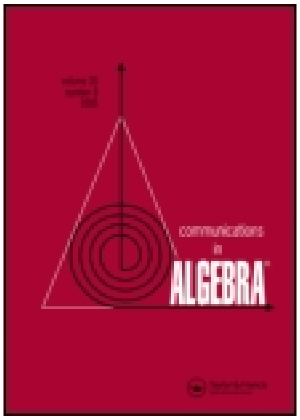

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Deformations of the Restricted Melikian Lie Algebra

Filippo Viviani [a]
[a] Institute of Mathematics, Humboldt Universität of Berlin, Berlin, Germany

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DEFORMATIONS OF THE RESTRICTED MELIKIAN LIE ALGEBRA

Filippo Viviani
Institute of Mathematics, Humboldt Universität of Berlin, Berlin, Germany

We compute the infinitesimal deformations of the restricted Melikian Lie algebra in characteristic 5.

Key Words: Cartan-type simple Lie algebras; Deformations; Restricted Lie algebras.

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1. INTRODUCTION

The restricted Melikian Lie algebra M is a restricted simple Lie algebra of dimension 125 defined over the prime field of characteristic $p = 5$. It was introduced by [6], and it is the only "exceptional" simple Lie algebra which appears in the classification of restricted simple Lie algebras over a field of characteristic $p \neq 2, 3$ (see 1 for $p > 7$ and 7, for $p = 5, 7$). The classification problem remains still open in characteristic 2 and 3, where several "exceptional" simple Lie algebras are known (see 9, p. 209).

This article is devoted to the study of the *infinitesimal deformations* of the restricted Melikian Lie algebra. The infinitesimal deformations have been computed for the other restricted simple Lie algebras in characteristic $p \geq 5$. [8] proved that the simple Lie algebras of classical type are rigid, in analogy of what happens in characteristic zero. On the other hand, the author computed the infinitesimal deformations of the four families (Witt–Jacobson, Special, Hamiltonian, Contact) of restricted simple algebras of Cartan-type (see [10, 11]), showing that these are never rigid.

By standard facts of deformation theory, the infinitesimal deformations of a Lie algebra are parametrized by the second cohomology of the Lie algebra with values in the adjoint representation. Assuming the notations from Section 2 about the restricted Melikian algebra M as well as the definition of the squaring operator Sq, we can state the main result of this article.

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Address correspondence to Filippo Viviani, via Marconi 36, Castorano 63030, Italy; E-mail: viviani@math.hu-berlin.de

Theorem 1.1. *The infinitesimal deformations of the Melikian algebra M are given by*

$$H^2(M, M) = \langle \mathrm{Sq}(1)\rangle_F \bigoplus_{i=1}^{2} \langle \mathrm{Sq}(D_i)\rangle_F \bigoplus_{i=1}^{2} \langle \mathrm{Sq}(\widetilde{D}_i)\rangle_F.$$

The strategy of our proof consists in exploiting the Hochschild–Serre spectral sequence relative to the subalgebra of negative elements of M. A similar strategy has been used by [5] in order to prove that the Melikian algebra does not admit filtered deformations. As a byproduct of our proof, we give a new proof (see Theorem 3.1) of the vanishing of the first cohomology group of the adjoint representation (5, Proposition 2.2.13).

The referee suggested another possible approach to prove the above Main Theorem. Namely, the restricted Melikian algebra M can be realized as a subalgebra of the restricted contact algebra $K(5)$ of rank 5 (see 6) in such a way that the negative elements of M coincide with the negative elements of $K(5)$. This allows to reduce the above result to the analogous result for the contact algebras (see 11).

The article is organized as follows. In Section 2 we recall, in order to fix the notations, the basic properties of the restricted Melikian algebra, the Hochschild–Serre spectral sequence and the squaring operators. In Section 3 we prove that the cocycle appearing in the Main Theorem 1.1 are independent in $H^2(M, M)$, and we outline the strategy to prove that they generate the whole second cohomology group. Each of the remaining four sections is devoted to carry over one of the four steps of this strategy.

2. NOTATIONS

2.1. The Restricted Melikian Algebra *M*

We fix a field F of characteristic $p = 5$. Let $A(2) = F[x_1, x_2]/(x_1^p, x_2^p)$ be the F-algebra of truncated polynomials in 2 variables, and let $W(2) = \mathrm{Der}_F A(2)$ the restricted Witt–Jacobson Lie algebra of rank 2. The Lie algebra $W(2)$ is a free $A(2)$-module with basis $D_1 := \frac{\partial}{\partial x_1}$ and $D_2 := \frac{\partial}{\partial x_2}$. Let $\widetilde{W(2)}$ be a copy of $W(2)$, and for an element $D \in W(2)$ we indicate with $\widetilde{D}$ the corresponding element inside $\widetilde{W(2)}$. The Melikian algebra M is defined as

$$M = A(2) \oplus W(2) \oplus \widetilde{W(2)},$$

with Lie bracket defined by the following rules (for all $D, E \in W(2)$ and $f, g \in A(2)$):

$$\begin{cases} [D, \widetilde{E}] := \widetilde{[D, E]} + 2\,\mathrm{div}(D)\widetilde{E}, \\ [D, f] := D(f) - 2\,\mathrm{div}(D)f, \\ [f_1\widetilde{D_1} + f_2\widetilde{D_2}, g_1\widetilde{D_1} + g_2\widetilde{D_2}] := f_1 g_2 - f_2 g_1, \\ [f, \widetilde{E}] := fE, \\ [f, g] := 2(gD_2(f) - fD_2(g))\widetilde{D_1} + 2(fD_1(g) - gD_1(f))\widetilde{D_2}, \end{cases}$$

where $\operatorname{div}(f_1D_1 + f_2D_2) := D_1(f_1) + D_2(f_2) \in A(2)$. The Melikian algebra M is a restricted simple Lie algebra of dimension 125 (see 9, Section 4.3) with a $\mathbb{Z}$-grading given by (for all $D, E \in W(2)$ and $f \in A(2)$):

$$\begin{cases} \deg_M(D) := 3\deg(D), \\ \deg_M(\widetilde{E}) := 3\deg(E) + 2, \\ \deg_M(f) := 3\deg(f) - 2. \end{cases}$$

The lowest terms of the gradation are

$$M_{-3} = FD_1 + FD_2, \qquad M_{-2} = F \cdot 1, \qquad M_{-1} = F\widetilde{D_1} + F\widetilde{D_2}, \qquad M_0 = \sum_{i,j=1,2} Fx_iD_j,$$

while the highest term is $M_{23} = x_1^4x_2^4(FD_1 + FD_2)$. Moreover, M has a $\mathbb{Z}/3\,\mathbb{Z}$-grading given by

$$M_{\bar{1}} := A(2), \qquad M_{\bar{0}} := W(2), \qquad M_{\bar{2}} := \widetilde{W(2)}.$$

The Melikian algebra M has a root space decomposition with respect to a canonical Cartan subalgebra.

Proposition 2.1.

(a) $T_M := \langle x_1D_1\rangle_F \oplus \langle x_2D_2\rangle_F$ *is a maximal torus of* M (*called the canonical maximal torus*).

(b) *The centralizer of* T_M *inside* M *is the subalgebra*

$$C_M = T_M \oplus \langle x_1^2x_2^2\rangle_F \oplus \langle x_1^4x_2^3\widetilde{D_1}\rangle_F \oplus \langle x_1^3x_2^4\widetilde{D_2}\rangle_F,$$

which is hence a Cartan subalgebra of M (*called the canonical Cartan subalgebra*).

(c) *Let* $\Phi_M := \operatorname{Hom}_{\mathbb{F}_5}(\bigoplus_{i=1}^2 \langle x_iD_i\rangle_{\mathbb{F}_5}, \mathbb{F}_5)$ *where* $\mathbb{F}_5$ *is the prime field of* F. *There is a Cartan decomposition* $M = \bigoplus_{\phi\in\Phi_M} M_\phi$, *where every summand* M_ϕ *has dimension* 5 *over* F. *Explicitly,*

$$\begin{cases} x^a \in M_{(a_1-2, a_2-2)}, \\ x^aD_i \in M_{(a_1-\delta_{1i}, a_2-\delta_{2i})}, \\ \widetilde{x^aD_i} \in M_{(a_1+2-\delta_{1i}, a_2+2-\delta_{2i})}. \end{cases}$$

In particular, if $E \in M_{(\phi_1,\phi_2)}$ *then* $\deg E \equiv 3(\phi_1 + \phi_2) \bmod 5$.

Proof. See [9, Section 4.3]. □

2.2. Cohomology of Lie Algebras

If $\mathfrak{g}$ is a Lie algebra over a field F and M is a $\mathfrak{g}$-module, then the cohomology groups $H^*(\mathfrak{g}, M)$ can be computed from the complex of n-dimensional cochains $C^n(\mathfrak{g}, M)$ $(n \geq 0)$, that are alternating n-linear functions $f : \Lambda^n(\mathfrak{g}) \to M$, with differential $d : C^n(\mathfrak{g}, M) \to C^{n+1}(\mathfrak{g}, M)$ defined by

$$\mathrm{d}f(\sigma_0, \ldots, \sigma_n) = \sum_{i=0}^{n}(-1)^i \sigma_i \cdot f(\sigma_0, \ldots, \widehat{\sigma_i}, \ldots, \sigma_n) + \sum_{p<q}(-1)^{p+q} f([\sigma_p, \sigma_q], \sigma_0, \ldots, \widehat{\sigma_p}, \ldots, \widehat{\sigma_q}, \ldots \sigma_n), \qquad (2.1)$$

where the sign ˆ means that the argument below must be omitted. Given $f \in C^n(\mathfrak{g}, M)$ and $\gamma \in \mathfrak{g}$, we denote with f_γ the restriction of f to $\gamma \in \mathfrak{g}$, that is the element of $C^{n-1}(\mathfrak{g}, M)$ given by

$$f_\gamma(\sigma_0, \ldots, \sigma_{n-1}) := f(\gamma, \sigma_0, \ldots, \sigma_{n-1}).$$

With this notation, the above differential satisfies the following useful formula (for any $\gamma \in \mathfrak{g}$ and $f \in C^n(\mathfrak{g}, M)$):

$$\mathrm{d}(\gamma \cdot f) = \gamma \cdot (\mathrm{d}f), \qquad (2.2)$$

$$(\mathrm{d}f)_\gamma = \gamma \cdot f - \mathrm{d}(f_\gamma), \qquad (2.3)$$

where each $C^n(\mathfrak{g}, M)$ is a $\mathfrak{g}$-module by means of the action

$$(\gamma \cdot f)(\sigma_1, \ldots, \sigma_n) = \gamma \cdot f(\sigma_1, \ldots, \sigma_n) - \sum_{i=1}^{n} f(\sigma_1, \ldots, [\gamma, \sigma_i], \ldots \sigma_n). \qquad (2.4)$$

As usual we indicate with $Z^n(\mathfrak{g}, M)$ the subspace of n-cocycles and with $B^n(\mathfrak{g}, M)$ the subspace of n-coboundaries. Therefore, $H^n(\mathfrak{g}, M) := Z^n(\mathfrak{g}, M)/B^n(\mathfrak{g}, M)$.

A useful tool to compute cohomology of Lie algebras is the Hochschild–Serre spectral sequence relative to a subalgebra. Given a subalgebra $\mathfrak{h} < \mathfrak{g}$, one can define a decreasing filtration $\{F^jC^n(\mathfrak{g}, M)\}_{j=0,\ldots,n+1}$ on the space of n-cochains:

$$F^jC^n(\mathfrak{g}, M) = \{f \in C^n(\mathfrak{g}, M) \mid f(\sigma_1, \ldots, \sigma_n) = 0 \text{ if } \sigma_1, \ldots, \sigma_{n+1-j} \in \mathfrak{h}\}.$$

This gives rise to a spectral sequence converging to the cohomology $H^n(\mathfrak{g}, M)$, whose first level is equal to (see 4):

$$E_1^{p,q} = H^q(\mathfrak{h}, C^p(\mathfrak{g}/\mathfrak{h}, M)) \Longrightarrow H^{p+q}(\mathfrak{g}, M). \qquad (2.5)$$

In the case where $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ (which we indicate as $\mathfrak{h} \lhd \mathfrak{g}$), the above spectral sequence becomes

$$E_2^{p,q} = H^p(\mathfrak{g}/\mathfrak{h}, H^q(\mathfrak{h}, M)) \Longrightarrow H^{p+q}(\mathfrak{g}, M). \qquad (2.6)$$

Moreover, for the second page of the first spectral sequence (2.5), we have the equality

$$E_2^{p,0} = H^p(\mathfrak{g}, \mathfrak{h}; M), \qquad (2.7)$$

where $H^*(\mathfrak{g}, \mathfrak{h}; M)$ are the relative cohomology groups defined (by 2) from the subcomplex $C^p(\mathfrak{g}, \mathfrak{h}; M) \subset C^p(\mathfrak{g}, M)$ consisting of cochains orthogonal to $\mathfrak{h}$, that is, cochains satisfying the two conditions:

$$f_{|\mathfrak{h}} = 0, \tag{2.8}$$

$$\mathrm{d}f_{|\mathfrak{h}} = 0 \qquad \text{or equivalently } \gamma \cdot f = 0 \qquad \text{for every } \gamma \in \mathfrak{h}. \tag{2.9}$$

Note that in the case where $\mathfrak{h} \lhd \mathfrak{g}$, the equality (2.7) is consistent with the second spectral sequence (2.6) because in that case we have $H^p(\mathfrak{g}, \mathfrak{h}, M) = H^p(\mathfrak{g}/\mathfrak{h}, M^{\mathfrak{h}})$.

Suppose that a torus T acts on both $\mathfrak{g}$ and M in a way that is compatible with the action of $\mathfrak{g}$ on M, which means that $t \cdot (g \cdot m) = (t \cdot g) \cdot m + t \cdot (g \cdot m)$ for every $t \in T$, $g \in \mathfrak{g}$ and $m \in M$. Then the action of T can be extended to the space of n-cochains by

$$(t \cdot f)(\sigma_1, \ldots, \sigma_n) = t \cdot f(\sigma_1, \ldots, \sigma_n) - \sum_{i=1}^{n} f(\sigma_1, \ldots, t \cdot \sigma_i, \ldots, \sigma_n).$$

It follows easily from the compatibility of the action of T and formula (2.3), that the action of T on the cochains commutes with the differential d. Therefore, since the action of a torus is always completely reducible, we get a decomposition in eigenspaces

$$H^n(\mathfrak{g}, M) = \bigoplus_{\phi \in \Phi} H^n(\mathfrak{g}, M)_\phi, \tag{2.10}$$

where $\Phi = \mathrm{Hom}_F(T, F)$ and $H^n(\mathfrak{g}, M)_\phi = \{[f] \in H^n(\mathfrak{g}, M) \mid t \cdot [f] = \phi(t)[f] \text{ if } t \in T\}$. A particular case of this situation occurs when $T \subset \mathfrak{g}$ and T acts on $\mathfrak{g}$ via the adjoint action and on M via restriction of the action of $\mathfrak{g}$. It is clear that this action is compatible and moreover the above decomposition reduces to

$$H^n(\mathfrak{g}, M) = H^n(\mathfrak{g}, M)_{\underline{0}},$$

where $\underline{0}$ is the trivial homomorphism (in this situation, we say that the cohomology reduces to *homogeneous* cohomology). Indeed, if we consider an element $f \in Z^n(\mathfrak{g}, M)_\phi$, then by applying formula (2.3) with $\gamma = t \in T$, we get

$$0 = (\mathrm{d}f)_t = t \cdot f - \mathrm{d}(f_t) = \phi(t) f - \mathrm{d}(f_t),$$

from which we see that the existence of a $t \in T$ such that $\phi(t) \neq 0$ forces f to be a coboundary.

Now suppose that $\mathfrak{g}$ and M are graded and that the action of $\mathfrak{g}$ respects these gradings, which means that $\mathfrak{g}_d \cdot M_e \subset M_{d+e}$ for all $e, d \geq 0$. Then the space of cochains can also be graded: a homogeneous cochain f of degree d is a cochain such that $f(\mathfrak{g}_{e_1} \times \cdots \times \mathfrak{g}_{e_n}) \subset M_{\sum e_i + d}$. With this definition, the differential becomes of degree 0, and therefore we get a degree decomposition

$$H^n(\mathfrak{g}, M) = \bigoplus_{d \in \mathbb{Z}} H^n(\mathfrak{g}, M)_d. \tag{2.11}$$

Finally, if the action of T is compatible with the grading, in the sense that T acts via degree 0 operators both on $\mathfrak{g}$ and on M, then the above two decompositions (2.10) and (2.11) are compatible and give rise to the refined weight-degree decomposition

$$H^n(\mathfrak{g}, M) = \bigoplus_{\phi\in\Phi}\bigoplus_{d\in\mathbb{Z}} H^n(\mathfrak{g}, M)_{\phi,d}. \tag{2.12}$$

We will use frequently the above weight-degree decomposition with respect to the action of the maximal torus $T_M \subset M_0$ of the restricted Melikian algebra M (see Proposition 2.1).

2.3. Squaring Operation

There is a canonical way to produce 2-cocycles in $Z^2(\mathfrak{g}, \mathfrak{g})$ over a field of characteristic $p > 0$, namely, the squaring operation (see 3). Given a derivation $\gamma \in Z^1(\mathfrak{g}, \mathfrak{g})$ (inner or not), one defines the squaring of γ to be

$$\mathrm{Sq}(\gamma)(x, y) = \sum_{i=1}^{p-1} \frac{[\gamma^i(x), \gamma^{p-i}(y)]}{i!(p-i)!} \in Z^2(\mathfrak{g}, \mathfrak{g}), \tag{2.13}$$

where γ^i is the i-iteration of γ. In [3] it is shown that $[\mathrm{Sq}(\gamma)] \in H^2(\mathfrak{g}, \mathfrak{g})$ is an obstruction to integrability of the derivation γ, that is, to the possibility of finding an automorphism of $\mathfrak{g}$ extending the infinitesimal automorphism given by γ.

3. STRATEGY OF THE PROOF OF THE MAIN THEOREM

First of all, we show that the five cocycles $\{\mathrm{Sq}(\widetilde{D}_i)\}_{i=1,2}$, $\{\mathrm{Sq}(1)\}$, $\{\mathrm{Sq}(D_i)\}_{i=1,2}$ are independent in $H^2(M, M)$. Observe that the first two cocycles have degree -5, the third has degree -10, and the last two have degree -15. Therefore, according to the decomposition (2.11), it is enough to show that the first two are independent, the third is nonzero, and the last two are independent in $H^2(M, M)$.

To prove the independence of the first two cocycles, we observe that (for $i \neq j$)

$$\mathrm{Sq}(\widetilde{D}_i)(x_iD_j, x_i\widetilde{D}_j) = D_i, \tag{3.1}$$

while for a cochain $g \in C^1(M, M)_{\underline{0},-5}$, we have that

$$\mathrm{d}g(x_iD_j, x_i\widetilde{D}_j) = [x_iD_j, g(x_i\widetilde{D}_j)] - [x_i\widetilde{D}_j, g(x_iD_j)] - g([x_iD_j, x_i\widetilde{D}_j]) = 0, \tag{3.2}$$

since $g(x_iD_j) = 0$ by degree reasons, $g(x_i\widetilde{D}_j) = 0$ by degree and homogeneity reasons and $[x_iD_j, x_i\widetilde{D}_j] = 0$.

The third cocyle is nonzero because

$$\begin{cases} \mathrm{Sq}(1)(x_i, x_ix_j^2D_i) = 2D_i, \\ \mathrm{Sq}(1)(x_i, x_j^3D_j) = D_i, \end{cases} \tag{3.3}$$

while for a cochain $g \in C^1(M, M)_{\underline{0},-10}$ we have that

$$\begin{cases} \mathrm{d}g(x_i, x_i x_j^2 D_i) = -g(x_i x_j^2), \\ \mathrm{d}g(x_i, x_j^3 D_j) = -g(x_i x_j^2). \end{cases} \tag{3.4}$$

Finally, the independence of the last two cocycles follows from

$$\begin{cases} \mathrm{Sq}(D_i)(x_i D_j, x_i^4 x_j D_j) = D_j, \\ \mathrm{Sq}(D_i)(x_i D_j, x_i^3 x_j^2 D_j) = 0, \end{cases} \tag{3.5}$$

together with the fact that for a cochain $g \in C^1(M, M)_{\underline{0},-15}$ we have that

$$\begin{cases} \mathrm{d}g\big(x_i D_j, x_i^4 x_j^2 D_i\big) = \big[x_i D_j, g\big(x_i^4 x_j D_j\big)\big], \\ \mathrm{d}g\big(x_i D_j, x_i^3 x_j^2 D_j\big) = -g\big(\big[x_i D_j, x_i^3 x_j^2 D_j\big]\big) = -2g\big(x_i^4 x_j D_j\big). \end{cases} \tag{3.6}$$

The remaining of this article is devoted to show that the above five cocycles generate the cohomology group $H^2(M, M)$, as stated in the Main Theorem 1.1. We outline here the strategy of the proof that will be carried over in the next sections. The proof is divided in five steps:

Step I: We prove in Corollary 4.4 that

$$H^2(M, M) = H^2(M, M_{<0}; M).$$

Step II: We prove in Proposition 5.1 that

$$H^2(M, M_{<0}; M) \subset H^2(M_{\geq 0}, M_{-3}),$$

where $M_{\geq 0}$ acts on $M_{-3} = \langle D_1, D_2 \rangle_F$ via projection onto $M_{\geq 0}/M_{\geq 1} = M_0$ followed by the adjoint representation of M_0 onto M_{-3}.

Step III: We prove in Corollary **??** that

$$H^2(M_{\geq 0}, M_{-3}) \subset \bigoplus_{i=1}^{2} \langle \mathrm{Sq}(\widetilde{D}_i) \rangle_F \oplus H^2(M_{\geq 1}, M_{-3})^{M_0},$$

where $M_{\geq 1}$ acts trivially on M_{-3}.

Step IV: We prove in Proposition 7.1 that

$$H^2(M_{\geq 1}, M_{-3})^{M_0} = \bigoplus_{i=1}^{2} \langle \mathrm{Sq}(D_i) \rangle_F \oplus \langle \mathrm{Sq}(1) \rangle_F.$$

As a byproduct of our Main Theorem, we obtain a new proof of the following result (5, Proposition 2.2.13; see also 9, Chapter 7).

Theorem 3.1 [5]. $H^1(M, M) = 0.$

Proof. The spectral sequence (4.1), together with Proposition 4.1, gives that

$$H^1(M, M) = E_2^{1,0} = H^1(M, M_{<0}; M).$$

The Proposition 5.1 gives that $H^1(M, M_{<0}; M) \subset H^1(M_{\geq 0}, M_{-3})$. Using the spectral sequence (6.1), together with Propositions 6.1 and 6.2, we get the vanishing of this last group. $\square$

4. STEP I: REDUCTION TO $M_{<0}$-RELATIVE COHOMOLOGY

In this section, we carry over the first step outlined in Section 3. To this aim, we consider the homogeneous Hochschild–Serre spectral sequence associated to the subalgebra $M_{<0} < M$ (see Section 2.2):

$$\left(E_1^{r,s}\right)_{\underline{0}} = H^s(M_{<0}, C^r(M/M_{<0}, M))_{\underline{0}} \Rightarrow H^{r+s}(M, M)_{\underline{0}} = H^{r+s}(M, M). \tag{4.1}$$

We adopt the following notation: given elements $E_1, \ldots, E_n \in M_{<0}$ and $G \in M$, we denote with $\delta^G_{E_1,\ldots,E_n}$ the cochain of $C^n(M_{<0}, M)$ whose only nonzero values are

$$\delta^G_{E_1,\ldots,E_n}(E_{\sigma(1)}, \ldots, E_{\sigma(n)}) = \operatorname{sgn}(\sigma)G,$$

for any permutation $\sigma \in S_n$.

Proposition 4.1. *In the above spectral sequence* (4.1), *we have*

$$\left(E_1^{0,1}\right)_{\underline{0}} = \left(E_1^{0,2}\right)_{\underline{0}} = 0.$$

Proof. Consider the Hochschild–Serre spectral sequence associated to the ideal $M_{-3} \lhd M_{\leq -2}$:

$$E_2^{r,s} = H^r(M_{\leq -2}/M_{-3}, H^s(M_{-3}, M)) \Rightarrow H^{r+s}(M_{\leq -2}, M). \tag{4.2}$$

The lowest term $M_{-3} = \langle D_1, D_2\rangle_F$ acts on $M = A(2) \oplus W(2) \oplus \widetilde{W(2)}$ via its natural action on $A(2)$ and via adjoint representation on $W(2)$ and $\widetilde{W(2)}$. Hence, according to [10, Proposition 3.4 and Corollary 3.5], we have that (for $s = 0, 1, 2$)

$$H^s(M_{-3}, M) = \begin{cases} M_{<0} & \text{if } s = 0, \\ \bigoplus_{G\in M_{<0}} \left\langle \delta_{D_1}^{x_1^4 G}, \delta_{D_2}^{x_2^4 G}\right\rangle_F & \text{if } s = 1, \\ \bigoplus_{G\in M_{<0}} \left\langle \delta_{D_1,D_2}^{x_1^4 x_2^4 G}\right\rangle_F & \text{if } s = 2. \end{cases}$$

Moreover, $M_{\leq -2}/M_{-3} = \langle 1\rangle_F$ acts on the above cohomology groups $H^s(M_3, M)$ via its adjoint action on $M_{<0} = \langle 1\rangle \oplus \langle D_1, D_2\rangle \oplus \langle \widetilde{D_1}, \widetilde{D_2}\rangle$, that is,

$$[1, 1] = 0, \qquad [1, D_i] = 0, \qquad [1, \widetilde{D_i}] = D_i.$$

Hence, using that the above Hochschild–Serre spectral sequence (4.2) is degenerate since $M_{\leq -2}/M_{-3} = \langle 1 \rangle_F$ has dimension 1, we deduce that

$$H^s(M_{\leq -2}, M) = \begin{cases} M_{-3} \oplus M_{-2} & \text{if } s = 0, \\ \bigoplus_{G \in M_{-3} \oplus M_{-2}} \langle \delta_{D_1}^{x_1^4 G}, \delta_{D_2}^{x_2^4 G} \rangle_F \bigoplus_{H \in M_{-2} \oplus M_{-1}} \langle \delta_1^H \rangle_F & \text{if } s = 1, \\ \bigoplus_{G \in M_{-3} \oplus M_{-2}} \langle \delta_{D_1, D_2}^{x_1^4 x_2^4 G} \rangle_F \bigoplus_{H \in M_{-2} \oplus M_{-1}} \langle \delta_{1, D_1}^{x_1^4 H}, \delta_{1, D_2}^{x_2^4 H} \rangle_F & \text{if } s = 2. \end{cases}$$

Finally, consider the homogeneous Hochschild–Serre spectral sequence associated to the ideal $M_{\leq -2} \lhd M_{<0}$:

$$(E_2^{r,s})_{\underline{0}} = H^r(M_{<0}/M_{\leq -2}, H^s(M_{\leq -2}, M))_{\underline{0}} \Rightarrow H^{r+s}(M_{<0}, M)_{\underline{0}}. \tag{4.3}$$

From the explicit description of above, one can easily check that the only nonzero terms and nonzero maps of the above spectral sequence that can contribute to $H^s(M_{<0}, M)_{\underline{0}}$ for $s = 1, 2$ are

$$\begin{aligned} \left(E_2^{0,1}\right)_{\underline{0}} &= \langle \delta_1^1 \rangle_F \to \left(E_2^{2,0}\right)_{\underline{0}} = \langle \delta^1_{\widetilde{D_1}, \widetilde{D_2}} \rangle_F, \\ \left(E_2^{0,2}\right)_{\underline{0}} &= \bigoplus_{i=1}^{2} \langle \delta_{D_i, 1}^{x_i^4} \rangle_F \to \left(E_2^{2,1}\right)_{\underline{0}} = \bigoplus_{i=1}^{2} \langle \delta^{x_i^4}_{D_i, \widetilde{D_1}, \widetilde{D_2}} \rangle_F, \end{aligned}$$

where the maps are given by the differentials. Using the relation $[\widetilde{D_1}, \widetilde{D_2}] = 1$, it is easy to see that all the above maps are isomorphisms, and hence the conclusion follows. □

In the next proposition, we need the following lemma.

Lemma 4.2. *We have that*

$$H^1(M_{<0}, M) = \bigoplus_{k,h} \langle \delta_{D_k}^{x_k^4 D_h} \rangle_F \bigoplus_{i \neq j} \langle \delta^{D_j}_{\widetilde{D_i}} \rangle_F \oplus \langle \delta^{D_1}_{\widetilde{D_1}} - \delta^{D_2}_{\widetilde{D_2}} \rangle_F.$$

Proof. From the (nonhomogeneous) Hochschild–Serre spectral sequence associated to the ideal $M_{\leq -2} \lhd M_{<0}$:

$$\left(E_2^{r,s}\right) = H^r(M_{<0}/M_{\leq -2}, H^s(M_{\leq -2}, M)) \Rightarrow H^{r+s}(M_{<0}, M),$$

we deduce the exact sequence

$$\begin{aligned} 0 &\to H^1(M_{<0}/M_{\leq -2}, M^{M_{\leq -2}}) \\ &\to H^1(M_{<0}, M) \to H^1(M_{\leq -2}, M)^{M_{<0}/M_{\leq -2}} \xrightarrow{\partial} H^2(M_{<0}/M_{\leq -2}, M^{M_{\leq -2}}). \end{aligned}$$

Using the computation of $H^s(M_{\leq -2}, M)$ for $s = 0, 1$ from the Proposition 4.1, it is easily seen that

$$\begin{cases} H^1\big(M_{<0}/M_{\leq -2}, M^{M_{\leq -2}}\big) = \dfrac{\bigoplus_{i,j}\langle \delta_{\widetilde{D_i}}^{D_j}\rangle_F}{\langle \mathrm{ad}(1)\rangle_F}, \\ H^1(M_{\leq -2}, M)^{M_{<0}/M_{\leq -2}} = \bigoplus_{h,k}\big\langle \delta_{D_h}^{x_h^4 D_k}\big\rangle_F \oplus \big\langle \delta_1^1\big\rangle_F, \\ H^2(M_{<0}/M_{\leq -2}, M^{M_{\leq -2}}) = \big\langle \delta^1_{(\widetilde{D_1}, \widetilde{D_2})}\big\rangle_F. \end{cases}$$

Using the relation $[\widetilde{D_1}, \widetilde{D_2}] = 1$, it is easy to see that $\partial(\delta_1^1) = \delta^1_{(\widetilde{D_1}, \widetilde{D_2})}$, which gives the conclusion. □

Proposition 4.3. *In the above spectral sequence* (4.1), *we have that* $(E_2^{1,1})_{\underline{0}} = 0$.

Proof. We have to show the injectivity of the level 1 differential map

$$d : \big(E_1^{1,1}\big)_{\underline{0}} \longrightarrow \big(E_1^{2,1}\big)_{\underline{0}}.$$

In the course of this proof, we adopt the following convention: given an element $f \in C^1(M_{<0}, C^s(M/M_{<0}, M)$, we write its value on $D \in M_{<0}$ as $f_D \in C^s(M/M_{<0}, M)$.

We want to show, by induction on the degree of $E \in M/M_{<0}$, that if $[df] = 0 \in H^1(M_{<0}, C^2(M/M_{<0}, M))$, then we can choose a representative $\tilde{f}$ of $[f] \in H^1(M_{<0}, C^1(M/M_{<0}, M))$ such that $\tilde{f}_D(E) = 0$ for every $D \in M_{<0}$. So suppose that we have already found a representative f such that $f_D(F) = 0$ for every $F \in M/M_{<0}$ of degree less than d and for every $D \in M_{<0}$. First of all, we can find a representative $\tilde{f}$ of $[f]$ such that

$$\begin{cases} \tilde{f}_{D_i}(E) = \displaystyle\sum_{j=1}^{2} \alpha_i^j x_i^4 D_j \qquad \text{for } i = 1, 2, \\ \tilde{f}_1(E) = 0, \\ \tilde{f}_{\widetilde{D_1}}(E) = \beta_1 D_2 + \gamma D_1, \\ \tilde{f}_{\widetilde{D_2}}(E) = \beta_2 D_1 - \gamma D_2, \end{cases} \tag{$*$}$$

for every $E \in M$ of degree d. Indeed, by the induction hypothesis, the cocycle condition for f is $\partial f_{D,D'}(E) = [D, f_{D'}(E)] - [D', f_D(E)] - f_{[D,D']}(E)$ for any $D, D' \in M_{<0}$. On the other hand, by choosing an element $h \in C^1(M/M_{<0}, M)$ that vanishes on the elements of degree less than d, we can add to f (without changing its cohomological class neither affecting the inductive assumption) the coboundary ∂h whose value on E are $\partial h_D(E) = [D, h(E)]$. Hence, for a fixed element E of degree d, the map $D \mapsto f_D(E)$ gives rise to an element of $H^1(M_{<0}, M)$ and, by Lemma 4.2, we can chose an element $h(E)$ as above such that the new cochain $\tilde{f} = f + \partial h$ verifies the condition $(*)$ of above.

Note that, by the homogeneity of $\tilde{f}$, we have the following pairwise disjoint possibilities for E:

$$\begin{cases} \alpha_i^j \neq 0 \quad \text{for } i = 1, 2 \Rightarrow E \in \left\{x_j x_{j+1}^2, x_j^4 x_{j+1} D_{j+1}, \widetilde{x_j^3 x_{j+1}^3} D_j, \widetilde{x_j^2 x_{j+1}^4} D_{j+1}\right\}, \\ \beta_j \neq 0 \Rightarrow E \in \left\{x_j x_{j+1}^4, x_{j+1}^2 D_j, x_j^4 x_{j+1}^3 D_{j+1}, \widetilde{x_j^3} D_j, x_j^2 \widetilde{x_{j+1}} D_{j+1}\right\}, \\ \gamma \neq 0 \Rightarrow E \in \left\{x_1^4 x_2^3 D_1, x_1^3 x_2^4 D_2, \widetilde{x_1^2 x_2} D_1, x_1 \widetilde{x_2^2} D_2\right\}. \end{cases}$$

Now we are going to use the condition that $[d\tilde{f}] = 0 \in (E_1^{2,1})_{\underline{0}}$, that is $d\tilde{f} = \partial g$ for some $g \in C^2(M/M_{<0}, M)_{\underline{0}}$. Explicitly, for $A, B \in M/M_{<0}$, we have that (for any $D \in M_{<0}$)

$$\partial g_D(A, B) = [D, g(A, B)] - g([D, A], B) - g(A, [D, B]), \tag{4.4}$$

$$d\tilde{f}_D(A, B) = \tilde{f}_D([A, B]) - [A, \tilde{f}_D(B)] + [B, \tilde{f}_D(A)] - \tilde{f}_{[D,A]}(B) + \tilde{f}_{[D,B]}(A), \tag{4.5}$$

where the last two terms in the first formula can be nonzero only if $[D, A] \in M_{\geq 0}$ and $[D, B] \in M_{\geq 0}$, respectively, where the last two terms in the second formula can be nonzero only if $[D, A] \in M_{<0}$ and $[D, B] \in M_{<0}$, respectively.

Suppose first that $\alpha_i^j \neq 0$ for a fixed j and for $i = 1, 2$. Then it is straightforward to check that for each E in the above list it is possible to find two elements $A, B \in M$ such that $[A, B] = E$, $\deg(B) = 0$, and A does not belong to the above list. Apply the above formulas for each such pair (A, B). Taking into account the inductive hypothesis on the degree and the homogeneity assumptions, the formula (4.5) becomes

$$d\tilde{f}_{D_i}(A, B) = \tilde{f}_{D_i}([A, B]) = \tilde{f}_{D_i}(E) = \alpha_i^j x_i^4 D_j,$$

while the formula (4.4) gives

$$\partial g_{D_i}(A, B) = [D_i, g(A, B)] - g([D_i, A], B).$$

The first term in the last expression is a derivation with respect to D_i, and therefore it cannot involve the monomial $x_i^4 D_j$. The same is true for the second term as it follows by applying the formulas (4.5) and (4.4) for the elements $\mathrm{ad}(D_i)^k(A)$ (with $k = 1, \ldots, p-1$) and B, and using the vanishing hypothesis:

$$\begin{cases} d\tilde{f}_{D_i}(\mathrm{ad}(D_i)^k(A), B) = 0, \\ \partial g_{D_i}(\mathrm{ad}(D_i)^k(A), B) = [D_i, g(\mathrm{ad}(D_i)^k(A), B)] - g(\mathrm{ad}(D_i)^{k+1}(A), B). \end{cases}$$

Therefore, we conclude that $\alpha_i^j = 0$, an absurd. The other cases $\beta_j \neq 0$ and $\gamma \neq 0$ are excluded using a similar argument. $\square$

Finally, we get the main result of this section.

Corollary 4.4. *We have that*

$$H^2(M, M) = H^2(M, M_{<0}; M).$$

Proof. From the spectral sequence (4.1), using the vanishing of $(E_1^{0,2})_{\underline{0}}$ (Proposition 4.1) and of $(E_1^{1,1})_{\underline{0}}$ (Proposition 4.3), we get that

$$H^2(M, M) = \left(E_\infty^{2,0}\right)_{\underline{0}} = \left(E_2^{2,0}\right)_{\underline{0}} = H^2(M, M_{<0}; M). \qquad \square$$

5. STEP II: REDUCTION TO $M_{\geq 0}$-COHOMOLOGY

In this section, we carry over the second step of proof of the main theorem (see Section 3).

Consider the action of $M_{\geq 0}$ on $M_{-3} = \langle D_1, D_2 \rangle_F$ obtained via projection onto $M_{\geq 0}/M_{\geq 1} = M_0$ followed by the adjoint representation of M_0 onto M_{-3}.

Proposition 5.1. *We have that*

$$\begin{cases} H^1(M, M_{<0}; M) \subset H^1(M_{\geq 0}, M_{-3}), \\ H^2(M, M_{<0}; M) \subset H^2(M_{\geq 0}, M_{-3}). \end{cases}$$

Proof. For every $s \in \mathbb{Z}_{\geq 0}$, consider the map

$$\phi_s : C^s(M, M_{<0}; M) \to C^s(M_{\geq 0}, M_{-3})$$

induced by the restriction to the subalgebra $M_{\geq 0} \subset M$ and by the projection $M \twoheadrightarrow M/M_{\geq -2} = M_{-3}$. It is straightforward to check that the maps ϕ_s commute with the differentials, and hence they define a map of complexes. Moreover, the orthogonality conditions with respect to the subalgebra $M_{<0}$ give the injectivity of the maps ϕ_s. Indeed, on one hand, the condition (2.8) says that an element $f \in C^s(M, M_{<0}; M)$ is determined by its restriction to $\wedge^s M_{\geq 0}$. On the other hand, the condition (2.9) implies that the values of f on an s-tuple are determined, up to elements of $M^{M_{<0}} = M_{-3}$, by induction on the total degree of the s-tuple.

The map ϕ_0 is an isomorphism since

$$C^0(M, M_{<0}; M) = M^{M_{<0}} = M_{-3} = C^0(M_{\geq 0}, M_{-3}).$$

From this, we get the first statement of the proposition. Moreover, it is easily checked that if $g \in C^1(M_{\geq 0}, M_{-3})$ is such that $\mathrm{d}g \in C^2(M, M_{<0}; M)$, then $g \in C^1(M, M_{<0}; M)$. This gives the second statement of the proposition. $\square$

6. STEP III: REDUCTION TO M_0-INVARIANT COHOMOLOGY

In this section, we carry over the third step of the proof of the Main Theorem (see Section 3). To this aim, we consider the Hochschild–Serre spectral sequence relative to the ideal $M_{\geq 1} \lhd M_{\geq 0}$:

$$E_2^{r,s} = H^r(M_0, H^s(M_{\geq 1}, M_{-3})) \Longrightarrow H^{r+s}(M_{\geq 0}, M_{-3}). \tag{6.1}$$

The first line $E_2^{*,0}$ of the above spectral sequence vanish.

Proposition 6.1. *In the above spectral sequence* (6.1), *we have for every* $r \geq 0$

$$E_2^{r,0} = H^r(M_0, M_{-3}) = 0.$$

Proof. Observe that, since the canonical maximal torus T_M is contained in M_0, we can restrict to homogeneous cohomology (see Section 2.2). But there are no homogeneous cochains in $C^r(M_0, M_{-3})$. Indeed the weights that occur in M_{-3} are $-\epsilon_i$ while the weights that occur in M_0 are $\underline{0}$ and $\epsilon_i - \epsilon_j$. Therefore the weights that occur in $M_0^{\otimes k}$ have degree congruent to 0 modulo 5, and hence they cannot be equal to $-\epsilon_i$. □

Next, we determine the groups $E_2^{r,1} = H^r(M_0, H^1(M_{\geq 1}, M_{-3}))$ for $r = 1, 2$ of the above spectral sequence (6.1).

Proposition 6.2. *In the above spectral sequence* (6.1), *we have that*

$$E_2^{r,1} = \begin{cases} 0 & \text{if } r = 0, \\ \langle \overline{\mathrm{Sq}(\widetilde{D_1})}, \overline{\mathrm{Sq}(\widetilde{D_2})} \rangle_F & \text{if } r = 1, \end{cases}$$

where $\overline{\mathrm{Sq}(\widetilde{D_i})}$ (*for* $i = 1, 2$) *is the restriction of* $\mathrm{Sq}(\widetilde{D_i})$ *to* $M_0 \times M_2$.

Proof. Using the Lemma 6.3 below, we have that

$$H^1(M_{\geq 1}, M_{-3}) = \left\{ f \in C^1(M_1 + M_2, M_{-3}) \mid f(\widetilde{x_1 D_1}) = -f(\widetilde{x_2 D_2}) \right\}.$$

Observe that, since the canonical maximal torus T_M is contained in M_0, we can restrict to homogeneous cohomology (see Section 2.2). The vanishing of $E_2^{0,1}$ follows directly from the fact that there are no homogeneous cochains in $C^1(M_0, H^1(M_{\geq 1}, M_{-3}))$.

The elements $\overline{\mathrm{Sq}(\widetilde{D_i})}$ for $i = 1, 2$ belong to $H^1(M_0, H^1(M_{\geq 1}, M_{-3}))$ and are non-zero in virtue of the formulas (3.1) and (3.2). Moreover, it is easy to see that, for homogeneity reasons, $C^1(M_0, H^1(M_{\geq 1}, M_{-3}))_{\underline{0}}$ is generated by $\overline{\mathrm{Sq}(\widetilde{D_i})}$ $(i = 1, 2)$, which gives the conclusion. □

Lemma 6.3. $[M_{\geq 1}, M_{\geq 1}] = M_{\geq 3} + \langle \widetilde{x_1 D_1} + \widetilde{x_2 D_2} \rangle_F$.

Proof. Clearly, $[M_{\geq 1}, M_{\geq 1}] \subset M_{\geq 2}$ and $[M_{\geq 1}, M_{\geq 1}] \cap M_2 = [M_1, M_1]$. Since $M_1 = \langle x_1, x_2 \rangle_F$, we have that $[M_1, M_1] = \langle [x_1, x_2] \rangle_F = \langle -2(\widetilde{x_1 D_1} + \widetilde{x_2 D_2}) \rangle_F$. Hence the proof is complete if we show that $M_{\geq 3} \subset [M_{\geq 1}, M_{\geq 1}]$. We will consider the $\mathbb{Z}/3\mathbb{Z}$-grading on M, and we will consider separately $M_{\geq 3} \cap M_{\bar{i}}$, with $i = 0, 1, 2$.

(i) $M_{\geq 3} \cap M_{\bar{0}} \subset [M_{\geq 1}, M_{\geq 1}]$ because of the formula $[x_j, \widetilde{x^a D_i}] = x_j x^a D_i$.

(ii) $M_{\geq 3} \cap M_{\bar{1}} = M_{\geq 4} \cap M_{\bar{1}} \subset [M_{\geq 1}, M_{\geq 1}]$.
Indeed, the formula $[x^a D_i, x_i] = (1 - 2a_i) x^a$ shows that x^a (with $|a| \geq 2$) belongs to $[M_{\geq 1}, M_{\geq 1}]$ provided that $x^a \neq x_1^3 x_2^3$, and this exceptional case is handled with the formula $[x_1^2 x_2^4 D_2, x_1] = -2 \cdot 4 x_1^3 x_2^3$.

(iii) $M_{\geq 3} \cap M_{\bar{2}} = M_{\geq 5} \cap M_{\bar{2}} \subset [M_{\geq 1}, M_{\geq 1}]$.
Indeed, the formula $[x_i^2 D_i, \widetilde{x^a D_j}] = [\widetilde{x_i^2 D_i, x^a D_j}] + 4x_i \widetilde{x^a D_j} = (a_i + 4 - \delta_{ij}) \widetilde{x^{a+\epsilon_i} D_j}$ shows the inclusion for the elements of $M_{\geq 5} \cap M_{\bar{2}}$ with the exception of the two elements of the form $\widetilde{x_i^2 x_j^4 D_j}$ for $i \neq j$, for which we use the formula $[x_i^2 D_j, \widetilde{x_i x_j^4 D_i}] = 4\widetilde{x_i^3 x_j^3 D_i} - 2\widetilde{x_i^2 x_j^4 D_j}$. □

Finally, we get the result we were interested in.

Corollary 6.4. *We have that*

$$H^2(M_{\geq 0}, M_{-3}) \subset \bigoplus_{i=1}^{2} \langle \mathrm{Sq}(\widetilde{D}_i) \rangle_F \oplus H^3(M_{\geq 1}, M_{-3})^{M_0}.$$

Proof. From the above spectral sequence (6.1), using the Propositions 6.1 and 6.2, we get the exact sequence

$$0 \to \langle \mathrm{Sq}(\widetilde{D}_1), \mathrm{Sq}(\widetilde{D}_2) \rangle_F \to H^2(M_{\geq 0}, M_{-3}) \to H^2(M_{\geq 1}, M_{-3})^{M_0},$$

which gives the conclusion. □

7. STEP IV: COMPUTATION OF M_0-INVARIANT COHOMOLOGY

In this section, we carry over the fourth and last step of the proof of the Main Theorem by proving the following proposition.

Proposition 7.1. *We have that*

$$H^2(M_{\geq 1}, M_{-3})^{M_0} = \bigoplus_{i=1}^{2} \langle \mathrm{Sq}(D_i) \rangle_F \oplus \langle \mathrm{Sq}(1) \rangle_F.$$

Proof. The strategy of the proof is to compute, step by step as d increases, the truncated invariant cohomology groups

$$H^2\left(\frac{M_{\geq 1}}{M_{\geq d+1}}, M_{-3}\right)^{M_0}.$$

Observe that if $d \geq 23$, then $M_{\geq d+1} = 0$, and hence we get the cohomology we are interested in.

The Lie algebra $M_{\geq 1}$ has a decreasing filtration $\{M_{\geq d}\}_{d=1,\ldots,23}$, and the adjoint action of M_0 respects this filtration. We consider one step of this filtration

$$M_d = \frac{M_{\geq d}}{M_{\geq d+1}} \lhd \frac{M_{\geq 1}}{M_{\geq d+1}}$$

and the related Hochschild–Serre spectral sequence

$$E_2^{r,s} = H^r\left(\frac{M_{\geq 1}}{M_{\geq d}}, H^s(M_d, M_{-3})\right) \Rightarrow H^{r+s}\left(\frac{M_{\geq 1}}{M_{\geq d+1}}, M_{-3}\right). \tag{7.1}$$

We fix a certain degree d, and we study, via the above spectral sequence, how the truncated cohomology groups change if we pass from d to $d+1$. It is easily checked that, by homogeneity, we have $H^2\big(\frac{M_{\geq 1}}{M_{\geq 3}}, M_{-3}\big)^{M_0} \subset C^2(M_1, M_{-3})_{\underline{0}} = 0$. Therefore, for the rest of this section, we suppose that $d \geq 3$. Observe also that, since M_d is in the center of $M_{\geq 1}/M_{\geq d+1}$, and M_{-3} is a trivial module, then $H^s\big(M_d, M_{-3}\big) = C^s(M_d, M_{-3})$, and $M_{\geq 1}/M_{\geq d}$ acts trivially on it.

Consider the following exact sequence deduced from the spectral sequence (7.1)

$$E_\infty^{1,0} = H^1\left(\frac{M_{\geq 1}}{M_{\geq d}}, M_{-3}\right) \hookrightarrow H^1\left(\frac{M_{\geq 1}}{M_{\geq d+1}}, M_{-3}\right) \twoheadrightarrow E_\infty^{0,1}.$$

From the above Lemma 6.3 (using that $d \geq 3$), we get that the first two terms are equal to

$$H^1\left(\frac{M_{\geq 1}}{M_{\geq d}}, M_{-3}\right) = H^1\left(\frac{M_{\geq 1}}{M_{\geq d+1}}, M_{-3}\right) = C^1\left(\frac{M_{\geq 1}}{M_{\geq 3} + \langle x_1\widetilde{D_1} + x_2\widetilde{D_2}\rangle_F}, M_{-3}\right)$$

and therefore we deduce that $E_\infty^{0,1} = 0$. Together with the vanishing $E_\infty^{0,2} = 0$ proved in the Lemma 7.2 below, we deduce the following exact diagram:

$$\begin{tikzcd}
C^1(M_d, M_{-3}) \arrow[d, hook, "\alpha"'] & & \\
H^2\left(\frac{M_{\geq 1}}{M_{\geq d}}, M_{-3}\right) \arrow[d] & & \\
E_\infty^{2,0} \arrow[r, hook] & H^2\left(\frac{M_{\geq 1}}{M_{\geq d+1}}, M_{-3}\right) \arrow[r, two heads] & E_\infty^{1,1}
\end{tikzcd}$$

We take the cohomology with respect to M_0 and use the Lemmas 7.3–7.5.

Observe that the cocycle $\mathrm{inv} \circ [-,-] \in (E_\infty^{1,1})^{M_0}$ which appears for $d = 15$ is annihilated by the differential of $\mathrm{inv} \in (E_2^{0,1})^{M_0} = C^1(M_d, M_{-3})^{M_0}$ for $d = 17$. Moreover, the element $\langle x_i D_j \to \delta_{ij}\mathrm{inv}\rangle \in H^1(M_0, C^1(M_{17}, M_{-3}))$ does not belong to the kernel of the differential map

$$\mathrm{d} : H^1(M_0, C^1(M_{17}, M_{-3})) = H^1\big(M_0, E_2^{0,1}\big) \to H^1\big(M_0, E_2^{2,0}\big).$$

Indeed the element $\langle x_i D_j \to \delta_{ij}\mathrm{inv}\rangle$ does not vanish on T_M, and the same is true for its image through the map d, while any coboundary of $H^1(M_0, E_2^{2,0})$ must vanish on T_M by homogeneity. Therefore, the only cocycles that contribute to the required cohomology group are $\{\mathrm{Sq}(1), \mathrm{Sq}(D_1), \mathrm{Sq}(D_2)\}$. □

The remaining part of this section is devoted to prove the lemmas that were used in the proof of the above proposition. In the first lemma, we show the vanishing of the term $E_\infty^{0,2}$ of the above spectral sequence (7.1).

Lemma 7.2. *In the above spectral sequence* (7.1), *we have* $E_3^{0,2} = 0$.

Proof. By definition, $E_3^{0,2}$ is the kernel of the map

$$\mathrm{d} : C^2(M_d, M_{-3}) = E_2^{0,2} \to E_2^{2,1} = H^2\left(\frac{M_{\geq 1}}{M_{\geq d}}, C^1(M_d, M_{-3})\right)$$

that sends a 2-cochain f to the element $\mathrm{d}f$ given by $\mathrm{d}f_{(E,F)}(G) = -f([E, F], G)$ whenever $\deg(E) + \deg(F) = d$, and 0 otherwise.

The subspace of coboundaries $B^2\left(\frac{M_{\geq 1}}{M_{\geq d}}, C^1(M_d, M_{-3})\right)$ is the image of the map

$$\partial : C^1\left(\frac{M_{\geq 1}}{M_{\geq d}}, C^1(M_d, M_{-3})\right) \to C^2\left(\frac{M_{\geq 1}}{M_{\geq d}}, C^1(M_d, M_{-3})\right)$$

that sends the element g to the element ∂g given by $\partial g_{(E,F)}(G) = -g_{[E,F]}(G)$. Hence ∂g vanishes on the pairs (E, F) for which $\deg(E) + \deg(F) = d$.

Therefore, if an element $f \in C^2(M_d, M_{-1})$ is in the kernel of d, that is, $\mathrm{d}f = \partial g$ for some g as before, then it should satisfy $f([E, F], G) = 0$ for every E, F, G such that $\deg(G) = d$ and $\deg(E) + \deg(F) = d$. By letting E vary in M_1 and F in M_{d-1}, the bracket $[E, F]$ varies in all M_d by Lemma 6.3 (note that we are assuming $d \geq 3$). Hence the preceding condition implies that $f = 0$. □

In the new two lemmas, we compute the M_0-invariants and the first M_0-cohomology group of the term $E_2^{0,1} = C^1(M_d, M_{-3})$ in the above spectral sequence (7.1).

Lemma 7.3. *Define* inv : $M_{17} \to M_{-3}$ *by* $(i \neq j)$:

$$\mathrm{inv}\big(x_i^3 x_j^3 \widetilde{D}_i\big) = \sigma(i) D_i \quad \text{and} \quad \mathrm{inv}\big(x_i^4 x_j^2 \widetilde{D}_i\big) = \sigma(i) D_j,$$

where $\sigma(i) = 1$ *or* -1 *if* $i = 1$ *or* $i = 2$, *respectively. Then*

$$C^1(M_d, M_{-3})^{M_0} = \begin{cases} \langle \mathrm{inv} \rangle & \text{if } d = 17, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By homogeneity, we can assume that $d \equiv 2 \mod 5$. We will consider the various cases separately.

$\boxed{\text{d=7}}$ Since $M_7 = A(2)_3 = \langle x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3 \rangle_F$ with weights $(1, 3)$, $(0, 4)$, $(4, 0)$, $(3, 1)$, respectively, a homogeneous cochain $g \in C^1(M_d, M_{-3})^{M_0}$ can take the nonzero values $g(x_i^2 x_j) = a_i D_j$ for $i \neq j$. We obtain the vanishing from the following M_0-invariance condition:

$$0 = (x_j D_i \circ g)(x_i^3) = -g(3x_i^2 x_j) = -3a_i D_j.$$

$\boxed{d=12}$ A homogeneous cochain $g \in C^1(M_{12}, M_{-3})_{\underline{0}}$ can take the nonzero values: $g(x_i^4 x_j D_j) = \eta_j^i D_i$ for $i \neq j$. We get the vanishing of g by mean of the following cocycle condition:

$$0 = \mathrm{d}g_{x_iD_j}\big(x_i^3 x_j^2 D_j\big) = -2g\big(x_i^4 x_j D_j\big) = -2\eta_j^i D_i. \tag{7.2}$$

$\boxed{d=17}$ $M_{17} = \widetilde{W(2)}_5 = \bigoplus_{i\neq j,k}\langle x_i^4 x_j^2 \widetilde{D_k}\rangle_F \bigoplus_{i\neq j,k}\langle x_i^3 x_j^3 \widetilde{D_k}\rangle$ with weights $\epsilon_i - \epsilon_j - \epsilon_k$ and $-\epsilon_k$, respectively. A homogeneous cochain $g \in C^1(M_d, M_{-3})^{M_0}$ can take the nonzero values: $g(x_i^3 x_j^3 \widetilde{D}_i) = \alpha_i D_i$ and $g(x_i^4 x_j^2 \widetilde{D}_i) = \beta_i D_j$ for $i \neq j$. The only M_0-invariance conditions are the following:

$$\begin{cases} (x_iD_j \circ g)\big(x_i^2 x_j^4 \widetilde{D}_i\big) = -g\big(4x_i^3 x_j^3 \widetilde{D}_i - x_i^2 x_j^4 \widetilde{D_j}\big) = (-4\alpha_i + \beta_j)D_i, \\ (x_iD_j \circ g)\big(x_i^2 x_j^4 \widetilde{D_j}\big) = \big[x_iD_j, g\big(x_i^2 x_j^4 \widetilde{D_j}\big)\big] - g\big(4x_i^3 x_j^3 \widetilde{D_j}\big) = (-\beta_j - 4\alpha_j)D_j, \\ (x_iD_j \circ g)\big(x_i^3 x_j^3 \widetilde{D}_i\big) = (-\alpha_i - 3\beta_i + \alpha_j)D_j, \end{cases}$$

from which it follows that $\alpha_i = \beta_i = -\alpha_j = -\beta_j$, that is, g is a multiple of inv.

$\boxed{d=22}$ Since $M_{22} = A(2)_8 = \langle x_1^4 x_2^4\rangle$ with weight $(2,2)$, there are no homogeneous cochains. □

Lemma 7.4. *We have that*

$$H^1(M_0, C^1(M_d, M_{-3})) = \begin{cases} \bigoplus_{i=1}^{2}\langle\overline{\mathrm{Sq}(D_i)}\rangle_F & \text{if } d = 12, \\ \langle x_iD_j \mapsto \delta_{ij}\mathrm{inv}\rangle_F & \text{if } d = 17, \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{\mathrm{Sq}(D_i)}$ *is the restriction of* $\mathrm{Sq}(D_i)$ *to* $M_0 \times M_{12}$ *and* $\mathrm{inv} : M_{17} \to M_3$ *is the cochain defined in Lemma* 7.3.

Proof. By homogeneity, we can assume that $d \equiv 2 \mod 5$ and consider the various cases separately.

$\boxed{d=7}$ Consider a homogeneous cochain $f \in C^1(M_0, C^1(M_7, M_{-3}))_{\underline{0}}$. By applying cocycle conditions of the form $0 = \mathrm{d}f_{(x_kD_k, x_iD_j)}$, it follows that $f_{x_kD_k} \in C^1(M_7, M_{-3})^{M_0} = 0$ (by the preceding lemma). By homogeneity, f can take only the following nonzero values for $i \neq j$: $f_{x_iD_j}(x_j^3) = b_i D_i$ and $f_{x_iD_j}(x_i x_j^2) = c_i D_j$.

By possibly modifying f with a coboundary dg (see Lemma 7.3), we can suppose that $b_i = b_j = 0$. The following cocycle condition

$$\begin{aligned} 0 = \mathrm{d}f_{(x_iD_j, x_jD_i)}\big(x_i^2 x_j\big) &= \big[x_iD_j, f_{x_jD_i}\big(x_j x_i^2\big)\big] - f_{x_jD_i}\big(x_i^3\big) + f_{x_iD_j}\big(2x_i x_j^2\big) \\ &= (-c_j - b_j + 2c_i)D_j = (-c_j + 2c_i)D_j, \end{aligned}$$

gives that $c_j = 2c_i = 4c_j$, and hence that $c_i = c_j = 0$, that is, $f = 0$.

$\boxed{d=12}$ First of all, the cocycles $\overline{\mathrm{Sq}(D_i)}$ for $i = 1, 2$ belong to $H^1(M_0, C^1(M_{12}, M_{-3}))_{\underline{0}}$ since they are restriction of global cocycles, and, moreover, they are independent as it follows from the formulas (3.5) and (3.6). It remains to show that the above cohomology group has dimension less than or equal to 2. Consider a homogeneous cocycle $f \in C^1(M_0, C^1(M_{12}, M_{-3}))_{\underline{0}}$. First of all, we observe that f must satisfy $f_{x_iD_i} = 0$. Indeed, by the formula (2.3), we have $0 = \mathrm{d}f_{|x_iD_i} = x_iD_i \circ f - \mathrm{d}(f_{|x_iD_i})$ from which, since the first term vanishes for homogeneity reasons, it follows that $f_{|x_iD_i} \in C^1(M_{p-1}, M_{-1})^{M_0}$ which is zero by the previous Lemma 7.3. Therefore, f can take only the following nonzero values (for $i \neq j$):

$$\begin{cases} f_{x_iD_j}\big(x_i^3x_j^2D_j\big) = \alpha_{ij}D_i, \\ f_{x_iD_j}\big(x_i^4x_jD_i\big) = \beta_{ij}D_i, \\ f_{x_iD_j}\big(x_i^4x_jD_j\big) = \gamma_{ij}D_j. \end{cases}$$

By possibly modifying f with a coboundary (see formula (7.2)), we can assume that $\underline{\alpha_{ij} = 0}$. The coefficients $\underline{\beta_{ij}}$ are determined by the coefficients γ_{ij} in virtue of the $\overline{\text{following cocycle condition}}$:

$$\begin{aligned} 0 = \mathrm{d}f_{(x_iD_j, x_jD_i)}\big(x_i^4x_jD_j\big) &= f_{x_iD_j}\big(-x_i^3x_j^2D_j\big) + f_{x_iD_j}\big(-x_i^4x_jD_i\big) + \\ &-\big[x_jD_i, f_{x_iD_j}\big(x_i^4x_jD_j\big)\big] = [-\alpha_{ij} - \beta_{ij} + \gamma_{ij}]D_i = [-\beta_{ij} + \gamma_{ij}]D_i. \end{aligned}$$

Therefore, the cohomology group depends on the two parameters γ_{12} and γ_{21}, and hence has dimension less than or equal to 2.

$\boxed{\text{d=17}}$ Consider a homogeneous cochain $f \in C^1(M_0, C^1(M_{17}, M_{-3}))_{\underline{0}}$. By imposing cocycle conditions of the form $0 = \mathrm{d}f_{(x_kD_k, x_iD_j)}$, one obtains that $f_{x_kD_k} \in C^1(M_{17}, M_{-3})^{M_0} = \langle \mathrm{inv} \rangle$ (by the preceding lemma). Put $f_{x_kD_k} = \mu_k \cdot \mathrm{inv}$. By homogeneity, the other nonzero values of f are (for $i \neq j$):

$$f_{x_iD_j}\big(x_i^2x_j^4\widetilde{D_i}\big) = \nu_iD_i, \qquad f_{x_iD_j}\big(x_i^2x_j^4\widetilde{D_j}\big) = \tau_iD_j, \qquad f_{x_iD_j}\big(x_i^3x_j^3\widetilde{D_i}\big) = \sigma_iD_j.$$

By possibly modifying f with a coboundary $\mathrm{d}g$, we can assume that $f_{x_1D_2} = 0$ (see Lemma 7.3). Considering all the cocycle conditions of the form $0 = \mathrm{d}f_{(x_1D_2, x_2D_1)} = (x_1D_2 \circ f_{x_2D_1}) - f_{x_1D_1} + f_{x_2D_2}$, one gets

$$\begin{cases} 0 = df_{(x_1D_2, x_2D_1)}\big(x_1^3x_2^3\widetilde{D_1}\big) = -3\tau_2 + \sigma_2 - (\mu_1 - \mu_2), \\ 0 = df_{(x_1D_2, x_2D_1)}\big(x_1^3x_2^3\widetilde{D_2}\big) = \sigma_2 + 3\nu_2 - (\mu_1 - \mu_2), \\ 0 = df_{(x_1D_2, x_2D_1)}\big(x_1^2x_2^4\widetilde{D_2}\big) = \sigma_2 + (\mu_1 - \mu_2), \\ 0 = df_{(x_1D_2, x_2D_1)}\big(x_1^4x_2^2\widetilde{D_1}\big) = \tau_2 - \nu_2 + (\mu_1 - \mu_2). \end{cases}$$

Since the 4×4 matrix associated to the preceding system of 4 equations in the 4 variables $\nu_2, \tau_2, \sigma_2, \mu_1 - \mu_2$ is invertible (it has determinant equal to 2), we conclude that $f_{x_2D_1} = 0$ and $f_{x_1D_1} = f_{x_2D_2}$ is a multiple of inv.

$\boxed{\text{d=22}}$ There are no homogeneous cochains. □

In the last lemma, we compute the M_0-invariants of the term $E_\infty^{1,1}$ of the above spectral sequence (7.1). In view of Lemma 6.3 and the hypothesis $d \geq 3$, we have that

$$E_2^{1,1} = C^1\left(\frac{M_1 \oplus M_2}{\langle x_1\widetilde{D_1} + x_2\widetilde{D_2}\rangle} \times M_d, M_{-3}\right).$$

Lemma 7.5. *In the above spectral sequence* (7.1), *we have that*

$$(E_\infty^{1,1})^{M_0} = \begin{cases} \langle\overline{\mathrm{Sq}(1)}\rangle_F & \text{if } d = 6, \\ \langle \mathrm{inv} \circ [-,-]\rangle_F & \text{if } d = 15, \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{\mathrm{Sq}(1)}$ *is the restriction of* $\mathrm{Sq}(1)$ *to* $M_1 \times M_6$, *and* $\mathrm{inv} \circ [-,-]$ *is defined by*

$$(\mathrm{inv} \circ [-,-])(E, F) = \mathrm{inv}([E, F]),$$

where $\mathrm{inv} : M_{17} \to M_{-3}$ *is the* M_0*-invariant map defined in Lemma* 7.3.

Proof. The term $(E_\infty^{1,1})^{M_0} = (E_3^{1,1})^{M_0}$ is the kernel of the map

$$\mathrm{d} : \left(E_2^{1,1}\right)^{M_0} \longrightarrow \left(E_2^{3,0}\right)^{M_0} \hookrightarrow E_2^{3,0} = H^3\left(\frac{M_{\geq 1}}{M_{\geq d}}, M_{-3}\right).$$

In order to avoid confusion, during this proof, we denote with $\partial f \in B^3\left(\frac{M_{\geq 1}}{M_{\geq d}}, M_{-3}\right)$ (instead of the usual $\mathrm{d}f$) the coboundary of an element $f \in C^2\left(\frac{M_{\geq 1}}{M_{\geq d}}, M_{-3}\right)$.

Note that $M_1 = \langle \widetilde{x_1}, \widetilde{x_2}\rangle_F$ with weights, respectively, $(4, 3)$ and $(3, 4)$, while $M_2/(x_1\widetilde{D_1} + x_2\widetilde{D_2}) = \langle x_1\widetilde{D_1} = -x_2\widetilde{D_2}, x_1\widetilde{D_2}, x_2\widetilde{D_1}\rangle_F$ with weights, respectively, $(2, 2)$, $(3, 1)$, and $(1, 3)$. Note also that after the identification $x_1\widetilde{D_1} = -x_2\widetilde{D_2}$, the action of $M_0 = W(2)_0$ on $M_2/(x_1\widetilde{D_1} + x_2\widetilde{D_2})$ is given by (for $i \neq j$) $[x_iD_j, x_i\widetilde{D_j}] = 0$, $[x_iD_j, x_1\widetilde{D_1}] = \sigma(j)x_i\widetilde{D_j}$ and $[x_iD_j, x_j\widetilde{D_i}] = -2\sigma(j)x_1\widetilde{D_1}$, where $\sigma(j) = 1, -1$ if, respectively, $j = 1, 2$.

By homogeneity, an M_0-invariant cochain of $E_2^{1,1}$ can assume nonzero values only on $M_1 \times M_d$ if $d \equiv 1 \mod 5$ or on $M_2 \times M_d$ if $d \equiv 0 \mod 5$. We will consider the various cases separately.

$\boxed{\mathrm{d}=5}$ $M_5 = \widetilde{W(2)}_1 = \bigoplus_{i,k}\langle x_i^2\widetilde{D_k}\rangle_F \bigoplus_k \langle x_1x_2\widetilde{D_k}\rangle_F$ with weights $(2, 2) + 2\epsilon_i - \epsilon_k$ and $(3, 3) - \epsilon_k$, respectively. A homogeneous cochain $g \in (E_2^{1,1})^{M_0}$ can take only the following nonzero values (for $i \neq j$):

$$\begin{cases} g\left(x_1\widetilde{D_1}, x_ix_j\widetilde{D_i}\right) = h_iD_i, & \qquad g\left(x_1\widetilde{D_1}, x_j^2\widetilde{D_j}\right) = k_iD_i, \\ g\left(x_i\widetilde{D_j}, x_j^2\widetilde{D_i}\right) = l_iD_i, \quad g\left(x_i\widetilde{D_j}, x_ix_j\widetilde{D_i}\right) = m_iD_j, & \qquad g\left(x_i\widetilde{D_j}, x_j^2\widetilde{D_j}\right) = n_iD_j. \end{cases}$$

Consider the following M_0-invariance conditions:

$$\begin{cases} 0 = (x_iD_j \circ g)\big(x_j\widetilde{D}_i, x_j^2\widetilde{D_j}\big) = 2(\sigma(j)k_i - m_j)D_i, \\ 0 = (x_iD_j \circ g)\big(x_j\widetilde{D}_i, x_ix_j\widetilde{D}_i\big) = (2\sigma(j)h_i - n_j + m_j)D_i, \\ 0 = (x_iD_j \circ g)\big(x_i\widetilde{D_j}, x_j^2\widetilde{D}_i\big) = (-l_i - 2m_i + n_i)D_j, \\ 0 = (x_iD_j \circ g)\big(x_j\widetilde{D}_i, x_ix_j\widetilde{D_j}\big) = (-m_j + 2\sigma(j)h_j - l_j)D_j, \\ 0 = (x_iD_j \circ g)\big(x_j\widetilde{D}_i, x_i^2\widetilde{D}_i\big) = (-n_j + 2\sigma(j)k_j + l_j)D_j. \end{cases}$$

From the first 3 equations, one obtains that $(m_j, n_j, l_j) = \sigma(j)(k_i, 2h_i + k_i, 2h_i - k_i)$, and substituting in the last two equations, one finds that $h_i = h_j := h$ and $k_i = k_j := k$.

Suppose now that g is in the kernel of the map d, that is, $\mathrm{d}g = -\partial f$ for some $f \in C^2(M_{\geq 1}/M_{\geq 5}, M_{-3})$. Applying $0 = \mathrm{d}g + \partial f$ to the triples $(x_i, x_j\widetilde{D_j}, x_ix_j)$ and $(x_j, x_i\widetilde{D_j}, x_ix_j)$ for $i \neq j$, we get the two conditions

$$\begin{cases} f\big(x_ix_j\widetilde{D_j}, x_ix_j\big) = 2\sigma(i)g\big(x_j\widetilde{D_j}, x_i^2\widetilde{D}_i\big) = -2kD_j, \\ f\big(x_ix_j\widetilde{D_j}, x_ix_j\big) = 2\sigma(j)g\big(x_i\widetilde{D_j}, x_j^2\widetilde{D_j}\big) = -2(2h + k)D_j, \end{cases}$$

from which it follows that $h = 0$. Considering now the triples $(x_i, x_r\widetilde{D_r}, x_ix_j)$ and $(x_i, x_r\widetilde{D_r}, x_j^2)$ (for $i \neq j$ and some $r = 1, 2$), we get

$$\begin{cases} f\big(x_ix_r\widetilde{D_r}, x_ix_j\big) = 2\sigma(i)g\big(x_r\widetilde{D_r}, x_i^2\widetilde{D}_i\big) = 2\sigma(i)\sigma(r)kD_j, \\ f\big(x_ix_r\widetilde{D_r}, x_j^2\big) = -\sigma(i)g\big(x_r\widetilde{D_r}, x_ix_j\widetilde{D}_i - 2x_j^2\widetilde{D_j}\big) = 2\sigma(i)\sigma(r)kD_i. \end{cases}$$

Finally, considering the triple $(x_i, x_i^2D_i, x_ix_jD_i)$, and using the two preceding relations, we get

$$\begin{aligned} 0 &= (\mathrm{d}g + \partial f)(x_i, x_i^2D_i, x_ix_jD_i) = f([x_i^2D_i, x_i], x_ix_jD_i) - f([x_ix_jD_i, x_i], x_i^2D_i) \\ &= 3f(x_jx_iD_i, x_i^2) - f(x_i^2D_i, x_ix_j) = -6kD_j - 2kD_j = 2kD_j, \end{aligned}$$

from which it follows that $k = 0$, that is $g = 0$.

$\boxed{\text{d=6}}$ First of all, the cocycle $\overline{\mathrm{Sq}(1)}$ is an element of $(E_\infty^{1,1})^{M_0}$ since it is the restriction of a global cocycle and is nonzero in virtue of formulas (3.3) and (3.4). Therefore, it remains to show that the dimension of $(E_\infty^{1,1})^{M_0}$ is at most 1. Consider a cochain $g \in (E_2^{1,1})^{M_0}$, which is in particular homogeneous. Since $M_6 = W(2)_2 = \bigoplus_{i,k}\langle x_i^2D_k\rangle_F \bigoplus_k\langle x_1x_2D_k\rangle_F$ with weights $2\epsilon_i - \epsilon_k$ and $(1, 1) - \epsilon_k$, respectively, g can take only the following nonzero values (for $i \neq j$):

$$\begin{cases} g\big(x_i, x_ix_j^2D_i\big) = c_iD_i, & g\big(x_i, x_i^2x_jD_i\big) = e_iD_j, \\ g\big(x_i, x_j^3D_j\big) = d_iD_i, & g\big(x_i, x_ix_j^2D_j\big) = f_iD_j. \end{cases}$$

Consider the following M_0-invariance conditions:

$$\begin{cases} 0 = (x_iD_j \circ g)\big(x_i, x_j^3D_i\big) = -g\big(x_i, 3x_ix_j^2D_i - x_j^3D_j\big) = (-3c_i + d_i)D_i, \\ 0 = (x_iD_j \circ g)\big(x_i, x_ix_j^2D_i\big) = (-c_i - 2e_i + f_i)D_j, \\ 0 = (x_iD_j \circ g)\big(x_i, x_j^3D_j\big) = [x_iD_j, d_iD_i] - g(x_i, 3x_ix_j^2D_j) = (-d_i - 3f_i)D_i, \\ 0 = (x_iD_j \circ g)\big(x_j, x_ix_j^2D_i\big) = (-c_i - 2f_j + e_j)D_i. \end{cases}$$

From the first 3 equations, one gets $(f_i, c_i, d_i) = (e_i, -e_i, 2e_i)$, and substituting into the last equation, one finds $e_i = e_j := e$. Therefore, $(E_\infty^{1,1})^{M_0}$ depends on one parameter and hence has dimension at most 1.

$\boxed{\text{d=10}}$ $M_{10} = A(2)_4 = \bigoplus_i \langle x_i^4\rangle_F \bigoplus_{i\neq j}\langle x_i^3x_j\rangle_F \bigoplus_{i\neq j}\langle x_i^2x_j^2\rangle_F$ with weights $2\epsilon_i - 2\epsilon_j$, $\epsilon_i - \epsilon_j$ and $(0,0)$, respectively. Consider a cochain $g \in (E_2^{1,1})^{M_0}$. Since M_0 acts transitively on M_2, to prove the vanishing of g it is enough to prove that $g(x_1\widetilde{D_2}, -) = 0$. By homogeneity the only possible nonzero such values are $g(x_1\widetilde{D_2}, x_1^3x_2)$ and $g(x_1\widetilde{D_2}, x_1^4)$, and the vanishing follows from the M_0-invariance conditions:

$$\begin{cases} 0 = (x_1D_2 \circ g)\big(x_1\widetilde{D_2}, x_1^2x_2^2\big) = -g\big(x_1\widetilde{D_2}, 2x_1^3x_2\big), \\ 0 = \big(x_1\widetilde{D_2} \circ g\big)\big(x_1\widetilde{D_2}, x_1^3x_2\big) = \big[x_1D_2, g(x_1\widetilde{D_2}, x_1^3x_2)\big] - g\big(x_1\widetilde{D_2}, x_1^4\big). \end{cases}$$

$\boxed{\text{d=15}}$ $M_{15} = W(2)_5 = \bigoplus_{i\neq j,k}\langle x_i^4x_j^2D_k\rangle_F \bigoplus_{i\neq j,k}\langle x_i^3x_j^3D_k\rangle_F$ with weights $-\epsilon_i + 2\epsilon_j - \epsilon_k$ and $3\epsilon_i + 3\epsilon_j - \epsilon_k$, respectively. A homogeneous cochain $g \in (E_2^{1,1})^{M_0}$ can take only the following nonzero values (for $i \neq j$):

$$\begin{cases} g\big(x_1\widetilde{D_1}, x_i^3x_j^3D_i\big) = p_iD_i, \qquad g\big(x_1\widetilde{D_1}, x_i^2x_j^4D_j\big) = q_iD_i, \\ g\big(x_i\widetilde{D_j}, x_i^2x_j^4D_i\big) = r_iD_i, \qquad g\big(x_i\widetilde{D_j}, x_i^3x_j^3D_i\big) = s_iD_j, \qquad g\big(x_i\widetilde{D_j}, x_i^2x_j^4D_j\big) = t_iD_j. \end{cases}$$

Consider the following M_0-invariance conditions:

$$\begin{cases} 0 = (x_iD_j \circ g)\big(x_j\widetilde{D}_i, x_i^2x_j^4D_j\big) = 2(\sigma(j)q_i - 2s_j)D_i, \\ 0 = (x_iD_j \circ g)\big(x_j\widetilde{D}_i, x_i^3x_j^3D_i\big) = (2\sigma(j)p_i - 3t_j + s_j)D_i, \\ 0 = (x_iD_j \circ g)\big(x_i\widetilde{D_j}, x_i^2x_j^4D_i\big) = (-r_i + s_i + t_i)D_j, \\ 0 = (x_iD_j \circ g)\big(x_j\widetilde{D}_i, x_i^3x_j^3D_j\big) = (-s_j + 2\sigma(j)p_j - 3r_j)D_j, \\ 0 = (x_iD_j \circ g)\big(x_j\widetilde{D}_i, x_i^4x_j^2D_i\big) = (-t_j + 2\sigma(j)q_j + r_j)D_j. \end{cases}$$

From the first 3 equations, one gets that $(s_j, t_j, r_j) = \sigma(j)(-2q_i, -p_i + q_i, -p_i - q_i)$, and substituting in the last two equations, one obtains $p_i = p_j := p$ and $q_i = q_j := q$.

Suppose now that g is in the kernel of d, that is, $\mathrm{d}g = -\partial f$ for some $f \in C^2(M_{\geq 1}/M_{\geq 15}, M_{-3})$. Applying $0 = \mathrm{d}g + \partial f$ to the triple $(x_i, x_j\widetilde{D_j}, x_i^4x_j\widetilde{D_i})$ for $i \neq j$, we get

$$0 = -f\big([x_i, x_j\widetilde{D_i}], x_i^4x_j\widetilde{D_i}\big) + g\big([x_i, x_i^4x_j\widetilde{D_i}], x_j\widetilde{D_i}\big) = f\big(x_ix_jD_i, x_i^4x_j\widetilde{D_i}\big).$$

Considering the triple $(x_j, x_i\widetilde{D}_i, x_i^4 x_j\widetilde{D}_i)$, and using the preceding vanishing, we obtain

$$0 = -f\big(x_i x_j\widetilde{D}_i, x_i^4 x_j\widetilde{D}_i\big) - g\big(x_i\widetilde{D}_i, x_i^4 x_j^2\widetilde{D}_i\big) = -\sigma(i)qD_j,$$

that is, $q = 0$. For $p = 1$, one obtains that $g = \mathrm{inv} \circ [-, -]$, which is clearly in the kernel of d since it is the restriction of a cocycle on $M_{\geq 1} \times M_{\geq 1}$.

$\boxed{\text{d=16}}$ $M_{16} = A(2)_6 = \bigoplus_{i\neq j}\langle x_i^2 x_j^4\rangle_F \oplus \langle x_1^3 x_2^3\rangle_F$ with weights $2\epsilon_j$ and $(1, 1)$, respectively. A homogeneous cochain $g \in (E_2^{1,1})^{M_0}$ can take the nonzero values (for $i \neq j$): $g(x_i, x_i^2 x_j^4) = a_i D_i$ and $g(x_i, x_1^3 x_1^3) = b_j D_j$ for $i \neq j$. We get that $a_i = a_j = b_i = b_j := a$ by considering the following M_0-invariant conditions:

$$\begin{cases} 0 = (x_i D_j \circ g)\big(x_i, x_i^2 x_j^4\big) = [x_i D_j, a_i D_i] - g\big(x_i, 4x_i^3 x_j^3\big) = (-a_i + b_i)D_j, \\ 0 = (x_i D_j \circ g)\big(x_j, x_i^2 x_j^4\big) = -g\big(x_i, x_i^2 x_j^4\big) - g\big(x_j, 4x_i^3 x_j^3\big) = (-a_i + b_j)D_i. \end{cases}$$

Now suppose that g is in the kernel of the map d, that is, $\mathrm{d}g = -\partial f$ for some $f \in C^2(M_{\geq 1}/M_{\geq 16}, M_{-3})$. Applying the relation $0 = \mathrm{d}g + \partial f$ to the two triples $(x_i, x_j, x_i^4 x_j^2 D_i)$ and $(x_i, x_j, x_i^3 x_j^3 D_i)$ for $i \neq j$, we get

$$\begin{cases} f\big(x_1\widetilde{D}_1 + x_2\widetilde{D}_2, x_i^4 x_j^2 D_i\big) = g\big([x_i, x_i^4 x_j^2 D_i], x_j\big) - g\big([x_j, x_i^4 x_j^2 D_i], x_i\big) = a, \\ f\big(x_1\widetilde{D}_1 + x_2\widetilde{D}_2, x_i^3 x_j^3 D_i\big) = g\big([x_i, x_i^3 x_j^3 D_i], x_j\big) - g\big([x_j, x_i^3 x_j^3 D_i], x_i\big) = a. \end{cases} \tag{$*$}$$

Considering the triples $(x_1, x_1\widetilde{D}_1 + x_2\widetilde{D}_2, x_1^3 x_2^2\widetilde{D}_1)$ and $(x_1, x_1\widetilde{D}_1 + x_2\widetilde{D}_2, x_1^2 x_2^3\widetilde{D}_2)$, and using $(*)$, we get

$$\begin{cases} f\big(x_1^2 D_1 + x_1 x_2 D_2, x_1^3 x_2^2\widetilde{D}_1\big) = -f\big(x_1\widetilde{D}_1 + x_2\widetilde{D}_2, x_1^4 x_2^2 D_1\big) - g\big(x_1, x_1^3 x_2^3\big) = -2a, \\ f\big(x_1^2 D_1 + x_1 x_2 D_2, x_1^2 x_2^3\widetilde{D}_2\big) = -f\big(x_1\widetilde{D}_1 + x_2\widetilde{D}_2, x_1^3 x_2^3 D_2\big) + g\big(x_1, x_1^3 x_2^3\big) = 0. \end{cases} \tag{$**$}$$

Finally, using the relations $(**)$, we get the vanishing of g by mean of the following:

$$\begin{aligned} 0 = (\mathrm{d}g + \partial f)\big(x_1, x_1^2 D_1 + x_1 x_2 D_2, x_1^2 x_2^3\big) &= -f\big(\big[x_1, x_1^2 D_1 + x_1 x_2 D_2\big], x_1^2 x_2^3\big) \\ &\quad + f\big(\big[x_1, x_1^2 x_2^3\big], x_1^2 D_1 + x_1 x_2 D_2\big) - g\big(\big[x_1^2 D_1 + x_1 x_2 D_2, x_1^2 x_2^3\big], x_1\big) \\ &= f\big(-x_1^3 x_2^2\widetilde{D}_1 + 2x_1^2 x_2^3\widetilde{D}_2, x_1^2 D_1 + x_1 x_2 D_2\big) - g\big(-x_1^3 x_2^3, x_1\big) = -2a - a = -3a. \end{aligned}$$

$\boxed{\text{d=20}}$ $M_{20} = \widetilde{W(2)}_6 = \bigoplus_{i\neq j,k}\langle x_i^4 x_j^3\widetilde{D}_k\rangle_F$ with weights $\epsilon_i - \epsilon_k$. A homogeneous cochain $g \in (E_2^{1,1})^{M_0}$ can take only the nonzero values $g(x_i\widetilde{D}_j, x_i^4 x_j^3\widetilde{D}_j) = \lambda_i D_i$. The vanishing follows from the M_0-invariance conditions

$$0 = (x_i D_j \circ g)\big(x_i\widetilde{D}_j, x_i^3 x_j^4\widetilde{D}_j\big) = -4g\big(x_i\widetilde{D}_j, x_i^4 x_j^3\widetilde{D}_j\big) = \lambda_i D_i.$$

$\boxed{\text{d=21}}$ Since $M_{21} = W(2)_7 = \langle x_1^4 x_2^4 D_1, x_1^4 x_2^4 D_2\rangle_F$ with weights $(3, 4)$ and $(4, 3)$, respectively, there are no homogeneous cochains. □

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