G. Bini *et al.* (2012) “On the Birational Geometry of $\bar{P}_{d,g}$,”
International Mathematics Research Notices, Vol. 2012, No. 4, pp. 740–780
Advance Access Publication March 31, 2011
doi:10.1093/imrn/rnr045

On the Birational Geometry of the Universal Picard Variety

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We compute the Kodaira dimension of the universal Picard variety $P_{d,g}$ parameterizing line bundles of degree d on curves of genus g under the assumption that $(d-g+1, 2g-2)=1$. We also give partial results for arbitrary degrees d and we investigate for which degrees the universal Picard varieties are birational.

1 Introduction

The study of the birational geometry of the moduli spaces has become a very active research area after the unexpected result of Harris–Mumford–Eisenbud [12, 22] that the moduli space M_g of curves of genus g is a variety of general type for $g \geq 24$, contradicting a long-standing conjecture of Severi on the unirationality of moduli of curves. More recently, also the birational geometry of other moduli spaces has been widely investigated: the moduli space of pointed curves [7, 26], the moduli space of Prym varieties [14], the moduli space of spin curves [13, 16, 28], to mention at least some contributions in this area.

Received June 17, 2010; Revised February 16, 2011; Accepted March 3, 2011
Communicated by Prof. Enrico Arbarello

The aim of this paper is to investigate the birational geometry of the universal Picard variety

$$P_{d,g} \to M_g,$$

parameterizing smooth curves of genus g together with a line bundle of degree d. The following result is due to Verra (see [40, Theorem 1.2]).

Theorem 1.1 ([40]). The variety $P_{d,g}$ is unirational for $g \le 9$ and any d. □

Our main result is the computation of the Kodaira dimension of $P_{d,g}$ with $g \ge 10$ under a technical assumption on the degree d. Recall that, since $P_{d,g}$ is singular and not projective, the Kodaira dimension of $P_{d,g}$, which we denote by $\kappa(P_{d,g})$, is defined as the Kodaira dimension of any smooth projective model of it (see [25, Example 2.1.5]). The previous result of Verra implies that $\kappa(P_{d,g}) = -\infty$ for $g \le 9$ and any d.

Theorem 1.2. Assume that $(d-g+1, 2g-2) = 1$ and $g \ge 10$. The Kodaira dimension of $P_{d,g}$ is equal to

$$\kappa(P_{d,g}) = \begin{cases} 0 & \text{if } g = 10, \\ 19 & \text{if } g = 11, \\ 3g-3 & \text{if } g \ge 12. \end{cases}$$
□

In Propositions 6.5 and 6.3, we also determine the Iitaka fibration (see [25, Definition 1.3.6]) of $P_{d,g}$ in the nontrivial cases, namely for $g \ge 11$. Without any assumption on the degree d, we obtain the following partial result:

Theorem 1.3. The Kodaira dimension of $P_{d,g}$ (for $g \ge 10$) satisfies the following inequalities:

$$\kappa(P_{d,g}) \le \begin{cases} 0 & \text{if } g = 10, \\ 19 & \text{if } g = 11, \\ 3g-3 & \text{if } g \ge 12. \end{cases}$$

Moreover, $\kappa(P_{d,g}) = 3g-3$ if $\kappa(M_g) \ge 0$ (and in particular for $g \ge 22$). □

Let us now explain the strategy that we use to prove the above results. The main tool we use is the GIT compactification constructed by Caporaso [3]

$$\phi_d : \bar{P}_{d,g} \to \bar{M}_g$$

of $P_{d,g}$ over the Deligne–Mumford moduli space $\bar{M}_g$ of stable curves of genus g. The projective normal variety $\bar{P}_{d,g}$ is a good moduli space for the stack $\overline{\mathcal{P}ic}_{d,g}$ (see [4, 30]), whose section over a scheme S is the groupoid $\overline{\mathcal{P}ic}_{d,g}(S)$ of families of quasistable curves of genus g

$$f : (\mathcal{C}, \mathcal{L}) \to S$$

endowed with a balanced line bundle $\mathcal{L}$ of degree d (see Section 2.1 for details). Furthermore, $\bar{P}_{d,g}$ is a coarse moduli scheme for $\overline{\mathcal{P}ic}_{d,g}$ if and only if $(d-g+1, 2g-2) = 1$, which is precisely the numerical hypothesis on the degree d in Theorem 1.2.

Albeit $\bar{P}_{d,g}$ is singular, we can prove (under the same assumption on the degree) that $\bar{P}_{d,g}$ has canonical singularities and therefore pluricanonical forms on the smooth locus lift to any desingularization:

Theorem 1.4. Assume that $(d-g+1, 2g-2) = 1$ and that $g \geq 4$. Then $\bar{P}_{d,g}$ has canonical singularities. In particular, if $\widetilde{\bar{P}_{d,g}}$ is a resolution of singularities of $\bar{P}_{d,g}$, then every pluricanonical form defined on the smooth locus $\bar{P}^{\text{reg}}_{d,g}$ of $\bar{P}_{d,g}$ extends holomorphically to $\widetilde{\bar{P}_{d,g}}$, that is, for all integers m we have

$$h^0(\bar{P}^{\text{reg}}_{d,g}, mK_{\bar{P}^{\text{reg}}_{d,g}}) = h^0(\widetilde{\bar{P}_{d,g}}, mK_{\widetilde{\bar{P}_{d,g}}}). \qquad \square$$

The proof of this theorem is given in Section 4. The restriction on the degree d comes from the fact that $\bar{P}_{d,g}$ has finite quotient singularities if and only if $(d-g+1, 2g-2) = 1$; hence only for such degrees d we can apply the Reid–Tai criterion for the canonicity of finite quotient singularities (see, e.g. [22, p. 27] or [27, Theorem 4.1.11]). Indeed, we establish in Theorem 4.8 a similar statement without any restriction on d for the open subset $\bar{P}^{\text{st}}_{d,g} \subset \bar{P}_{d,g}$ of GIT-stable points of $\bar{P}_{d,g}$, which coincides with $\bar{P}_{d,g}$ if and only if $(d-g+1, 2g-2) = 1$. In proving Theorem 4.8 (from which Theorem 1.4 follows), we determine the nonsmooth locus of $\bar{P}^{\text{st}}_{d,g}$ in Proposition 4.7. Note that a proof of Theorem 1.4 for all degrees d would imply the validity of Theorem 1.2 without any assumptions on the degree d.

The above Theorem 1.4 is crucial for our purposes because it allows us to compute the Kodaira dimension of $\bar{P}_{d,g}$ as the Iitaka dimension (see [25, Definition 2.1.3]) of the canonical divisor $K_{\bar{P}_{d,g}}$ on the modular variety $\bar{P}_{d,g}$, instead of working on some (a priori non modular) desingularization of $\bar{P}_{d,g}$. The class of $K_{\bar{P}_{d,g}}$ is given by the following:

Theorem 1.5. For any $g \geq 4$, we have

$$K_{\bar{P}_{d,g}} = \phi_d^*(14\lambda - 2\delta),$$

where λ and δ denote the Hodge and the boundary class on $\bar{M}_g$, respectively. □

The proof of this theorem is given in Section 5. We first compute in Theorem 5.1 the canonical class of $\overline{\mathcal{P}ic}_{d,g}$ through a careful application of the Grothendieck–Riemann–Roch theorem to the universal family over $\overline{\mathcal{P}ic}_{d,g}$. Then we show that the pull-back of $K_{\bar{P}_{d,g}}$ via the canonical map $p: \overline{\mathcal{P}ic}_{d,g} \to \bar{P}_{d,g}$ is equal to $K_{\overline{\mathcal{P}ic}_{d,g}}$. Note that this is in contrast with what happens for $\bar{M}_g$ (or for the moduli space of Prym or spin curves), where the pull-back of the canonical class of the coarse moduli space is equal to the canonical class of the moduli stack plus some (small) corrections at the boundary.

Theorem 1.5 allows us to compute the Iitaka dimension of $K_{\bar{P}_{d,g}}$ as the Iitaka dimension of the divisor $14\lambda - 2\delta$ on $\bar{M}_g$ (because ϕ_d is a regular fibration). By exploiting the rich available knowledge on the birational geometry of $\bar{M}_g$, we prove the following:

Theorem 1.6. The Iitaka dimension of $K_{\bar{P}_{d,g}}$ on $\bar{P}_{d,g}$ is equal to

$$\kappa(K_{\bar{P}_{d,g}}) = \kappa(14\lambda - 2\delta) = \begin{cases} -\infty & \text{if } g \leq 9, \\ 0 & \text{if } g = 10, \\ 19 & \text{if } g = 11, \\ 3g-3 & \text{if } g \geq 12. \end{cases}$$

□

The proof of the above theorem is given in Section 6 by combining Propositions 6.1, 6.3–6.5.

With the above results it is now easy to prove Theorems 1.2 and 1.3. Indeed, note that we always have the inequality

$$\kappa(P_{d,g}) \leq \kappa(K_{\bar{P}_{d,g}}), \tag{1}$$

with equality if $(d-g+1, 2g-2)=1$ by Theorem 1.4. From (1) and Theorem 1.6, we deduce Theorem 1.2 and the first part of Theorem 1.3. The second part of Theorem 1.3 follows from Proposition 3.2, which is proved in Section 3 via a careful analysis of the regular fibration $\phi_d : \bar{P}_{d,g} \to \bar{M}_g$.

In the final Section 7, inspired by Caporaso [3, Lemma 8.1], we investigate for which values of d and d' the varieties $P_{d,g}$ and $P_{d',g}$ are birational. We prove the following:

Theorem 1.7. Assume that $g \geq 22$ or $g \geq 12$ and $(d-g+1, 2g-2)=1$. Then $P_{d,g}$ is birational to $P_{d',g}$ if and only if $d' \equiv \pm d \mod (2g-2)$. In this case, $P_{d,g}$ is isomorphic to $P_{d',g}$. □

This follows from Theorem 7.3, where we also determine the possible birational maps between the varieties $P_{d,g}$ for g big enough. From the same result, we obtain a description of the group of birational self-maps of $P_{d,g}$ (see Corollary 7.5) and we deduce that the boundary of $\bar{P}_{d,g}$ is preserved by any automorphism of $\bar{P}_{d,g}$ (see Corollary 7.6).

While this work was being written down, Farkas and Verra posted on the arXiv the preprint [17], where they determine, among other things, the Kodaira dimension of $P_{g,g}$ (note that the degree g satisfies the assumptions of our Theorem 1.2, so that their result is a particular case of our main theorem). However, their strategy is different from ours and it seems to apply only in the special case $d=g$. Indeed, the authors of [17] consider the global Abel–Jacobi map

$$A_{g,d} : M_{g,d}/S_d \to P_{d,g}$$

obtained by sending a curve C together with a collection of unordered points $\{p_1, \dots, p_d\}$ into the pair $(C, \mathcal{O}_C(p_1 + \cdots + p_d))$. It is well known that the map $A_{g,d}$ is a birational isomorphism in degree $d=g$ (and only in this case). Using this fact, Farkas and Verra determine the Kodaira dimension of $P_{g,g}$ by studying the pluricanonical forms on the Deligne–Mumford–Knudsen compactification $\bar{M}_{g,g}/S_g$ (instead of the Caporaso compactification $\bar{P}_{g,g}$, as we do in this paper).

Throughout this paper, we work over the complex field $\mathbb{C}$. Moreover, we fix two integers $g \geq 2$ and d.

2 Preliminaries

2.1 The stack $\overline{\mathcal{P}ic}_{d,g}$ and the scheme $\bar{P}_{d,g}$

In this subsection, we recall the definition of the stack $\overline{\mathcal{P}ic}_{d,g}$ and its good moduli space $\bar{P}_{d,g}$, and collect some of their properties to be used later on.

Let $\mathcal{P}ic_{d,g}$ be the universal Picard stack over the moduli stack $\mathcal{M}_g$ of smooth curves of genus g. The fiber $\mathcal{P}ic_{d,g}(S)$ of $\mathcal{P}ic_{d,g}$ over a scheme S is the groupoid whose objects are families of smooth curves $\mathcal{C} \to S$ endowed with a line bundle $\mathcal{L}$ over $\mathcal{C}$ of relative degree d over S and whose arrows are the obvious isomorphisms. $\mathcal{P}ic_{d,g}$ is a smooth irreducible (Artin) algebraic stack of dimension $4g-4$ endowed with a natural forgetful map $\Phi_d : \mathcal{P}ic_{d,g} \to \mathcal{M}_g$. The stack $\mathcal{P}ic_{d,g}$ admits a good moduli scheme $P_{d,g}$ of dimension $4g-3$ which has a natural forgetful map $\phi_d : P_{d,g} \to M_g$ onto the coarse moduli scheme of smooth curves of genus g. We have the following commutative diagram:

$$\begin{tikzcd} \mathcal{P}ic_{d,g} \arrow[r] \arrow[d, "\Phi_d"'] & P_{d,g} \arrow[d, "\phi_d"] \\ \mathcal{M}_g \arrow[r] & M_g \end{tikzcd} \tag{2}$$

Warning 2.1. The fact that $\mathcal{P}ic_{d,g}$ has dimension $4g-4$ (and not $4g-3$ as $P_{d,g}$) is due to the fact that on each object $(\mathcal{C} \to S, \mathcal{L})$ of $\mathcal{P}ic_{d,g}(S)$ there is an action of the multiplicative group $\mathbb{G}_m$ via scalar multiplication on $\mathcal{L}$. Therefore, the map Φ_d factors as

$$\Phi_d : \mathcal{P}ic_{d,g} \to \mathcal{P}ic_{d,g}^{\mathbb{G}_m} \to P_{d,g},$$

where $\mathcal{P}ic_{d,g}^{\mathbb{G}_m}$ (which is denoted by $\mathcal{P}ic_{d,g} /\!\!/ \mathbb{G}_m$ by some authors) is the $\mathbb{G}_m$-rigidification of $\mathcal{P}ic_{d,g}$ along the subgroup $\mathbb{G}_m$. Note that $\mathcal{P}ic_{d,g}^{\mathbb{G}_m}$ is a Deligne–Mumford stack of dimension $4g-3$ while $\mathcal{P}ic_{d,g}$ is an Artin stack of dimension $4g-4$ which is not Deligne–Mumford. However, we will never need the rigidified stack $\mathcal{P}ic_{d,g}^{\mathbb{G}_m}$ in this work so that we refer to [30, Section 4] for more details (note that in [30] our stack $\mathcal{P}ic_{d,g}$ is denoted by $\mathcal{G}_{d,g}$ while its rigidification $\mathcal{P}ic_{d,g}^{\mathbb{G}_m}$ is denoted by $\mathcal{P}_{d,g}$). □

The stack $\mathcal{P}ic_{d,g}$ and the scheme $P_{d,g}$ have been compactified in a modular way in [3, 4, 30]. To describe these compactifications, we need to recall some definitions.

Definition 2.2. A connected, projective nodal curve C is said to be *quasistable* if it is (Deligne-Mumford) semistable and the exceptional components of C do not meet. □

Given a quasi-stable curve C, we will denote by C_{exc} the subcurve of C (called exceptional subcurve) given by the union of all the exceptional components of C; by $\bar{C} := \overline{C \setminus C_{\text{exc}}}$ its complementary subcurve (called nonexceptional subcurve) and by C^{st} the stabilization of C. Moreover, we will denote by $\gamma(\bar{C})$ the number of connected components of $\bar{C}$.

Definition 2.3. Let C be a quasistable curve of genus $g \geq 2$ and L a degree d line bundle on C.

(i) We say that L is *balanced* if

- for every subcurve Z of C the following ("basic inequality") holds

$$\frac{d \deg_Z(\omega_{C|Z})}{2g-2} - \frac{k_Z}{2} \leq \deg_Z L \leq \frac{d \deg_Z(\omega_{C|Z})}{2g-2} + \frac{k_Z}{2}, \tag{3}$$

  where k_Z is the number of intersection points of Z with the complementary subcurve $Z^c := \overline{C \setminus Z}$.
- $\deg_E L = 1$ for every exceptional component E of C.

(ii) We say that L is *strictly balanced* if it is balanced and if for each proper subcurve Z of C for which one of the two inequalities in (3) is not strict, then the intersection $Z \cap Z^c$ is contained C_{exc}.

(iii) We say that L is *stably balanced* if it is balanced and if for each proper subcurve Z of C for which one of the two inequalities in (3) is not strict, then either Z or Z^c is entirely contained in C_{exc}. □

The above Definitions 2.3(i) and 2.3(iii) are taken from [6, Definition 5.1.1] (see also [4, Definition 4.6]) and they are equivalent, respectively, to the definitions of semistable in [3, Section 5.5] and G-stable in [3, Section 6.2]. The Definition 2.3(ii) is taken from [5, Section 4.1] and it is equivalent to the definition of extremal in [3, Section 5.2].

There is an equivalence relation of the set of balanced line bundles on a quasi-stable curve C.

Definition 2.4. Given two balanced line bundles L and L' on a quasi-stable curve C, we say that L and L' are equivalent, and we write $(C, L) \equiv (C, L')$, if $L_{|\tilde{C}} \cong L'_{|\tilde{C}}$. The equivalence class of a pair (C, L) is denoted by $[(C, L)]$. □

Note that the above equivalence relation $\equiv$ clearly preserves the multidegree of the line bundles, hence it preserves the condition of being strictly balanced or stably balanced.

Remark 2.5. In the GIT construction of $\bar{P}_{d,g}$ given in [3], the equivalence classes $[(C, L)]$ such that C is quasi-stable and L is balanced (resp. strictly balanced, resp. stably balanced) correspond to the GIT-semistable (resp. GIT-polystable, resp. GIT-stable) orbits (see [3, Proposition 6.1, Lemma 6.1] and also [6, Theorem 5.1.6]). □

The relationship between stably balanced and strictly balanced line bundles is given by the following lemma:

Lemma 2.6. A line bundle L on a quasi-stable curve C is stably balanced if and only if it is strictly balanced and $\tilde{C}$ is connected. □

Proof. Assume first that L is strictly balanced and that $\tilde{C}$ is connected. Let Z be a proper subcurve of C such that one of the two inequalities in (3) is not strict. Then $Z \cap Z^c \subset C_{\text{exc}}$ because L is strictly balanced by hypothesis. Therefore, the nonexceptional subcurve $\tilde{C}$ can be written as a disjoint union of the two subcurves $Z \cap \tilde{C}$ and $Z^c \cap \tilde{C}$. Since $\tilde{C}$ is connected by hypothesis, we must have that either $Z \cap \tilde{C} = \emptyset$ or $Z^c \cap \tilde{C} = \emptyset$, which implies that either $Z \subseteq C_{\text{exc}}$ or $Z^c \subseteq C_{\text{exc}}$, respectively. This shows that L is stably balanced.

Conversely, assume that L is stably balanced. Clearly this implies that L is strictly balanced. Assume, by contradiction, that $\tilde{C}$ is not connected. Then we can find two proper disjoint subcurves D_1 and D_2 of C that are not contained in C_{exc} and such that $E := (D_1 \cup D_2)^c$ is the union of $r \geq 1$ exceptional components of C. It is easily checked that

$$\begin{aligned} \deg_{D_1\cup E}(\omega_C) &= \deg_{D_1}(\omega_C), \\ k_{D_1\cup E} &= k_{D_1} = r, \qquad (*) \\ \deg_{D_1\cup E} L &= \deg_{D_1} L + r. \end{aligned}$$

Applying the inequality (3) to the subcurves D_1 and $D_1 \cup E$, we get

$$\begin{aligned} -\frac{r}{2} = -\frac{k_{D_1}}{2} &\leq \deg_{D_1} L - d\frac{\deg_{D_1}(\omega_C)}{2g-1} \\ &= \deg_{D_1\cup E} L - r - d\frac{\deg_{D_1\cup E}(\omega_C)}{2g-2} \leq \frac{k_{D_1\cup E}}{2} - r = -\frac{r}{2}. \end{aligned}$$

Therefore one of the inequalities (3) is strict for the subcurve D_1 and this contradicts the fact that L is strictly balanced since $\emptyset \neq D_1 \not\subseteq C_{\text{exc}}$ by construction. ■

Let $\overline{\mathcal{P}ic}_{d,g}$ be the category whose objects are families of quasistable curves $\mathcal{C} \to S$ endowed with a line bundle $\mathcal{L}$ of relative degree d whose restriction to each geometric fiber is balanced and whose arrows are Cartesian diagrams of such families. Cleary $\overline{\mathcal{P}ic}_{d,g}$ is a category fibered in groupoids over the category of schemes. The following theorem summarizes some of the properties of $\overline{\mathcal{P}ic}_{d,g}$ and of its good moduli space $\bar{P}_{d,g}$ known thanks to Caporaso and Melo (note that our stacks $\mathcal{P}ic_{d,g}$ and $\overline{\mathcal{P}ic}_{d,g}$ are called $\mathcal{G}_{d,g}$ and $\bar{\mathcal{G}}_{d,g}$ in [30]).

Theorem 2.7 ([3, 4, 30])**.**

(1) $\overline{\mathcal{P}ic}_{d,g}$ is an irreducible, smooth and universally closed Artin stack of finite type over $\mathbb{C}$ and of dimension $4g-4$. It contains the stack $\mathcal{P}ic_{d,g}$ as a dense open substack.

(2) $\overline{\mathcal{P}ic}_{d,g}$ admits a good moduli space $\bar{P}_{d,g}$, that is a normal irreducible projective variety of dimension $4g-3$. The geometric points of $\bar{P}_{d,g}$ correspond bijectively to the equivalence classes of pairs (C, L) where C is a quasi-stable curve of genus g and L is a strictly balanced line bundle of degree d.

(3) $\bar{P}_{d,g}$ is a coarse moduli scheme for $\overline{\mathcal{P}ic}_{d,g}$ if and only if $(d+1-g, 2g-2) = 1$. In this case $\bar{P}_{d,g}$ has only finite quotient singularities. □

The construction of the scheme $\bar{P}_{d,g}$ as a GIT-quotient is due to Caporaso [3]; the construction of the stack $\overline{\mathcal{P}ic}_{d,g}$ is due to Caporaso [4] in the case $(d+1-g, 2g-2)=1$ and to Melo [30] in the general case. Note that we have a natural commutative diagram compactifying the diagram (2):

$$\begin{tikzcd} \overline{\mathcal{P}ic}_{d,g} \arrow[r] \arrow[d, "\Phi_d"'] & \bar{P}_{d,g} \arrow[d, "\phi_d"] \\ \bar{\mathcal{M}}_g \arrow[r] & \bar{M}_g \end{tikzcd} \tag{4}$$

Notation 2.8. From now on, for the ease of notation, whenever we write $(C, L) \in \bar{P}_{d,g}$ we mean that L is a strictly balanced line bundle on the quasi-stable curve C, considered up to the equivalence relation of Definition 2.4. □

Next we introduce an open subset of $\bar{P}_{d,g}$ that will play a special role in the sequel.

Definition 2.9. We denote by $\bar{P}^{\mathrm{st}}_{d,g}$ the open subset of $\bar{P}_{d,g}$ consisting of pairs $(C, L) \in \bar{P}_{d,g}$, where L is stably balanced. □

By Remark 2.5, $\bar{P}^{\mathrm{st}}_{d,g}$ is the open subset of $\bar{P}_{d,g}$ where the GIT quotient is geometric. In [3, Lemma 2.2], it is proved that the semistable locus (called H_d in [3]) inside the Hilbert scheme whose GIT quotient gives $\bar{P}_{d,g}$ is smooth. From this, it follows that $\bar{P}^{\mathrm{st}}_{d,g}$ has finite quotient singularities (see (12) for an explicit local description). Moreover, $\bar{P}^{\mathrm{st}}_{d,g} = \bar{P}_{d,g}$ if and only if $(d+1-g, 2g-2)=1$ by [3, Proposition 6.2].

Albeit $\bar{P}_{d,g}$ has not necessarily finite quotient singularities, we have the following useful result (see the proof of [18, Corollary 1]):

Theorem 2.10 ([18]). $\bar{P}_{d,g}$ is a $\mathbb{Q}$-factorial variety. □

In view of the above result, we will identify throughout this paper $\mathbb{Q}$-Weil divisors and $\mathbb{Q}$-Cartier divisors on $\bar{P}_{d,g}$.

2.2 The automorphism group $\mathrm{Aut}(C, L)$

For later use, we describe the automorphism group of a pair (C, L) consisting of a quasi-stable curve C and a balanced line bundle L on C. An automorphism of (C, L)

is given by a pair (σ, ψ) such that $\sigma \in \mathrm{Aut}(C)$ and ψ is an isomorphism between the line bundles L and $\sigma^*(L)$. The group of automorphisms of (C, L) is denoted by $\mathrm{Aut}(C, L)$. We get a natural forgetful homomorphism

$$\begin{aligned} F : \mathrm{Aut}(C, L) &\to \mathrm{Aut}(C), \\ (\sigma, \psi) &\mapsto \sigma, \end{aligned} \tag{5}$$

whose kernel is the multiplicative group $\mathbb{G}_m$, acting as fiberwise multiplication on L, and whose image is the subgroup of $\sigma \in \mathrm{Aut}(C)$ such that $\sigma^*(L) \cong L$. The quotient $\mathrm{Aut}(C, L)/\mathbb{G}_m$ is denoted by $\overline{\mathrm{Aut}(C, L)}$ and is called the *reduced* automorphism group of (C, L). Note that $\mathrm{Aut}(C, L)$ depends only on the equivalence class $[(L, C)]$ (see Definition 2.4).

By composing the above homomorphism F of (5) with the natural homomorphism $\mathrm{Aut}(C) \to \mathrm{Aut}(C^{\mathrm{st}})$ induced by the stabilization map $C \to C^{\mathrm{st}}$, we get a homomorphism

$$G : \mathrm{Aut}(C, L) \to \mathrm{Aut}(C^{\mathrm{st}}),$$

whose kernel is described in the next lemma.

Lemma 2.11. We have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{G}_m^{\gamma(\bar{C})} \arrow[r] \arrow[d] & \mathrm{Aut}(C, L) \arrow[r, "G"] \arrow[d] & \mathrm{Aut}(C^{\mathrm{st}}) \arrow[d, equal] \\ 0 \arrow[r] & \mathbb{G}_m^{\gamma(\bar{C})}/\mathbb{G}_m \arrow[r] & \overline{\mathrm{Aut}(C, L)} \arrow[r, "\bar{G}"] & \mathrm{Aut}(C^{\mathrm{st}}) \end{tikzcd}$$

where $\mathbb{G}_m \subseteq \mathbb{G}_m^{\gamma(\bar{C})}$ is the diagonal embedding. □

Proof. The exactness of the first row is proved using an argument similar to that used in the proof of [6, Lemma 2.3.2]. We sketch the argument for the sake of completeness. Let $E_1, \ldots, E_m$ be the exceptional components of C and let $X_1, \ldots, X_{\gamma(\bar{C})}$ be the connected components of $\bar{C}$. We identify each E_i to a copy of $\mathbb{P}^1$ attached to the rest of the curve at the points 0 and ∞. An element $(\sigma, \psi) \in \mathrm{Aut}(C, L)$ belongs to the kernel of the map $\mathrm{Aut}(C, L) \to \mathrm{Aut}(C^{\mathrm{st}})$ if and only if $\sigma_{|\bar{C}} = \mathrm{id}_{\bar{C}}$ and σ acts as multiplication by $m_i \in \mathbb{G}_m(k) = k^*$ on the exceptional component E_i. If we restrict the isomorphism $\psi : L \xrightarrow{\cong} \sigma^*(L)$ to $\bar{C}$, we get that ψ is the fiberwise multiplication by $l_j \in \mathbb{G}_m(k) = k^*$ on each line bundle $L_{|X_j}$. The scalars m_i are uniquely determined by the scalars l_j: if $0 \in E_i$ lies on the component X_j and $\infty \in E_i$ lies on the component X_h (possibly with $j = h$), then by the compatibility between σ and ψ we get that $m_i = l_j/l_h$ (see the proof of [6, Lemma 2.3.2]). Therefore, the element (σ, ψ) is uniquely determined by the scalars $l_1, \ldots, l_{\gamma(\bar{C})}$ and we are done.

From the above proof, it is clear that the homomorphisms corresponding to the diagonal embedding $\mathbb{G}_m \hookrightarrow \mathbb{G}_m^{\gamma(\bar{C})}$ are exactly the fiberwise automorphisms on L, hence the exactness of the second row follows. ■

Corollary 2.12. If (C, L) is stably balanced then $\overline{\mathrm{Aut}(C, L)}$ is a subgroup of $\mathrm{Aut}(C^{\mathrm{st}})$. In particular, $\overline{\mathrm{Aut}(C, L)}$ is a finite group. □

Proof. The first assertion follows from the last row of the diagram in Lemma 2.11 together with the fact that if L is stably balanced on C then $\bar{C}$ must be connected by Lemma 2.6. The last assertion follows from the first one together with the well-known fact that the automorphism group of a stable curve is finite. ■

2.3 The local structure of $\bar{P}_{d,g}$

The complete local ring $\hat{\mathcal{O}}_{\bar{P}_{d,g},(C,L)}$ of $\bar{P}_{d,g}$ at a point (C, L) can be described using the deformation theory of pairs (C, L), which can be found in [37, Section 3.3.3] for C smooth and has been extended to the singular case in [41].

Let us recall the results of [41]. We denote by $\mathcal{P}^1_C(L)$ the sheaf of one jets (or sheaf of principle parts) of L on C [41, Section 2]. The sheaf $\mathcal{P}^1_C(L)$ fits into an exact sequence (see [41, Equation (2.1)])

$$0 \to \Omega^1_C \to \mathcal{P}^1_C(L) \otimes L^{-1} \to \mathcal{O}_C \to 0. \tag{6}$$

Explicitly, the above extension (6) can be described as follows (see [37, p. 145]). Let $\mathcal{O}^*_C \to \Omega^1_C$ be the homomorphism of sheaves given by sending $u \in \Gamma(U, \mathcal{O}^*_C)$ into $\frac{du}{u} \in \Gamma(U, \Omega^1_C)$ for any open subset $U \subseteq C$. By passing to cohomology, we get a group homomorphism θ_C : $\mathrm{Pic}(C) = H^1(C, \mathcal{O}^*_C) \to H^1(C, \Omega^1_C)$. By using the identification $H^1(C, \Omega^1_C) \cong \mathrm{Ext}^1(\mathcal{O}_C, \Omega^1_C)$, the map θ_C sends the line bundle L to the class of the extension (6).

Wang [41] proves that the sheaf $\mathcal{P}^1_C(L)$ controls the tangent and obstruction theory of the pair (C, L). Let us denote by $\mathrm{Def}_{(C,L)}$ the functor of infinitesimal deformations of the pair (C, L) (see [37, p. 146]) and by $T_{\mathrm{Def}_{(C,L)}}$ the tangent space to $\mathrm{Def}_{(C,L)}$ (in the sense of [37, Lemma 2.2.1]).

Theorem 2.13 ([41]).

(i) We have that $T_{\mathrm{Def}_{(C,L)}} = \mathrm{Ext}^1(\mathcal{P}^1_C(L), L)$.

(ii) An obstruction space for $\mathrm{Def}_{(C,L)}$ is given by $\mathrm{Ext}^2(\mathcal{P}^1_C(L), L)$. □

Proof. Part (i) is [41, Theorem 3.1(1)]; Part (ii) is [41, Theorem 4.6(a)]. ■

Moreover, the infinitesimal automorphisms of the pair (C, L) are governed by $\mathrm{Ext}^0(\mathcal{P}^1_C(L), L)$, as shown by the following lemma.

Lemma 2.14. The tangent space of $\mathrm{Aut}(C, L)$ at the identity is equal to $\mathrm{Ext}^0(\mathcal{P}^1_C(L), L)$. □

Proof. The lemma is certainly well known to the experts, at least in the case where C is smooth. However, we include a proof for the lack of a suitable reference.

According to the discussion in Section 2.2, we have an exact sequence of groups

$$0 \to \mathbb{G}_m \to \mathrm{Aut}(C, L) \to \mathrm{Stab}_L(\mathrm{Aut}(C)) \to 0,$$

where $\mathrm{Stab}_L(\mathrm{Aut}(C))$ is the stabilizer of L in $\mathrm{Aut}(C)$, that is, the subgroup of $\mathrm{Aut}(C)$ consisting of all the automorphisms σ of C such that $\sigma^*(L) \cong L$. In other words, $\mathrm{Stab}_L(\mathrm{Aut}(C))$ is the image of $\mathrm{Aut}(C, L)$ via the map F of (5). By passing to the tangent spaces at the origin, we get

$$0 \to T_0\mathbb{G}_m = k \to T_0\mathrm{Aut}(C, L) \to T_0\mathrm{Stab}_L(\mathrm{Aut}(C)) \to 0, \tag{7}$$

where we have denoted by 0 the identity element in each of the above groups.

On the other hand, by dualizing (6), we get the short exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{P}^1(L)^\vee \otimes L \to T_C \to 0. \tag{8}$$

Passing to cohomology, we get the exact sequence

$$0 \to H^0(C, \mathcal{O}_C) = k \to H^0(C, \mathcal{P}^1(L)^\vee \otimes L) \xrightarrow{p} H^0(C, T_C). \tag{9}$$

Compare now the exact sequences (7) and (9). Since $T_0\mathrm{Aut}(C) = H^0(C, T_C)$ by [37, Proposition 2.6.2] and clearly $\mathrm{Ext}^0(\mathcal{P}^1(L), L) = H^0(C, \mathcal{P}^1(L)^\vee \otimes L)$, it is enough to show that

$$T_0\mathrm{Stab}_L(\mathrm{Aut}(C)) = \mathrm{Im}(p). \tag{10}$$

Let $\mathcal{U} = \{U_\alpha\}$ be an affine open covering of C trivializing L and let $f_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathcal{O}_C^*)$ the transition functions of L with respect to $\mathcal{U}$, where as usual $U_{\alpha\beta} := U_\alpha \cap U_\beta$. Then $\theta_C(L) \in H^1(C, \Omega_C^1)$ is represented by the Čech 1-cocycle $(\frac{df_{\alpha\beta}}{f_{\alpha\beta}}) \in Z^1(\mathcal{U}, \Omega_C^1)$ (see [37, p. 145]). From the exact sequence (8), it follows that the sheaf $(\mathcal{P}^1(L)^\vee \otimes L)_{|U_\alpha}$ is isomorphic to $(\mathcal{O}_C)_{U_\alpha} \oplus (T_C)_{|U_\alpha}$ and an element of $\mathrm{Ext}^0(\mathcal{P}^1(L), L) = H^0(C, \mathcal{P}^1(L)^\vee \otimes L)$ is represented by a Čech 0-cochain

$$(k_\alpha, d_\alpha) \in C^0(\mathcal{U}, \mathcal{P}^1(L)^\vee \otimes L) = C^0(\mathcal{U}, \mathcal{O}_C) \oplus C^0(\mathcal{U}, T_C),$$

which satisfies the cocycle conditions: $d_\alpha = d_\beta$ and $k_\beta - k_\alpha = \frac{d_\alpha(f_{\alpha\beta})}{f_{\alpha\beta}}$ on $U_{\alpha\beta} = U_\alpha \cap U_\beta$ (see [37, p. 145]). Since the $f_{\alpha\beta}$'s are the transition functions of L, we conclude that the image of p consists of all the Čech 0-cocycles $(d_\alpha) \in Z^0(\mathcal{U}, T_C)$ corresponding to the infinitesimal automorphisms of C which preserve the line bundle L. In other words, (10) is satisfied and we are done. ■

We can now compute the dimension of the vector spaces $\mathrm{Ext}^i(\mathcal{P}_C^1(L), L)$.

Lemma 2.15. We have that

$$\dim \mathrm{Ext}^i(\mathcal{P}_C^1(L), L) = 0 \quad \text{for } i \geq 2,$$
$$\dim \mathrm{Ext}^1(\mathcal{P}_C^1(L), L) = 4g - 4 + \gamma(\bar{C}),$$
$$\dim \mathrm{Ext}^0(\mathcal{P}_C^1(L), L) = \gamma(\bar{C}).$$

□

Proof. By applying the functor $\mathrm{Hom}(-, \mathcal{O}_C)$ to the exact sequence (6) and using that $\mathrm{Ext}^{\geq 2}(\mathcal{O}_C, \mathcal{O}_C) = H^{\geq 2}(C, \mathcal{O}_C) = 0$ since C is a curve and that $\mathrm{Ext}^{\geq 2}(\Omega_C^1, \mathcal{O}_C) = 0$ by [9, Lemma 1.3], we get the vanishing $\mathrm{Ext}^{\geq 2}(\mathcal{P}_C^1(L), L) = 0$. The fact that $\mathrm{Ext}^0(\mathcal{P}_C^1(L), L) = \gamma(\bar{C})$ follows from Lemmas2.11 and 2.14. Finally, from the exact sequence (6), we get

$$\begin{aligned}
&\dim \mathrm{Ext}^0(\mathcal{P}_C^1(L), L) - \dim \mathrm{Ext}^1(\mathcal{P}_C^1(L), L) \\
&\quad = \dim \mathrm{Ext}^0(\mathcal{O}_C, \mathcal{O}_C) - \dim \mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) + \dim \mathrm{Ext}^0(\Omega_C^1, \mathcal{O}_C) - \dim \mathrm{Ext}^1(\Omega_C^1, \mathcal{O}_C) \\
&\quad = -(g-1) - (3g-3) = -(4g-4),
\end{aligned}$$

from which we conclude. ■

We can now prove that the functor $\mathrm{Def}_{(C,L)}$ has a semiuniversal formal element (in the sense of [37, Definition 2.2.6]).

Proposition 2.16.

(i) The functor $\mathrm{Def}_{(C,L)}$ has a semiuniversal formal element $\underline{\mathrm{Def}}_{(C,L)}$.

(ii) $\underline{\mathrm{Def}}_{(C,L)}$ is equal to the formal spectrum of $k[[x_1, \ldots, x_{4g-4+\gamma(\bar{C})}]]$. □

Proof. Part (i) is proved in [37, Theorem 3.3.11(i)] in the case where C is smooth. The proof of [37] consists in showing that Schlessinger's conditions are satisfied and this extends to the case where C is nodal: the crucial point of the proof is showing that $T_{\mathrm{Def}_{(C,L)}}$ is finite dimensional, and this follows in our case from Theorem 2.13(i).

From Theorem 2.13 and Lemma 2.15, it follows that $\underline{\mathrm{Def}}_{(C,L)}$ is formally smooth and that the dimension of the tangent space at its unique closed point is $4g-4+\gamma(\bar{C})$, from which part (ii) follows. ■

Now we can describe the complete local ring $\hat{\mathcal{O}}_{\bar{P}_{d,g},(C,L)}$ of $\bar{P}_{d,g}$ at a point (C,L). Note that the automorphism group $\mathrm{Aut}(C,L)$ acts on $\underline{\mathrm{Def}}_{(C,L)}$ (hence on $\mathbb{C}[[x_1, \ldots, x_{4g-4+\gamma(\bar{C})}]]$) by the semiuniversality of $\underline{\mathrm{Def}}_{(C,L)}$. By a standard argument based on Luna's étale slice theorem (see [29, p. 97] and also [24, Section II; 10, Section 7.4]), the formal spectrum (which we denote by Spf) of the complete local ring $\hat{\mathcal{O}}_{\bar{P}_{d,g},(C,L)}$ of $\bar{P}_{d,g}$ at the point $(C,L)\in\bar{P}_{d,g}$ is given by

$$\mathrm{Spf}\hat{\mathcal{O}}_{\bar{P}_{d,g},(C,L)} = \underline{\mathrm{Def}}_{(C,L)}/\mathrm{Aut}(C,L), \tag{11}$$

where $\underline{\mathrm{Def}}_{(C,L)}/\mathrm{Aut}(C,L)$ is the quotient of $\underline{\mathrm{Def}}_{(C,L)}$ with respect to the natural action of $\mathrm{Aut}(C,L)$. In other words, $\underline{\mathrm{Def}}_{(C,L)}/\mathrm{Aut}(C,L)$ is equal to the formal spectrum of the ring of invariants $\mathbb{C}[[x_1, \ldots, x_{4g-4+\gamma(\bar{C})}]]^{\mathrm{Aut}(C,L)}$ (see Proposition 2.16).

Clearly, the scalar automorphisms $\mathbb{G}_m \subseteq \mathrm{Aut}(C,L)$ act trivially on $\underline{\mathrm{Def}}_{(C,L)}$ and thus we get the alternative description:

$$\mathrm{Spf}\hat{\mathcal{O}}_{\bar{P}_{d,g},(C,L)} = \underline{\mathrm{Def}}_{(C,L)}/\overline{\mathrm{Aut}(C,L)}. \tag{12}$$

Note that, from the above description and Lemma 2.11, it follows that $\bar{P}^{\mathrm{st}}_{d,g}$ is the open subset of $\bar{P}_{d,g}$ consisting of pairs $(C,L)\in\bar{P}_{d,g}$ such that $\bar{C}$ is connected.

We can similarly describe the morphism $\phi_d : \bar{P}_{d,g} \to \bar{M}_g$ locally at $(C, L) \in \bar{P}_{d,g}$. Denote by $\underline{\mathrm{Def}}_C$ (resp. $\underline{\mathrm{Def}}_{C^{\mathrm{st}}}$) the semiuniversal formal element associated to the infinitesimal deformation functor Def_C (resp. $\mathrm{Def}_{C^{\mathrm{st}}}$) of C (resp. C^{st}), see [37, Corollary 2.4.2].

Locally at $(C, L) \in \bar{P}_{d,g}$, the morphism $\phi_d : \bar{P}_{d,g} \to \bar{M}_g$ is given by

$$\mathrm{Spf}\hat{\mathcal{O}}_{\bar{P}_{d,g},(C,L)} = \underline{\mathrm{Def}}_{(C,L)}/\overline{\mathrm{Aut}(C, L)} \to \mathrm{Spf}\hat{\mathcal{O}}_{\bar{M}_g, C^{\mathrm{st}}} = \underline{\mathrm{Def}}_{C^{\mathrm{st}}}/\mathrm{Aut}(C^{\mathrm{st}}), \tag{13}$$

where the homomorphism of groups $\overline{\mathrm{Aut}(C, L)} \to \mathrm{Aut}(C^{\mathrm{st}})$ is the one given by Lemma 2.11 and the morphism $\underline{\mathrm{Def}}_{(C,L)} \to \underline{\mathrm{Def}}_{C^{\mathrm{st}}}$ is the composition of the forgetful morphism $\underline{\mathrm{Def}}_{(C,L)} \to \underline{\mathrm{Def}}_C$ with the stabilization morphism $\underline{\mathrm{Def}}_C \to \underline{\mathrm{Def}}_{C^{\mathrm{st}}}$. The induced morphism at the level of tangent spaces

$$T_{\mathrm{Def}_{(C,L)}} = \mathrm{Ext}^1(\mathcal{P}^1_C(L), L) \to T_{\mathrm{Def}_{C^{\mathrm{st}}}} = \mathrm{Ext}^1(\Omega^1_{C^{\mathrm{st}}}, \mathcal{O}_{C^{\mathrm{st}}}) \tag{14}$$

is given by composing the morphism $\mathrm{Ext}^1(\mathcal{P}^1_C(L), L) \to \mathrm{Ext}^1(\Omega^1_C \otimes L, L) = \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C)$ induced by the exact sequence (6) with the morphism $\mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \to \mathrm{Ext}^1(\Omega_{C^{\mathrm{st}}}, \mathcal{O}_{C^{\mathrm{st}}})$ induced by the stabilization map $C \to C^{\mathrm{st}}$. More precisely, let $f : C \to C^{\mathrm{st}}$ be the stabilization morphism and denote by Lf^* (resp. Rf_*) the left-derived functor of f^* (resp. the right-derived functor of f_*). We have a natural map

$$\mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \to \mathrm{Ext}^1(Lf^*\Omega^1_{C^{\mathrm{st}}}, \mathcal{O}_C) = \mathrm{Ext}^1(\Omega^1_{C^{\mathrm{st}}}, \mathcal{O}_{C^{\mathrm{st}}}). \tag{15}$$

The first map in (15) is induced by the composite map $Lf^*\Omega^1_{C^{\mathrm{st}}} \to L^0 f^*\Omega^1_{C^{\mathrm{st}}} = f^*\Omega^1_{C^{\mathrm{st}}} \to \Omega^1_C$ in the derived category of coherent sheaves on C. The equality in (15) follows from the adjointness of the functors Lf^* and Rf_* between the derived category of coherent sheaves on C and on C^{st}, together with the fact that $Rf_*\mathcal{O}_C \cong \mathcal{O}_{C^{\mathrm{st}}}$ because f is a sequence of blow-ups with projective spaces as fibers.

3 The Fibration $\phi_d : \bar{P}_{d,g} \to \bar{M}_g$

The aim of this section is to prove Proposition 3.2, which gives the second part of Theorem 1.3.

To this aim, we analyze the natural morphism $\phi_d : \bar{P}_{d,g} \to \bar{M}_g$. Note that ϕ_d is a regular fibration (i.e., a proper, surjective morphism with connected fibers), whose general fiber is the degree-d Jacobian $\mathrm{Pic}^d(C)$ of a general $[C] \in \bar{M}_g$. Following Kawamata (see [23, Section 1 and Corollary 7.3]), we define the variation $\mathrm{Var}(\phi_d)$ of ϕ_d to be

$$\mathrm{Var}(\phi_d) = \dim \bar{M}_g - \dim \mathrm{Ker}(\delta_C), \tag{16}$$

where

$$\delta_C : T_{\bar{M}_g, C} = H^1(C, T_C) \to H^1(\mathrm{Pic}^d(C), T_{\mathrm{Pic}^d(C)}) \tag{17}$$

is the Kodaira–Spencer map associated to ϕ_d at a general point $C \in \bar{M}_g$.

Lemma 3.1. The algebraic fiber space $\phi_d : \bar{P}_{d,g} \to \bar{M}_g$ has a maximal variation, that is, $\mathrm{Var}(\phi_d) = 3g - 3$. □

Proof. By (16), we have to prove the injectivity of the Kodaira–Spencer map δ_C for a general curve $C \in \bar{M}_g$.

We will reinterpret the above Kodaira–Spencer map as a composition of certain maps that were studied in [35, Section 2], in their analysis of the local Torelli problems for curves. We need to recall their setting, with the simplification that, since we are only interested in the general curve, we can work directly with the coarse moduli spaces M_g and A_g (where, as usual, A_g denotes the coarse moduli space of principally polarized abelian varieties of dimension g), without having to pass to their n-level covers. Consider the following commutative diagram (see [35, p. 169]):

$$\begin{tikzcd}
T_{M_g,C} \arrow[r, "k_C", "\cong"'] \arrow[d, "dt_g"'] & H^1(C, T_C) \arrow[d, "du"] \\
 & H^1(C, u^* T_{\mathrm{Pic}^d(C)}) \\
T_{A_g, \mathrm{Pic}^d(C)} \arrow[dr, "k_{\mathrm{Pic}^d(C)}"] & \\
 & H^1(\mathrm{Pic}^d(C), T_{\mathrm{Pic}^d(C)}) \arrow[uu, "u^*"', "\cong"]
\end{tikzcd}$$

where $u : C \to \mathrm{Pic}^d(C)$ is an Abel–Jacobi map (well defined only up to translation), $t_g : M_g \to A_g$ is the classical Torelli map, k_C is the Kodaira–Spencer map in C associated to the universal family over an open subset of M_g containing C and $k_{\mathrm{Pic}^d(C)}$ is the Kodaira–Spencer map in $\mathrm{Pic}^d(C)$ associated to the universal family over an open subset of A_g

containing $\mathrm{Pic}^d(C)$. The map k_C is an isomorphism (see, e. g. [35, Theorem 2.2]) and the map u^* is an isomorphism since $T_{\mathrm{Pic}^d(C)}$ is the trivial bundle of rank g and

$$u^* : H^1(\mathrm{Pic}^d(C), \mathcal{O}_{\mathrm{Pic}^d(C)}) \xrightarrow{\cong} H^1(C, \mathcal{O}_C).$$

It is easy to see that

$$\delta_C = k_{\mathrm{Pic}^d(C)} \circ dt_g = (u^*)^{-1} \circ du \circ k_C.$$

Therefore, the injectivity of δ_C is equivalent to the injectivity of du. According to [35, Theorem 2.6], the map du is the dual of the multiplication map

$$\mu : H^0(C, \omega_C) \otimes H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 2}),$$

which is well known (Noether's theorem) to be surjective if $g = 2$ or if $g \geq 3$ and C is not hyperelliptic. Since we assumed C to be generic, we deduce the injectivity of δ_C and we are done. ■

Proposition 3.2.

(1) We have that $\kappa(\bar{P}_{d,g}) \leq 3g - 3$.

(2) If $\kappa(\bar{M}_g) \geq 0$, then $\kappa(\bar{P}_{d,g}) = 3g - 3$ and the map $\phi_d : \bar{P}_{d,g} \to \bar{M}_g$ is the Iitaka fibration of $\bar{P}_{d,g}$. □

Proof. The subaddivity of the Kodaira dimension (see [39, Theorem 6.12]) applied to the regular fibration ϕ_d gives that

$$\kappa(\bar{P}_{d,g}) \leq \dim \bar{M}_g + \kappa(\phi_d^{-1}(C)),$$

for a general $C \in \bar{M}_g$. Since, for a general $C \in \bar{M}_g$, the fiber $\phi_d^{-1}(C) = \mathrm{Pic}^d(C)$ is an abelian variety, we have that $\kappa(\phi_d^{-1}(C)) = 0$, which proves part (1).

Assume now that $\kappa(\bar{M}_g) \geq 0$. Observe that $\mathrm{Pic}^d(C)$ is a good minimal model, since it is smooth and the canonical $K_{\mathrm{Pic}^d(C)}$ is trivial and thus clearly semi-ample. Therefore, the Iitaka conjecture (in the stronger form of [23, p. 1]) does hold true by [23, Corollary 1.2] and gives that

$$\kappa(\bar{P}_{d,g}) \geq \kappa(\phi_d^{-1}(C)) + \max\{\kappa(\bar{M}_g), \mathrm{Var}(\phi_d)\} = 3g - 3,$$

using the above Lemma 3.1. This, combined with part (1), proves that $\kappa(\bar{P}_{d,g}) = 3g - 3$.

The last part follows from the birational characterization of the Iitaka fibration (see, e. g. [39, Theorem 6.11]) since $\phi_d: \bar{P}_{d,g} \to \bar{M}_g$ is an algebraic fiber space such that $\dim \bar{M}_g = \kappa(\bar{P}_{d,g})$ and the generic fiber $\phi_d^{-1}(C) = \mathrm{Pic}^d(C)$ is smooth and irreducible of Kodaira dimension zero. ■

4 The Singularities of $\bar{P}_{d,g}^{\mathrm{st}}$

The purpose of this section is to study the singularities of $\bar{P}_{d,g}$ with the aim of proving Theorem 1.4. More generally, we will prove a similar statement (see Theorem 4.8) for the open subvariety $\bar{P}_{d,g}^{\mathrm{st}} \subseteq \bar{P}_{d,g}$ of (2.1) and for any degree d. Since $\bar{P}_{d,g}^{\mathrm{st}} = \bar{P}_{d,g}$ if (and only if) $(d+1-g, 2g-2) = 1$ (by [3, Proposition 6.2]), Theorem 1.4 is a special case of Theorem 4.8. The need of restricting ourselves to the open subset $\bar{P}_{d,g}^{\mathrm{st}}$ is due to the fact that $\bar{P}_{d,g}^{\mathrm{st}}$ has finite quotient singularities and therefore we can apply the Reid–Tai criterion for the canonicity of finite quotient singularities (see, e.g. [22, pp. 27–28] or [27, Theorem 4.1.11]). For simplicity, we assume throughout this section that $g \geq 4$ in order to avoid problems with the hyperelliptic locus in $\bar{M}_g$.

In the analysis of the singularities of $\bar{P}_{d,g}^{\mathrm{st}}$, the pairs $(C, L) \in \bar{P}_{d,g}^{\mathrm{st}}$ such that C contains an elliptic tail will play a special role. Let us give some definitions.

Definition 4.1. A connected subcurve E of a quasi-stable curve C is called an *elliptic tail* if it has arithmetic genus 1 and meets the rest of the curve in exactly one node P which is called an elliptic tail node. □

The following remark is straightforward.

Remark 4.2. If a quasi-stable curve C has an elliptic tail $E \subseteq C$ then the image $E^{\mathrm{st}} := \mathrm{st}(E) \subseteq C^{\mathrm{st}}$ of E via the stabilization morphism $\mathrm{st}: C \to C^{\mathrm{st}}$ is an elliptic tail of C^{st}. Conversely, if C^{st} has an elliptic tail $E' \subseteq C^{\mathrm{st}}$, then $E := \mathrm{st}^{-1}(E')$ is an elliptic tail of C such that $E' = E^{\mathrm{st}}$. □

In the next lemma, we describe the pairs $(C, L) \in \bar{P}_{d,g}^{\mathrm{st}}$ such that C has an elliptic tail $E \subseteq C$.

Lemma 4.3. Let $(C, L) \in \bar{P}_{d,g}^{\mathrm{st}}$ such that C has an elliptic tail $E \subseteq C$. Then we have:

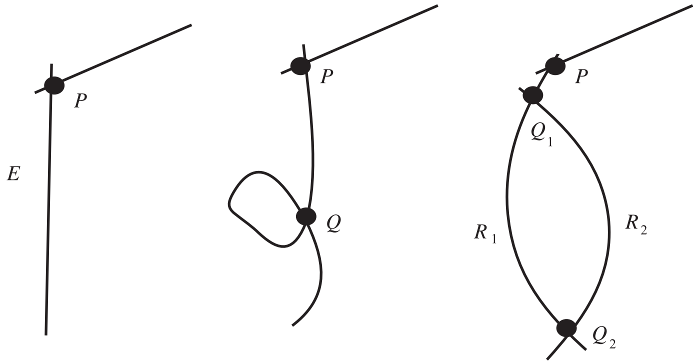

Fig. 1. The possible elliptic tails of a quasi-stable curve C such that $(C, L) \in \bar{P}^{\mathrm{st}}_{d,g}$ for some line bundle L.

(i) $d \not\equiv g-1 \mod (2g-2)$ and the degree $\deg_E(L)$ of L on E is the unique integer d_E such that

$$-\frac{1}{2} < d_E - \frac{d}{2g-2} < \frac{1}{2}; \tag{18}$$

(ii) the elliptic tail node P does not belong to C_{exc};

(iii) E is either smooth or it is a rational (irreducible) curve with one node Q or it is formed by two smooth rational curves R_1 and R_2 meeting in two points Q_1 and Q_2, as depicted in Figure 1. □

Proof. The equation (18) follows from the basic inequality (3) applied to the subcurve $E \subseteq C$ together with the fact that the inequalities must be strict since clearly $E \not\subseteq C_{\mathrm{ex}}$ and L is stably balanced by the hypothesis that $(C, L) \in \bar{P}^{\mathrm{st}}_{d,g}$. The fact that $d \not\equiv g-1 \mod (2g-2)$ follows from the fact that there exists an integer d_E satisfying the strict inequalities in (18). This proves part (i).

Next we turn to Part (ii). By contradiction, if $P \in R$ where R is an exceptional component of C, then R^c is a disjoint union of two subcurves of C each of which contains some components of $\bar{C}$. Therefore $\bar{C}$ is disconnected and this contradicts the hypothesis that $(C, L) \in \bar{P}^{\mathrm{st}}_{d,g}$ by Lemma 2.6.

Part (iii) follows from the fact that C is quasi-stable together with Part (ii) and the fact that E^{st} is either a smooth elliptic curve (which occurs for Type I) or a rational irreducible curve with one node Q (which occurs for Types II and III). ■

The elements $(C, L) \in \bar{P}_{d,g}$ such that C has an elliptic tail E have special automorphisms that will play a key role in the sequel.

Definition 4.4. Given an element $(C, L) \in \bar{P}_{d,g}$ such that C has an elliptic tail E, an automorphism $\phi = (\sigma, \psi) \in \mathrm{Aut}(C, L)$ (or its image in $\overline{\mathrm{Aut}(C, L)}$) is called an *elliptic tail automorphism* of (C, L) of order $n \geq 1$ (with respect to the elliptic tail $E \subseteq C$) if σ is the identity on $\overline{C \setminus E}$ and $\sigma_{|E}$ has order n. □

The assumption that (C, L) belongs to $\bar{P}^{\mathrm{st}}_{d,g}$ puts some constraints on the possible elliptic tail automorphisms that can occur. Indeed, under this assumption, using that the map $\bar{G} : \overline{\mathrm{Aut}(C, L)} \to \mathrm{Aut}(C^{\mathrm{st}})$ is injective (see Corollary 2.12), we deduce immediately the following:

Remark 4.5. Given $(C, L) \in \bar{P}^{\mathrm{st}}_{d,g}$, an element $\phi \in \mathrm{Aut}(C, L)$ is an elliptic tail automorphism of order $n \geq 1$ with respect to the elliptic tail $E \subseteq C$ if and only if $G(\phi) \in \mathrm{Aut}(C^{\mathrm{st}})$ (see the notation of Lemma 2.11) is an elliptic tail automorphism of order n of C^{st} with respect to the elliptic tail $E^{\mathrm{st}} \subseteq C^{\mathrm{st}}$, that is, $G(\phi)$ is the identity on $\overline{C^{\mathrm{st}} \setminus E^{\mathrm{st}}}$ and $G(\phi)_{|E^{\mathrm{st}}}$ has order n. □

Using this remark, we can give a complete description of the possible elliptic tail automorphisms of elements $(C, L) \in \bar{P}^{\mathrm{st}}_{d,g}$.

Lemma 4.6. Assume that $(C, L) \in \bar{P}^{\mathrm{st}}_{d,g}$ and that C has an elliptic tail $E \subseteq C$. For an elliptic tail automorphism $\phi = (\sigma, \psi) \in \mathrm{Aut}(C, L)$ of (C, L) of order $n > 1$ with respect to the elliptic tail $E \subseteq C$, the restriction $\sigma_{|E}$ of σ to E must satisfy the following conditions (according to whether the elliptic tail E is of Type I, II or III as in Lemma 4.3(iii)):

(i) Type I: $\sigma_{|E}$ is an automorphism of E fixing P and $n = 2$ or $n = 4$ (which can occur if and only if E has j-invariant equal to 1728) or $n = 3, 6$ (which can occur if and only if E has j-invariant equal to 0).

(ii) Type II: $\sigma_{|E}$ is an automorphism of order $n = 2$ fixing P and Q. If we call $\nu : E^{\nu} \to E$ the normalization map and identify E^{ν} with $\mathbb{P}^1$ in such a way that $\nu^{-1}(P) = \infty$ and $\nu^{-1}(Q) = \{1, -1\}$, then the automorphism $\sigma_{|E}$ is induced by the automorphism $x \mapsto -x$ on $\mathbb{P}^1$.

(iii) Type III: $\sigma_{|E}$ is an automorphism of order $n = 2$ such that, if we identify R_i (for $i = 1, 2$) with $\mathbb{P}^1$ in such a way that Q_1 and Q_2 get identified with 1 and -1 (on both copies of $\mathbb{P}^1$) and $P \in R_1$ gets identified with ∞, then $\sigma_{|R_i}$ (for $i = 1, 2$) is equal to the automorphism $x \mapsto -x$ on $\mathbb{P}^1$. In particular, $\sigma_{|E}$ fixes P and exchanges Q_1 with Q_2. □

Proof. Parts (i) and (ii) follow easily from Remark 4.5 together with the fact that $E^{\mathrm{st}} \cong E$ for Types I and II and the well-known description of the elliptic tail automorphisms of stable curves (see, e.g. [27, Remark 4.2.2]).

In order to prove Part (iii), observe that in this case $E^{\mathrm{st}} \subseteq C^{\mathrm{st}}$ is a rational curve with one node. Therefore, there exists a unique elliptic tail automorphism of C^{st} with respect to E^{st}, namely the automorphism σ whose restriction $\sigma_{|E}$ is described in Part (ii). We conclude by Remark 4.5 together with the fact that the elliptic tail automorphism of (C, L) described in Part (iii) is the unique (by Corollary 2.12) lift to $\overline{\mathrm{Aut}(C, L)}$ of the elliptic tail automorphism of C^{st} with respect to E^{st} described in Part (ii). ■

We can now determine the singular locus of $\bar{P}^{\mathrm{st}}_{d,g}$.

Proposition 4.7. The singular locus of $\bar{P}^{\mathrm{st}}_{d,g}$ (for $g \geq 4$) is exactly the locus of pairs (C, L) such that $\overline{\mathrm{Aut}(C, L)}$ is not trivial. □

Proof. Near a point $(C, L) \in \bar{P}^{\mathrm{st}}_{d,g}$, using the local description (12) and Corollary 2.12, the scheme $\bar{P}^{\mathrm{st}}_{d,g}$ is isomorphic to the finite quotient

$$T_{\mathrm{Def}_{(C,L)}}/\overline{\mathrm{Aut}(C, L)},$$

where $T_{\mathrm{Def}_{(C,L)}}$ is a $\mathbb{C}$-vector space of dimension $4g-3$ (by Proposition 2.16(ii)) and $\overline{\mathrm{Aut}(C, L)}$ can be naturally identified with a finite subgroup of $\mathrm{GL}(T_{\mathrm{Def}_{(C,L)}})$.

By a well-known result of Prill (see [36]), it is enough to prove that $\overline{\mathrm{Aut}(C, L)} \subseteq \mathrm{GL}(T_{\mathrm{Def}_{(C,L)}})$ does not contain quasi-reflections, that is, elements ϕ such that 1 is an eigenvalue of ϕ with multiplicity equal to $4g-4$ or, equivalently, such that the fixed locus $\mathrm{Fix}(\phi)$ of ϕ is a divisor inside $T_{\mathrm{Def}_{(C,L)}}$.

Consider the morphism $\phi_d : \bar{P}_{d,g} \to \bar{M}_g$ which, according to (14), locally looks like

$$T_{\mathrm{Def}_{(C,L)}}/\overline{\mathrm{Aut}(C, L)} \to T_{\mathrm{Def}_{C^{\mathrm{st}}}}/\mathrm{Aut}(C^{\mathrm{st}}),$$

where $T_{\mathrm{Def}_{(C,L)}} \twoheadrightarrow T_{\mathrm{Def}_{C^{\mathrm{st}}}}$ is surjective with kernel V of dimension g and $\mathrm{Aut}(C^{\mathrm{st}})$ can be naturally identified with a finite subgroup of $\mathrm{GL}(T_{\mathrm{Def}_{C^{\mathrm{st}}}})$.

Assume, by contradiction, that $\phi \in \overline{\mathrm{Aut}(C, L)} \subseteq \mathrm{GL}(T_{\mathrm{Def}_{(C,L)}})$ is a quasi-reflection. By the above local description of the morphism ϕ_d, there are two possibilities for the image $\bar{G}(\phi)$ of ϕ in $\mathrm{Aut}(C^{\mathrm{st}}) \subseteq \mathrm{GL}(T_{\mathrm{Def}_{C^{\mathrm{st}}}})$ via the homomorphism $\bar{G}$ of Lemma 2.11:

(i) 1 is an eigenvalue of multiplicity $3g-3$ for $\bar{G}(\phi)$, that is, $\bar{G}(\phi) = \mathrm{id} \in \mathrm{Aut}(C^{\mathrm{st}})$;

(ii) 1 is an eigenvalue of multiplicity $3g-4$ for $\bar{G}(\phi)$, that is, $\bar{G}(\phi)$ is a quasi-reflection for $\mathrm{Aut}(C^{\mathrm{st}}) \subseteq \mathrm{GL}(T_{\mathrm{Def}_{C^{\mathrm{st}}}})$.

In case (i), we conclude that $\phi = \mathrm{id} \in \mathrm{Aut}(C, L)$ since $\bar{G}$ is injective for an element $(C, L) \in \bar{P}^{\mathrm{st}}_{d,g}$ by Corollary 2.12. This contradicts the fact that ϕ is a quasi-reflection.

In case (ii), it is well known (see, e.g. [27, Corollary 4.2.6]) that C^{st} must have an elliptic tail E and $\bar{G}(\phi)$ must be equal to the elliptic tail automorphism i of C^{st} of order 2 with respect to E (see Lemma 4.6). Since $i = \bar{G}(\phi)$ admits a lifting to $\overline{\mathrm{Aut}(C, L)}$, namely ϕ, the line bundle L on C must be such that the restriction $L_{|E}$ of L to E is a suitable translate of a 2-torsion point of $\mathrm{Pic}^0(E)$ (using some identification $\mathrm{Pic}^{d_E}(E) \cong \mathrm{Pic}^0(E)$ and the fact that i acts on $\mathrm{Pic}^0(E)$ sending η into η^{-1}). Therefore, the fixed locus $\mathrm{Fix}(\phi)$ of ϕ inside $T_{\mathrm{Def}_{(C,L)}}$ has codimension at least 2, hence ϕ is not a quasi-reflection. ■

By applying the Reid–Tai criterion for canonical singularities, we can prove the following result.

Theorem 4.8. Assume $g \geq 4$. Then the stable locus $\bar{P}^{\mathrm{st}}_{d,g}$ has canonical singularities. In particular, if $\widetilde{\bar{P}^{\mathrm{st}}_{d,g}}$ is a resolution of singularities of $\bar{P}^{\mathrm{st}}_{d,g}$, then every pluricanonical form defined on the smooth locus $(\bar{P}^{\mathrm{st}}_{d,g})^{\mathrm{reg}}$ of $\bar{P}^{\mathrm{st}}_{d,g}$ extends holomorphically to $\widetilde{\bar{P}^{\mathrm{st}}_{d,g}}$, that is, for all integers m we have

$$h^0((\bar{P}^{\mathrm{st}}_{d,g})^{\mathrm{reg}}, mK_{(\bar{P}^{\mathrm{st}}_{d,g})^{\mathrm{reg}}}) = h^0(\widetilde{\bar{P}^{\mathrm{st}}_{d,g}}, mK_{\widetilde{\bar{P}^{\mathrm{st}}_{d,g}}}). \qquad \square$$

Proof. We use the notation introduced in the proof of Proposition 4.7.

Given an element $\phi \in \overline{\mathrm{Aut}(C, L)} \subseteq \mathrm{GL}(T_{\mathrm{Def}_{(C,L)}})$ of order n, we can choose suitable coordinates of $T_{\mathrm{Def}_{(C,L)}}$ and a primitive nth root of unity ζ, such that the action of ϕ on $T_{\mathrm{Def}_{(C,L)}}$ is given by the sum $M(\phi) \oplus N(\phi)$ of two matrices (with $0 \leq a_i < n$ for $1 \leq i \leq 4g-3$):

$$M(\phi) = \begin{pmatrix} \zeta^{a_1} & & 0 \\ & \ddots & \\ 0 & & \zeta^{a_{3g-3}} \end{pmatrix} \quad \text{and} \quad N(\phi) = \begin{pmatrix} \zeta^{a_{3g-2}} & & 0 \\ & \ddots & \\ 0 & & \zeta^{a_{4g-3}} \end{pmatrix}$$

in such a way that the action of ϕ on V is given by $N(\phi)$ and the action of $\bar{G}(\phi)$ on $T_{\mathrm{Def}_{C^{\mathrm{st}}}}$ is given by $M(\phi)$.

Recall that, according to the Reid–Tai criterion for the canonicity of finite quotient singularities (see, e.g. [22, pp. 27–28] or [27, Theorem 4.1.11]), a point (C, L)

is a canonical singularity if and only if for every $\phi \in \overline{\mathrm{Aut}(C, L)}$ of some order n and every nth root of unity ζ we have

$$\sum_{i=1}^{4g-3} \frac{a_i}{n} \geq 1. \tag{19}$$

Note that this is true because $\overline{\mathrm{Aut}(C, L)}$ does not contain quasi-reflections (see the proof of Proposition 4.7).

Denote, as usual, by Δ_1 the divisor of $\bar{M}_g$ consisting of curves having an elliptic tail. If $C^{\mathrm{st}} \notin \Delta_1$ or $C^{\mathrm{st}} \in \Delta_1$ but $\bar{G}(\phi)$ is not an elliptic tail automorphism (or equivalently, by Remark 4.5, ϕ is not an elliptic tail automorphism) then by [22, Theorem 2] we get

$$\sum_{i=1}^{4g-3} \frac{a_i}{n} \geq \sum_{i=1}^{3g-3} \frac{a_i}{n} \geq 1,$$

and we are done in this case.

If $C^{\mathrm{st}} \in \Delta_1$ and $\bar{G}(\phi)$ is an elliptic tail automorphism with respect to the elliptic tail $E^{\mathrm{st}} \subset C^{\mathrm{st}}$ (where E^{st} is equal to the image via $\mathrm{st}: C \to C^{\mathrm{st}}$ of the elliptic tail $E \subset C$ as in Remark 4.2) then we choose, as in [27, Proposition 4.2.5], the first two coordinates t_1 and t_2 of $T_{\mathrm{Def}_{C^{\mathrm{st}}}}$ in such a way that (in the notation of Lemma 4.3): t_1 corresponds to the elliptic tail node P and t_2 correspond to Q if E^{st} is singular and is a coordinate for $T_{(E,P)}(M_{1,1})$ if E is smooth. In [27, Proposition 4.2.5], it is proved that the matrix $M(\phi)$ is given by (depending on the choice of the primitive nth root of unity ζ):

$$M(\phi) = \begin{cases} \begin{pmatrix} \zeta^1 & & \\ & \zeta^0 & \\ & & \mathbb{I} \end{pmatrix} & \text{if } n=2, \\ \begin{pmatrix} \zeta^1 & & \\ & \zeta^2 & \\ & & \mathbb{I} \end{pmatrix} \text{ or } \begin{pmatrix} \zeta^3 & & \\ & \zeta^2 & \\ & & \mathbb{I} \end{pmatrix} & \text{if } n=4, \\ \begin{pmatrix} \zeta^1 & & \\ & \zeta^2 & \\ & & \mathbb{I} \end{pmatrix} \text{ or } \begin{pmatrix} \zeta^2 & & \\ & \zeta^1 & \\ & & \mathbb{I} \end{pmatrix} & \text{if } n=3, \\ \begin{pmatrix} \zeta^5 & & \\ & \zeta^4 & \\ & & \mathbb{I} \end{pmatrix} \text{ or } \begin{pmatrix} \zeta^1 & & \\ & \zeta^2 & \\ & & \mathbb{I} \end{pmatrix} & \text{if } n=6, \end{cases} \tag{20}$$

where $\mathbb{I}$ is the suitable unit matrix.

Let us now turn to the matrix $N(\phi)$. We choose the first coordinate s_1 on V so that it is a coordinate for $T_{L_{|E}}(\mathrm{Pic}^{d_E}(E))$, where d_E is defined in Lemma 4.3(i). In order to compute the action of ϕ on s_1, we distinguish three cases according to whether the elliptic tail $E \subset C$ is of Type I, II or III (see Lemma 4.3(iii) and Figure 1).

If E is of Type I, that is, E is smooth, then we can identify E with $\mathrm{Pic}^{d_E}(E)$ sending $q \in E$ into $\mathcal{O}_E(q + (d_E - 1)P) \in \mathrm{Pic}^{d_E}(E)$. Since ϕ acts on $\mathrm{Pic}^{d_E}(E)$ via pull-back, if the action of $\bar{G}(\phi)$ on $T_P(E)$ is given by the multiplication by a root of unity ζ, then the action of ϕ on $T_{L_{|E}}(\mathrm{Pic}^{d_E}(E))$ is given by the multiplication by ζ^{-1}. Therefore, the matrix $N(\phi)$ is equal to (with respect to the same choice of the primitive nth root of unity ζ as in the above matrix $M(\phi)$):

$$N(\phi) = \begin{cases} \begin{pmatrix} \zeta^1 & \\ & \mathbb{I} \end{pmatrix} & \text{if } n=2, \\ \begin{pmatrix} \zeta^3 & \\ & \mathbb{I} \end{pmatrix} \text{ or } \begin{pmatrix} \zeta^1 & \\ & \mathbb{I} \end{pmatrix} & \text{if } n=4, \\ \begin{pmatrix} \zeta^2 & \\ & \mathbb{I} \end{pmatrix} \text{ or } \begin{pmatrix} \zeta^1 & \\ & \mathbb{I} \end{pmatrix} & \text{if } n=3, \\ \begin{pmatrix} \zeta^1 & \\ & \mathbb{I} \end{pmatrix} \text{ or } \begin{pmatrix} \zeta^5 & \\ & \mathbb{I} \end{pmatrix} & \text{if } n=6. \end{cases} \tag{21}$$

If E is of type II, that is, E is an irreducible rational curve with one node Q (as in Figure 1), then $\mathrm{Pic}^{d_E}(E) \cong \mathbb{G}_m$. Explicitly, if we consider the normalization morphism $\nu : E^\nu \cong \mathbb{P}^1 \to E$ and let $\nu^{-1}(Q) = \{u, v\}$, then any $\lambda \in \mathbb{G}_m(k)$ determines a unique line bundle $L_\lambda \in \mathrm{Pic}^{d_E}(E)$ whose local sections are the local sections s of $\mathcal{O}_{\mathbb{P}^1}(d_E)$ such that $s(u) = \lambda s(v)$. Since $\phi_{|E}$ is induced by an involution of E^ν that exchanges u and v (by Lemma 4.6(ii)), then clearly ϕ will send L_λ into $L_{\lambda^{-1}}$. This implies that the action of ϕ on $T_{L_{|E}}(\mathrm{Pic}^{d_E}(E))$ is given by multiplication by -1, hence the matrix $N(\phi)$ is also in this case given by (21) with $n=2$.

If E is of type III, that is, E is made of two irreducible rational components R_1 and R_2 meeting in two points Q_1 and Q_2 (as in Figure 1), then again $\mathrm{Pic}^{d_E}(E) \cong \mathbb{G}_m$. Explicitly, if we consider the normalization morphism $\nu : E^\nu = R_1 \coprod R_2 \to E$ and let $\nu^{-1}(Q_i) = \{u_i, v_i\}$ with $u_i \in R_1$ and $v_i \in R_2$ (for $i = 1, 2$), then any $\lambda \in \mathbb{G}_m(k)$ determines a unique line bundle $L_\lambda \in \mathrm{Pic}^{d_E}(E)$ whose local sections are pairs of local sections (s_1, s_2) of $(\mathcal{O}_{R_1}(d_E - 1), \mathcal{O}_{R_2}(1))$ such that

$$\frac{s_1(u_1)}{s_1(u_2)} = \lambda \frac{s_2(v_1)}{s_2(v_2)}.$$

Since $\phi_{|E}$ is induced by an involution of E^ν that exchanges u_1 with u_2 and v_1 with v_2 (by Lemma 4.6(iii)), then clearly ϕ will send L_λ into $L_{\lambda^{-1}}$. This implies that the action of ϕ on $T_{L_{|E}}(\mathrm{Pic}^{d_E}(E))$ is given by multiplication by -1, hence the matrix $N(\phi)$ is also in this case given by (21) with $n=2$.

An easy inspection of the matrices $M(\phi)$ in (20) and $N(\phi)$ in (21) reveals that the condition (19) is always satisfied, which shows that $\bar{P}^{\mathrm{st}}_{d,g}$ has canonical singularities.

The last assertion of the theorem follows from the well-known fact that canonical singularities do not impose adjoint conditions on the pluricanonical forms. ■

5 The canonical class of $\overline{\mathcal{P}ic}_{d,g}$ and of $\bar{P}_{d,g}$

The aim of this section is to prove Theorem 1.5. To achieve that, we first determine the canonical class of the stack $\overline{\mathcal{P}ic}_{d,g}$.

Theorem 5.1. The canonical class of $\overline{\mathcal{P}ic}_{d,g}$ is equal to

$$K_{\overline{\mathcal{P}ic}_{d,g}} = \Phi_d^*(14\lambda - 2\delta),$$

where λ and δ are the Hodge and total boundary class on $\bar{\mathcal{M}}_g$. □

Proof. Let $\pi : \overline{\mathcal{P}ic}_{d,g,1} \to \overline{\mathcal{P}ic}_{d,g}$ the universal family over $\overline{\mathcal{P}ic}_{d,g}$ and $\mathcal{L}_d$ the universal line bundle over $\overline{\mathcal{P}ic}_{d,g,1}$ (see [31] for a modular desc-ription of $\overline{\mathcal{P}ic}_{d,g,1}$). Denote by Ω_π and ω_π the sheaf of relative Kähler differentials and the relative dualizing sheaf, respectively. Let $d\colon \mathcal{O}^*_{\overline{\mathcal{P}ic}_{d,g,1}} \to \Omega_\pi$ be the universal derivation and consider the map induced in cohomology $\theta : \mathrm{Pic}(\overline{\mathcal{P}ic}_{d,g,1}) \to H^1(\overline{\mathcal{P}ic}_{d,g,1}, \Omega_\pi)$. Since $H^1(\overline{\mathcal{P}ic}_{d,g,1}, \Omega_\pi) \cong \mathrm{Ext}^1(\mathcal{O}_{\overline{\mathcal{P}ic}_{d,g,1}}, \Omega_\pi)$, the map θ sends the line bundle $\mathcal{L}_d$ on $\overline{\mathcal{P}ic}_{d,g,1}$ into the class of an extension

$$0 \to \Omega_\pi \to \mathcal{E} \to \mathcal{O}_{\overline{\mathcal{P}ic}_{d,g,1}} \to 0. \tag{22}$$

The restriction of the above extension (22) to a geometric fiber (C, L) of π is the extension (6) as it follows from the discussion in Section 2.3.

From this and the analysis of the deformation theory of the pair (C, L) carried out in Section 2.3, it follows that the tangent space of $\overline{\mathcal{P}ic}_{d,g}$ at a geometric point (C, L) is equal to $\mathrm{Ext}^1(\mathcal{E}_{|C}, \mathcal{O}_C) - \mathrm{Ext}^0(\mathcal{E}_{|C}, \mathcal{O}_C)$ Therefore, using relative duality for π, it follows that the canonical class $K_{\overline{\mathcal{P}ic}_{d,g}}$ of $\overline{\mathcal{P}ic}_{d,g}$ is equal to

$$K_{\overline{\mathcal{P}ic}_{d,g}} = c_1(\pi_!(\mathcal{E} \otimes \omega_\pi)).$$

To compute this class, we apply the Grothendieck–Riemann–Roch Theorem for quotient stacks [11] relative to the morphism π:

$$\mathrm{ch}(\pi_!(\omega_\pi \otimes \mathcal{E})) = \pi_*(\mathrm{ch}(\omega_\pi \otimes \mathcal{E}) \cdot \mathrm{Td}(\Omega_\pi)^{-1}). \tag{23}$$

Let us compute the degree one part of the right-hand side of (23). We set $\tilde{K} := c_1(\omega_\pi)$ and $\tilde{\eta} := c_2(\Omega_\pi)$. Note that, as remarked in [21, p. 158], we have $\tilde{K} = c_1(\Omega_\pi)$.

The first three terms of inverse of the Todd class of Ω_π are equal to

$$\mathrm{Td}(\Omega_\pi)^{-1} = 1 - \frac{c_1(\Omega_\pi)}{2} + \frac{c_1^2(\Omega_\pi) + c_2(\Omega_\pi)}{12} + \cdots = 1 - \frac{\tilde{K}}{2} + \frac{\tilde{K}^2 + \tilde{\eta}}{12} + \cdots. \tag{24}$$

Using (22), we get

$$\mathrm{ch}(\mathcal{E} \otimes \omega_\pi) = \mathrm{ch}(\mathcal{E}) \cdot \mathrm{ch}(\omega_\pi) = (\mathrm{ch}(\Omega_\pi) + \mathrm{ch}(\mathcal{O}_{\overline{\mathcal{P}ic}_{d,g,1}})) \cdot \mathrm{ch}(\omega_\pi) = (\mathrm{ch}(\Omega_\pi) + 1) \cdot \mathrm{ch}(\omega_\pi). \tag{25}$$

Moreover, we have

$$\mathrm{ch}(\omega_\pi) = 1 + c_1(\omega_\pi) + \frac{c_1(\omega_\pi)}{2} + \cdots = 1 + \tilde{K} + \frac{\tilde{K}^2}{2} + \cdots,$$
$$\mathrm{ch}(\Omega_\pi) = 1 + c_1(\Omega_\pi) + \frac{c_1(\Omega_\pi)}{2} - c_2(\Omega_\pi) + \cdots = 1 + \tilde{K} + \frac{\tilde{K}^2}{2} - \tilde{\eta} + \cdots.$$

Substituting into (25), we arrive at

$$\mathrm{ch}(\mathcal{E} \otimes \omega_\pi) = 2 + 3\tilde{K} + \tfrac{5}{2}\tilde{K}^2 - \tilde{\eta} + \cdots. \tag{26}$$

Combining (24) and (26), we get

$$[\mathrm{ch}(\omega_\pi \otimes \mathcal{E}) \cdot \mathrm{Td}(\Omega_\pi)^{-1}]_2 = \frac{\tilde{K}^2 + \tilde{\eta}}{6} - \frac{3}{2}\tilde{K}^2 + \frac{5}{2}\tilde{K}^2 - \tilde{\eta} = \frac{7}{6}\tilde{K}^2 - \frac{5}{6}\tilde{\eta},$$

hence, from (23), we deduce

$$K_{\overline{\mathcal{P}ic}_{d,g}} = \tfrac{7}{6}\pi_*(\tilde{K}^2) - \tfrac{5}{6}\pi_*(\tilde{\eta}). \tag{27}$$

Let us now apply the Grothendieck–Riemann–Roch theorem to the sheaf ω_π. Since $R^1\pi_*\omega_\pi = \mathcal{O}_{\overline{\mathcal{P}ic}_{d,g}}$ by relative duality, we get

$$c_1(\pi_*\omega_\pi) = \pi_*\left(\frac{\tilde{K}^2+\tilde{\eta}}{12} - \frac{1}{2}\tilde{K}^2 + \frac{1}{2}\tilde{K}^2\right) = \frac{1}{12}\pi_*(\tilde{K}^2) + \frac{1}{12}\pi_*(\tilde{\eta}).$$

If we set $\tilde{\lambda} := c_1(\pi_*\omega_\pi)$ and $\tilde{\delta} := \pi_*(\tilde{\eta})$, then the previous relation becomes $12\tilde{\lambda} = \pi_*(\tilde{K}^2) + \tilde{\delta}$. Substituting into (27), we obtain

$$K_{\overline{\mathcal{P}ic}_{d,g}} = 14\tilde{\lambda} - 2\tilde{\delta}.$$

The lemma below completes the proof. ■

Lemma 5.2. With the notation of Theorem 5.1, we have

$$\tilde{\lambda} = \Phi_d^*(\lambda) \quad \text{and} \quad \tilde{\delta} = \Phi_d^*(\delta).$$ □

Proof. Consider the diagram

$$\begin{tikzcd}
\overline{\mathcal{P}ic}_{d,g,1} \arrow[r, "\Phi_{d,1}"] \arrow[d, "\pi"'] & \bar{\mathcal{M}}_{g,1} \arrow[d, "\bar{\pi}"] \\
\overline{\mathcal{P}ic}_{d,g} \arrow[r, "\Phi_d"] & \bar{\mathcal{M}}_g
\end{tikzcd} \tag{28}$$

Recall that the classes λ and δ on $\bar{\mathcal{M}}_g$ are defined as

$$\lambda := c_1(\bar{\pi}_*(\omega_{\bar{\pi}})) \quad \text{and} \quad \delta := \bar{\pi}_*(c_2(\Omega_{\bar{\pi}})),$$

where $\Omega_{\bar{\pi}}$ and $\omega_{\bar{\pi}}$ are the sheaf of relative Kähler differentials and the relative dualizing sheaf of $\bar{\pi}$, respectively.

The map Φ_d sends an element $(\mathcal{C} \to S, \mathcal{L}) \in \overline{\mathcal{P}ic}_{d,g}(S)$ into the stabilization $\mathcal{C}^{\mathrm{st}} \to S \in \bar{\mathcal{M}}_g(S)$. Recall that for every quasi-stable (or more generally semistable) curve C with a stabilization morphism $\psi : C \to C^{\mathrm{st}}$, the pull-back via ψ induces a natural isomorphism $\psi^* : H^0(C^{\mathrm{st}}, \omega_{C^{\mathrm{st}}}) \xrightarrow{\cong} H^0(C, \omega_C)$. Therefore, we have $\Phi_d^*(\bar{\pi}_*(\omega_{\bar{\pi}})) = \pi_*(\omega_\pi)$ and, by taking the first Chern classes, we get $\tilde{\lambda} = \Phi_d^*(\lambda)$.

On the other hand, since the class $\tilde{\delta}$ is the total boundary class of $\overline{\mathcal{P}ic}_{d,g}$ and δ is the total boundary class of $\bar{\mathcal{M}}_g$ (see [22, pp. 49–50]), it is clear that $\tilde{\delta} = \Phi_d^*(\delta)$. ■

Proof of Theorem 1.5. Let $p\colon \overline{\mathcal{P}ic}_{d,g} \to \bar{P}_{d,g}$ the natural map from the stack $\overline{\mathcal{P}ic}_{d,g}$ to its good moduli space. In view of Theorem 5.1, it is enough to show that $p^*(K_{\bar{P}_{d,g}}) = K_{\overline{\mathcal{P}ic}_{d,g}}$.

Clearly, the two classes agree on the interior $\mathcal{P}ic_{d,g}$, since for C varying in an open subset of M_g whose complement has codimension at least 2 (since $g \geq 4$), by Lemma 2.11 we have that $\overline{\mathrm{Aut}(C,L)} \subseteq \mathrm{Aut}(C) = \{\mathrm{id}\}$.

Let us now look at the boundary of $\bar{P}_{d,g}$. By [18, Proposition 4], the boundary of $\bar{P}_{d,g}$ is the union of the irreducible divisors $D_i := \phi_d^{-1}(\Delta_i)$, for $i = 0, \ldots, [g/2]$. Let $k_{d,g} := (2g-2, d-g+1)$. A general element of $(C,L) \in D_i$ looks as follows (see e.g [3, Ex. 7.1, 7.2; 30, Proposition 2]):

(1) If $i = 0$, then C is a general irreducible nodal curve with one node and L is a general line bundle of degree d on C.

(2) If $i > 0$ and $2g-2$ does not divide $(2i-1)\cdot k_{d,g}$, then C is a stable curve consisting of two general smooth curves C_1 and C_2 of genera, respectively, i and $g-i$ meeting in one point and L is a general line bundle of multidegree $(\deg_{C_1}(L), \deg_{C_2}(L)) = (d_1, d_2 = d - d_1)$ where d_1 is the unique integer such that
$$\left| d_1 - \frac{d(2i-1)}{2g-2} \right| < \frac{1}{2}.$$

(3) If $i > 0$ and $2g-2$ divides $(2i-1)\cdot k_{d,g}$, then C is a quasi-stable curve consisting of two general smooth curves C_1 and C_2 of genera, respectively, i and $g-i$ joined by a rational curve $R \cong \mathbb{P}^1$ and L is a general line bundle whose multidegree is such that $\deg_R L = 1$ and
$$d_1 := \deg_{C_1} L = \frac{d(2i-1)}{(2g-2)} - \frac{1}{2},$$
$$d_2 := \deg_{C_2} L = \frac{d(2g-2i-1)}{(2g-2)} - \frac{1}{2}.$$

We claim that the automorphism group of a general point $(C,L) \in D_i$ is equal to
$$\mathrm{Aut}(C,L) = \begin{cases} \mathbb{G}_m & \text{in cases } 1 \text{ and } 2, \\ \mathbb{G}_m^2 & \text{in case } 3. \end{cases} \tag{29}$$

Indeed, by the explicit description above, $\gamma(\bar{C})=1$ in cases (1) and (2), and $\gamma(\bar{C})=2$ in case (3). Therefore, the claim will follow from Lemma 2.11 if we show that the image of $\mathrm{Aut}(C,L)\to\mathrm{Aut}(C^{\mathrm{st}})$ is trivial. This is trivially true if $i\neq 1$ since in this case C^{st} is a general curve in Δ_i, hence $\mathrm{Aut}(C^{\mathrm{st}})=\{\mathrm{id}\}$. If $i=1$, then $\mathrm{Aut}(C^{\mathrm{st}})=\mathbb{Z}/2\mathbb{Z}$ generated by the elliptic tail involution σ with respect to the elliptic tail C_1 (see Remark 4.5 and the notation there). However, in this case, σ comes from an automorphism of the pair (C,L) if and only if $L_{|C_1}(-d_1\cdot P)$ is a 2-torsion point of $\mathrm{Pic}^0(C_1)$, where $P=C_1\cap R$ is the elliptic tail node of C_1. Clearly, this is not the case for a general strictly balanced line bundle L on C.

In cases (1) and (2), $\underline{\mathrm{Def}}_{(C,L)}$ has dimension $4g-3$ and $\mathrm{Aut}(C,L)=\mathbb{G}_m$ acts trivially on it (see Section 2.3). Therefore, the morphism p looks locally at (C,L) as $\tilde{p}:[\underline{\mathrm{Def}}_{(C,L)}/\mathbb{G}_m]=\underline{\mathrm{Def}}_{(C,L)}\times B\mathbb{G}_m\to\underline{\mathrm{Def}}_{(C,L)}$. It is clear that in this case $\tilde{p}^*(K_{\underline{\mathrm{Def}}_{(C,L)}})=K_{\underline{\mathrm{Def}}_{(C,L)}\times B\mathbb{G}_m}$.

In case (3), $\underline{\mathrm{Def}}_{(C,L)}$ has dimension $4g-2$ (see 2.3). If we choose the first two coordinates x and y of $\underline{\mathrm{Def}}_{(C,L)}$ in such a way that they correspond to the local deformations of the two nodes $P_1:=C_1\cap R$ and $P_2:=C_2\cap R$, then the action of $(\mu,\nu)\in\mathrm{Aut}(C,L)=\mathbb{G}_m^2$ on the first two coordinates of $\underline{\mathrm{Def}}_{(C,L)}$ is given by

$$(\mu,\nu)\cdot(x,y)=(\mu\nu^{-1}x,\mu^{-1}\nu y), \tag{30}$$

while it is trivial on the other coordinates. Therefore, neglecting the trivial coordinates, at (C,L) the morphism p looks locally as

$$\mathcal{X}:=[\mathrm{Spf}\mathbb{C}[[x,y]]/\mathbb{G}_m^2]\xrightarrow{\tilde{p}}\mathrm{Spf}\mathbb{C}[[x,y]]/\mathbb{G}_m^2:=X.$$

Since the ring of invariants for the action (30) of $\mathbb{G}_m^2$ on $\mathbb{C}[[x,y]]$ is generated by xy, the quotient X is isomorphic to $\mathrm{Spf}\mathbb{C}[[xy]]$. The quotient map $Y:=\mathrm{Spf}\mathbb{C}[[x,y]]\xrightarrow{q}X=\mathrm{Spf}\mathbb{C}[[xy]]$ induces a pull-back map

$$q^*:\langle \mathrm{d}(xy)\rangle=\Omega_X^1\to\Omega_Y^1=\langle \mathrm{d}x,\mathrm{d}y\rangle,$$
$$\mathrm{d}(xy)\mapsto x\,\mathrm{d}y+y\,\mathrm{d}x.$$

On the other hand, the cotangent complex of $\mathcal{X}$ is equal to the $\mathbb{G}_m^2$-equivariant cotangent complex of $Y:=\mathrm{Spf}\mathbb{C}[[x,y]]$ (see, e.g. [32, p. 37]), which in our case looks like:

$$\mathbb{L}:\langle \mathrm{d}x,\mathrm{d}y\rangle=\Omega_Y^1\xrightarrow{f}\mathcal{O}_Y\otimes\mathrm{Lie}(\mathbb{G}_m^2)^*=\mathcal{O}_Y\otimes\left\langle\frac{\mathrm{d}\lambda}{\lambda},\frac{\mathrm{d}\mu}{\mu}\right\rangle.$$

The map f is the dual of the infinitesimal action of $\mathrm{Lie}(\mathbb{G}_m^2)$ on Y, hence, by the explicit action (30), we compute

$$f(\mathrm{d}x) = x\frac{\mathrm{d}\lambda}{\lambda} - x\frac{\mathrm{d}\mu}{\mu},$$

$$f(\mathrm{d}y) = y\frac{\mathrm{d}\mu}{\mu} - y\frac{\mathrm{d}\lambda}{\lambda}.$$

Since the image of q^* is equal to the kernel of f, we deduce that $\tilde{p}^*(K_X) = K_{\mathcal{X}}$, which concludes the proof. ■

6 The Iitaka Dimension of $K_{\bar{P}_{d,g}}$

The aim of this section is to prove Theorem 1.6. Since $K_{\bar{P}_{d,g}} = \phi_d^*(14\lambda - 2\delta)$ (by Theorem 1.5) and ϕ_d has connected fibers, we have

$$\kappa(K_{\bar{P}_{d,g}}) = \kappa(14\lambda - 2\delta). \tag{31}$$

Therefore, we are reduced to study the Iitaka dimension of the divisor $14\lambda - 2\delta$ on $\bar{M}_g$. Note that the slope of $14\lambda - 2\delta$ is equal to 7. Hence, if the slope $s(\bar{M}_g)$ of $\bar{M}_g$ (in the sense of Harris–Morrison [20]) is strictly less than 7 then we conclude that $14\lambda - 2\delta$ is big, that is, that $\kappa(14\lambda - 2\delta) = 3g - 3$; while if $s(\bar{M}_g) > 7$ then $14\lambda - 2\delta$ is not pseudo-effective and $\kappa(14\lambda - 2\delta) = -\infty$ (see the discussion at the beginning of [20]).

Proposition 6.1. If $g \leq 9$, then $\kappa(K_{\bar{P}_{d,g}}) = -\infty$. □

Proof. This follows by the fact that $s(\bar{M}_g) > 7$ for $g \leq 9$ (see [38]). ■

Remark 6.2. By combining the above Proposition 6.1 with the inequality (1), we obtain another proof of the fact that $\kappa(P_{d,g}) = -\infty$ for $g \leq 9$ and any d (which of course follows from the stronger Theorem 1.1). □

For $g \geq 12$ we can prove the following:

Proposition 6.3. If $g \geq 12$, then $\kappa(K_{\bar{P}_{d,g}}) = 3g - 3$ and the fibration $\phi_d\colon \bar{P}_{d,g} \to \bar{M}_g$ is the Iitaka fibration of $K_{\bar{P}_{d,g}}$. □

Proof. We have already observed, in the proof of Proposition 3.2, that if $\kappa(K_{\bar{P}_{d,g}}) = 3g-3$ then $\phi_d : \bar{P}_{d,g} \to \bar{M}_g$ is the Iitaka fibration (see [25, Definition 2.1.34]) of $K_{\bar{P}_{d,g}}$. Therefore, it is enough to prove the first assertion. As pointed out before, this will follow if we show that $s(\bar{M}_g) < 7$ for $g \geq 12$.

By computing the class of the Brill–Noether divisor $D^1_{\frac{g+1}{2}}$, Harris and Mumford proved in [22] that

$$s(\bar{M}_g) \leq 6 + \frac{12}{g+1} \quad \text{if } g \text{ is odd.}$$

Since $6 + \frac{12}{g+1} < 7$ if and only if $g > 11$, we get that

$$s(\bar{M}_g) < 7 \quad \text{if } g \text{ is odd and } g \geq 13. \tag{32}$$

By computing the class of the Petri divisor $E^1_{\frac{g}{2}+1}$, Eisenbud and Harris in [12, Theorem 2] proved that

$$s(\bar{M}_g) \leq 6 + \frac{14g+4}{g(g+2)} \quad \text{if } g \text{ is even.}$$

Since $6 + \frac{14g+4}{g(g+2)} < 7$ if and only if $g > 13$, we get that

$$s(\bar{M}_g) < 7 \quad \text{if } g \text{ is even and } g \geq 14. \tag{33}$$

By computing the slope of some effective divisors on $\bar{M}_g$ associated to curves equipped with secant-exceptional linear series, Cotterill in [8, Section 6.2] showed in particular that

$$s(\bar{M}_{12}) \leq 6979 \cdots < 7. \tag{34}$$

Equations (32)–(34) together imply the result. ■

The cases $g = 10$ and 11 requires a special care since it is known that in this case $s(\bar{M}_g) = 7$ (see [15, Corollary 1.3; 38]). We start with the case $g = 10$.

Proposition 6.4. If $g = 10$, then $\kappa(K_{\bar{P}_{d,g}}) = 0$. □

Proof. Farkas and Popa proved in [15, Theorem 1.6] that the effective irreducible divisor F (which is denoted by $\bar{K}$ in [15]) given by the closure of the locus of smooth curves of genus 10 lying on a $K3$ surface has a class equal to

$$F = 7\lambda - \delta_0 - 5\delta_1 - 9\delta_2 - 12\delta_3 - 14\delta_4 - B_5\delta_5, \tag{35}$$

with $B_5 \geq 6$. Since it is easily checked that $14\lambda - 2\delta$ is the sum of $2F$ and an effective boundary divisor, we get, using (31), that $\kappa(K_{\bar{P}_{d,g}}) = \kappa(14\lambda - 2\delta) \geq \kappa(2F) = \kappa(F) \geq 0$.

It remains to prove that $h^0(\bar{M}_{10}, m(14\lambda - 2\delta)) = 1$ for any m sufficiently divisible.

Claim 1. If m is sufficiently divisible, then $2mF$ is contained in the base locus of $|m(14\lambda - 2\delta)|$. □

Take $D \in |m(14\lambda - 2\delta)|$ and let r be the multiplicity of F inside D. Consider a Lefschetz pencil of curves of genus 10 lying on a general K3 surface of degree 18 in $\mathbb{P}^{10}$. This gives rise to an irreducible curve B in the moduli space $\bar{M}_{10}$. Such pencils B fill the divisor F, by results of Mukai [33]. Therefore B is not contained in the support of $D - rF$, hence $(D - rF) \cdot B \geq 0$. Using the well-known formulas $\lambda \cdot B = g + 1 = 11$, $\delta_0 \cdot B = 6(g+3) = 78$, $\delta_i \cdot B = 0$ for $i \geq 1$ (see, e. g. [15, Lemma 2.1]), together with the expression (35), we get that

$$0 \leq (D - rF) \cdot B = (2m - r)[(7\lambda - \delta_0) \cdot B] = r - 2m,$$

which concludes the proof of the claim.

From the previous claim, it follows that

$$h^0(\bar{M}_{10}, m(14\lambda - 2\delta)) = h^0(\bar{M}_{10}, m(14\lambda - 2\delta) - 2mF).$$

Note that $m(14\lambda - 2\delta) - 2mF = m(\sum_{i\geq 1} a_i\delta_i)$ for some $a_i \geq 0$. Therefore, the proof of the theorem is concluded by the following:

Claim 2. Let Δ be an effective divisor in $\bar{M}_g$ (for $g \geq 3$) whose class in $\mathrm{Pic}(\bar{M}_g)_{\mathbb{Q}}$ is equal to $\sum_{i\geq 0} a_i\delta_i$, with $a_i \geq 0$. Then $h^0(\bar{M}_g, m\Delta) = 1$ for any m sufficiently divisible. □

Take $E \in |m\Delta|$. We have to show that $E = m\Delta$.

If E meets the interior M_g of $\bar{M}_g$, then, from the well-known result that $\mathrm{Pic}(M_g)_{\mathbb{Q}}$ is generated by λ and λ is ample on M_g, we get that the class of E in $\mathrm{Pic}(\bar{M}_g)_{\mathbb{Q}}$ is equal to $a\lambda + \sum_{i\geq 0} b_i\delta_i$ with $a > 0$ and $b_i \in \mathbb{Z}$. However, the class of E in $\mathrm{Pic}(\bar{M}_g)_{\mathbb{Q}}$ is also equal to the class of $m\Delta$, which is $\sum_{i\geq 0} ma_i\delta_i$. This produces a nontrivial relation between λ and the boundary classes δ_i, which contradicts the well-known result that $\mathrm{Pic}(\bar{M}_g)_{\mathbb{Q}}$ is freely generated by λ and the boundary classes δ_i for $g \geq 3$ (see [1]).

Therefore, E must be entirely contained in the boundary $\bar{M}_g \setminus M_g = \bigcup_{i\geq 0} \Delta_i$ of $\bar{M}_g$. This implies that $E = \sum_{i\geq 0} b_i\Delta_i$ for some $b_i \geq 0$. Looking at the classes of $\sum_{i\geq 0} b_i\Delta_i$ and $m\Delta$ in $\mathrm{Pic}(\bar{M}_g)_{\mathbb{Q}}$ and using the independence of the boundary classes δ_i in $\mathrm{Pic}(\bar{M}_g)_{\mathbb{Q}}$, we deduce that $E = m\Delta$, as required. ■

We finally examine the case $g = 11$. As usual, denote by $\mathcal{F}_g$ $(g \geq 3)$ the moduli space of K3 surfaces endowed with a polarization of degree $2g - 2$. By work of Mukai [34], there exists a fibration

$$\psi : \bar{M}_{11} \dashrightarrow \mathcal{F}_{11},$$

sending a general curve C of genus g into $(S, \mathcal{O}_S(C))$, where S is the unique K3 surface containing C.

Proposition 6.5. If $g = 11$, then $\kappa(K_{\bar{P}_{d,g}}) = 19$ and the Iitaka fibration of $K_{\bar{P}_{d,g}}$ is the composition

$$\bar{P}_{d,11} \xrightarrow{\phi_d} \bar{M}_{11} \overset{\psi}{\dashrightarrow} \mathcal{F}_{11}.$$ □

Proof. Farkas and Popa proved in [15, Proposition 6.2] that the Iitaka dimension of the divisor

$$E := 7\lambda - \delta_0 - 5\delta_1 - 9\delta_2 - 8\delta_3 - 7\delta_4 - 7\delta_5$$

is 19. Since it is easily checked that $14\lambda - 2\delta$ is the sum of $2E$ and an effective boundary divisor, we get, using (31), that $\kappa(K_{\bar{P}_{d,g}}) = \kappa(14\lambda - 2\delta) \geq \kappa(2E) = \kappa(E) = 19$.

Consider now a general point $(S, L) \in \mathcal{F}_{11}$. The fiber of ψ over (S, L) is the open subset of the complete linear series $|L| \cong \mathbb{P}^{11}$ consisting of smooth connected curves. Pick a Lefschetz pencil on S and consider the associated curve B inside $\bar{M}_{11}$. It is well known (see e. g. [15, Lemma 2.1]) that $\lambda \cdot B = 12$, $\delta_0 \cdot B = 84$ and $\delta_i \cdot B = 0$ for every $i > 1$. This easily implies that

$$(14\lambda - 2\delta) \cdot B = 0. \tag{36}$$

Consider now the Iitaka fibration of the divisor $K_{\bar{P}_{d,11}}$, which we denote by

$$i_{K_{\bar{P}_{d,11}}} : \bar{P}_{d,11} \dashrightarrow I(K_{\bar{P}_{d,11}}).$$

Since $K_{\bar{P}_{d,11}} = \phi_d^*(14\lambda - 2\delta)$, the Iitaka fibration $i_{14\lambda-2\delta}$ of $K_{\bar{P}_{d,11}}$ is the composition of the Iitaka fibration of $14\lambda - 2\delta$ with ϕ_d, that is, we have a natural diagram (up to birationality)

$$\begin{tikzcd}
\bar{P}_{d,11} \arrow[rr, "\phi_d"] \arrow[dr, dashed, "i_{K_{\bar{P}_{d,11}}}"'] & & \bar{M}_{11} \arrow[dl, dashed, "i_{14\lambda-2\delta}"] \\
& I(K_{\bar{P}_{d,11}}) = I(14\lambda - 2\delta) &
\end{tikzcd}$$

Now, equation (36) implies that the Iitaka fibration $i_{14\lambda-2\delta}$ contracts the general fiber $\psi^{-1}(S, L) \subset |L| \cong \mathbb{P}^{11}$. Therefore, the Iitaka fibration $i_{14\lambda-2\delta}$ factors through the fibration ψ:

$$\begin{tikzcd}
& \bar{M}_{11} \arrow[dl, dashed, "\psi"'] \arrow[dr, dashed, "i_{14\lambda-2\delta}"] & \\
\mathcal{F}_{11} \arrow[rr, dashed, "\rho"] & & I(14\lambda - 2\delta)
\end{tikzcd}$$

Recall that $\dim \mathcal{F}_{11} = 19$. On other hand, by the usual properties of the Iitaka fibration and what proved before, we have that $\dim I(14\lambda - 2\delta) = \kappa(14\lambda - 2\delta) \geq 19$. Since ρ is dominant and has connected general fiber, this implies that ρ is a birational isomorphism, hence we are done. ■

7 Birationalities among Different $P_{d,g}$'s

In this section, inspired by [3, Lemma 8.1], we investigate the following:

Question 7.1. For what values of d and d' is $P_{d,g}$ birational to $P_{d',g}$? How do the birational maps among them look like? □

Note that if $d' = d + n(2g-2)$ for some $n \in \mathbb{Z}$, then we have the isomorphism

$$\begin{aligned} \psi_n^1 : P_{d,g} &\xrightarrow{\cong} P_{d',g}, \\ (C, L) &\mapsto (C, L \otimes \omega_C^n), \end{aligned} \tag{37}$$

while if $d' = -d + n(2g-2)$ for some $n \in \mathbb{Z}$ we have the isomorphism

$$\begin{aligned} \psi_n^2 : P_{d,g} &\xrightarrow{\cong} P_{d',g}, \\ (C, L) &\mapsto (C, L^{-1} \otimes \omega_C^n). \end{aligned} \tag{38}$$

Clearly the maps ψ_n^1 and ψ_n^2 commute with the projections ϕ_d and $\phi_{d'}$ onto M_g. Indeed, the converse is also true, as it follows from an argument of Caporaso (see [3, Lemma 8.1] and also, for further details, [5, Proposition 3.2.2]).

Theorem 7.2 ([3]). If $\eta : P_{d,g} \dashrightarrow P_{d',g}$ is a birational map over M_g, that is, a map η inducing a commutative diagram

$$\begin{tikzcd} P_{d,g} \arrow[r, dashed, "\eta"] \arrow[d, "\phi_d"'] & P_{d',g} \arrow[d, "\phi_{d'}"] \\ M_g \arrow[r, "\mathrm{id}"] & M_g \end{tikzcd}$$

then there exists $n \in \mathbb{Z}$ such that either $d' = d + n(2g-2)$ and $\eta = \psi_n^1$ or $d' = -d + n(2g-2)$ and $\eta = \psi_n^2$. $\square$

By using our results on the Kodaira dimension of $P_{d,g}$, we can improve Theorem 7.2 at least for genus big enough.

Theorem 7.3. Assume that $g \geq 22$ or $g \geq 12$ and $(d-g+1, 2g-2) = 1$. Let $\eta : P_{d,g} \dashrightarrow P_{d',g}$ be a birational map. Then there exists $n \in \mathbb{Z}$ such that either $d' = d + n(2g-2)$ and $\eta = \psi_n^1$ or $d' = -d + n(2g-2)$ and $\eta = \psi_n^2$. In particular, η is an isomorphism. $\square$

Proof. By Theorems 1.2 and 1.3, the assumptions of the statement imply that $\kappa(P_{d,g}) = 3g-3$, hence that $\kappa(P_{d',g}) = 3g-3$. From the proof of Proposition 3.2, it follows that $\phi_d : P_{d,g} \to M_g$ is the Iitaka fibration of $P_{d,g}$ and similarly for $P_{d',g}$. Since the Iitaka fibration is a birational invariant, the map η induces a birational map $\xi : M_g \dashrightarrow M_g$ such that the following diagram commutes:

$$\begin{tikzcd} P_{d,g} \arrow[r, dashed, "\eta"] \arrow[d, "\phi_d"'] & P_{d',g} \arrow[d, "\phi_{d'}"] \\ M_g \arrow[r, dashed, "\xi"] & M_g \end{tikzcd}$$

The map ξ sends a very general curve $C \in M_g$ to a very general curve $C' \in M_g$ so that the restriction of η induces a birational map $J(C) \cong \mathrm{Pic}^d(C) \dashrightarrow \mathrm{Pic}^{d'}(C') \cong J(C')$. By Lemma 7.4, we get that $C \cong C'$, hence $\xi = \mathrm{id}$. Finally, we conclude by Theorem 7.2. ■

Lemma 7.4. If C and C' are very general curves in M_g such that $J(C)$ is birational to $J(C')$, then $C \cong C'$. □

Proof. Let $\epsilon : J(C) \dashrightarrow J(C')$ be a birational map. Since $J(C)$ and $J(C')$ are abelian varieties, then it is well known that ϵ extends to an isomorphism $\epsilon : J(C) \xrightarrow{\cong} J(C')$. Since C (resp. C') are very general curves in M_g, we may assume (by [2, Corollary 17.5.2]) that $\mathrm{NS}(J(C)) = \mathbb{Z}$ (resp. $\mathrm{NS}(J(C')) = \mathbb{Z}$) generated by the class of the theta divisor $[\Theta_C]$ (resp. $[\Theta_{C'}]$). Therefore, we must have $\epsilon^*([\Theta_{C'}]) = \pm[\Theta_C]$. Moreover, since $[\Theta_C]$ is ample and the pull-back morphism ϵ^* preserves ampleness, we get that actually $\epsilon^*([\Theta_{C'}]) = [\Theta_C]$. We conclude that $C \cong C'$ by the classical Torelli theorem. ■

From the previous result, we can deduce two corollaries. The first one concerns the group of birational self maps $\mathrm{Bir}(P_{d,g})$ and the group of automorphisms $\mathrm{Aut}(P_{d,g})$ of $P_{d,g}$.

Corollary 7.5. With the same assumptions as in Theorem 7.3, we have

$$\mathrm{Bir}(P_{d,g}) = \mathrm{Aut}(P_{d,g}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } d = n(g-1) \text{ for some } n \in \mathbb{Z}, \\ \{\mathrm{id}\} & \text{otherwise,} \end{cases}$$

where in the first case the generator of the cyclic group $\mathbb{Z}/2\mathbb{Z}$ is ψ_n^2. □

Proof. The claim follows from Theorem 7.3, since the only maps ψ_n^1 and ψ_n^2 having domain $P_{d,g}$ and codomain $P_{d,g}$ are $\psi_0^1 = \mathrm{id}$ and ψ_n^2 if $d = n(g-1)$, and in this case $(\psi_n^2) \circ (\psi_n^2) = \mathrm{id}$. ■

The second corollary is analogous to [19, Corollary (0.12)], which states that the boundary $\partial \bar{M}_g$ of the moduli space $\bar{M}_g$ of stable curves of genus $g \geq 2$ is preserved by any automorphism of $\bar{M}_g$.

Corollary 7.6. Same assumptions as in Theorem 7.3. Then any automorphism $\phi : \bar{P}_{d,g} \to \bar{P}_{d,g}$ preserves the boundary $\partial \bar{P}_{d,g} := \bar{P}_{d,g} \setminus P_{d,g}$. □

Proof. The restriction $\eta := \phi_{|P_{d,g}}$ of ϕ to $P_{d,g}$ defines a birational self map of $P_{d,g}$. By Corollary 7.5, η is an automorphism of $P_{d,g}$. Therefore, ϕ maps $P_{d,g}$ isomorphically onto $P_{d,g}$, hence it preserves the boundary $\partial \bar{P}_{d,g}$. ■

Remark 7.7. Under the same assumptions as in Theorem 7.3, Corollary 7.6 implies that we have a restriction map

$$\mathrm{res} : \mathrm{Aut}(\bar{P}_{d,g}) \to \mathrm{Aut}(P_{d,g}).$$

The map res is injective since $\bar{P}_{d,g}$ is separated. In [3, Lemma 8.1], it is claimed that the map ψ_n^2 of (38) extends to a map $\bar{P}_{d,g} \to \bar{P}_{d',g}$ (the analogous statement for ψ_n^1 is easy to prove). This fact, together with Corollary 7.5, would imply that res is an isomorphism. □

Finally, note that if one could remove our technical assumption on the degree in Theorem 1.2, then Theorem 7.3 and Corollaries 7.5 and 7.6 would follow for $g \geq 12$ without any hypothesis on the degree.

Acknowledgements

We are grateful to Silvia Brannetti, Lucia Caporaso, Eduardo Esteves, and Margarida Melo for stimulating discussions on these topics. We thank Gavril Farkas and Sandro Verra for pointing out a gap in the proof of a previous version of Theorem 7.3. We thank the two referees of the paper for their precious remarks that helped in improving the exposition. In particular, we are grateful to one of the referees for pointing out a mistake in a previous version of Theorems 1.4 and 4.8.

Funding

G. B. was partially supported by FIRST Universit à di Milanoand by MIUR Cofin 2008-Varietà algebriche: geometria, aritmetica, strutture di Hodge. C. F. was partially supported by GNSAGA of INdAM and by MIUR Cofin 2008-Geometria delle varietà algebriche e dei loro spazi di moduli. F. V. was partially supported by FCT-Ciência2008 and by MIUR Cofin 2008-Geometria delle varietà algebriche e dei loro spazi di moduli.

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