 2$) canali binari simmetrici con parametro ε.