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Egalitarian roommate allocations: Complexity and stability ☆

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ABSTRACT

We study two roommate assignment problems, called Ordinal Roommate Allocation and Cardinal Roommate Allocation, where students have preferences over roommates, rooms have varying capacities, and the goal is to maximize the minimum payoff of the students (under two distinct notions of payoff). Both problems are NP-hard when room sizes are unrestricted. In contrast, the Ordinal Roommate Allocation problem becomes tractable when the maximum room capacity is fixed, while the Cardinal Roommate Allocation problem remains NP-hard even with bounded room capacity and number of preferences. We then analyze the problems through the lens of stability, considering envy-freeness and a weaker notion we call swap-resistance. Not all instances guarantee an envy-free outcome, and it is shown to be NP-hard to determine which ones do. However, swap-resistance is always achievable using an efficient algorithm. We discuss connections and distinctions between our work and existing research about utilitarian matchings and stable roommate problems.

1. Introduction

Alice is a teacher organizing a schooltrip that includes an overnight stay. Thinking that it would be a nice idea, she had asked her students to write down who they would like to share the room with, expressing up to three preferences each. Now Alice has to assign the students to the hotel rooms while satisfying their preferences as much as possible, and realizes that this is no easy task: for instance, Angela would like to share the room with Daisy and Erika, but Daisy's preferences are Fiona and Grace, while Erika asked to be with Daisy, Grace, and Helena. Additionally, some rooms can accommodate 3 students, while others can only accommodate 2. Clearly, a criterion to be optimized is needed. In order to reduce dissatisfaction between the students, Alice adopts an egalitarian viewpoint, striving to maximize the satisfaction level (payoff) of the least satisfied student. Alice starts to wonder about the complexity of the underlying optimization problem, and she also wonders whether the students will be satisfied enough to respect the allocation during the night, instead of initiating a chaotic room exchange.

We consider two versions of this problem, depending on whether the preference lists of the students are *(strictly) ordered* or *unordered*. In the ordered version, which we call the *Ordinal Roommate Allocation* problem, the payoff of each student t is defined as the largest integer k such that the first k students on t's preference list are all assigned to the same room as t. In accordance with the egalitarian perspective, the goal is to maximize the payoff of the least satisfied student. In the unordered version, which we call

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Cardinal Roommate Allocation problem, the payoff of each student t is defined simply as the number of students from t's list that share the room with t. The goal is again to maximize the lowest payoff.

We also consider the problem from the point of view of the game-theoretic stability of the given room allocation. A strong notion of stability of an allocation is *envy-freeness*, which roughly speaking means that no student would rather prefer to be in another student's place. A weaker notion of stability of an allocation is that there is no way for a student to improve her payoff by swapping room with someone else, without negatively affecting the payoff of some other student; we call this notion *swap-resistance*.

1.1. Our contribution

We show that both the Ordinal and Cardinal Roommate Allocation problems are strongly NP-hard when the rooms have arbitrary size. When the maximal capacity of rooms is constant, we show that the Ordinal Roommate Allocation problem is polynomial-time solvable, while the Cardinal Roommate Allocation problem remains strongly NP-hard even for rooms of maximal capacity 5 and at most 3 preferences per student, or for rooms of maximal capacity 3 and arbitrarily many preferences. On the other hand, the Cardinal Roommate Allocation problem is shown to be polynomial-time solvable for rooms of capacity 2, or when both the maximal room capacity is constant and each student expresses only one preference.

We also consider two stability notions, envy-freeness and swap-resistance. Not all instances of the roommate problems we consider are guaranteed to admit an envy-free allocation; we show that recognizing which instances do is NP-hard. On the other hand, we show swap-resistance to be always attainable by means of an efficient local search algorithm. This algorithm can be applied to any allocation that is optimal for the ordinal or cardinal criterion to yield an allocation that is simultaneously optimal for the same criterion and swap-resistant.

1.2. Related work

In the classic *perfect matching* problem on a graph [2], one seeks to partition the vertices of a graph into pairs in a way that respects the underlying graph structure; namely, each pair should correspond to an edge of the graph. As is well-known, this problem was one of the motivating problems behind the notion of polynomial-time complexity, and by now it has a very vast literature, see for example the monograph by Lovasz and Plummer [14] and references therein. Not surprisingly, there is a close connection between the perfect matching problem and Ordinal/Cardinal Roommate Allocation problems when each room can host two students (see for example Theorem 3.5 and Proposition 4.6). However, this work is focused on the more general case where rooms can accommodate different numbers of students, beyond just two.

An allocation problem can be approached from the optimization perspective of a central planner (such as the teacher organizing the rooms), or from the game-theoretic perspective of the individuals receiving allocations. In the classic *Roommates problem* [6,9,15], an even number of students wish to divide up into pairs of roommates and students have preferences on each other. A set of pairings is called *stable* if under it there are no two students who are not roommates and who prefer each other to their respective roommates. It was observed already by Gale and Shapley [6] that not every instance of the roommates problem admits a stable pairing. Irving [11] describes a polynomial-time algorithm to test whether a stable pairing exists for a given instance. Ng and Hirschberg [17] and Huang [10] study stability in a roommate problem when rooms have capacity 3.

The notion of stability adopted in the above works, called *core stability* in the game-theoretic literature, is based on the idea that any pair of students can deviate if their preferences encourage them to do so. However, when allocated resources are scarce, not every deviation can be enforced and therefore other notions of stability may be more natural and meaningful than core stability. Bogomolnaia and Jackson [1] define an abstract class of *hedonic coalition formation* games, where each player's payoff is completely determined by the identity of other members of the coalition. Our setting can be seen as a special case and therefore we borrow their notion of an *envy-free* allocation. We note that the paper by Bogomolnaia and Jackson is not algorithmic in nature and in particular is not concerned with the complexity of recognizing which coalitional games admit envy-free solutions.

Boehmer and Elkind [3] discuss envy-freeness and exchange stability in the context of stable roommate allocations with diversity preferences. Their contribution is similar in spirit to the present paper, but the roommate problem they consider has several key differences; the main one being their assumptions that the agents are of two *types* and that the agents' preferences depend solely on the fraction of agents of their own type among their roommates.

It is important to note that there is a rather different criterion that is also sometimes called egalitarian in the context of the stable matching and stable roommate problems [13,8,12,5]. This criterion concerns stable matchings in which the *average* cost incurred by the students is minimized – where the cost is defined as the rank of each student's roommate in her preference list. Perhaps a more accurate name for this criterion would have been *utilitarian* stable matching according to standard economic terminology [16, Chapter 3]. When a stable matching exists, Cseh, Irving and Manlove [5] show that finding such utilitarian stable matchings is NP-hard even when preference lists have length at most 3.

1.3. Overview of the sections

We set up the model and notation in Section 2. The computational complexity of finding optimal egalitarian allocations is discussed in Section 3. Section 4 discusses the additional notions of swap-resistant and envy-free allocations and their complexity.

2. Roommate allocations

Consider a finite set N of n elements, called *students*, and a finite set M of m elements, called *rooms*. Each room $j \in M$ has an associated integer *capacity* $c_j \geq 1$. Let $c \overset{\text{def}}{=} \max_{j \in M} c_j$.

Definition 2.1. An *allocation* is a function $a : N \to M$ such that $|a^{-1}(j)| \leq c_j$ for each room $j \in M$.

Thus, an allocation is a mapping of students to rooms that does not violate the capacity of any room. Clearly, allocations exist if and only if $\sum_{j \in M} c_j \geq n$, that is, when total room capacity is enough to fit all the students. The set of allocation functions (for given N, M, and capacities $\{c_j\}$) will be denoted $\mathcal{A}$. In the following we always assume the easily checked condition $\mathcal{A} \neq \emptyset$ to avoid triviality.

Students have preferences over other students. We investigate ordinal preferences, where students rank their preferred roommates, and approval preferences, where students express approval over potential roommates.

Definition 2.2. An *ordinal preference list* p_i for student $i \in N$ is a nonempty ordered list of distinct elements of $N \setminus \{i\}$.

Definition 2.3. An *approval preference list* p_i for student $i \in N$ is a nonempty unordered list of distinct elements of $N \setminus \{i\}$, or equivalently, a nonempty subset of $N \setminus \{i\}$.

Note that preference lists are not required to be complete; in both cases, the absence of student i' from student i's list signifies that i does not benefit from having i' as a roommate. Let p denote the maximum length of a preference list. We denote the k-th element of list p_i by $p_i[k]$.

Definition 2.4. The *ordinal payoff* of student $i \in N$ under an allocation $a \in \mathcal{A}$ is the largest $k \geq 0$ such that the first k students from i's list are allocated to the same room as i:

$$u_i^O(a) = \max\{k \geq 0 \,:\, k \leq |p_i| \wedge a(p_i[1]) = a(p_i[2]) = \ldots = a(p_i[k]) = a(i)\}.$$

Definition 2.5. The *cardinal payoff* of student $i \in N$ under an allocation $a \in \mathcal{A}$ is the largest $k \geq 0$ such that at least k students from i's list are allocated to the same room as i, that is, the number of students approved by i that end up in the same room as i:

$$u_i^C(a) = |a^{-1}(a(i)) \cap p_i|.$$

Definition 2.6. The *Ordinal Roommate Allocation (ORA)* problem is to find, given room capacities and ordinal preferences, an allocation maximizing the minimum ordinal payoff:

$$\max_{a \in \mathcal{A}} \min_{i \in N} u_i^O(a).$$

Definition 2.7. The *Cardinal Roommate Allocation (CRA)* problem is to find, given room capacities and approval preferences, an allocation maximizing the minimum cardinal payoff:

$$\max_{a \in \mathcal{A}} \min_{i \in N} u_i^C(a).$$

The standard decision problems corresponding to the above optimization problems are the following.

ORDINAL ROOMMATE ALLOCATION – DECISION VERSION (ORA-D)
INSTANCE: Sets N, M, room capacities $\{c_j\}$, ordinal preference lists $\{p_i\}$, and an integer $k \geq 1$.
QUESTION: Is there some allocation $a \in \mathcal{A}$ such that

$$\min_{i \in N} u_i^O(a) \geq k \text{ ?}$$

CARDINAL ROOMMATE ALLOCATION – DECISION VERSION (CRA-D)
INSTANCE: Sets N, M, room capacities $\{c_j\}$, cardinal preference lists $\{p_i\}$, and an integer $k \geq 1$.
QUESTION: Is there some allocation $a \in \mathcal{A}$ such that

$$\min_{i \in N} u_i^C(a) \geq k \text{ ?}$$

It is useful to introduce terminology to indicate when $u_i(a) \geq k$ for a student $i \in N$.

Definition 2.8. Under a given allocation a for ORA-D or CRA-D and given the threshold $k \geq 0$, we say that a student i is *satisfied* if her payoff is at least k (that is, $u_i^O(a) \geq k$ for ORA-D, or $u_i^C(a) \geq k$ for CRA-D).

Notice that the decision problems are equivalent to asking whether an allocation exists that satisfies all students. The following proposition establishes some relation between the number of students n, number of rooms m, and maximal room capacity c.

Proposition 2.1. *Without loss of generality, in both problems ORA-D and CRA-D one can assume:*

1. $mc \geq n$,
2. $m \leq n$
3. $c < n$.

Proof. 1. Since $\sum_{j \in M} c_j \leq mc$, if $mc < n$, the total capacity of the rooms is less than the number of students and no allocation exists, contradicting the assumption that the set of allocations is nonempty.
2. If $m > n$, we can remove the $m - n$ rooms with the lowest capacities and obtain an equivalent smaller instance. Indeed, if an allocation uses any of the $m - n$ lowest capacity rooms, it can be transformed into an allocation that is at least as good for each student by "upgrading" her room size to an equal or larger one, so that only the (at most) n largest capacity rooms are used.
3. If $c \geq n$, by assigning all students to the same room of capacity c we obtain an allocation that is optimal from the point of view of all students, as all students share the same room. □

2.1. Social digraph

A fundamental notion for our analysis will be that of the social digraph associated to an instance, which is the digraph that can be inferred from the students' preferences.

Definition 2.9. The *social digraph* is the digraph $G = (N, E)$ where $(i, i') \in E$ if and only if $i' \in p_i$.

Note that the social digraph has out-degree at least 1, due to the assumption that all preference lists are nonempty. The out-neighbors of student i in the social digraph are also called the *friends* of i.

When dealing with ordinal preferences, it is also useful to consider an additional digraph in which we truncate each neighborhood of the social digraph to the first k preferences of the corresponding student.

Definition 2.10. Given $k \geq 1$, the *truncated social digraph* of level k is the digraph $G_k = (N, E_k)$ where $(i, i') \in E_k$ if and only if i' is among the first k preferences in i's list.

The following observation will be particularly useful for the Ordinal Roommates problem.

Lemma 2.2. *Given an ORA-D instance, if $a \in \mathcal{A}$ satisfies all students, then the students corresponding to any weakly connected component of the truncated social digraph G_k must be allocated to the same room by a.*

Proof. Consider any two students i, i' in the same weakly connected component of the truncated social digraph. Because i and i' are weakly connected, there must be a path $v_1, v_2, v_3, \ldots, v_q$ in the digraph where $v_1 = i$, $v_q = i'$, and for any pair (v_l, v_{l+1}), either v_l is in v_{l+1}'s truncated preference list or vice versa. But since all students are satisfied by a, each pair (v_l, v_{l+1}) is allocated by a to the same room. Therefore i and i' are allocated to the same room. □

2.2. 3-partition, exact-cover-by-3-sets and partition into triangles

The following three problems are well-known NP-complete decision problems. They can be found as problems SP15, SP2, and GT11, respectively, in Garey and Johnson's list [7]. They will be used in our NP-hardness proofs.

3-PARTITION
INSTANCE: A finite set X of $3m$ elements, a bound $B \in \mathbb{Z}^+$, and a "size" $s(x) \in \mathbb{Z}^+$ for each $x \in X$, such that each $s(x)$ satisfies $B/4 < s(x) < B/2$ and such that $\sum_{x \in X} s(x) = mB$.
QUESTION: Can X be partitioned into m disjoint sets $S_1, S_2, \ldots, S_m$ such that, for $1 \leq i \leq m$, $\sum_{x \in S_i} s(x) = B$?
Comment: NP-complete in the strong sense.

EXACT COVER BY 3-SETS (X3C)
INSTANCE: Set X with $|X| = 3q$ and a collection C of 3-element subsets of X.
QUESTION: Does C contain an exact cover for X, i.e., a subcollection $C' \subseteq C$ such that every element of X occurs in exactly one member of C'?
Comment: Remains NP-complete if no element occurs in more than three subsets.

PARTITION INTO TRIANGLES
INSTANCE: An undirected graph $G = (V, E)$, with $|V| = 3q$ for a positive integer q.
QUESTION: Is there a partition of V into q disjoint sets $V_1, V_2, \dots, V_q$ of three vertices such that, for each $V_i = \{v_{i[1]}, v_{i[2]}, v_{i[3]}\}$, the three edges $\{v_{i[1]}, v_{i[2]}\}$, $\{v_{i[1]}, v_{i[3]}\}$, and $\{v_{i[2]}, v_{i[3]}\}$ all belong to E?
Comment: Remains NP-complete if the maximum degree of the graph is 4 [4,18].

3. Optimal egalitarian allocations

In this section we consider the Ordinal and Cardinal Roommate Allocation problems (from the point of view of a centralized allocation authority) and study their computational complexity.

3.1. Ordinal roommate allocations

Theorem 3.1. *If the maximal room capacity c is arbitrary, ORA-D is strongly* NP*-complete.*

Proof. We describe a reduction from 3-PARTITION (recall the definition from Section 2.2). Let X be the set of elements, s the size function, and B the limit in the 3-PARTITION instance. We assume that $s(x) > 1$ for each $x \in X$, which is without loss of generality (one can increase the size of every element by 1 and B by 3 to obtain an equivalent instance that satisfies the assumption).

We construct a digraph G on $\sum_{x \in X} s(x)$ nodes. The digraph G will have $|X|$ strongly connected components. For each element $x \in X$ of the 3-PARTITION instance, G has a strongly connected component consisting of an oriented cycle of length $s(x)$. Consider then an instance of ORA-D having social digraph G, $m = |X|/3$ rooms of capacity B, and threshold $k = 1$.

Observe that to satisfy all students' preferences, by Lemma 2.2 the students corresponding to any cycle of G must be allocated to a common room, but this is also sufficient for each of those students' payoff to be at least 1. Therefore, if a 3-PARTITION solution exists, the corresponding allocation satisfies all the students and is a solution to ORA-D.

Vice versa, observe that if an allocation exists satisfying all the students, then in every room there are exactly B students. Indeed, the number of students in each room is at most B; moreover, by counting, if some room had less than B students, some other room would have more than B, which would contradict the definition of allocation. Therefore, the partition corresponding to such an allocation is a solution to the 3-PARTITION instance.

Since 3-PARTITION is strongly NP-hard, we can assume unary encoding for the numbers $s(a)$ without loss of generality; hence, the above reduction is a polynomial time reduction, and the NP-hardness of ORA-D follows. Finally, membership in NP is straightforward: a function $a : N \to M$ can be represented by n integers of length $O(\log m)$, and after guessing such a function, we can efficiently check that it represents an allocation (by ensuring that $|a^{-1}(j)| \le c_j$ for each $j \in M$), evaluate the payoff of each student in time polynomial in n and m, and ensure that each payoff is greater or equal than the threshold k. □

Theorem 3.2. *If the maximal room capacity is constant, ORA-D is solvable in polynomial time.*

Proof. Recall that in order to satisfy all preferences, the students corresponding to nodes of any weakly connected component of the truncated social digraph must be allocated to the same room (Lemma 2.2). Observe that a partition of an integer $r \in \{1, 2, \dots, c\}$ can be considered as a "configuration" for a room of capacity at least r, in the following sense: given a partition of r, the integers in the partition correspond to the sizes of the weakly connected components of the digraph allocated to the room. Since c is constant, the total number of integer partitions of the numbers $1, 2, \dots, c$ is constant as well, call it $f(c)$. Therefore, the number of potential guesses for the room configurations is at most $(m+1)^{f(c)}$ where m is the number of rooms, as for every configuration there can be between 0 and m rooms in that configuration.

Given a guess of the room configurations, all that remains to do is checking whether for every size of weakly connected component of the social digraph there is a corresponding size in the integer partition corresponding to the guess for the room configurations. Namely, if the integers in the guessed room configurations are $g_1, g_2, \dots$ and the sizes of the weakly connected components are $w_1, w_2, \dots$, one needs to check whether the multiset of integers w is contained in the multiset of integers g. This can be done efficiently (e.g. in time $O(m \log m)$) by sorting, as each of the m rooms has capacity at most $c = O(1)$ and by Proposition 2.1, $n \le mc = O(m)$. □

We illustrate the algorithm in the proof of Theorem 3.2 with an example.

Example 3.1. Consider an ORA instance with 12 students, 3 rooms of capacity 4 each, $k = 1$, and preferences yielding the truncated social digraph illustrated in Fig. 1. The digraph has 4 weakly connected components of size 4, 2, 2, and 4. The partitions of the integers from 1 to 4 are:

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

$$3 = 2 + 1 = 1 + 1 + 1$$

$$2 = 1 + 1$$

$$1 = 1.$$

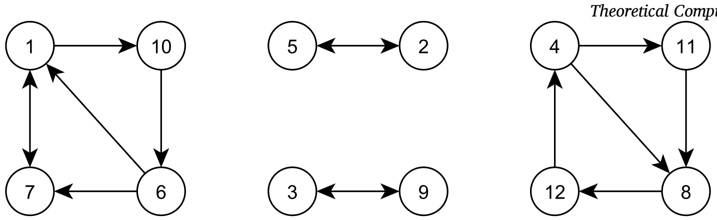

Fig. 1. An example truncated social digraph of an ORA-D instance.

Table 1
Room configuration guesses for the example in Fig. 1.

Guess	4	(2,2)	3	2
1	3 rooms	–	–	–
2	–	3 rooms	–	–
3	–	–	3 rooms	–
4	–	–	–	3 rooms
5	2 rooms	1 room	–	–
6	2 rooms	–	1 room	–
7	2 rooms	–	–	1 room
8	1 room	2 rooms	–	–
9	–	2 rooms	1 room	–
10	–	2 rooms	–	1 room
11	1 room	–	2 rooms	–
12	–	1 room	2 rooms	–
13	–	–	2 rooms	1 room
14	1 room	–	–	2 rooms
15	–	1 room	–	2 rooms
16	–	–	1 room	2 rooms

We can discard partitions containing any 1, as they would correspond to students that have no friends in their allocated room, which would be infeasible since $k = 1$. Thus every room will be in one of the configurations 4, 3, 2, or $(2, 2)$. By enumerating guesses of the number of rooms for each of these remaining configuration (Table 1) we can see that the only feasible guess is the fifth one (row 5), with the rooms in configuration 4, 4, and $(2, 2)$ respectively, yielding the following satisfactory allocation:

$$a^{-1}(1) = \{1, 6, 7, 10\}, \quad a^{-1}(2) = \{4, 8, 11, 12\}, \quad a^{-1}(3) = \{2, 3, 5, 9\}.$$

3.2. *Cardinal roommate allocations*

Differently from the ordinal case, the Cardinal Roommate Allocation problem is strongly NP-hard even when c is bounded above by a constant.

Theorem 3.3. *CRA-D is strongly* NP*-complete, even when the maximal room capacity is* 5 *and every student expresses at most 3 preferences.*

Proof. We reduce to CRA-D the special case of the X3C problem in which every element belongs to at most three sets (Section 2.2).

Let (X, C) be an instance of Exact Cover by 3-Sets such that $|X| = 3q$ for some $q \geq 1$, C is a collection of 3-element subsets of X, and each element of X belongs to at most 3 subsets in C. We construct a CRA-D instance with $2|C| + |X|$ students, as follows:

- The first $2|C|$ students, called *set students*, are denoted s^+, s^- for each subset $s \in C$. The preference list of s^+ (resp., s^-) consists solely of s^- (resp., s^+).
- The other $|X|$ students, called *element students*, are denoted by $x_1, \ldots, x_{3q}$. The preference list of x_i consists of all s^+ such that $x_i \in s$.
- Finally, the CRA-D instance consists of q rooms of capacity 5 and $|C| - q$ rooms of capacity 2; the threshold k equals 1.

Recall that a student is satisfied by an allocation if at least k of her preferences are met by the allocation. The problem is to decide whether an allocation exists that is satisfactory for all students.

First, we argue that in any satisfactory allocation, all rooms of size 2 are allocated to set students. Suppose this is not the case for some room; then either such a room contains 2 element students, in which case none of them is satisfied (all element students have only set students in their preferences), or it contains an element student and a set student, in which case the set student is not satisfied, as it is separated from her only preference. Thus, any room of capacity 2 contains a pair of set students of the form s^+, s^-.

There remain $2|C| - 2(|C| - q) = 2q$ set students and $|X| = 3q$ element students that are necessarily allocated to q rooms of capacity 5. Notice that all these q rooms must be used to full capacity in any satisfactory allocation.

Now by a counting argument we argue that in each room of size 5 there are exactly 3 element students and 2 set students (and the latter have each other as preference). We already know that in the q rooms of size 5 must be somehow distributed $3q$ element students and $2q$ set students. Hence the average number of set students per 5-room is 2. If some 5-room had fewer than the average number of set students, then either it has 1, who would be unsatisfied, being separated from her friend, or it has 0 set students and 5 element students, who would be unsatisfied. If some 5-room had more than the average number of set students, obviously some other room would have less than the average, and one still gets a contradiction.

Therefore, in any satisfactory allocation, one of the two set students in each 5-room must be in the preference list of all the three element students in that room. Hence, the collection of set students s^+ that are allocated in 5-rooms forms an exact cover of X.

Conversely, if an exact cover exists, let us allocate a 5-room to each set student pair s^+, s^- with s in the exact cover and to the 3 element students covered by s. All other set students are paired in the 2-rooms (each s^+ with the corresponding s^-). The resulting allocation is satisfactory. This concludes the NP-hardness claim.

Finally, membership of CRA-D in NP can be argued as in the proof of Theorem 3.1, the only difference being the payoff function used. □

The social digraph in the proof of Theorem 3.3 is asymmetric. One might wonder whether the hardness is tied to the asymmetry, but the answer is negative.

Theorem 3.4. *CRA-D is strongly* NP*-complete, even when the maximal room capacity is 3, the social digraph is symmetric, and every student expresses at most* 4 *preferences.*

Proof. The reduction is from PARTITION INTO TRIANGLES on graphs of maximum degree 4 [4,18] (recall Section 2.2). Given an instance of such problem – that is, an n-node undirected graph G of maximum degree 4 – construct a CRA-D instance where there is a student for each vertex of G, the preferences of i are given by i's neighbors in G, and there are $n/3$ rooms of size 3 (the assumption that n is a multiple of 3 can be made without loss of generality, as otherwise the original instance is clearly infeasible). Let $k = 2$. Since the graph has maximum degree 4, each student expresses at most 4 preferences.

Since all the rooms have size 3, any allocation induces a partition of the graph into subgraphs of order 3. Moreover, each student has payoff at least 2 if and only if 2 of her friends share the same room as her, which implies that each room can be identified with a triangle in G. Therefore, a partition of G into triangles must exist whenever a satisfactory allocation exists.

Conversely, any partition of G into triangles corresponds to an allocation where each student has payoff at least 2, and is therefore satisfied. The construction of the CRA-D instance can be carried out in polynomial time given G. Therefore, deciding if a satisfactory allocation exists is strongly NP-hard. □

We end this section with two easy cases; the first one is reducible to matching, the second one to ORA-D.

Theorem 3.5. *If all rooms have capacity 2, CRA-D is solvable in polynomial time.*

Proof. Observe that if $k \geq 2$, no satisfactory allocation exists, as all students would need to share the room with at least two friends, but the rooms lack the capacity. Therefore assume $k = 1$ and consider the (full) social digraph. We can also, without loss of generality, assume that for any pair of students (i, i'), $i' \in p_i$ if and only if $i \in p_{i'}$: namely, any arc (i, i') such that $i \notin p_{i'}$ can be safely dropped from the digraph, as any allocation that assigned i and i' to the same room would be unsatisfactory anyway (for student i').

Thus, in this setting CRA-D is equivalent to the question of whether the social digraph (which is now symmetric without loss of generality) admits a perfect matching. This can be settled in polynomial time with, say, Edmonds' algorithm [2]. □

Theorem 3.6. *If the maximal room capacity is constant and every student expresses a single preference, CRA-D is solvable in polynomial time.*

Proof. Observe that with a single preference per student, the two utility functions u_i^C and u_i^O coincide. Hence, in this case the two problems ORA-D and CRA-D are equivalent and the claim follows by Theorem 3.2. □

4. Swap-resistant and envy-free allocations

Section 3 is from the point of view of a central authority (the teacher) that strives to set up an allocation that is as satisfactory as possible. Looking at the problem from the point of view of the allocated elements (the students) suggests introducing some notion of whether an allocation is stable, that is, whether it has no incentives to potential deviations. In this section, we consider two notions of stability and investigate their complexity. Because of the room capacity constraint, we only consider deviations where two students are swapped.

Definition 4.1. Given an allocation $a \in \mathcal{A}$, an *augmenting swap for a* is a pair of distinct students $i, i' \in N$ such that the allocation $b \in \mathcal{A}$ defined by $b(s) = a(s)$ for all $s \in N \setminus \{i, i'\}$, $b(i) = a(i')$, $b(i') = a(i)$ satisfies

$u_s(b) \geq u_s(a)$ for all $s \in N$,

$u_s(b) > u_s(a)$ for some $s \in N$.

In natural language, an augmenting swap increases the payoff of at least one student while not decreasing the payoff of any student.

Definition 4.2. An allocation $a \in \mathcal{A}$ is *swap-resistant* if there is no augmenting swap for a.

Lemma 4.1. *Any allocation can be transformed into a swap-resistant allocation by at most pn augmenting swaps. A swap-resistant allocation can be found in polynomial time.*

Proof. Whether an allocation admits an augmenting swap can be checked by enumerating all pairs of students and comparing the payoffs of all students before the swap with the payoffs after the swap.

As long as the current allocation a has an augmenting swap, we apply the swap to obtain a new allocation b and we charge it to one student s such that $u_s(b) > u_s(a)$. We continue in this way as long as the current allocation admits an augmenting swap. Since payoffs are between 0 and p, each student can be charged at most p times; hence there will be at most pn swaps in total and the process will terminate. The last allocation will be swap-resistant by construction. □

The process of transforming an allocation into a swap-resistant allocation does not decrease the payoff of any student. In particular, the algorithm of Lemma 4.1 can be applied to any optimal allocation while preserving its optimality.

Theorem 4.2. *Whenever the Ordinal (resp., Cardinal) Roommate Allocation problem is solvable in polynomial time, an optimal and swap-resistant allocation can also be found in polynomial time.*

Proof. Let a^* be an optimal allocation for the problem at hand obtained by applying the polynomial time algorithm from the hypothesis. By Lemma 4.1, after applying at most pn swaps we obtain a swap-resistant allocation a such that $u_i(a) \geq u_i(a^*)$ for all $i \in N$. But a^* is optimal, therefore a must be optimal as well. □

A stronger notion than swap-resistance is that of envy-freeness.[1]

Definition 4.3. An allocation $a \in \mathcal{A}$ is *envy-free* if for all distinct $i, i' \in N$, the allocation $b \in \mathcal{A}$ defined by $b(s) = a(s)$ for $s \in N \setminus \{i, i'\}$, $b(i) = a(i')$, $b(i') = a(i)$ satisfies

$$u_i(a) \geq u_i(b).$$

In natural language: after swapping any i and i', the new roommates of i do not make i more satisfied than her old roommates did.

Proposition 4.3. *In the Cardinal Roommate Allocation problem, an envy-free allocation may not exist, even when the social digraph is symmetric. In the Ordinal Roommate Allocation problem, an envy-free allocation may not exist, even when the truncated social digraph of level 1 is symmetric.*

Proof. We describe the proof for the Cardinal Roommate Allocation problem, as the proof for the ordinal case is essentially the same. Consider any connected graph G without a perfect matching and construct a CRA instance having G as its social digraph and where all rooms are double rooms. Since G has no perfect matching, in any allocation there will be some student t that does not share a room with any of her neighbors in G. This student will envy any roommate of one of t's neighbors. □

We argue that finding an envy-free allocation is in general NP-hard.

Theorem 4.4. *The problem of determining whether an envy-free allocation exists for an ORA or CRA instance is strongly* NP-*complete, even when each student only expresses one preference.*

Proof. Consider the family of ORA instances where each student only expresses one preference, the social digraph is a set of directed cycles, and all rooms have capacity at least 2. In any such instance, the only way for a student to not be envious is to be allocated to the same room as her only preference – as otherwise she would envy her preference's roommate. In other words, the set of envy-free allocations is a subset of the set of satisfying allocations, and the converse holds as well, as no student can ever have a payoff larger

[1] Our definition can be seen as a special case of the notion of envy-freeness is used by Bogomolnaia and Jackson [1, Footnote 20] in the context of coalition formation.

than 1 in this setting. Therefore, determining whether an envy-free allocation exists is strongly NP-hard by the same reasoning as in the proof of Theorem 3.1. Membership in NP follows from the fact that, after guessing a function $a : N \to M$, it only remains to: 1) check that a is an allocation, by computing the number of students assigned to each room; 2) compute the payoff of each student under allocation a; 3) check that a is envy-free by computing the payoff of each student in every allocation b that only differs from a by the swap of two students, as per Definition 4.3. As there are $O(n^2)$ possible swaps, these steps can be carried out in polynomial time.

The proof for CRA instances is the same as for ORA instances, since when only one preference is expressed, the ordinal and cardinal payoff functions coincide. □

In the construction used in the proof of Theorem 4.4, the maximal room capacity might be large (non-constant). But at least for the Cardinal Roommate Allocation problem, finding an envy-free allocation is NP-hard even when all rooms have capacity 3.

Theorem 4.5. *The problem of determining whether an envy-free allocation exists for a CRA instance is strongly* NP*-hard, even when all rooms have capacity 3.*

Proof. We reduce from PARTITION INTO TRIANGLES. Let G be an instance of PARTITION INTO TRIANGLES with $3q$ vertices. We construct a CRA instance as follows. As in the proof of Theorem 3.4, for each vertex from G, we add a student which has all its neighbors in G in her preference list. Further, we add three students s_1, s_2, s_3 which are included in the preference list of every agent; thus, s_1, s_2, and s_3 list precisely the other two agents from $\{s_1, s_2, s_3\}$ except for themselves in their preferences, while for each student from G the preferences include s_1, s_2, s_3, and her neighbors in G. Finally, there are $q + 1$ rooms, each having capacity 3. Now s_1, s_2, and s_3 must be allocated to the same room, as otherwise one of them, say s_1, is in a room without s_2 and s_3 and thus envies any student in the same room as s_2. This implies that the minimum satisfaction of every student is at least two, as a student with lower satisfaction would envy s_1. Consequently, an envy-free allocation corresponds to a partition of G into triangles.

Vice versa, to any partition of G into triangles we associate the allocation obtained by replicating the partition on q of the rooms and additionally allocating s_1, s_2, s_3 into the $(q + 1)$-th room. This allocation is envy-free, as every student shares the room with exactly two of her preferences and there is no margin of improvement (as all rooms have capacity 3). □

On the other hand, for double rooms and a symmetric social digraph, envy-free allocations can be efficiently detected via reductions to perfect matching.

Proposition 4.6. *In the Cardinal Roommate Allocation problem, if all room capacities equal 2 and the social digraph G is symmetric, an envy-free allocation exists if and only if G has a perfect matching. In the Ordinal Roommate Allocation problem, if all room capacities equal 2 and the truncated social digraph of level 1, G_1, is symmetric, an envy-free allocation exists if and only if G_1 has a perfect matching.*

Proof. One direction follows from the proof of Proposition 4.3: when G (respectively, G_1 for the ordinal case) does not admit a perfect matching, in any allocation there must be some student that does not share the room with any of her neighbors, and is therefore envious of someone else. For the other direction, if G (resp., G_1) has a perfect matching consider the corresponding allocation (where each matched pair is allocated to a separate room) and observe that the payoff of every student under this allocation is equal to 1. Hence, no student i can improve her payoff after swapping with some other student j, as the largest payoff achievable in any allocation is also equal to 1 (due to capacities). □

5. Conclusions

In this work we have considered egalitarian roommate allocations under two notions of payoffs: ordinal payoffs, which are suitable when preferences are ordered, and cardinal payoffs, which are more suitable for approval preferences. One implication of our results is that when optimal allocations are sought, approval preferences can be harder to deal with than ordered preferences.

The ordinal (ORA) model has the advantage of being tractable when the maximal room capacity is constant. On the other hand, the optimality property in ORA is rather strong, as the ordinal payoff for a student s is the maximum k such that s shares a room with their first k choices. So, for example, if s shares a room R of capacity 4 with their 1st, 99th and 100th choices, s obtains a payoff of 1, whereas if s shares R with their 2nd, 3rd and 4th choices, s obtains a payoff of 0. This may be too restrictive in certain applications, and it could be more interesting to consider a more flexible interpretation of an allocation based on s's ordinal preference list.

We have also studied the complexity of finding envy-free allocations or swap-resistant allocations under several scenarios. One interesting case that we did not settle is the complexity of finding envy-free allocations under ordinal payoffs when the maximal room capacity is constant; we leave this as an open problem.

CRediT authorship contribution statement

Vincenzo Bonifaci: Writing – review & editing, Writing – original draft, Supervision, Methodology, Conceptualization. **Helena Rivera Dallorto:** Writing – original draft.

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

References

[1] A. Bogomolnaia, M.O. Jackson, The stability of hedonic coalition structures, Games Econ. Behav. 38 (2) (2002) 201–230, https://doi.org/10.1006/game.2001.0877.
[2] J. Edmonds, Paths, trees, and flowers, Can. J. Math. 17 (1965) 449–467, https://doi.org/10.4153/CJM-1965-045-4.
[3] N. Boehmer, E. Elkind, Stable roommate problem with diversity preferences, in: Proceedings of the 29th International Joint Conference on Artificial Intelligence (IJCAI 2020), IJCAI, 2020, pp. 96–102.
[4] A. Caprara, R. Rizzi, Packing triangles in bounded degree graphs, Inf. Process. Lett. 84 (4) (2002) 175–180, https://doi.org/10.1016/S0020-0190(02)00274-0.
[5] Á. Cseh, R.W. Irving, D.F. Manlove, The stable roommates problem with short lists, Theory Comput. Syst. 63 (1) (2019) 128–149, https://doi.org/10.1007/s00224-017-9810-9.
[6] D. Gale, L.S. Shapley, College admissions and the stability of marriage, Am. Math. Mon. 69 (1) (1962) 9–15, https://doi.org/10.2307/2312726.
[7] M.R. Garey, D.S. Johnson, Computers and Intractability, W.H. Freeman, 1979.
[8] D. Gusfield, Three fast algorithms for four problems in stable marriage, SIAM J. Comput. 16 (1) (1987) 111–128, https://doi.org/10.1137/0216010.
[9] D. Gusfield, R.W. Irving, The Stable Marriage Problem: Structure and Algorithms, The MIT Press, 1989.
[10] C.-C. Huang, Two's company, three's a crowd: stable family and threesome roommates problems, in: L. Arge, M. Hoffmann, E. Welzl (Eds.), Proceedings of the 15th European Symposium on Algorithms (ESA 2007), Springer, 2007, pp. 558–569.
[11] R.W. Irving, An efficient algorithm for the "stable roommates" problem, J. Algorithms 6 (4) (1985) 577–595, https://doi.org/10.1016/0196-6774(85)90033-1.
[12] R.W. Irving, P. Leather, D. Gusfield, An efficient algorithm for the 'optimal' stable marriage, J. ACM 34 (3) (1987) 532–543, https://doi.org/10.1145/28869.28871.
[13] D.E. Knuth, Stable Marriage and Its Relation to Other Combinatorial Problems, American Mathematical Society, 1997.
[14] L. Lovász, M.D. Plummer, Matching Theory, American Mathematical Society, 2009.
[15] D.F. Manlove, Algorithmics of Matching under Preferences, Wiley, 2013.
[16] H.J. Moulin, Fair Division and Collective Welfare, The MIT Press, 2003.
[17] C. Ng, D.S. Hirschberg, Three-dimensional stable matching problems, SIAM J. Discrete Math. 4 (2) (1991) 245–252, https://doi.org/10.1137/0404023.
[18] J.M.M. van Rooij, M.E. van Kooten Niekerk, H.L. Bodlaender, Partition into triangles on bounded degree graphs, Theory Comput. Syst. 52 (4) (2013) 687–718, https://doi.org/10.1007/s00224-012-9412-5.